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# Minimal tori in $S^3$ whose lines of curvature lie in $S^2$

by

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# Minimal tori in $S^3$ whose lines of curvature lie in $S^2$

Dedicated to Professor Morio Obata on his 60th birthday

*Kotaro YAMADA*

**0.Introduction.** Let $\varphi : \Sigma \to S^3$ be a minimal immersion of a compact orientable surface $\Sigma$ into the unit 3-sphere $S^3$. It is valuable to study the set of such immersions with given genus of $\Sigma$. For example, when $\Sigma$ is of genus 0, *i.e.* $\Sigma$ is a 2-sphere, $\varphi$ must be a totally geodesic immersion of $S^2$ into $S^3$[L,A].

Assume $\Sigma$ is a torus. In this case, there is a well-known minimal isometric *embedding* of the flat square torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2})$ into $S^3$ called *the Clifford immersion.* Though there are many minimal immersions of a torus into $S^3$, they are not embedded. Thus, it is conjectured that the only minimal embedding of a torus into $S^3$ is the Clifford one[Y].

To study this, we consider minimal immersions of a torus into $S^3$ having the following property:

(*) Each line of curvature of the immersion is closed and lies in some totally geodesic 2-sphere in $S^3$.

The main theorem of this paper is the following:

THEOREM.

(1) *There exist infinitely many minimal immersions of a torus into $S^3$ satisfying (*).*

(2) *A minimal immersion of a torus in $S^3$ satisfying (*) is not embedding provided that it is not congruent with the Clifford one.*

**1.Preliminaries.** Let $\varphi : \Sigma \to S^3$ be a smooth immersion of a surface into the euclidean 3-sphere. The first fundamental form of $\varphi$ is the induced metric $g = \varphi^* < , >$, where $< , >$ is the standard metric of $S^3$. The second fundamental form $h$ of $\varphi$ is defined as $h(X,Y) = - < \overline{\nabla}_X \nu, Y >$ for all vectors X and Y tangent to $\varphi$, where $\nu$ is the unit normal vector field of $\varphi$ and $\overline{\nabla}$ is the canonical connection of $S^3$.

The existence of isothermal coordinates shows us that there exist local coordinates $(u,v)$ of $\Sigma$ in which $g$ is written as

$$g = e^{\sigma}(du^2 + dv^2), \tag{1-1}$$

where $\sigma$ is a smooth function of $u$ and $v$. Write the second fundamental form in these coordinates as

$$h = Ldu^2 + 2Mdudv + Ndv^2, \tag{1-2}$$

where $L, M$ and $N$ are functions of $u$ and $v$.

The mean curvature of $\varphi$ is the function $H$ on $\Sigma$ defined by

$$H = \frac{1}{2}e^{-\sigma}(L+N) \tag{1-3}$$

in the present isothermal coordinates. The immersion $\varphi$ called *minimal* when $H$ is identically 0 *i.e.* $N = -L$ in (1–2).

In these coordinates, the equation of Gauss is

$$-\frac{1}{2}e^{-\sigma}\Delta\sigma = (LN - M^2)e^{-2\sigma} + 1, \qquad \text{where} \quad \Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}. \tag{1-4}$$

Consider the complex function $f$ of $z = u + iv$

$$f(z) = M + iN. \tag{1-5}$$

When $\varphi$ is minimal, the equation of Codazzi holds if and only if $f$ is a holomorphic function of $z$.

**2.Fundamental equation.** Suppose $\varphi : \Sigma \to S^3$ be a minimal immersion of a torus. On taking the universal cover of $\Sigma$ , $\varphi$ is lifted to the minimal immersion $\tilde{\varphi} : \mathbf{R}^2 \to S^3$. Since the induced metric $\tilde{g} = \tilde{\varphi}^* < , >$ is conformal to the flat metric of $\mathbf{R}^2$ [B], there exist *global* coordinates $(u, v)$ in which the first fundamental form is

$$\tilde{g} = e^{\sigma}(du^2 + dv^2), \tag{2–1}$$

where $\sigma$ is a smooth function on $\mathbf{R}^2$ which is invariant by the deck transformations of the cover $\mathbf{R}^2 \to \Sigma$, *i.e.* $\sigma$ is a doubly periodic function. The second fundamental form of $\tilde{\varphi}$ is written as (1–2) , where $L, M$ and $N$ are doubly periodic functions defined on $\mathbf{R}^2$.

Since $\tilde{\varphi}$ is minimal, the complex function $f$ in (1–5) is holomorphic on whole the complex plane. Hence by Liouville's theorem, $L, M$ and $N$ must be constant on $\mathbf{R}^2$. Then, by a suitable change of coordinates, we may assume the second fundamental form is diagonalized as

$$h = L(du^2 - dv^2),$$

where $L$ is a positive constant. Replacing $u, v$ and $\sigma$ by $u/\sqrt{L}, v/\sqrt{L}$ and $\sigma + \log L$ respectively, we have the first fundamental form (2–1) and the second fundamental form

$$h = du^2 - dv^2. \tag{2–2}$$

By (2–1) and (2–2), the equation of Gauss (1–4) is

$$\Delta\sigma = -4\sinh\sigma. \tag{2–3}$$

Conversely, by the fundamental theorem of the theory of surfaces [S], we have the following proposition:

PROPOSITION 2.1.

(1) *If $\varphi : \Sigma \to S^3$ is a minimal immersion of a torus, and $\tilde{\varphi} : \mathbf{R}^2 \to S^3$ is the lift of $\varphi$ to the universal cover of $\Sigma$, then there exist coordinates $(u, v)$ of $\mathbf{R}^2$ in which the first and the second fundamental forms of $\tilde{\varphi}$ is written as (2–1) and (2–2) respectively, and the function $\sigma$ in (2–1) satisfies (2–3).*

(2) *If a smooth function $\sigma$ on $\mathbf{R}^2$ satisfies (2–3), then there exists a minimal immersion $\varphi_\sigma : \mathbf{R}^2 \to S^3$ whose first and second fundamental forms are (2–1) and (2–2) respectively. Moreover, such an immersion is unique up to congruence.*

*Remark.* Even if $\sigma$ in (2–3) is doubly periodic, the corresponding immersion $\varphi_\sigma$ is not necessarily doubly periodic. To study minimal immersions of a torus into $S^3$, we must search for doubly periodic solutions of (2–3) whose corresponding immersions are also doubly periodic.

The trivial solution of (2–3) is $\sigma = 0$. In this case, the corresponding minimal immersion $\varphi_0$ is an isometric minimal immersion of $\mathbf{R}^2$ with flat metric which is written explicitly as

$$\varphi_0(u, v) = \left(\frac{1}{\sqrt{2}}\cos\sqrt{2}u, \frac{1}{\sqrt{2}}\sin\sqrt{2}u, \frac{1}{\sqrt{2}}\cos\sqrt{2}v, \frac{1}{\sqrt{2}}\sin\sqrt{2}v\right) \in S^3,$$

where $S^3 = \{(x^0, x^1, x^2, x^3) \in \mathbf{R}^4; \sum_{i=0}^{3}(x^i)^2 = 1\}$. Since $\varphi_0$ is doubly periodic, it gives the minimal isometric immersion of the flat torus $\mathbf{R}^2/\Gamma$ into $S^3$, where $\Gamma$ is the lattice on $\mathbf{R}^2$ generated by $\{(0, \sqrt{2}\pi), (\sqrt{2}\pi, 0)\}$. This immersion is called *the Clifford immersion*, which has the following properties:

(1) It is the only isometric minimal immersion of the flat torus into $S^3$ up to congruence.

(2) The immersion is 1 to 1, *i.e.* it is embedding.

(3) The area of the immersed torus is $2\pi^2$.

(4) The immersion is given by the first eigenfunctions of the laplacian of $\mathbf{R}^2/\Gamma$. In other words, the first eigenvalue of the laplacian of $\mathbf{R}^2/\Gamma$ is 2.

**3.Lines of curvature.** Suppose $\varphi : \mathbf{R}^2 \to S^3$ be a minimal immersion with the first and second fundamental forms (2–1) and (2–2) respectively.

Vector fields $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ give the principal directions of $h$, and their integral curves are the lines of curvature of $\varphi$. Let

$$c_u(v) = \varphi(u,v), \qquad c_v(u) = \varphi(u,v). \tag{3–1}$$

Then curves $c_u$ and $c_v$ in $S^3$ are lines of curvature of $\varphi$ parameterized by $v$ and $u$ respectively. The following lemma is easy to show.

LEMMA 3.1.

(1) *The curve $c_u$ has the curvature*

$$\kappa_u = \frac{1}{2}e^{-\sigma/2}\{(\partial_u\sigma)^2 + 4e^{-\sigma}\}^{1/2}$$

*and the tortion*

$$\tau_u = e^{-\sigma/2}[\{\partial_v(\frac{e^{-\sigma/2}\partial_u\sigma}{2\kappa_u})\}^2 + \{\partial_v(\frac{e^{-\sigma}}{\kappa_u})\}^2]^{1/2}.$$

(2) *The curve $c_v$ has the curvature*

$$\kappa_v = \frac{1}{2}e^{-\sigma/2}\{(\partial_v\sigma)^2 + 4e^{-\sigma}\}^{1/2}$$

*and the tortion*

$$\tau_v = e^{-\sigma/2}[\{\partial_u(\frac{e^{-\sigma/2}\partial_v\sigma}{2\kappa_v})\}^2 + \{\partial_u(\frac{e^{-\sigma}}{\kappa_v})\}^2]^{1/2}.$$

LEMMA 3.2. *Each line of curvature of $\varphi$ lies in some totally geodesic 2-sphere in $S^3$ if and only if $\sigma$ is the following form:*

$$\sigma(u,v) = \log\{U(u) + V(v)\}^2 \tag{3–2}$$

*where $U$ and $V$ are smooth functions on* $\mathbf{R}$.

PROOF: Suppose $\sigma$ is as in (3–2). So, it is an easy consequence of Lemma 3.1 that $\tau_u$ and $\tau_v$ are identically 0 for any $u$ and $v$. Then each $c_u$ and $c_v$ lies in some totally geodesic 2-sphere in $S^3$.

Conversely, if each $\tau_u$ is identically 0, $\partial_v(\frac{e^{-\sigma}}{\kappa_u})$ must be identically 0. Hence $4(e^{\sigma}\kappa_u)^2 = (\partial_u e^{\sigma/2})^2 + 1$ must depend only on $v$. Let $\partial_u e^{\sigma/2} = V(v)$. Then $e^{\sigma/2} = U(u) + V(v)$ for some function $U(u)$ and the conclusion follows.□

PROPOSITION 3.3. *Let $\varphi : \mathbf{R}^2 \to S^3$ be a minimal immersion with the first and second fundamental forms* (1–6) *and* (1–7) *respectively. Then each line of curvature of $\varphi$ lies in some totally geodesic 2-sphere in $S^3$ if and only if $\sigma(u,v)$ depends only on one variable $u$ or $v$.*

PROOF: If $\sigma$ depends only on $u$ or $v$, $c_u$ and $c_v$ are curves without tortion because of Lemma 3.1.

Assume each $c_u$ or $c_v$ lies in a totally geodesic $S^2$. Then $\sigma$ is written as (3–2). Substituting (3–2) in (1–8), we have

$$U''(U+V) + V''(U+V) + (U')^2 + (V')^2 = 1 - (U+V)^4,$$

where $U' = \frac{dU}{du}, V' = \frac{dV}{dv}$ *etc.* Differentiating this equation by $u$ and $v$,

$$U'''V' + U'V''' = -12(U+V)^2U'V'.$$

If $U'V' \neq 0$, then

$$(\frac{U'''}{U'}) + (\frac{V'''}{V'}) = -12(U+V)^2.$$

Differentiating the above, we obtain $U'V' = 0$, so $U'V'$ must be identically 0. Hence $U$ or $V$ is a constant function.□

**4.Differential equation.** In this section, we construct a family of minimal immersions of $\mathbf{R}^2$ into $S^3$ whose lines of curvature lie in some totally geodesic 2-spheres in $S^3$.

Let $\varphi : \mathbf{R} \to S^3$ be one of such immersions. So, by Proposition 2.1. and Proposition 3.3., there exist coordinates $(u, v)$ of $\mathbf{R}^2$ such that

(1) The first fundamental form of $\varphi$ is

$$g = e^{\sigma}(du^2 + dv^2). \tag{4–1}$$

(2) The second fundamental form of $\varphi$ is

$$h = du^2 - dv^2. \tag{4–2}$$

(3) The function $\sigma$ depends on only $v$.

(4) The function $\sigma(v)$ satisfies the ordinally differential equation:

$$\frac{d^2\sigma}{dv^2} = -4\sinh\sigma. \tag{4–3}$$

Conversely, for each solution of (4–3), there exists a minimal immersion of $\mathbf{R}^2$ whose first and second fundamental forms are (4–1) and (4–2) respectively.

The equation (4–3) has an integral:

$$\frac{1}{2}\left(\frac{d\sigma}{dv}\right)^2 + 4\cosh\sigma = 4\alpha,$$

where $\alpha$ is an integral constant. Then for each $\alpha \in [1, \infty)$, there exists a unique solution $\sigma_\alpha$ such that:

$$\frac{1}{2}\left(\frac{d\sigma_\alpha}{dv}\right)^2 + 4\cosh\sigma_\alpha = 4\alpha, \tag{4–4}$$

(4–5) $$\sigma_\alpha(0) = \log a, \qquad \text{where} \quad a = \alpha + \sqrt{\alpha^2 - 1},$$

(4–6) $$\frac{d^2\sigma_\alpha}{dv^2}(0) \geq 0.$$

LEMMA 4.1.

*The solutions $\{\sigma_\alpha; \alpha \in [1, \infty)\}$ have the following properties:*

(1) $\sigma_1 = 0$.

(2) *For each $\alpha \in (1, \infty)$, $\sigma_\alpha$ is a periodic function with period:*

$$T(\alpha) = \frac{2}{\sqrt{a}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - (1 - a^{-2}) \sin^2 x}}.$$

(3)

$$\sigma_\alpha(v) = \sigma_\alpha(-v), \quad \frac{d\sigma_\alpha}{dv}(v) = \frac{d\sigma_\alpha}{dv}(-v).$$

(4) $-\log a \leq \sigma_\alpha \leq \log a$.

(5) *$\sigma_\alpha$ is simply decreasing on $[0, \frac{T(\alpha)}{2}]$, and increasing on $[\frac{T(\alpha)}{2}, T(\alpha)]$.*

PROOF: (1) and (3) are immediate consequences of (4–4).

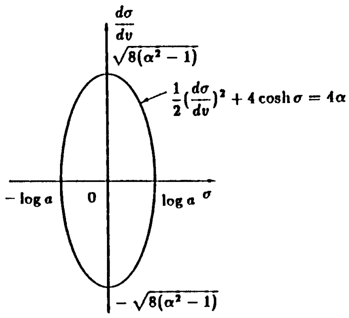



**Fig.1**

**Fig.1** is the phase curve of the solution $\sigma_\alpha$ of the equation (4–4). The tangent vectors of this curve $(\frac{d\sigma_\alpha}{dv}, \frac{d^2\sigma_\alpha}{dv^2})$ never vanishes, so $\sigma_\alpha$ is periodic with period

$$\begin{aligned} T(\alpha) &= \int_0^{T(\alpha)} dv = 2\int_{-\log a}^{\log a} \frac{d\sigma_\alpha}{d\sigma_\alpha/dv} \\ &= 2\int_{-\log a}^{\log a} \frac{d\sigma}{\sqrt{8(\alpha - \cosh\sigma)}} \\ &= \frac{2}{\sqrt{a}}\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-(1-a^{-2})\sin^2 x}}, \end{aligned}$$

and thus (2) is proved.

By **Fig.1**, (4) and (5) are also proved.□

By Proposition 1.1., there exists the immersion $\varphi_\alpha$ of $\mathbf{R}^2$ into $S^3$ defined by (4–1),(4–2) and $\sigma = \sigma_\alpha$. Since $\sigma_1 = 0$, the immersion $\varphi_1$ is the Clifford immersion.

*Remark.* Though the period of the Clifford immersion in a direction $v$ is $\sqrt{2}\pi$, $\lim_{\alpha\downarrow 1} T(\alpha) = \pi$. This shows that the Clifford immersion is *isolated* in the family $\{\varphi_\alpha\}$ as an immersion of the torus.

Consider the lines of curvature of $\varphi_\alpha$,

$$c_u^\alpha(v) = \varphi_\alpha(u,v), \qquad c_v^\alpha(u) = \varphi_\alpha(u,v).$$

Since they lie in some totally geodesic 2-spheres in $S^3$, we may consider each of $c_u^\alpha$ and $c_v^\alpha$ as a curve in $S^2 \subset \mathbf{R}^3$. By Lemma 3.1., we obtain the following lemma.

LEMMA 4.2.

(1) $c_v^\alpha$ *is the curve with the curvature*

$$\kappa_v^\alpha = \sqrt{2\alpha e^{-\sigma_\alpha} - 1}$$

*in* $S^2$.

(2) $c_v^\alpha$ *is a small circle with radius* $e^{\frac{\sigma_\alpha}{2}}/\sqrt{2\alpha}$ *in* $\mathbf{R}^3$.

(3) $\varphi_\alpha$ *gives a minimal immersion of the cylinder whose fundamental domain is*

$$C_\alpha = \{(u,v); 0 \le u < \sqrt{\frac{2}{\alpha}}\pi\} \subset \mathbf{R}^2.$$

(4) $c_u^\alpha$ *is the curve with the curvature*

$$\kappa_u^\alpha = e^{-\sigma_\alpha}.$$

(5) *The curves* $c_u^\alpha$ *are congruent with each other.*

**5.Proof of the main theorem.** Suppose that $\sigma_\alpha$ and $\varphi_\alpha$ be as in the previous section. By Lemma 4.2.(3), $\varphi_\alpha$ gives an isometric immersion of a cylinder. Hence $\varphi_\alpha$ gives an immersion of a torus with closed lines of curvature if and only if the curve $c_u^\alpha$ is closed with some integral times of the period of $\sigma_\alpha$.

The theorem mentioned in **Introduction** is an immediate consequence of the following proposition:

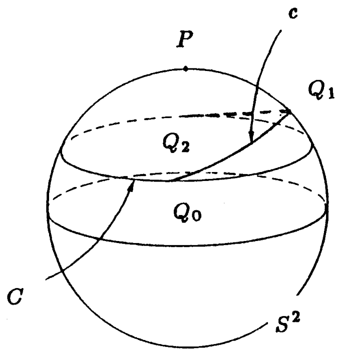
PROPOSITION 5.1.

(1) *There exist countably many* $\alpha$*'s in* $(1,\infty)$ *such that the curve* $c_u^\alpha$ *is closed with period* $k_\alpha T(\alpha)$ *for some positive number* $k_\alpha > 2$.

(2) *If* $\alpha \in (1,\infty)$ *is as in* (1), $c_u^\alpha$ *has a self intersection in the period* $[0, \kappa_\alpha T(\alpha)]$.

Take $\alpha \in (1,\infty)$, and let $T = T(\alpha)$ and $\sigma = \sigma_\alpha$. Consider $c = c_u^\alpha|_{[0,T(\alpha)]}$ as a curve in $S^2 \subset \mathbf{R}^3$. Let $C$ be the circle of curvature in $S^2$ of $c$ at $v = 0$, and $P$ be its center. Because (4–5), $C$ is the small circle of $S^2$ with radius

$$R(\alpha) = \frac{1}{\sqrt{1+a^2}}.$$



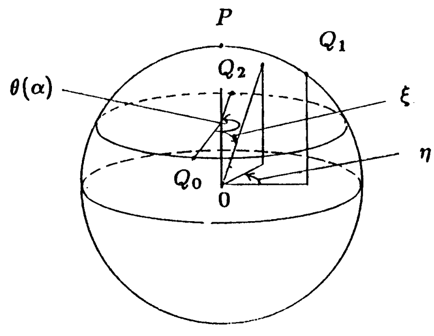




**Fig.2** **Fig.3**

By Lemma 4.1., $\kappa_u^\alpha(v) \geq -a$ and $\kappa_u^\alpha(0) = -a$. So $c$ lies in the interior of $C$. Since $\kappa_u^\alpha$ is an odd periodic function with period $T$, $c|_{[0,T/2]}$ and $c|_{[T/2,T]}$ are symmetric with respect to a great circle of $S^2$ through $P$ and $c(T/2)$. So $c$ is tangent to $C$ at $v = T$(**Fig.2**).

Take polar coordinates $(\xi, \eta)$ of $S^2$ as **Fig.3**, and let $c(v) = (\xi(v), \eta(v))$, where $\xi(0) = \arcsin R(\alpha)$ and $\eta(T/2) = 0$. Define a function $\theta(\alpha)$ as

$$\text{(5–1)} \qquad \theta(\alpha) = \int_0^T \frac{d\eta(v)}{dv}\, dv.$$

LEMMA 5.2.

(1) *$\theta(\alpha)$ is a continuous function of $\alpha$.*

(2) $$\pi \leq \frac{\pi}{R(\alpha)} < \theta(\alpha) < 2\pi\sqrt{\frac{\alpha}{\alpha+1}} \leq 2\pi.$$

(3) $$\lim_{\alpha \downarrow 1} \theta(\alpha) = \sqrt{2}\pi, \qquad \lim_{\alpha \uparrow \infty} \theta(\alpha) = \pi.$$

PROOF: The first claim holds, because $\{\varphi_\alpha\}$ gives the $C^\infty$ family of minimal immersions.

Let $Q_0 = c(0), Q_1 = c(T/2)$ and $Q_2 = c(T)$ be as in **Fig.2**. The length of $c$ is shorter than that of arc $Q_0Q_2$, so $\pi < R(\alpha)\theta(\alpha)$.

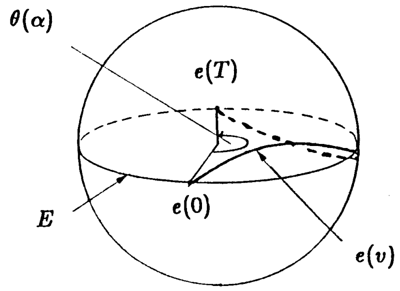



**Fig.4**

Consider $c$ as a curve in $\mathbf{R}^3$ with curvature

$$\tilde{\kappa}_u^\alpha = \sqrt{(\kappa_u^\alpha)^2 + 1} = \sqrt{e^{-2\sigma_\alpha} + 1}.$$

Let $e$ be the Gauss map of $c$ as in **Fig.4** , *i.e.* the unit tangent vector of $c$. So $\theta(\alpha)$ is the length of the projection of $e$ onto the equator $E$ which contains $e(0)$ and $e(T)$. Then,

$$\begin{aligned}
\theta(\alpha) &\leq \text{length of } e(v) \\
&= \int_0^T \tilde{\kappa}_u^\alpha(v) \left\| \frac{dc}{dv}(v) \right\| dv \\
&= \int_{-\log a}^{\log a} \sqrt{\frac{\cosh\sigma}{\alpha - \cosh\sigma}}\, d\sigma \\
&= \sqrt{\frac{8}{\alpha}} \int_0^{\frac{\pi}{2}} \frac{dx}{(1 - \frac{\alpha-1}{\alpha}\sin^2 x)\sqrt{1 - \frac{\alpha-1}{2\alpha}\sin^2 x}} \\
&\leq 2\pi\sqrt{\frac{\alpha}{\alpha+1}}.
\end{aligned}$$

Thus (2) is proved.

The first claim of (3) is an easy consequence of (2), because $\lim_{\alpha\downarrow 1} R(\alpha) = \frac{1}{\sqrt{2}}$. To prove the second, we consider

$$S^2 = \{(x^0, x^1, x^2) \in \mathbf{R}^3; (x^0)^2 + (x^1)^2 + (x^2)^2 = 1\} \subset \mathbf{R}^3.$$

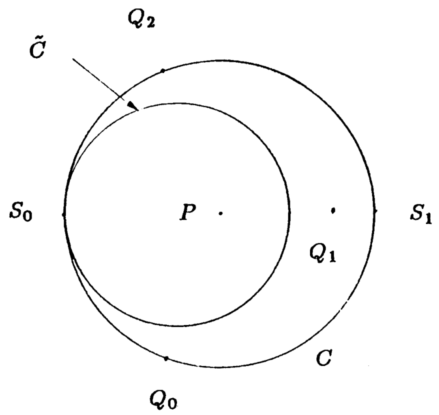



**Fig.5**

Let $S_0$ and $S_1$ be points of $C$ such that the great circle through $S_0$,$S_1$ gives the axis of symmetry of $c$ as **Fig.5**.

Take the coordinates of $\mathbf{R}^3$ such that:

The coordinates of $P$ are $(0,0,1)$.

The coordinates of $S_1$ are $(R(\alpha), 0, \sqrt{1-R(\alpha)^2})$.

The coordinates of $Q_1$ are $(R(\alpha)\cos\frac{\theta(\alpha)}{2}, R(\alpha)\sin\frac{\theta(\alpha)}{2}, \sqrt{1-R(\alpha)^2})$.

Let $d$ and $\tilde{d}$ be the distance functions of $S^2$ and $\mathbf{R}^3$ respectively. The curve $c|_{[0,T/2]}$ joining $Q_0$ and $Q_1$ has length

$$\int_0^T \left\| \frac{dc}{dv} \right\| dv = 2\int_{-\log a}^{\log a} \frac{e^{\sigma/2}\, d\sigma}{\sqrt{8(\alpha - \cosh\sigma)}} = \frac{\pi}{2},$$

so $d(Q_0, Q_1) \leq \frac{\pi}{2}$. In terms of $\tilde{d}$,

$$\sqrt{2} = 2\arcsin\frac{\pi}{4} \geq \tilde{d}(Q_0, Q_1).$$

On putting $\rho = \arcsin R(\alpha) + d(S_1, Q_1)$, $Q_1$ have the coordinates $(\cos\rho, 0, \sin\rho)$. Then

$$\begin{aligned} 2 &\geq \{\tilde{d}(Q_0, Q_1)\}^2 \\ &= 2 - 2R\cos\rho\cos\frac{\theta(\alpha)}{2} - 2\sqrt{1-R^2}\sin\rho, \qquad \text{where} \quad R = R(\alpha), \end{aligned}$$

hence

$$\cos\frac{\theta(\alpha)}{2} \leq -\frac{\sqrt{1-R^2}}{R}\tan\rho. \tag{5–2}$$

Let

$$I = \{v \in [0, \frac{T}{2}]; -\log a \leq \sigma_\alpha(v) \leq \log\frac{a}{\alpha}\}$$

and

$$\ell(\alpha) = \text{length of } c|_I.$$

So,

$$\begin{aligned}\ell(\alpha) &= \int_{-\log a}^{\log\frac{a}{\alpha}} \frac{e^{\sigma/2}\,d\sigma}{\sqrt{8(\alpha-\cosh\sigma)}}\\ &= \frac{1}{2}\arccos\frac{\alpha(\alpha-1)-\sqrt{\alpha^2-1}}{\alpha\sqrt{\alpha^2-1}},\end{aligned}$$

and

$$\lim_{\alpha\uparrow\infty}\ell(\alpha) = 0. \tag{5–3}$$

Take a small circle $\tilde{C}$ of $S^2$ with radius $1/\sqrt{(\frac{\alpha}{a})^2+1}$ (in $\mathbf{R}^3$), which is tangent to $C$ at $S_0$ and lies in the interior of $C$. Since $\kappa_v^\alpha$ is simply increasing on $[0, \frac{T}{2}]$,

$$\kappa_u^\alpha|_{[0,\frac{T}{2}]\setminus I} \leq \frac{\alpha}{a},$$

and $c|_{[0,\frac{T}{2}]\setminus I}$ lies in the exterior of $\tilde{C}$. So,

$$\begin{aligned}\rho &= \arcsin R + d(S_1, Q_1)\\ &\leq \arcsin R + \{\text{diameter of } C - \text{diameter of } \tilde{C} + \ell(\alpha)\}\\ &= \arccos R + 2\arcsin R - 2\arcsin\frac{1}{\sqrt{(\frac{\alpha}{a})^2+1}} + \ell(\alpha).\end{aligned}$$

Let $\tilde{\rho}(\alpha)$ be the right hand side of the above, so by (5–3), we have

$$\lim_{\alpha\uparrow\infty}\tilde{\rho}(\alpha) = \pi - 2\arcsin\frac{2}{\sqrt{5}} < \frac{\pi}{2}. \tag{5–4}$$

By (5–2),

$$\theta(\alpha) \leq 2\arccos\{-\frac{\sqrt{1-R^2}}{R}\tan\rho\}$$
$$\leq 2\arccos\{-\frac{\sqrt{1-R^2}}{R}\tan\tilde{\rho}\}.$$

So,

$$\lim_{\alpha\uparrow\infty}\theta(\alpha) \leq \pi,$$

because $\lim_{\alpha\uparrow\infty} R(\alpha) = 1$ and (5–3). By (2), $\theta(\alpha) \geq \pi$, hence

$$\lim_{\alpha\uparrow\infty}\theta(\alpha) = \pi$$

and (3) is proved.□

PROOF OF PROPOSITION 5.1.: By Lemma 5.1.(1) and (3), $f(\alpha) = \frac{\theta(\alpha)}{2\pi}$ is a continuous function defined on $(1,\infty)$ such that $\lim_{\alpha\downarrow 1} f(\alpha) = \frac{1}{\sqrt{2}}$ and $\lim_{\alpha\uparrow\infty} f(\alpha) = \frac{1}{2}$. Hence there exist countably many $\alpha$'s such that $f(\alpha)$ is a rational numbers. For such $\alpha$, let

$$f(\alpha) = \frac{m_\alpha}{k_\alpha}, \qquad \text{where} \quad m_\alpha, k_\alpha \in \mathbf{Z}^+ \quad \text{and} \quad (m_\alpha, k_\alpha) = 1.$$

So, $k_\alpha\theta(\alpha) \equiv 0 (\text{mod } 2\pi)$ and $c_u^\alpha$ is closed in $k_\alpha$ times of $T(\alpha)$.

Since $\frac{1}{2} < f(\alpha) < 1$, $2 < k_\alpha$ and the rotation number of $c_u^\alpha|_{[0,k_\alpha T(\alpha)]}$ is greater than 2. Hence $c_u^\alpha$ has a self intersection in the period of the immersion $\varphi_\alpha$ for $\alpha \in (1,\infty)$. This completes the proof of the proposition.□

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