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Local smooth isometric embeddings of low dimensional Riemannian manifolds into Euclidean spaces

by

Gen Nakamura
Yoshiaki Maeda

Gen Nakamura

Department of Mathematics
Faculty of Science
Josai University

Keyakidai 1-1
Sakado, 350-02 Japan

Yoshiaki Maeda

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku
Yokohama, 223 Japan

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

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Department of Mathematics, Faculty of Science,

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and

Department of Mathematics, Faculty of Science and Technology,

Keio University

Introduction

In the study of isometric embedding problem of a Riemannian n-manifold into Euclidean space, the following has been studied as an old and debated problem ([Y]):

Problem. *Given a Riemannian n-manifold* (M^n, ds^2) *, is there exist a local isometric embedding* (M^n, ds^2) *into Euclidean space* $\mathbf{R}^N$*, where* $N = n(n+1)/2$ *?*

In particular, Jacobowicz [J] showed that there always exists a local isometric embedding (M^n, ds^2) into $\mathbf{R}^N$, if (M^n, ds^2) is real analytic. In contrast to the above, it would be an interesting problem to consider the above problem in the smooth category. Namely, the problem which will be discussed here is to obtain an existence of the smooth local isometric embedding for n=2,3. As a typical result, the following theorem is well-known (cf. Jacobowicz [J]) :

Theorem A. *Let* (M^2,ds^2) *be a* C^∞ *Riemannian 2-manifold and* $p_o \in M$ *a fixed point such that the Gaussian curvature does not vanish at* p_0. *Then, there exists a local* C^∞ *isometric embedding of a neighborhood* U_o*of* p_o *into* $\mathbf{R}^3$,

In this paper, we shall generalize the above theorem for n=3 as follows :

Theorem B *Let* (M^3,ds^2) *be a* C^∞ *Riemannian 3-manifold and* $p_0 \in M$ *a fixed point such that the curvature tensor* $R(p_0)$ *does not vanish. Then, there exists a local* C^∞ *isometric embedding of a neighborhood* U_o*of* p_o *into* $\mathbf{R}^6$.

We remark that Bryant-Griffiths-Yang [BGY] proved the special case of Theorem B under the assumption :

(*) $Sig\ R(x_o) \neq (0,0),\ (0,1)$

Here the signature $SigR(x_0)$ of $R(x_o)$ is defined by considering $R(x_0)$ as an symmetric linear operator acting on the space of 2-froms (cf. § 2).

They proved the Nash-Moser type theorem for a non-linear PDE whose linearized PDE is either symmetric hyperbolic or strongly hyperbolic, and as an application, they solved the local isometric embedding problem under the assumption (*). On the other hand, without the assumption (*) the linearized equation of the isometric embedding equation is merely real principal type (cf. [BGY], and § 1). Therefore, the key to the proof of Theorem 1 is to establish the local solvability of the non-linear PDE whose linearized operator is a system of real principal type.

In our scheme, we can also recover the result of Lin [L] :

Theorem C. *Let* (M^2,ds^2) *be an Riemannian 2-manifold and fix a point* $p_o \in M$.

Assume that the Gaussian curvature K of M^2 *satisfies*

$$K(p_o) = 0 \quad \text{and} \quad dK(p_o) \neq 0 \ .$$

Then, there exists a local C^∞ *isometric embedding of a neighborhood* U_o *of* p_o *into* $\mathbf{R}^3$.

As an application of the following local solvability of the real principal non-linear PDE, we shall give a unified proof of Theorems A-C.

Theorem D. *Let* $\Phi(u)$ *be a* $\mathbf{R}^N$ *valued non-linear partial differential operator of order m applied to a* C^∞ *map* $u : \mathbf{R}^n \to \mathbf{R}^N$ *with appropriate smoothness and boundedness conditions. Assume that* $\Phi(u)$ *is Frechet differentiable in any Sobolev space of order greater than* $m + \frac{n}{2}$ *and denotes its derivative by* $\Phi'(u)$. *Let* $x_o \in \mathbf{R}^n$ *and* $u_o \in C^\infty(U_o, \mathbf{R}^N)$. *Assume that* $\Phi'(u_o)$ *is an* $N \times N-$ *system of real principal type at* x_o *(see § 1 for the precise definition). Then, there exists a neighborhood* $U_1 \subset U$ *of* x_o, $s_o \in \mathbf{Z}_+$, *and* $\eta > 0$ *such that the following property holds: For any* $g \in C^\infty(U_1)$ *satisfying*

$$(0.1) \qquad \|g - \Phi(u_o)\|_{H^{s_o}(U_1)} < \eta \ ,$$

there exists $u \in C^\infty(\mathbf{R}^n, \mathbf{R}^N)$ *such that*

$$(0.2) \qquad \Phi(u) = g \quad in\ U_1.$$

As in § 3, our proof of Theorem D is based on the Nash-Moser type implicit function theorem. Since the linearized equation

$$(0.3) \qquad \Phi'(u)v = h$$

in an open neighborhood U_1 of x_o does not have the uniqueness property for the solutions, we can not obtain the higher regularity estimate of the solutions in a certain space of functions by differentiating (0.3). To avoid this difficulty, we construct an exact right inverse $Q(u)$ of $\Phi'(u)$ with the so called tame estimate, and modifying the Nash-Moser's iteration scheme, we get a smooth solution u of

(0.2).

Although the construction of $Q(u)$ can be done by following the arguments of [H] and [DH], it is troublesome to prove $Q(u)$ is a right inverse of $\Phi'(u)$ in a certain common neighborhood U_1 of x_0 for any u and get the tame estimate for $Q(u)$. For these proofs, we have made a device on the arguments of [H] and [DH] so that the dependency of $Q(u)$ on $\Phi'(u)$ can be easily seen. If we push our argument on the local solvability of non-linear P.D.E. a little further, we can get various kind of local solvability theorems on non-linear P.D.Es whose linearized equations are locally solvable with the aid of Fourier integral operators, and these theorems are expected to have applications to some geometrical problems. We shall write them in the forthcomming paper.

§ 1. Linearized PDE for the isometric embedding equation

Let $U(u^1, u^2, \ldots, u^n)$ be a coordinate neighborhood around the given point $x_0 \in M$ which we take as the origin. Then, to obtain an isometric embedding of (M, g) into $\mathbf{R}^N$ is to show the existence of the solution of the following non-linear PDE in an open neighborhood $U_1 \subset U$ of x_0:

$$(1.1) \quad \sum_{A=1}^{N} \frac{\partial x^A}{\partial u^i} \cdot \frac{\partial x^A}{\partial u^j} = g_{ij}(u^1, \ldots, u^n) \quad i,j = 1, \ldots, n \quad ,$$

where $(g_{ij}(u))$ is the component of the Riemannian metric g in U.

We shall consider the linearized PDE corresponding to (1.1), that is, let $x^A(u)$ be a local C^∞ embedding of U into $\mathbf{R}^N$ and consider the following PDE for the unknown functions $(y^A(u))$:

$$(1.2) \quad \sum_{A=1}^{N} \frac{\partial x^A}{\partial u^i} \cdot \frac{\partial y^A}{\partial u^j} + \frac{\partial y^A}{\partial u^i} \cdot \frac{\partial x^A}{\partial u^j} = k_{ij}(u) \quad i,j = 1, \ldots, n \quad ,$$

where $(k_{ij}(u))$ is a smooth symmetric contravariant 2-tensor on U.

Choosing a unit normal frame field $\{N_\lambda(u)\}_{\lambda = n+1, \ldots, N}$ on U, we set

$$y^A(u) = \sum_{i=1}^{n} y_i \frac{\partial x^A}{\partial u^i} + \sum_{\lambda=n+1}^{N} y_\lambda \cdot N_\lambda^A \quad .$$

Denote the covariant derivatives and the second fundamental tensor for the isometric embedding (1.1) by ∇_i and $H_{ij\lambda}(u)$, respectively. Here, ∇_i is defined for a contravariant 1-tensor $\xi_j(u)$ on U as follows:

(1.3) $\quad \nabla_i \xi_j(u) = \partial_{u^i} \xi_j(u) - \sum_{k=1}^{n} \{^k_{ij}\}(u) \xi_k(u)$

where $\{^k_{ij}\}$ is the Christoffel tensor defined by

$$\{^k_{ij}\} = \frac{1}{2}\{\partial_{u^i} g_{jk} + \partial_{u^j} g_{ik} - \partial_{u^k} g_{ij}\}.$$

Also, $H_{ij\lambda}(u)$ is defined by the following Gauss- Codazzi equations associated with (1.1):

(1.4) $\quad \nabla_j \dfrac{\partial x^A}{\partial u^i} = \sum_\lambda H^{\lambda}_{ij} N^A_\lambda$

$$\nabla_j N^A_\lambda(u) = - \sum_\lambda H^{j}_{i\lambda} \frac{\partial x^A}{\partial u^j}$$

where the lifting and lowering the index are done by the given Riemannian metric.

Rewriting (1.3) by using (1.4), we easily get

Lemma 1.1

(1.5) $\quad \nabla_i y_j + \nabla_j y_i = 2\sum_{\lambda=n+1}^{N} y_\lambda H^{\lambda}_{ij}(u) + k_{ij}(u) \qquad i,j = 1,\ldots,n$,

Definition 1.2. (i) The second fundamental tensor $(H^{\lambda}_{ij}(x_o))_{\lambda=1,\ldots,n}$ is called *non-degenerate at* x_o if $\{(H^{\lambda}_{ij})(x_o)\}$ are linearly independent in the vector space of all contravariant symmetric 2-tensors at x_o.

(ii) An isometric embedding is called *non-degenerate* if the corresponding second fundamental form is non-degenerate at each point of U.

Although the second fundamental tensor $H_{ij\lambda}(u)$ depends on choosing the normal frame $N^A_\lambda(u)$, the above definition does not depends on their choice.

Now, we review the terminology of PDE. Let T^*M denote the cotangent bundle of M. For a positive integer N, let P be an N×N system of classical pseudo-differential operator on M with principal symbol $p(x,\xi)$.

Definition 1.3. (i) P is called *a system of real principal type at* $(x_0,\xi_0)\in T^*M-\{0\}$ if there exists a conic neighborhood Γ of (x_0,ξ_0), an N×N classical symbol $p(x,\xi)$, and a real valued classical scalar symbol $q(x,\xi)$ such that

$$(1.6)\qquad p(x,\xi)p(x,\xi) = q(x,\xi)Id_N \quad in \quad \Gamma \ ,$$

and the principal part $q(x,\xi)$ of $q(x,\xi)$ satisfies the condition that dq and $\theta = \sum_{i=1}\xi^i dx^i$ are linearly independent on $\Gamma\cap\{(x,\xi)\ ;\ q(x,\xi)=0\}$,

where Id_N is the N×N identity matrix.

(ii) P is called *a system of real principal type at* $x_0\in M$ *(resp. over an open set U of M)* if P is a system of principal type at each point of $\pi^{-1}(p_0)-\{0\}$ (resp. $\pi^{-1}(U-\{0\})$, where $\pi:T^*M\to M$ is the projection.

Remark. (i) The condition for q in (1.6) obviously holds if $grad_\xi q\neq 0$ on $\Gamma\cap\{(x;\xi);q(x,\xi)=0\}$. We refer this by saying P is a system of real principal type at $(x_0,\xi_0)\in T^*M-\{0\}$ in the *strong sense*. By modifying Definition 1.3, (ii) in an obvious manner, we obtain the corresponding definitions at a point $x_0\in M$ and on an open set U of M.

(ii) As we see in Lemma 1.7, it should be remarked that only in the case of Theorem C, the linearized PDE of the isometric embedding equation is not real principal type in the strong sence.

Corresponding to Definition 1.2, we define as follows:

Definition 1.4. The isometric embedding (1.1) is called *a real principal type* at $x_o\in M$ (resp. on U) if the linearized equation (1.3) is real principal type at x_o (resp. on U).

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Before constructing a non-degenerate, principal type embedding in § 2, we study a second fundamental tensor at a fixed point x_o. First, we recall the Gauss-equation for the isometric embedding:

$$(1.7) \qquad R_{ijkl}(u) = \sum_\lambda H^\lambda_{il}(u)\cdot H^\lambda_{jk}(u) - H^\lambda_{ik}(u)\cdot H^\lambda_{jl}(u)$$

which is a restriction in seeking an isometric embedding.

Now, we have the following:

Proposition 1.5. *Let (M^n, ds^2) be a Riemannian n-manifold $(n \leq 3)$, which satisfies the assumptions in Theorems A-C. Then, there exists a $(N-n)-$tuple of $n \times n$ symmetric matrix $(H^\lambda_{ij})(0)_{\lambda=n+1,\dots,N}$ such that*

(i) for $(H_{ij}\lambda)(0)_{\lambda = n+1,\dots N}$, (1.5) is real principal type at the origin $0 \in \mathbf{R}^n$.

(ii) $(H^\lambda_{ij})(0)_{\lambda=n+1,\dots,N}$ satisfies (1.7) at the origin 0.

Furthermore,

(iii) $(H^\lambda_{ij})(0)_{\lambda = n+1,\dots N}$, is non-degenarate at the origin $0 \in \mathbf{R}^n$ under the assumptions in Theorems A-B

To prove Proposition 1.5 we need several lemmas. For simplicity, we put

$$(1.8) \qquad q(x,\xi) = \det \begin{pmatrix} \xi_1 & 0 & \cdots & 0 & \xi_2 & \cdots & \cdot \\ 0 & \xi_2 & \cdots & \cdot & \xi_1 & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \xi_n \\ 0 & \cdot & \cdots & \xi_n & \cdot & \cdots & \xi_{n-1} \\ H^\lambda_{11}(x) & H^\lambda_{22}(x) & \cdots & H^\lambda_{nn} & 2H^\lambda_{12}(x) & \cdots & 2H^\lambda_{n-1n}(x) \end{pmatrix},$$

which is the determinant of the symbol of the operator (1.5). Then, we construct a $(n+1)\times(n+1)$ symmetric matrix $(H^\lambda_{ij})(0)$ which satisfies Proposition 1.5 for each dimension n=2,3. First, we get

Lemma 1.6 *Under the assumption of Theorem A, Proposition 1.4 is valid.*

Proof. Denote $(H_{ij\lambda})(0)$ simply by (H_{ij}). Then, (1.8) can be rewritten as

$$(1.9) \qquad q(x,\xi) = \det \begin{pmatrix} \xi_1 & 0 & \xi_2 \\ 0 & \xi_2 & \xi_1 \\ H_{11} & H_{22} & 2H_{12} \end{pmatrix}$$

which clearly satisfies $d_\xi q(x,\xi) \neq 0$.

Lemma 1.7. *Under the assumptions of Theorem B, Proposition 1.4 is valid.*

Proof. Since $dK(x) \neq 0$ on a neighborhood U, we may assume that $K(x) = x^1$ in U, by a change of coordinate. Now, put

$$(1.10) \qquad H_{11}(x) = x^1 \ , H_{12}(x) = 0 \ , H_{22}(x) = 1.$$

Then, we have $K(x) = H_{11}(x)H_{22}(x) - H_{12}(x) = x^1$. By considering $q(x,\xi)$ in (1.9), we have

$$q(x,\xi) = -(\xi_1^2 + x^1\xi_2^2) \ ,$$

and

$$(1.11) \qquad dq = (\partial_{\xi_1}q, \partial_{\xi_2}q, \partial_{x^1}q, d_{x^2}q)$$

$$= (2\xi_1, -2x^1\xi_2, -\xi_2^2, 0) \ ,$$

which gives Lemma 1.7.

Next, we shall consider the case n=3 (cf. [BGY]) :

Lemma 1.8. *Under the assumptions of Theorem C, Proposition 1.5 is valid.*

Proof. Consider the following symmetric 3×3 matrix :

$$(1.12) \qquad G = (G_{ab}) \quad a,b = 1,2,3.$$

where we have set

$$G_{11} = R_{2323} \ , G_{22} = R_{3131} \ , G_{33} = R_{1212} \ ,$$
$$G_{12} = G_{21} = R_{2331} \ , G_{23} = G_{31} = R_{2312} \ , G_{13} = G_{31} = R_{3112} \ .$$

Then, G can been seen as a linear mapping from the space of curvature like tensors into the space of symmetric 3×3 matrices. Notice that $GL(3,\mathbf{R})$ acts on both spaces naturally. Therefore, given the signature of the curvature tensor , we get

Lemma 1.8 if we can obtain the symmetric matrix $(H_{ij\lambda})$ such that the signature of the both side of (1.7) coincide.

To see this we seek the second fundamental form H in the following form

$$(1.13) \qquad H = (H_{ij}^{\lambda})(0) = \begin{pmatrix} w_1 & \lambda w_3 & \lambda w_2 \\ \lambda w_3 & w_2 & \lambda w_1 \\ \lambda w_2 & \lambda w_1 & w_3 \end{pmatrix}$$

where w_1, w_2, $w_3 \in \mathbf{R}^3$, $\lambda \in \mathbf{R}$. It can be easily seen that, if $\{w_a\}_{a=1,2,3}$ is linearly independent, then $(H_{ij\lambda}(0))$ is nondegenerate at 0. Denote by $r(H)$ a 3×3 matrix whose component G_{ij} is defined by (1.6) for $i,j=1,2,3$. By a direct computation, we get

$$(1.14) \qquad r(H) = \begin{pmatrix} b_1-\lambda^2 a_1 & -\lambda a_3+\lambda^2 b_3 & -\lambda a_2+\lambda^2 b_2 \\ -\lambda a_3+\lambda^2 b_3 & b_2-\lambda^2 a_2 & -\lambda a_1+\lambda^2 b_1 \\ -\lambda a_2+\lambda^2 b_2 & -\lambda a_1+\lambda^2 b_1 & b_1-\lambda^2 a_3 \end{pmatrix}$$

where $a_i = \langle w_i, w_i \rangle$, $b_1 = \langle w_2, w_3 \rangle$, $b_2 = \langle w_3, w_1 \rangle$, and $b_3 = \langle w_1, w_2 \rangle$.

Now, for a given $\lambda \in \mathbf{R}$, we set

$$(1.15) \qquad L_\lambda(H) = L_\lambda \begin{pmatrix} a_1 & b_3 & b_2 \\ b_3 & a_2 & b_1 \\ b_2 & b_1 & a_3 \end{pmatrix} = \begin{pmatrix} b_1-\lambda^2 a_1 & -\lambda a_3+\lambda^2 b_3 & -\lambda a_2+\lambda^2 b_2 \\ -\lambda a_3+\lambda^2 b_3 & b_2-\lambda^2 a_2 & -\lambda a_1+\lambda^2 b_1 \\ -\lambda a_2+\lambda^2 b_2 & -\lambda a_1+\lambda^2 b_1 & b_1-\lambda^2 a_3 \end{pmatrix}$$

Note that L_λ is invertible for $\lambda = -1/2, 0, 1$, because $L_\lambda^{-1} = (\lambda^4-\lambda)^{-1} L_\lambda$.

To complete the proof of Lemma 1.8, we show the following :

Lemma 1.9. *Let G be a* 3×3 *symmetric matrix such that the signature of G is different from* (0,0). *Then, there exist H of the form (1.13) such that* $L_\lambda(H) = G$.

Proof. We seek H for all cases classified by the signature of G.

First, put

$$\bar{H} = \begin{pmatrix} 1 & b & b \\ b & 1 & b \\ b & b & 1 \end{pmatrix}$$

Then, the eigenvalue of $\bar{H}$ are 1-b and 1+2b whose multiplicities are 1 and 2. Varying $\lambda \in (-1/2,0)\cup(0,1)$, we obtain that for the cases that the signature of G = (0,3),(3,0) (2,1),(1,2) and (2,0), there exist $\bar{H}$ such that $L_\lambda(\bar{H}) = G$.

We put

$$\bar{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \bar{H} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

for $sig(G) = (1,1)$, $(0,1)$, $(1,0)$ respectively. Then it is easily obtained that $sig(L_\lambda(\bar{H})) = sig(G)$, which implies Lemma 1.8.

§ 2 . Construction of a real principal embedding

In this section we shall construct a real principal type embedding whose induced metric is sufficiently closed to the given Riemannian metric ds^2. Namely, we prove the following:

Proposition 2.1. *Let* (M^n, ds^2) *be a Riemannian manifold and* $n = 2,3$. *Under the assumptions of Theorem A-C, given any* $\eta > 0$ *and any positive integer s, there exists an open neighborhood* U_0 *of* p_o, *(N-n)-tuple of local* C^∞ *unit vector fields* $(N_\lambda(u))_{\lambda=n+1,\ldots,N}$ *on* U_0, *and non-degenerate* C^∞ *embedding* $(x^A(u))_{A=1,\ldots,N}$ *of* U_0 *into* $\mathbf{R}^N$ *with* $N = n(n+1)/2$ *such that*

(i) $(N_\lambda(u))_{\lambda=n+1,..,N}$ *is a unit normal frame field on* U_0 *of the embedding*

$(x^A)_{A=1,\ldots,N}$,

(ii) $(H_{ij\lambda})(0)$ *are the second fundamental form of* (x^A) *at* p_0 *with respect to* $(N_\lambda(u))$,

(iii) the embedding $x^A(u)$ *is a real principal type on* U_o.

(iv) $\| g_{ij}(u) - \sum_{A=1}^{N} \frac{\partial x^A}{\partial u^i} \cdot \frac{\partial x^A}{\partial u^j} \|_{H^r(U_o)} < \eta$.

To prove Proposition 2.1, we expand the Riemannian metric g into the Taylor series. Namely , let $g(u)=(g_{ij}(u))$ be the Riemannian metric in the coordinate neighborhood $U(u_1,\ldots,u_n)$. We put the Taylor expansion of $g(u)$ as follows:

$$(2.1) \quad g_{ij}^{(k)}(u) = \delta_{ij} + \frac{i}{2!}\sum A_{ij\alpha_1\alpha_2}u^{\alpha_1}u^{\alpha_2} + \cdots + \frac{1}{k!}\sum A_{ij\alpha_1,\ldots,\alpha_k}u^{\alpha_1}\cdots u^{\alpha_k} + \cdots ,$$

Now, we seek the mapping

$$i : U(u^1, \ldots ,u^n) \to \mathbf{R}^N$$

by the following form:

$$(2.2) \quad i(u^1, \ldots ,u^n) = (x^A(u))$$
$$= \sum I_a^A u^a + \frac{1}{2!}\sum I_{ab}^A u^a u^b + \ldots + \frac{1}{k!}\sum I_{a_1,\ldots,a_k}^A u^{a_1}\cdots u^{a_k} + \ldots ,$$

satisfying

$(H-I)$ $i(0) = 0$,

$(H-II)$ $di(0) = Id.$ (*the identity*) .

$(H-III)$ $e_\lambda(0) = (0,\ldots,\overset{(\lambda)}{1},0,\ldots0)$ $(\lambda = n+1,\ldots,N)$.

(H-IV) Given non-degenerate (H_{ij}^λ) which satisfies (1.7), we have $H_{ij}^\lambda(0) = H_{ij}^\lambda$.

Now, we seek (2.2) by the following relation:

$$(2.3) \quad \sum_A \frac{\partial x_{(k)}^A}{\partial u^i} \cdot \frac{\partial x_{(k)}^A}{\partial u^j} = g_{ij}(u) \quad mod\ O(x^k)$$

Therefore, we have

$$(2.4) \quad I_a^A = \delta_a^A \quad (a=1,\ldots,n) .$$

By using $\{^{\beta}_{bc}\}(0)=0$, we have

$$(2.5) \quad I_{ca}^\mu = H_{ca}^\mu \ (\mu = n+1,\ldots,N) , \ I_{ca}^b = 0 (b=1,\ldots,n) .$$

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Substituting (2.1),(2.2) into (2.3) and equating the corresponding coefficients, we can solve (2.3) formally. Note that the equations thus obtained for the unknown I^A's are underdetermined. So, we only have to restrict the symmetric part of each I^A's and choose the anti-symmetric part of those arbitrarily. Therefore, we get Proposition 2.1.

§ 3. Local solvability of real principal type partial differential system.

We first give a general theorem on the local solvability of a non-linear partial differential operator.

Let $U_o \subset \mathbf{R}^n$ be an open neighborhood of $x_o \in \mathbf{R}^n$ and $u_o : U_o \to \mathbf{R}^N$ be a C^∞ map (Hereafter we also denote by u_o a C^∞ map from $\mathbf{R}^n \to \mathbf{R}^N$ obtained by cutting off the support of u_o.) Let $\Phi(u) = \Phi(x, \overline{D}^m u)$ be a $\mathbf{R}^N$ -valued nonlinear partial differential operator of order m applied to $u \in C^\infty(\mathbf{R}^n, \mathbf{R}^N)$, where

$$\overline{D}^m u = (D^\alpha u \; ; |\alpha| \leq m) \; , \; D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \; ,$$

$$D_i = -\sqrt{-1}\frac{\partial}{\partial x_i} \; , \; \alpha = (\alpha_1, \; . \; . \; . \; , \alpha_n) \; .$$

Theorem 3.1. *Under the above situations, we assume that* $\Phi(x,w) \in B^\infty(\mathbf{R}^n \times \{|w| \leq w_0\})$ *for any* $w_0 > 0$, *which assures the Frechet differentiability of* $\Phi(u)$ *with respect to* $u \in H^s(\mathbf{R}^n)$ $(s > m + \frac{n}{2})$, *and the following condition is valid for* $\Phi'(u)$. *Namely, there exist an open neighborhood* $U_1 \subset U_o$ *of* $x_o \in \mathbf{R}^n$ *with a smooth boundary,* $\alpha > m + \frac{n}{2}$, $\delta (0 < \delta < 1)$ *and* $d \geq 0$ *such that, for any* $u (\|u - u_0\|_\alpha \leq \delta)$, $s \in \mathbf{R}$ *and* $h \in H^s(\mathbf{R}^n)$, *the equation*

(3.1) $\quad \Phi'(u)v = h \quad$ *in* U_1

admits a solution $v \in H^{s-d}(\mathbf{R}^n)$ *with the so called tame estimate*

(3.2) $\qquad \|v\|_{s-d} \leq C_s(\|h\|_s + \|u\|_s \|h\|_d)$, *where C_s is independent of u and h. Then, there exist $s_0 \in \mathbf{Z}_+$ and $\eta > 0$ such that, for any $g \in C^\infty(U_1)$,*

$$\|g - \Phi(u_0)\|_{s_0}^0 \equiv \|g - \Phi(u_0)\|_{H^{s_0}(U_1)} < \eta \quad ,$$

the equation

$$(3.3) \qquad \Phi(u) = g \quad \text{in } U_1$$

admits a solution $u \in C^\infty(\mathbf{R}^n, \mathbf{R}^N)$. Here $B^\infty(\mathbf{R}^n \times \{|w| \leq w_0\})$ denotes the set of all $\mathbf{R}^N$ valued C^∞ functions with bounded derivatives.

Proof. Put

$$\Psi = \Phi(u+u_0) - \Phi(u_0) \ , f = g - \Phi(u_0)$$

and try to seek a solution $u \in C^\infty(\mathbf{R}^n, \mathbf{R}^N)$ of the equation

$$(3.4) \qquad \Psi(u) = f \quad \text{in } U_1 \ .$$

Then, $u + u_0$ will clearly be a solution of the equation (3.3).

Rewriting the conditions (3.1) and (3.2) in terms of $\Phi'(u)$, we obtain that, for any u ($\|u\|_\alpha \leq \delta$), $s \geq d$ and $h \in H^s(\mathbf{R}^n)$,

$$(3.5) \qquad \Psi'(u)v = h \quad \text{in } U_1 \ ,$$

admits a solution $v \in H^{s-d}(\mathbf{R}^n)$ with the estimate

$$(3.6) \qquad \|v\|_{s-d} \leq C_s(\|h\|_s + \|u\|_s \|h\|_d) \ ,$$

where C_s is independent of u and h. For the later reference, we denote this v by $\Psi'(u)^{-1}h$.

In order to solve (3.4), consider a series of functions $\{u_n\}$ defined by

$$(3.7) \qquad \begin{cases} u_1 = 0 \ , \\ u_{n+1} = u_n + s_{\theta_n}\rho_n \quad (n \geq 1) \ , \end{cases}$$

where s_θ ($\theta \geq 1$) are the usual smoothing operators for the Banach scale $H^s(\mathbf{R}^n)$ ($s \in \mathbf{R}$), $\theta_n = \theta^{\tau^{n+n_0}}$ with $\tau = \frac{4}{3}$ and large enough $\theta > 1$, n_0, and $\rho_n = \Psi'(u_n)^{-1} \epsilon R g_n$

with

(3.8) $\quad g_n = f - \Psi(u_n)$.

Here

$$\epsilon : H^s(U_1) \to H^s(\mathbf{R}^n) \ , \ R : H^s(\mathbf{R}^n) \to H^s(U_1)$$

are the extension operator and the restriction operator, respectively.

First of all, we prove the following inequalities $(i)_j$, $(ii)_j$, $(iii)_{j-1}$ $(j \ge 1)$ by induction on j. Namely, there exists an appropriately large integer $s_0 \ge m$ and small η $(0<\eta<1)$ such that, for any f $(\|f\|^0_{s_0}<\eta)$ and nonnegative integer j,

$(i)_j$ $\quad \|u_j\|_\alpha \le \delta$, $(ii)_j$ $\quad \|g_j\|^0_\alpha \le M\theta_j^{-\mu}\|f\|^0_{s_0}$,

$(iii)_j$ $\quad \|\rho_j\|_\alpha \le M\theta_j^{-\alpha}\|f\|^0_{s_0}$

are valid for some positive numbers M, μ, a independent of f and j.

For $j=1$, the validity of these inequalities immediately follow by setting

(3.9) $\quad M = \theta_1^\mu$.

Assuming their validity for $j \le n$, we shall next prove that these inequalities also hold for $j=n+1$.

Lemma 3.2. *Suppose (3.10)* $\alpha \ge [\frac{n}{2}]+m+1$ *and* $(i)_n$ *is valid. Then,*

(3.11) $\quad \|g_n\|^0_s \le C_s\{\|g_1\|^0_s + \|u_n\|^0_{s+m}\}$ for *any* $s \ge 0$.

Here, $[\frac{n}{2}]$ denote the largest integer which is not greater than $n/2$.

Proof. This follows from applying the Sobolev's embedding theorem and the following Moser's lemma to

$$g_n - g_1 = -\Psi(u_n) \equiv -\Psi(x, \overline{D}^m u_n) \ .$$

Lemma 3.3 (Morser's Lemma). *Let* Ω *be a domain in* $\mathbf{R}^n$ *whose boundary is a compact* C^∞ *hypersurface and* $F(x,v) \in B^\infty(\Omega \times \{v \in \mathbf{R}^l; |v| \le v_0\})$ *with the property* $F(x,0)=0$. *Then, for any* $s \ge 0$ *and* $v \in H^s(\Omega)$, $\|v\|_{L^\infty(\Omega)} \le v_0$, *we have*

(3.12) $\|F(\cdot,v(\cdot))\| \le C(s,|F|_{B^{s+1}})\|v\|_{H^s(\Omega)}$,

where $|F|_{B^{s+1}}$ *denotes the maximum of all the supnorm in* $\Omega\times\{|v|\le v_0\}$ *of all the derivatives of* $F(x,v)$ *up to order* $s+1$.

Lemma 3.4. *Suppose (3.13)* $\alpha\ge m+d$ *and* $(i)_j$ *is valid for* $1\le j\le n$. *Then, for any integer* $s^*\ge m+d$, *there exists* $\theta>1$ *such that*

(3.14) $\|u_{n+1}\|_s \le \theta_{n+1}^{\frac{m+d}{\tau-1}+1}\|g_1\|_s^0$,

for any s $(m+d\le s\le s^*)$.

Proof. For the rest of the proof of Theorem 3.1, we use a common symbol K to denote several constants which depend only on s, δ and η. By (3.6), (3.10) and taking account of $\|\epsilon Rg_j\|_s \le K\|g_i\|_s^0$ for any $s\ge 0$, we have

(3.15)
$$\begin{aligned}\|u_{i+1}\|_s &\le \|u_i\|_s + \|S_{\theta_i}\rho_i\|_s\\ &\le \|u_i\|_s + K\theta_i^{m+d}\|\rho_i\|_{s-m-d}\\ &\le \|u_i\|_s + K\theta_i^{m+d}(\|\epsilon Rg_i\|_{s+m} + \|\epsilon g_i\|_d\|u_i\|_{s-m})\\ &\le \|u_i\|_s + K\theta_i^{m+d}(\|g_i\|_{s-m}^0 + \|g_i\|_d^0\|u_i\|_{s-m})\\ &\le K\theta_i^{m+d}(\|u_i\|_s + \|f\|_{s-m}^0)\end{aligned}$$

for any $i(1\le i\le n)$. Then, using (3.15) recursively and taking θ large enough

(3.16) $\|u_{n+1}\|_s$

$$\begin{aligned}&\le K^n(\textstyle\prod_{i=1}^n\theta_i)^{m+d}\|f\|_s^0\{1 + (K\theta_1)^{-m-d} + (K\theta_1)^{-m-d}(K\theta_2)^{-m-d} +\\ &\qquad \cdots + \textstyle\prod_{i=1}^{n-1}(K\theta_i)^{-m-d}\}\\ &\le K^n\theta_{n+1}^{\frac{m+d}{\tau-1}}\|f\|_s^0\\ &\le \theta_{n+1}^{\frac{m+d}{\tau-1}+1}\|f\|_s^0\end{aligned}$$

holds for any s $(m+d \le s \le s^*)$.

Now, we assume $(i)_j$, $(ii)_j$ $(1 \le j \le n)$ are valid and try to prove $(i)_{n+1}$, $(ii)_{n+1}$, $(iii)_n$. Note that, from this assumption which we have just made, Lemma 3.2 and Lemma 3.4 hold.

We first prove $(ii)_{n+1}$. From (3.8),

$$(3.17) \qquad g_{n+1} - g_n = -(\Psi(u_{n+1}) - \Psi(u_n)).$$

Combine (3.17) with

$$(3.18) \qquad \Psi(u_{n+1}) - \Psi(u_n) - \Psi'(u_n)(u_{n+1} - u_n)$$

$$= \int_0^1 \theta \Psi''(\theta u_n + (1-\theta)u_{n+1})(u_{n+1} - u_n) d\theta$$

Then, we have

$$(3.19) \qquad g_{n+1} = g_n - \Psi'(u_n)(u_{n+1} - u_n) + Q_n \ ,$$

where

$$(3.20) \qquad Q_n$$

$$= -\int_0^1 \theta \cdot \Psi''(\theta u_n + (1-\theta)u_{n+1})(u_{n+1} - u_n, u_{n+1} - u_n)\, d\theta \ .$$

Since

$$(3.21) \qquad \Psi'(u_n)\rho_n = g_n \ \ in\ U_1 \ ,$$

(3.7) and (3.19) imply

$$(3.22) \qquad g_{n+1} = \Psi'(u_n)(1 - S_{\theta_n})\rho_n + Q_n \ \ in\ U_1 \ ,$$

From Lemma 3.2, Lemma 3.4 and (3.6), we have

$$(3.23) \qquad \|\Psi'(u_n)(1 - S_{\theta_n})\rho_n\|_d^0 \le K_1 \theta_n^{\frac{m+d}{\tau-1}+2m-s_0+2d+1} \|g_1\|_{s_0}^0 \ ,$$

$$(3.24) \qquad \|S_{\theta_n}\rho_n\|_\alpha \le MK_2\theta_n^{\alpha-\mu}\|g_1\|_\alpha^0 < MK_2\theta_n^{\alpha-\mu}\eta \ .$$

Here and below in this proof of Theorem 3.1, K_1 and K_2 denote general constants depending only on s_0 and α, respectively. If we take (3.25) $\mu>\alpha$, a sufficiently large θ and a sufficiently small η such that

(3.26) $MK_1\theta_n^{\alpha-\mu}\eta \le \delta$,

the estimate

(3.27) $\|\theta u_n + (1-\theta)u_{n+1}\| \le \delta \quad (0\le\theta\le 1)$

follows from (3.7),(3.24), $(i)_n$. Then,

(3.28) $\|Q_n\|_d^0 \le K_1\theta_n^{2(\alpha-\mu)}\|g_1\|_{s_0}^0$

by taking account of (3.24),(3.27) and applying the well known fact :

For any integer $s\ge[\frac{n}{2}]+1$, there exists $C_s>0$ such that

$$\|v\cdot w\|_{H^s(\Omega)} \le C_s(\|v\|_{L^\infty(\Omega)}\|w\|_{H^s(\Omega)} + \|w\|_{L^\infty(\Omega)}\|v\|_{H^s(\Omega)})$$

for any $v,w\in H^s(\Omega)$, where Ω is the one in the previous Moser's lemma. Therefore, from (3.22),(3.23), (3.28), we have

(3.29) $\|g_{n+1}\|_d^0 \le K_1\theta_n^{\beta}\|g_1\|_{s_0}^0$,

(3.30) $\beta = \max(\frac{m+d}{\tau-1} + 2m - s_0 + 2d + 1, 2(\alpha-\mu))$.

These immediately yield $(ii)_{n+1}$ if we take θ large enough and let s_0, μ satisfy

(3.31) $s_0 \ge 5(m+d) + 2(\alpha-\mu) + 1$,

(3.32) $\mu > 3\alpha$

so that $\beta < -\mu\tau$.

Secondly, we prove $(iii)_n$. By Lemma 3.2 , Lemma 3.4 , (3.6) and $(ii)_n$,

(3.33) $\|\rho_n\|_0 \le K_1\theta_n^{-\mu}\|g_1\|_{s_0}^0$,

(3.34) $\|\rho_n\|_{s_0-d-m} \le K_1\theta_n^{\frac{m+d}{\tau-1}+1}\|g_1\|_{s_0}^0$.

Suppose

(3.35) $\alpha < s_0 - m - d$

and interpolate (3.33).(3.34), then we have

(3.36) $\|\rho_n\|_\alpha \leq K_3 \theta_n^{\gamma} \|g_1\|_{s_0}^0$

with

$$(3.37) \quad \gamma = \frac{-\mu(s_0 - m - d - \alpha) + \alpha(\frac{m+d}{\tau - 1} + 1)}{s_0 - m - d}$$

Here and below in the proof of Theorem 3.1, K_3 denotes a general constant depending only on α, s_0. Now, suppose μ satisfies

$$(3.38) \quad \mu > \frac{(3\alpha - 2a)(m+d) + 2as_0 + \alpha}{s_0 - m - d - \alpha}$$

which is equivalent to $\gamma < -2a$, $(iii)_n$ is valid for sufficiently large θ.

Thirdly, we prove $(i)_{n+1}$. By (3.7) and $(iii)_j$ $(j \leq n)$, we have

$$(3.39) \quad \|u_{n+1}\|_\alpha \leq \textstyle\sum_{j=1}^n \|u_{j+1} - u_j\|_\alpha$$

$$\leq K_2 \textstyle\sum_{j=1}^n \|\rho_j\|_\alpha$$

$$\leq K_2 M \|g_1\|_{s_0}^0 \textstyle\sum_{j=1}^n \theta_j^{-a} \leq K_2 M \eta$$

for sufficiently large θ. Hence, $(i)_{n+1}$ is true provided that

(3.40) $K_2 M \eta \leq \delta$.

In the above argument, the conditions which we have required are (3.10), (3.13), (3.25), (3.31), (3.35), (3.38) and (3.40). It is easy to check the existence of α, s_0, a, μ, η which satisfy these conditions. Thus we have proved $(i)_j$, $(ii)_j$, $(iii)_j$ $(j \geq 1)$.

Next, we show the existence of a solution u of (3.4). By(3.39),

(3.41) $\|u_{j+1} - u_j\|_\alpha \leq K_2 M \theta_j^{-a} \|g_1\|_{s_0}^0.$

Thus, $\{u_j\}$ is a Cauchy sequence in $H^{\alpha}(\mathbf{R}^n)$, because θ_j monotonically increases to infinity as $j \to \infty$. Consequently, there exists $u \in H^{\alpha}(\mathbf{R}^n)$ such that

$$\lim_{j\to\infty} u_j = u \quad in\ H^{\alpha}(\mathbf{R}_n).$$

On the other hand, $(ii)_j$ implies $g_j \to 0 (j \to \infty)$ in $H^d(U_1)$. Hence, reminding (3.10) and letting n tend to infinity in (3.8), we can see u is a solution of (3.4).

Finally, we prove $u \in C^{\infty}(\mathbf{R}^n, \mathbf{R}^N)$. It is enough to prove that $\{u_j\}$ converges in $H^{\alpha'}(\mathbf{R}^n)$ for any α'. by the same arguments which let to (3.15) and (3.34), we have

$$(3.42) \qquad \|u_{n+1}\|_s \le K\theta_n^{m+d}(\|u_n\|_s + \|g_1\|_s^0) \ ,$$

$$(3.43) \qquad \|\rho_n\|_{s-m-d} \le K(\|g_1\|_{s-m}^0 + \|u_n\|_s) \quad \text{for } any\ s \ge m+d \ .$$

Since $\theta_n^{m+d} \ge 1$, (3.42) implies

$$(3.44) \qquad 1 + \|u_{n+1}\|_s \le K(1 + \|g_1\|_s^0)\theta_n^{m+d}(1 + \|u_n\|_s) \ .$$

Using (3.44) recursively and reminding $u_1 = 0$,

$$(3.45) \qquad 1 + \|u_{n+1}\|_s \le \prod_{l=1}^{n}(K(1 + \|g_1\|_s^0)\theta_l^{m+d}) \ .$$

From the definition of θ_l, we can easily prove

$$(3.46) \qquad \prod_{l=1}^{n}\theta_l^{2(m+d)} \le \theta_{n+1}^{6(m+d)} \ .$$

Therefore, from (3.45) and (3.46), we have

$$(3.47) \qquad 1 + \|u_{n+1}\|_s \le \theta_{n+1}^{6(m+d)}\prod_{l=1}^{n}(K(1 + \|g_1\|_s^0)\theta_l^{-(m+d)}) \ .$$

Since $\theta_l^{-(m+d)} \downarrow 0 \ (l \to \infty)$,

$$(3.48) \qquad \prod_{l=1}^{n}(K(1 + \|g_1\|_s^0)\theta_l^{-(m+d)}) \le K \quad (n \ge 1) \ .$$

Here, note that we are fixing g_1 and recall the convention about the general constant K. Thus,

$$(3.49)_{n+1} \qquad 1 + \|u_{n+1}\|_s \le K\theta_{n+1}^{6(m+d)} \ .$$

Combining (3.43) and $(3.49)_n$, we obtain

(3.50) $\quad \|\rho_n\|_{s-m-d} \leq K\theta_n^{6(m+d)}$.

Now, suppose $s \geq m+d$, $\alpha' < s-m-d$, the interpolation of (3.33) and (3.50) yields

(3.51) $\quad \|\rho_n\|_{\alpha'} \leq K_4\theta_n^{\sigma}$,

(3.52) $$\sigma = \frac{-\mu(s-m-d-\alpha') + 6\alpha'(m+d)}{s-m-d} ,$$

where K_4 is a constant depending only on s_0, s, δ and η. Clearly, by taking s large enough in (3.51), (3.52), we have

(3.53) $\quad \|\rho_n\|_{\alpha'} \leq K_4\theta_n^{-b}$ for *some* $b>0$.

By the same argument used before, we can prove that $\{u_j\}$ is also a Cauchy sequence in $H^{\alpha'}(\mathbf{R}_n)$. Since α' can be taken arbitrarily large, we can conclude $u \in C^\infty(\mathbf{R}^n,\mathbf{R}^N)$ with the aid of Sobolev's embedding theorem. This completes the proof of Theorem 3.1.

Theorem 3.5. *Let x_o, U_o, $\Phi(u)$ be the same as those in Theorem 3.1 with all the assumptions in Theorem 3.1 except those related to (3.1),(3.2). Suppose the Frechet derivative $\Phi'(u_0)$ of $\Phi(u)$ at u_0 is a system of real principal type at x_o. Then, there exist an open neighborhood $U_1 \subset U_0$ of x_o with smooth boundary, $s_o \in \mathbf{Z}_+$ and $\eta>0$ such that, for any $g \in C^\infty(U_1)$, $\|g - \Phi(u_o)\|_{H^{s_0}(U_1)} < \eta$, the equation (3.3) admits a solution $u \in C^\infty(\mathbf{R}^n,\mathbf{R}^N)$.*

Proof of Theorem 3.5. From Theorem 3.1, it is enough to prove the existence of U_1, α, δ, d which satisfy (3.1),(3.2),(3.3). To begin with we set

(3.54) $\quad L(u) = \Phi'(u)$,

(3.55) $\quad l(x,\xi;u) =$ *the principal symbol of* $\det L(x,\xi;u)$.

and

(3.56) $\quad M(x,D;u) = L(x,D;u)^{co}L(x,D;u)$

$$= l(x,D;u)I + R(x,D;u) \quad , R(x,D;u) \in S^{mN-1} ,$$

where S^{mN-1} denotes the usual Hormander class $S^{mN-1}_{1,0}$. Moreover, we have the following lemma for the local solvability of the operator (3.56).

Lemma 3.6. *Suppose there are an open neighborhood* $U_1 \subset U_o$ *of* x_0, $m+\frac{n}{2}<d_1$, $d_2 \in Z_+$, $d' = \max(d_1,d_2) < \alpha' \in Z_+$, $\delta_1 > 0$ *such that for any* u $(\|u-u_o\|_{\alpha'} \leq \delta_1)$, *there exist an operator* $Q(u)$ *and* $\phi_j \in C_o^\infty(U_0)$ $(1 \leq j \leq 3)$ *with the properties :*

(3.59) $\quad \phi_1 = 1 \ \ in \ U_1 \ ,$

(3.60) $\quad \phi_1 \subset \phi_2 \subset \phi_3 \ ,$

(3.61) $\quad \|Q(u)h\|_{s-d_1} \leq C_s(\|h\|_s + \|u\|_s\|h\|_{d_1}) \quad (d_1 \leq s \in Z_+ \ ; h \in H^s(R_n)),$

(3.62) $\quad \|K(u)\| \leq \frac{1}{2}$

(3.63) $\quad \|(1 - K(u))^{-1}h\|_s \leq C_s(\|h\|_s + \|u\|_s\|h\|_{d_2})(s \in Z_+ \ ; h \in H^s(R^n)) \ ,$

where

(3.64) $\quad K(u) = \phi_2(I - M(u)Q(u))\phi_3 \ ,$

$\|K(u)\|$ *denotes the operator norm of* $K(u):L^2(R^n) \to L^2(R^n)$ *and* $\phi_1 \subset \phi_2$ *means that* $\phi_2 = 1$ *on supp* ϕ_1. *Then,*

(3.65) $\quad \phi_1\{M(u)Q(u)(1 - K(u))^{-1}\phi_2 - I\} = 0 \ ,$

(3.66) $\quad \|Q(u)(1 - K(u))^{-1}\phi_2 h\|_{s-d'} \leq C_s(\|h\|_s + \|u\|_s\|h\|_{d'})$

$$(d' \leq s \in Z_+ \ ; h \in H^s(R^n)),$$

and these immediately imply the followings. Namely, for any u $(\|u-u_o\|_\alpha \leq \delta_1)$, *we have*

(3.67) $\quad \phi_1[L(u)\{\,{}^{co}L(u)Q(u)(1 - K(u))^{-1}\phi_2\} - I] = 0$

and

$$(3.68) \qquad \| {}^{co}L(u)Q(u)(1 - K(u))^{-1}\phi_2 h\|_{s-d} \le c_s(\|h\|_s + \|u\|_s\|h\|_d)$$

$$(d \le s \in \mathbf{Z}_+ \ ;\ h \in H^s(\mathbf{R}^n)),$$

where

$$(3.69) \qquad \alpha = \alpha' + [\frac{n}{2}] + 1 \ ,$$

$$(3.70) \qquad d = d' + \max\{mN-1, m+[\frac{n}{2}]+1\}$$

The proof of Lemma 3.6 can be done by direct computations. So, we omit the proof.

Now to complete the proof of Theorem 3.5, we only have to take ν appropriately large and set

$$(3.71) \qquad Q(u) = \sum_{j=0}^{J} Q_j(u)A_j \ ,$$

where ν, A_j, $Q_j(u)$ $(0 \le j \le J)$ are those given in Theorem A.5 and Theorem A.6 and $Q(u)$ corresponds to the one in Lemma 3.6.

§ 4. Unified proof of Theorems A-C.

Finally, we shall make a brief comment on the unified proof of Theorems A-C stated in the introduction.

Recall that we have already get in § 2 a real principal type embedding which is approximately near to a given Riemannian metric g. Thus, Theorem 3.5 immediately implies Theorems A-C.

Appendix.

In this appendix, we shall construct a local right inverse of the real principal type linear PDE of the form

$$(1) \qquad Pu = f \ .$$

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by following the work of Hormander [H] and Duistermaat- Hormander [DH]. However, to use it to our iteration scheme, we have to estimate the dependence of the local right inverse on P.

More precisely, in Appendix A, we shall shorten the proof of [H],[DH] by using a geometrical idea based on the proof of Darboux's lemma due to Moser (cf [M],[W] and [AB]), because those proof are too long and complicated for the purpose of analyzing the dependency of the local right inverse on P.

Let $\overset{o}{T}{}^*\mathbf{R}^n$ be the cotangent bundle of $\mathbf{R}^n$ minus zero section and θ the canonical 1-form on $\overset{o}{T}{}^*\mathbf{R}^n$ which can be expressed by $\theta = \sum_{i=1}^{n}\xi^i dx^i$, by using the coordinate of $(x^1, \ldots, x^n, \xi^1, \ldots, \xi^n)$ on $\overset{o}{T}{}^*\mathbf{R}^n$, and $\Gamma \subset \overset{o}{T}{}^*\mathbf{R}^n$ be an open conic set. Given $p(x,\xi) \in C^\infty(\Gamma)$, we say that $p(x,\xi)$ is a *positively homogeneous symbol of degree one* if it satisfies

(2) $\quad p(x,\lambda\xi) = \lambda p(x,\xi) \quad$ for *any* $\lambda > 0$.

and denote by $S^1(\overset{o}{T}{}^*\mathbf{R}^n)$ the set of positively homogeneous symbols of degree one.

Take a point $z_0 \in \overset{o}{T}{}^*\mathbf{R}^n$ and fix it. Recall that $p(x,\xi) \in S^1$ is *real principal type at* $z_0 \in \overset{o}{T}{}^*\mathbf{R}^n$ if it satisfies

(*) dp and θ are linearly independent at z_0 if $p(z_0) = 0$.

The above condition (*) is open in the class $S^1(\overset{o}{T}{}^*\mathbf{R}^n)$ in the topology defined by the Frechet norms $\|p\|_k$; for positive integers $k > 0$,

(3) $$\|p\|_k = \sup_{(x,\xi)\in \overset{o}{T}{}^*\mathbf{R}^n, |\alpha|+|\beta|\le k} |\xi|^{k-|\alpha|} |\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \quad .$$

First, we prove the following :

Theorem A.1. *Let* $p(x,\xi) \in S^1$ *be real principal type at* $z_0 \in \overset{o}{T}{}^*\mathbf{R}^n$. *Given any* *positive integer m and any positive* ϵ, *there exist* $r>0$, $\gamma_1>0$ *and* $k\in\mathbf{Z}_+$ *such that the following properties hold : For any* $\bar{p}\in S^1(\overset{o}{T}{}^*\mathbf{R}^n)$, *satisfying*

(4) $\quad \|\bar{p}-p\|_k \leq \gamma_1$,

there exists a conic neighborhood $\Gamma(r) = \{(x,\xi);\ |x-x_0|+|\xi-\xi_0|\leq r\}$ of $z_0 = (x_0,\xi_0)$ which satisfies the following: There exist 2n-tuple smooth functions $\{X_1(\bar{p}),\ldots,X_n(\bar{p}),\ \Xi_1(\bar{p}),\ldots,\Xi_n(\bar{p})\}$ on $\Gamma(r)$ such that

(*i*) $\{dX_1(\bar{p}),\ldots,dX_n(\bar{p}),\Xi_1(\bar{p}),\ldots,\Xi_n(\bar{p})\}$ *are linear independent on* $\Gamma(r)$.

(*ii*) $\{X_j(\bar{p})\ ,\ X_i(\bar{p})\} = \{\Xi_j(\bar{p})\ ,\ \Xi_i(\bar{p})\} = 0\ ,\ \{X_j(\bar{p})\ ,\ \Xi_i(\bar{p})\} = \delta_{ij}$.
for all $i,j=1,\ldots,n$.

(*iii*) $\quad \Xi_1(\bar{p})(x,\xi) = \bar{p}(x,\xi)$ *on* $\Gamma(r)$.

(iv) $X_i(\bar{p})(x,\xi)$ *and* $\Xi_i(\bar{p})(x,\xi)$ *are positively homogeneous of degree 0 and 1 respectively.*

(v) $X_i(\bar{p})$ *and* $\Xi_i(\bar{p})$ *satisfying the following :*

$$(5)\qquad \sup_{\substack{|\alpha|+|\beta|\leq m,0\leq i\leq n\\ (x,\xi)\in\Gamma(r)}} \left|\partial_x^\alpha\partial_\xi^\beta\begin{Bmatrix} X_i(\bar{p}) & - X_i(p) \\ \Xi_i(\bar{p}) & - \Xi_i(p)\end{Bmatrix}\right| <\epsilon \ .$$

To show Theorem A.1, we prepare some basic lemmas stated below. Let $p(x,\xi)\in C^\infty(\overset{o}{T}{}^*\mathbf{R}^n)$ be a symbol of positively homogeneous degree one. Then, in terms of local coordinate (x,ξ), Hamiltonian vector field H_p is given by

$$(6)\qquad H_p = \sum_{i=1}^n \frac{\partial p}{\partial\xi^i}\frac{\partial}{\partial x^i} - \sum_{i=1}^n \frac{\partial p}{\partial x^i}\frac{\partial}{\partial\xi^i}$$

on $\overset{o}{T}{}^*\mathbf{R}^n$. Thus, H_p is real principal type at z_0 if H_p is not parallel to the radial vector field $\Xi = \sum_{i=1}^n \xi^i\frac{\partial}{\partial\xi^i}$

Let $S^*\mathbf{R}^n$ be the unit sphere bundle defined by the equivalent relation

(7) $\quad (x,\xi)\sim(x',\xi') \longleftrightarrow x = x'\ ,\xi = \lambda\xi'$ for *some* $\lambda>0$.

and denote by π the projection $\pi:\overset{o}{T}{}^*\mathbf{R}^n \to S^*\mathbf{R}^n$.

Fix a point $z_0 \in \overset{o}{T}{}^*\mathbf{R}^n$, $z_0 = (x_0, \xi_0)$ and put $\pi(z_0) = \hat{z}_0$. Now, take a conic neighborhood Γ of z_0 and an open neighborhood U of $\hat{z}_0$ such that $\pi(\Gamma) = U$. In each Γ and U, introduce coordinates $(x^1,\dots,x^n,\xi^1,\dots,\xi^n)$ and $(y^1,\dots,y^n,\eta^2,\dots,\eta^n)$ around z_0 and $\hat{z}_0$, respectively, such that $z_0 = (0,\dots,1,\dots,0)$, $\hat{z}_0 = (0,\dots\dots0)$, $\Gamma = U \times \mathbf{R}_+$, and the projection $\pi:\overset{o}{T}{}^*\mathbf{R}^n \to S^*\mathbf{R}^n$ can be expressed by

(8) $\quad y^i = x^i \; (i=1,\dots,n) \quad , \quad \eta^a = \xi^a/\xi^1 \; (a=2,\dots,n) \quad .$

Put $\xi^1 = t$, and consider the contact 1-form ω on $S^*\mathbf{R}^n$, defined by

(9) $\quad \omega = dy^1 + \sum_{a=2}^n \eta^a dy^a$

in terms of the local coordinate (y,η). Then, we have $\theta = t\omega$ and ω satisfies

(10) $\quad (d\omega)^n \wedge \omega \neq 0 \quad .$

Now, by the homogeneity of p, H_p can be projected down by π and define a contact vector field u on $S^*\mathbf{R}^n$. Here, the vector field u on $S^*\mathbf{R}^n$ is called the *contact vector field* which satisfies

(11) $\quad L_u\omega = d(\omega \lrcorner u) + d\omega \lrcorner u = f\omega$

for some smooth function f, where L_u is the Lie derivative with respect to u. We also define a smooth vector field ξ_ω on $S^*\mathbf{R}^n$ by

(12) $\quad \omega \lrcorner \xi_\omega = 1 \quad , d\omega \lrcorner \xi_\omega = 0 \; .$

which is called the *characteristic vector field* for the contact structure ω. Moreover, we define the subbundle $\mathbf{E}_\omega$ by $\omega = 0$. Let $\hat{\mathbf{E}}_\omega$ be the annihilator of ξ_ω in the cotangent bundle of $S^*\mathbf{R}^n$. Then, we have

(13) $\quad T(S^*\mathbf{R}^n) = \mathbf{R}\xi_\omega + \mathbf{E}_\omega \quad , \quad T^*(S^*\mathbf{R}^n) = \mathbf{R}\omega + \hat{\mathbf{E}}_\omega \quad .$

By (10), the mapping $d\omega : v \to d\omega \lrcorner v$ is an isomorphism of $\mathbf{E}_\omega$ onto $\hat{\mathbf{E}}_\omega$. So, we denote by $d\omega^{-1}$ its inverse mapping.

Given a pair (f,u) of a smooth function f and a contact vector field u on $S^*\mathbf{R}^n$, satisfying $L_u\omega = f\omega$, we write

(14) $\quad u = q\xi_\omega + v \quad ,v\in\Gamma(\mathbf{E}_\omega)$,

where $\Gamma(\mathbf{E}_\omega)$ is the set of smooth sections on $\mathbf{E}_\omega$. Then, we have

$$L_u\omega + f\omega = dq + d\omega\lrcorner v + f\omega = 0 \quad .$$

Thus, $\xi_\omega q = f$, and we see that $dq - (\xi_\omega q)\omega$ is contained in $\Gamma(\hat{\mathbf{E}}\omega)$ for any smooth function q. Therefore, we have the following correspondence :

(15) $\quad q \longleftarrow\!\!-\!-\!\!\rightarrow (-\xi_\omega q \ , u_q = (\xi_\omega q)\xi_\omega + d\omega^{-1}(dh - (\xi_\omega q)\omega)$.

Putting

(16) $\quad q(y,\eta) = p(y^1,\ldots,y^n,1,\eta^2,\ldots,\eta^n)$

and

$$p(x,\xi) = \xi^1 q(x^1,\ldots,x^n,\frac{\xi^2}{\xi^1},\ldots,\frac{\xi^n}{\xi^1})$$

for $p(x,\xi)\in C^\infty(\Gamma)$ and $q(y,\eta)\in C^\infty(U)$ by using the coordinates in U and Γ, we get the correspondence between the Hamiltonian vector field on Γ and the contact vector field on U. Using the local coordinates (y,η), it is easily seen that u_q and H_p can be written in the forms :

(17) $\quad u_q$

$$= \sum_{a=2}^n\{\frac{\partial q}{\partial y^a} - \eta^a\frac{\partial q}{\partial y^1}\}\frac{\partial}{\partial\eta^a} - \sum\{\frac{\partial q}{\partial\eta^a}\}\frac{\partial}{\partial y^a} + \{\sum_{a=2}^n\frac{\eta^a}{2}\frac{\partial q}{\partial\eta^a} - q\}\frac{\partial}{\partial y^1} \quad .$$

and

(18) $\quad H_p = u_q + t\dfrac{\partial q}{\partial y^1}\dfrac{\partial}{\partial t} \quad .$

Namely, by this correspondence, we easily get

Lemma A.2. *Let $p\in\mathbf{S}^1(\Gamma)$ and assume that $p(x,\xi)$ is real principal type at z_0. Then, u_q does not vanish at $\hat{z}_0$, $(dy^1\lrcorner u_q)(\hat{z}_o) = 0$.*

For a positive r, we shall denote $D(r)$ by the open set of $\hat{z}_0$ defined by

$$D(r) = \{\hat{z}=(y,\eta)\in U \mid |y| + |\eta|<r\}$$

Now, we put a topology in $C^\infty(U)$ by the Frechet norm $\|q\|_k$: for positive integers k,

$$\|q\|_k = \sup_{|\alpha|+|\beta|\le k,(y,\eta)\in U} |\partial_y^\alpha \partial_\eta^\beta q(y,\eta)| \tag{19}$$

We denote $C_R^\infty(U)$ by the set of $q\in C^\infty(U)$ such that u_q satisfies the conclusion of Lemma A2.

Denote by V a 2n-2 dimensional subspace which is transversal to $u_q(\hat{z}_o)$. Remark that for any $\bar{p}(x,\xi)\in S^1(\overset{o}{T}{}^*\mathbf{R}^n)$ which is sufficiently close to p in the norm (3), $\bar{q}(y,\eta) = \bar{p}(y,1,\eta_2, \ldots ,\eta_n)$ is contained in $C_R^\infty(U)$ and $u_{\bar{q}}(\hat{z}_o)$ is transversal to V.

First, we get the following :

Lemma A.3. *Let $q(y,\eta)\in C_R^\infty(U)$. Given any $m\in \mathbf{Z}_+$, and any $\epsilon>0$, there exist positive constants $r_1=r_1(q,m,\epsilon)$ $k_1=k_1(q,m,\epsilon)$, $\delta_1=\delta_1(q,m,\epsilon)$ such that the following properties hold: For any $\bar{q}\in C_R^\infty(U)$ which satisfies $\|\bar{q} - q\|_{k_1}\le \delta_1$, there exist (2n-1)-tuple smooth functions on $D(r_1)$, $\{Y_1(\bar{q}), \ldots ,Y_n(\bar{q}),E_1(qbar),\ldots,E_1(\bar{q})\}$ which depends on $\bar{q}\in C_R^\infty(U)$ and satisfies*

(*i*) $\{dY_1(\bar{q}),\ldots,dY_n(\bar{q}),dE_2(\bar{q}),\ldots,dE_n(\bar{q})\}$ *are linearly independent on $D(r_1)$.*

(*ii*) $u_{\bar{q}} = \dfrac{\partial}{\partial Y^1(\bar{q})}$ *on*

(*iii*) *There exists a smooth function $\bar{F}(\bar{q})$ on $D(r_1)$ such that*

$$\omega = \bar{F}(\bar{q})(dY_1(\bar{q}) + \textstyle\sum_{a=2}^n E_a(\bar{q})dY_a(\bar{q}))$$

on $D(r_1)$.

$$(iv) \quad \sup_{\substack{|\alpha|\le m,2\le a\le n,1\le i\le n \\ (y,\eta)\in D(r_1(q,m,\epsilon))}} \left| \partial^{\alpha}_{(y,\eta)} \begin{Bmatrix} Y^i(\bar{q}) - Y^i(q) \\ E^a(\bar{q}) - E^a(q) \\ F(\bar{q}) - F(q) \end{Bmatrix} \right| < \epsilon$$

for any $q \in C^{\infty}_R(U)$ *which satisfies*

$$\|\bar{q} - q\|_{k_1} \le \delta_1.$$

Proof. Since $u_q(\hat{z})$ is transversal to V at $\hat{z}_o$, any point $z' \in V$ can be written as

$$z' = (0, Y^2, \ldots, Y^n, E^2, \ldots, E^n).$$

by a linear change of coordinates. Now, we define the mapping

$$\Phi(Y^1, Y^2, \ldots, Y^n, E^1, \ldots, E^n) = (\exp\ tu_q)(z') = (y^1, \ldots, y^n, \eta^2, \ldots, \eta^n) \ .$$

where $Z(t) = (\exp\ tu_q)(z')$ is the integral curve of u_q which is defined by the solution of the following ordinary differential equation:

$$(20) \qquad \frac{\partial Z(t)}{\partial t} = u_q(Z(t)) \quad Z(0) = z' \ .$$

Then, we have

$$\Phi(0,0,\ldots,0) = 0 \quad , d\Phi(0,0,\ldots,0) = id \ .$$

Therefore, by using the inverse function theorem, there exists a positive constant $r_1 = r_1(q)$ such that Φ^{-1} is a $C^{\infty}-$ diffeomorphism on $D(r_1)$. Letting $(y,\eta) = \Phi^{-1}(t, a_1, \ldots, a_{2n-2})$, we get (i)-(iii) of Lemma A3. If we take $q \in C^{\infty}_R(U)$ sufficiently close to q, then the above V is also transversal to u_q at the origin. Thus, by using the smooth dependency of parameter in the ordinary differential equation, we get (iii) in Lemma A.3.

By Lemma A.3, we see that for $\bar{q} \in C^{\infty}_R(U)$, $\{Y_1(\bar{q}), \ldots, E_n(\bar{q})\}$ gives a local coordinate on $D(r_1)$. In terms of this coordinate, we can rewrite the contact 1-form ω as

$$(21) \qquad \omega = \omega_1(\bar{q}) + dY^1(\bar{q}) \sum_{i=2}^{n} \omega_i(\bar{q}) dY^i(\bar{q}) + \sum_{a=2}^{n} \omega'_a(\bar{q}) dE^a(\bar{q}) \ .$$

Observing the proof of Lemma A3, we remark that $\omega_1(\bar{q})(0,0) = 0$.

On the other hand, we have

(22) $$\omega_i(\bar{q}) = \omega_i(\bar{q})(0,Y^2(\bar{q}),\ldots,Y^n(\bar{q}),E^2(\bar{q}),\ldots,E^n(\bar{q})) \times \exp\int_0^{Y^1(\bar{q})} h(\bar{q})(t,Y^2(\bar{q}),\ldots,E^n(\bar{q})k)dt \quad ,$$

(23) $$\omega_a(\bar{q}) = \omega_a(\bar{q})(0,Y^2(\bar{q}),\ldots,Y^n(\bar{q}),E^2(\bar{q}),\ldots,E^n(\bar{q})) \times \exp\int_0^{Y^1(\bar{q})} h(\bar{q})(t,Y^2(\bar{q}),\ldots,E^n(\bar{q}))dt \quad ,$$

because $L_{u_{\bar{q}}}\omega = h(\bar{q})\omega$ and $u_{\bar{q}} = \dfrac{\partial}{\partial Y_1(\bar{q})}$. Put

(24) $$F = \exp\int_0^{Y^1(\bar{q})} h(\bar{q})(t,Y^2(\bar{q}),\ldots,E^n(\bar{q}))dt \quad ,$$

and

(25) $$\omega(\bar{q}) = \bar{F}(\bar{q})\omega'(\bar{q}) \quad .$$

Then, the coefficients of ω' does not contain the variable $Y^1(\bar{q})$.

From now on we shall reduce ω' into a canonical contact form by a contact transformation. Denote by $\xi_{\omega'}$ the characteristic vector field for ω' defined by (11). Then, it is clear that the coefficients of $\xi_{\omega'}$ does not involve the variable Y^1. Therefore, by the similar method in Lemma A.2, we get the following :

Lemma A.4 *Let $q \in C_R^\infty(U)$. Given any positive integer m and any positive ϵ, there exist positive constants r_2, γ_2, $k \in \mathbf{Z}_+$ and a new (2n-2)-tuple of smooth functions depending on $\bar{q}$ ($\|\bar{q}-q\|_k \le \gamma_2$)*

$\{Y^1(\bar{q}),Y^2(\bar{q}),\ldots,Y^n(\bar{q}),E^2(\bar{q}), \ldots ,E^n(\bar{q})\}$ *on* $D(r_2)$ *such that*

(i) $\{dY^1(\bar{q}),\ldots,dE^2(\bar{q}),\ldots,dE^n(\bar{q})\}$ *is linearly independent on* $D(r_2)$.

(ii) $$u_{\bar{q}} = \frac{\partial}{\partial Y^2(\bar{q})} \quad \text{and} \quad \xi_{\omega'(\bar{q})} = \frac{\partial}{\partial Y^1(\bar{q})} \quad .$$

and

(iii) $$\sup_{\substack{|\alpha|+|\beta|\le m,1\le i\le n\\ 2\le a\le n,(y,\eta)\in D(r_2)}} \left| \partial_y^\alpha \partial_\eta^\beta \begin{Bmatrix} Y^i(\bar{q}) & - Y^i(q) \\ E^a(\bar{q}) & - E^a(q) \end{Bmatrix} \right| < \epsilon \quad ,$$

for $\|\bar{q} - q\|_k < \gamma_2$.

Furthermore, we write $\omega'(\bar{q})$ in terms of the newly chosen coordinates $\{Y^1(\bar{q}), \ldots, Y^n(\bar{q}), E^2(\bar{q}), \ldots, E^n(\bar{q})\}$ as

(26) $\omega'(\bar{q})k = dY^1(\bar{q}) + \lambda_2(\bar{q})dY^2(\bar{q}) + \sum_{i=2}^n \lambda_i(\bar{q})dY^i(\bar{q}) + \sum_{a=2}^n \mu_a(\bar{q})dE^a(\bar{q})$

Again λ_i, $i=2,\ldots,n$ and μ_a, $a=2,..,n$ does not contain the variable $Y^2(\bar{q})$. Since $(d(\omega'(\bar{q})))^{2n-2}\wedge\omega'(\bar{q})\neq 0$, we see that

$\{dY^1(\bar{q}), \ldots, dY^n(\bar{q}), d\lambda_2(\bar{q}), dE^3(\bar{q}), \ldots, dE^n(\bar{q})\}$

is linearly independent on $D(r_2)$. So, we substruct the coordinate $E^2(\bar{q})$ by $\lambda^2(\bar{q})$ and for simplicity denote it by $E^2(\bar{q})$ again. Moreover, since $L_{\xi_{\omega'}(\bar{q})}\omega'(\bar{q})=0$, we see that $\lambda_i(\bar{q})$, $i=2,\ldots,n$ and $\mu_a(\bar{q})$, $a=2,\ldots,n$ dose not contain the variable $Y^1(\bar{q})$. Write $d(\omega'(\bar{q})) = dE^2(\bar{q})\wedge dY^2(\bar{q}) + \Omega(\bar{q})$. Thus, $\Omega(\bar{q})$ can be regarded as a symplectic 2-form on a neighborhood of the origin of $\mathbf{R}^{2n-4}$ and $L_{\frac{\partial}{\partial Y^1}} d\omega' = 0$.

Therefore, we apply Darboux's Lemma. Namely, we consider the standard closed 2-from on $D''(r'_2) = \{(Y^3,\ldots,E^n) \mid |Y|+|E|<r'_2\}$

(27) $\Omega_* = \sum_{a=2}^n dY^a\wedge dE^a$.

By a linear symplectic transformation of (Y,E), we may assume that $\Omega(\bar{q})(0,0) = \Omega(\bar{q})$, and this linear symplectic transformation smoothly depend on $\bar{q}$ in the topology of matrix.

Let $\Omega_t(\bar{q}) = \Omega_* + t(\Omega(\bar{q}) - \Omega_*), 0\le t\le 1$. For each t, $\Omega_t(\bar{q})(0,0)=\Omega_*$ is nondegenerate. Hence, there is a neighborhood $D''(r'_3)$ of $(0,0)\in\mathbf{R}^{2n-4}$. on which $\Omega_t(\bar{q})$ is nondegenerate for all $0\le t\le 1$. Thus, by the Poincare lemma, $\Omega(\bar{q}) = d(\alpha(\bar{q}))$ for a 1-form $\alpha(\bar{q})$. Namely, define $\alpha(\bar{q}) = \sum_{l=1}^{2n-4}\alpha_l(\bar{q})(u)du^l$ by

(28) $\alpha^l(\bar{q})(u) = \int_0^1 t\Omega(\bar{q})(tu)(u,e_l)dt$

at a point $u = (Y^3, \ldots, E^n)$, where e_l is the unit vector tangent to the Y or $E-$ axis . Also, we suppose $\alpha(\bar{q})(0,0)=0$. Define a smooth vector field $Z(t)$ by

$\Omega_t(\bar{q}) \lrcorner Z(\bar{q})(t) = -\alpha(\bar{q})$, which is possible since $\Omega_t(\bar{q})$ is nondegenerate. Moreover, since $Z(t)(\bar{q})(0,0) = (0,0)$ by the local existence theory, there exists a positive r'_4 and a flow $\mathbf{F}_t$ of $Z(\bar{q})(t)$ on $D''(r'_4)$. So, we get

(29) $\quad \frac{d}{dt}(\mathbf{F}_t^* \Omega_t(\bar{q})) = \mathbf{F}_t^*(-d\alpha(\bar{q}) + \Omega(\bar{q})) = 0.$

Therefore, $\mathbf{F}_1^* \Omega_1(\bar{q}) = d\omega'(\bar{q})$. Now, putting

(30) $\quad Y^2(\bar{q}) = Y^2 \,,\; E^2(\bar{q}) = E^2 \,,\; (Y^2(\bar{q}), \ldots, E^n(\bar{q})) = \mathbf{F}_1(Y^2, \ldots, E^n) \;.$

we have

(31) $\quad d\omega'(\bar{q}) = dE^2 \wedge dY^2 + \sum_{a=3}^n dE^a \wedge dY^a \,,$

Hence, there exists a smooth function $k(Y^1, Y, E)$ such that

(32) $\quad \omega' = dY^1 + dk(Y^1, Y, E) + \sum_{a=2}^n E^a dY^a \;.$

Put

(33) $\quad Y^1 = Y^1 + k \;,$

Then, $\{Y^a, E^a\}$, $a = 2, \ldots, n$ satisfies (i)-(iii) in Lemma A3. It is easy to get the estimate (iv) by a direct computation in the above arguments. Notice that by (17), we have $q(Y,E) = E^1$ on $D(r_5)$ for some positive r_5. Put (X, Ξ) on $\pi^{-1}(D(r_5))$. Then, by (8) we get the desired coordinate stated in Theorem A1.

Now, we will reduce the operator P in (1) to the canonical one given in Theorem A5, which will be a direct and a routine computation by making use of Fourier integral operators. Also, we will observe the smooth dependence in the symbol function $p(x,\xi)$ of P on these procedures. These arguments would be similar to [OMY 1-2],[OMYK 3-7] in the technical stand point of view.

Now, we take a operator P in (1) which is real principal type and fixed it. Remark that the non-characteristic points of $\bar{p} \in S^1(\overset{o}{T}{}^*\mathbf{R}^n)$ over a certain open neighborhood of $(x_o, \xi_o) \in \overset{o}{T}{}^*\mathbf{R}^n$ doesn't diminish under any perturbation $\bar{P}$ of P in $S^1(\overset{o}{T}{}^*\mathbf{R}^n)$ with respect to the norm $\|\cdot\|_k$, we can prove the following theorem by applying Egorov's theorem (cf. [E]).

Theorem A.5. *Let* P, $\bar{P}\in S^1(\overset{o}{T^*\mathbf{R}^n})$ *be as (4). Then, there exist an open neighborhood* U_o *of* x_0 *and open conic sets* Γ_j $(0\le j\le J)$ *which satisfy the following:*

(i) $\mathbf{R}^n-\{0\} = \cup_{j=0}^{J}\Gamma_j$.

(ii) given a positive integer ν, *there exist* $\gamma_2>0$, β_2, $\sigma_2\in\mathbf{Z}_+$ $(\beta_2>\sigma_2)$ *and Fourier integral operators* $\bar{V}_j$, $\bar{W}_j$ $(1\le j\le J)$ *such that, for any* u $(\|u-u_o\|_{\beta_2}\le\gamma_2)$, $\bar{P}\neq 0$ *in* Γ_0,

(34) $\bar{V}_j\bar{P}\bar{W}_j = D_{y^1}$, $\bar{V}_j\bar{W}_j = \bar{W}_j\bar{V}_j = 1$ $(mod\ S^{-\nu})$ *in* Γ_j ,

(35) $\|\bar{V}_jh\|_{s-\sigma_2}, \|\bar{W}_jh\|_{s-\sigma_2} \le C_s(\|h\|_s + \|h\|_{\sigma_2}\|u\|_s)$ $(\sigma_2\le s\in\mathbf{Z}_+ ; h\in H^s(\mathbf{R}^n))$,

where $y_j = X_j(\bar{p})$, $\eta_j = \Xi_j(\bar{p})$ $(1\le j\le n)$ *is the canonical transformation given in Theorem A.1 provided that* $\Gamma_j=\Gamma(r)$.

Remark. (i) To derive the estimate (49), we don't need anything new. It follows from the expansion formula for $D_x^\alpha\bar{V}_j$ $(|\alpha|\le s-\sigma_2)$, the boundedness theorem of Fourier integral operators and a simple interpolation argument.

(ii) Moreover, if we apply the argument of [DH pp. 199-200] to $M(x,D;u)$ given by (3.6), we can replace $\bar{P}$ in Theorem A.1 by $M(x,D;u)$.

By integrating $M(x,D;u)$ in Γ_0 and D_{y^1} in each Γ_j, we can easily prove the following :

Theorem A.6. *Let* U_o, Γ_j $(0\le j\le J)$ *be the same as those in Theorem A.2 and* $M(u)= M(x,D;u)$ *be the one given by (3.56). Then, there exists a partition of unity* $A_j(x;\xi)\in S^1$ $(0\le j\le J)$ *subordinated to* $U_o\times\Gamma_j$ $(0\le j\le J)$ *which satisfies the following : Given a positive integer* ν, *there exist* $\gamma_3>0$, β_3, $\sigma_3\in\mathbf{Z}_+$ $(\beta_3>\sigma_3)$ *and linear operators* $Q_j=Q_j(u)$ $(0\le j\le J)$ *such that, for any* u $(\|u-u_0\|_{\beta_3}\le\gamma_3)$,

(36) $M(u)Q_j(u)A_j \equiv A_j \quad (mod S^{-\nu})$,

(37) $\|Q_j(u)h\|_{s-\sigma_3} \leq C_s(\|h\|_s + \|h\|_{\sigma_3}\|u\|_s) \quad (\sigma_3 \leq s \in \mathbf{Z}_+ \; ; h \in H^s(\mathbf{R}^n))$.

references

[BGY] R. Bryant, P.A. Griffiths and D. Yang, Characteristics and existence of isometric embeddings, Duke Mathematical Journal, 3(1983),893-994.

[DH] J.J. Duistermaat and L. Hormander, Fourier integral operators II, Acta Math., 128(1972),183-269.

[E] Ju.V. Egorov, Canonical transformations and pseudo-differential operators, Trans. Moscow Math. Soc., 24(1971), 1-28.

[H] L. Hormander, Fourier integral operators I, Acta Math., 127(1971), 79-183.

[J] H. Jacobowitz, Local isometric embeddings of Riemannian and pseudo-Riemannian manifolds, Memoirs of the Amer. Math. Soc. 97(1970).

[L] C.S. Lin The local isometric embedding $\mathbf{R}^3$ of two dimensinal Riemannian manifolds with Gaussian curvature changing sign cleanly, Ph.D. dissertation at the Courant Institute (1983).

[OMY 1] H. Omori, Y. Maeda and A. Yoshioka, On regular Frechet-Lie groups,I, Some differential geometrical expressions of Fourier-integral operators on a Riemannian manifold, Tokyo J. of Math., 3(1980),353-390.

[OMY 2] H. Omori, Y. Maeda and A. Yoshioka, On regular Frechet-Lie groups, II, Composition rules of Fourier-integral operators on a Riemannian manifold, Tokyo J. Math., 4(1981), 221-253.

[OMYK 3] H. Omori, Y. Maeda, A. Yoshioka and O. Kobayashi, On regular Frechet-Lie groups, III, A second cohomology class related to the Lie algebra of pseudo-differential operators of order one, Tokyo J. Math., 4(1981), 255-277.

[OMYK 4] H. Omori, Y. Maeda, A. Yoshioka and O. Kobayashi, On regular Frechet-Lie groups, IV, Definition and fundamental theorems, Tokyo J. Math., 5(1982), 365-398.

[OMYK 5] H. Omori, Y. Maeda, A. Yoshioka and O. Kobayashi, On regular Frechet-Lie groups, V, Several basic properties, Tokyo J. Math., 6(1983), 39-64.

[OMYK 6] A. Yoshioka, Y. Maeda, H. Omori and O. Kobayashi, On regular Frechet-Lie groups, VII, The group generated by pseudo-differential operators of negative ordr, Tokyo J. Math., 7(1984), 315-336.

[OMYK 7] Y. Maeda, H. Omori, A. Yoshioka and O. Kobayashi, On regular Frechet-Lie groups, VIII, Primordial operators and Fourier integral operators, Tokyo J. Math., 8(1985), 1-47.

[Y] S.T. Yau, Seminar on differential geometry, Princeton University Press and University of Tokyo Press, 1982.