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Limit Theorems related to the Interface of the three-dimensional Ising model

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Ising Model

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1. Introduction

We consider the three-dimensional Ising model, in a box , $V_{L,M} = \{t = (t_1,t_2,t_3) \in \mathbf{Z}^3; 0 \le t_1,t_2 \le L, -M \le t_3 \le M+1\}$, with ± boundary condition ; spin variables are fixed to be +1 and -1 on the upper and the lower half part of the boundary of the box respectively. This boundary condition yields a surface which decomposes the box into the upper region and the lower region surrounded by + and - spins respectively. We call such a surface an interface. In the ground state, i.e. T=0 (or $\beta = \infty$) , the interface is perfectly flat. However, the interface λ will be deformed as the temperature goes up from zero, and this deformation is characterised by a family of elementary shapes $\mathbf{w}=(w_1, \ldots, w_n)$ called standard walls [1],[2]. Dobrushin showed that at sufficiently low temperature the interface doesn't fluctuate and becomes " rigid " in the following sence ; the probability that the interface passes through a point which isn't on the perfect flat surface at T=0 tends to zero , as $\beta \to \infty$,uniformely in $V_{L,M}$. This implies that the limit Gibbs state becomes non-translational invariant state [1].

In this paper we investigate the fluctuation on this "nearly flat " interface ,at sufficiently low temperature , by considering the ramdom field $X^L(t,s)$ given by ,

$$X^L_{(t,s)}(\mathrm{w}) = \frac{1}{\sigma(\beta)L} \sum_{\substack{w \in \mathrm{w} \\ w \subset [0,tL]\times[0,sL]}} F(w),$$

where F(w) is the real valued function of w satisfying some conditions which will be stated in § 3, and the sum runs over all $w \in \mathrm{w}$ in the region , $[0,tL]\times[0,sL]$,of the interface . (See Fig. 1)

Letting M $\to\infty$ we induce the probability distribution $P_L(\lambda)$ of the interface λ in $V_L^* = \{t \in \mathbf{Z}^3 \,;\, 0 \le t_1, t_2 \le L\}$. This probability distribution $P_L(\cdot)$ is described as the Gibbs .ate for the system of standard walls with long range interaction.

We shall prove that the finite dimensional distribution of $X^L_{(t,s)}$ comverges to the corresponding distribution of the Brownian sheet as $L\to\infty$ if the temperature is sufficiently low .

We use the method of low-temperature expansion or polymer expansion developped by Gallavotti,Martin-Lof, Miracle-Sole [3,4,5] and Del Grosso [6]. This method is known to be very useful for studying the probabilistic behaviour of the phase-separation line or the analyticity of the correlation functions. For instance ,Higuchi [7] proved that the phase sepalation line in the two dimensional Ising model converged to the Brownian Bridge.Also Bricmont ,Lebowitz and Pfister proved the analyticity of the spin correlation functions for three-dimensional Ising model and Widom-Rowlinson model [8].

To prove the convergence of the finite dimensional distribution of $X^L(t,s)$ we express the probability distribution of the interface in terms of the contour expansion [3] and apply the method of polymer expansion to the system of standard walls . Our method strongly rely on [7,8].

We introduce the continuous randomfield $Y^L(t,s)$ by refining $X^L(t,s)$ to state our result in more mathematical or more probabilistic form , and we prove the convergence of $Y^L(t,s)$ to the Brownian Sheet as the distribution on the path space $\mathbf{C}([0,1]^2 \to \mathbf{R})$ by checking the moment condition for tightness.

2. The interface and it's probability distribution

2.1. 3-dimensional Ising Model

Consider a 3-dimensional cubic lattice $\mathbf{Z}^3$. We arrange either a (+)-particle or (-)-particle at each site $t \in \mathbf{Z}^3$. A configuration space in $\mathbf{Z}^3$ is defined by

$$\Omega = \{ +1 , -1 \}^{\mathbf{Z}^3}.$$

Let us now consider the system enclosed in V

$$V = V_{L,M} = \{ t = (t_1,t_2,t_3) \in \mathbf{Z}^3 ; 0 \le t_1 , t_2 \le L , -M \le t_3 \le M+1\},$$

with the boundary condition $\omega_\pm$ given by,

$$\omega_\pm(t) = \begin{cases} +1 & \text{if } t_3>0 \\ -1 & \text{if } t_3 \le 0 \end{cases}$$

We associate to each configuration $\xi \in \Omega_V = \{+1,-1\}^V$ the interaction energy $H_V(\xi|\omega_\pm)$ given by

$$H_V(\xi|\omega_\pm) = -J \sum_{i,j \in V|i-j|=1} \xi(i)\xi(j) - J \sum_{i \in V, j \in V^c, |i-j|=1} \xi(i)\omega_\pm(j), \quad J > 0 ,$$

where $|i-j|$ is the Eucledean distance between i and j .

The Gibbs state on Ω_V for the interaction energy $H_V(\xi|\omega_\pm)$ is defined by

$$P_{V,\pm}(\xi) = \frac{1}{Z_{V,\pm}} \exp\{ -\beta H_V(\xi|\omega_\pm)\}, \tag{2-1}$$

where $\beta > 0$.

As in the two dimensional Ising model , it is convenient to describe the configuration ξ by the family of contours. For a given configuration $\xi \in \Omega_V$ with $\omega_\pm$ we put a unit square perpendicular to the bond <i,j> with $\xi(i)\xi(j) = -1$, or $\xi(i)\omega_\pm(j) = -1$

(j $\in \partial$ V), and passing through the middle point of <i,j> . Then the set of such unit squares is decomposed , in a unique way , into a finite number of closed polyhedrons $\{\Gamma_1,\ldots,\Gamma_n\}$ and an open surface λ which is pinned in the boundary ∂ V $\cap$ S , S = $\{t \in Z^3\ ;\ t_3 = 1/2\}$. By this surface λ the box V is divided into the upper region V_λ^u and the lower region V_λ^l . Owing to the choice of the boundary condition $\omega_\pm$ the interior of V_λ^u is surrounded by (+)- particles and the interior of V_λ^l is surrounded by (-)- particles . (See Fig.1) The surface λ is called an interface or the phase separation surface and each Γ contour . We say a family $\{\lambda,\Gamma_1, \ldots ,\Gamma_n\}$ of an interface and contours is *admissible* if it corresponds to a configuration $\xi \in \Omega_V$

For simplicity we put 2J=1 . Then the probability distribution (2-1) is described by ,

$$P_{V,\pm}(\xi) = \frac{1}{Z_V} \exp\{-\beta|\lambda| - \beta\sum_{i=1}^{n}|\Gamma_i|\}, \tag{2-2}$$

if ξ is given by $\{\lambda,\Gamma_1, \ldots ,\Gamma_n\}$, where $|\lambda|$ and $|\Gamma_i|$ are areas of λ and Γ_i *respectively* .

For the study of the probabilistic behavior of the interface λ we first derive the probability distribution of λ from (2-2) ; it is given by

$$P_V(\lambda) = \frac{1}{Z_V} \exp\{-\beta|\lambda|\}\cdot Z_{V_\lambda^u,+} \cdot Z_{V_\lambda^l,-}, \tag{2-3}$$

where $Z_{V_\lambda^u,+}$ and $Z_{V_\lambda^l,-}$ are the partition functions in V_λ^u and V_λ^l with (+) and (-) boundary conditions respectively. From the symmetricity $Z_{V_\lambda^l,-} = Z_{V_\lambda^l,+}$ and it is explicitely given by ,

$$Z_{V_\lambda l,+} = \sum_{\{\Gamma_1,\ldots,\Gamma_k\}\subset V_\lambda l} \exp\{-\beta\sum_{i=1}^{k}|\Gamma_i|\},$$

where the sum runs over all admissible family of contours in the interior of V_λ^l .

To describe the probability distribution (2- 3) in terms of polymer functionals first induced by Gallavotti-Martinlof we shall review the algebraic formalism for contour expansion. (See [4],[5],[6] for details)

2.2. Contour Expansion

Let $\mathcal{C}$ be the set of all contours in $\mathbf{Z}^3$. $\mathcal{X}$ is the space of mapping X from $\mathcal{C}$ to the set of non-negative integers **N** satisfying,

$$||X|| = \sum_{\Gamma \in \mathcal{C}} X(\Gamma) < \infty.$$

$\mathcal{F}$ is the space of real valued functions on $\mathcal{X}$ defined by ,

$$= \{\Psi: \ \to \mathbf{R}\ s.t.\ |\Psi|_n = sup_{||X||=n} |\Psi| < \infty \quad \text{for } each\ \ n \geq 1\}$$

and the convolution product is defined on $\mathcal{F}$ as follows ,

$$\Psi_1 {*} \Psi_2(X) = \sum_{(X_1,X_2)\,:\,X_1+X_2=X} \Psi_1(X_1)\Psi_2(X_2) \frac{X!}{X_1!X_2!}$$

where the sum is taken over all ordered pair (X_1,X_2) , $X=X_1+X_2$, and X! = $\prod_{\Gamma \in \mathcal{C}} X(\Gamma)!$.

Subspaces $\mathcal{F}_0$ and $\mathcal{F}_1$ of $\mathcal{F}$ are defined by ,

$$\mathcal{F}_0 = \{\Psi \in \mathcal{F} ;\ \Psi(\emptyset) = 0\}$$

and

$$\mathcal{F}_1 = \{\Psi \in \mathcal{F};\ \Psi(\emptyset) = 1\}$$

The exponential mapping $Exp\Psi$ and the logalithm mapping $Log\Psi$ are defined on $\mathcal{F}_0$ and $\mathcal{F}_1$ respectively by ,

$$Exp\Psi(X) = \sum_{n=0}^{\infty} \frac{1}{n!} \Psi^{*n}(X) \qquad \Psi \in \mathcal{F}_0$$

$$Log\Psi(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \Psi_0^{*n}(X) \qquad \Psi = \Psi_0 + e \in \mathcal{F}_1,$$

where $\Psi \in \mathcal{F}_1$ is uniquely expresed as the sum of $\Psi_0 \in \mathcal{F}_0$ and the function e given by ,

$$e(X) = \begin{cases} 1 & \text{if } X = \emptyset \\ 0 & 0 \ \ otherwise \end{cases}$$

As $||X|| < \infty$ the sums above are finite sums .

It follows from the elementaly calculation that

$$Log\ Exp\Psi = \Psi \quad \text{for } \Psi \in \mathfrak{F}_0 \tag{1}$$

$$Exp\ Log\Psi = \Psi \quad \text{for } \Psi \in \mathfrak{F}_1 \ . \tag{2}$$

If $\chi \in \mathfrak{F}$ is multiplicative ,i.e. $\chi(X_1 + X_2) = \chi(X_1)\cdot\chi(X_2)$, and the following condition is satisfied for Ψ_1 and $\Psi_2 \in \mathfrak{F}$,

$$\sum_{X \in \mathfrak{X}} \frac{|\Psi_k(X)\chi(X)|}{X!} < \infty \quad (k = 1,2),$$

then

$$\sum_{X \in \mathfrak{X}} \frac{(\Psi_1 * \Psi_2)(X)}{X!}\chi(X) = \sum_{X \in \mathfrak{X}} \frac{\Psi_1(X)\chi(X)}{X!} \cdot \sum_{X \in \mathfrak{X}} \frac{\Psi_2(X)\chi(X)}{X!} \ . \tag{3}$$

Using this relation we can prove the following lemma .

Lemma 2-1 Put $\Psi^T = \log\Psi$ for $\Psi \in \mathfrak{F}_1$. If $\chi \in \mathfrak{F}$ is multiplicative and

$$\sum_{X \in \mathfrak{X}} \frac{|\Psi^T(X)\chi(X)|}{X!} < \infty,$$

then

$$\sum_{X \in \mathfrak{X}} \frac{|\Psi(X)\chi(X)|}{X!} < \infty$$

and

$$\sum_{X \in \mathfrak{X}} \frac{\Psi(X)\chi(X)}{X!} = \exp\{ \sum_{X \in \mathfrak{X}} \frac{\Psi^T(X)\chi(X)}{X!} \}. \tag{2-4}$$

For each $X \in \mathfrak{X}$ we define a mapping $D_X : \mathfrak{F} \to \mathfrak{F}$ by $(D_X\Psi)(Y) = \Psi(X+Y)$ $\Psi \in \mathfrak{F}$, $Y \in \mathfrak{X}$. We identify $\gamma \in \mathfrak{G}$ with the element of $\mathfrak{X}$ given by

$$\gamma(\Gamma) = \begin{cases} 1 & \text{if } \Gamma = \gamma \\ 0 & otherwise \end{cases} .$$

Then the mapping D_γ is a derivation in the following sense .

Lemma 2-2

(i) $D_\gamma(\Psi_1 * \Psi_2) = (D_\gamma \Psi_1) * \Psi_2 + \Psi_1 * (D_\gamma \Psi_2)$

(ii) $D_\gamma(\mathrm{Exp}\,\Psi) = (D_\gamma \Psi) * \mathrm{Exp}\,\Psi$

Now we shall see how the above algebraic formalism can be applied to the analysis of our system.

We say $X \in \mathcal{X}$ is admissible if X! = 1 and there exists a particle configuration which has supp X as the set of contours. When X is admissible we can say X stands for the particle configuration in $\mathbf{Z}^3$.

We introduce special functions Ψ_β and $\alpha \in \mathcal{G}_1$,

$$\Psi_\beta(X) = \begin{cases} \exp(-\beta|X|) & \text{if } X \text{ is admissible} \\ 0 & \text{otherwise} \end{cases}$$

$$\alpha(X) = \begin{cases} 1 & \text{if } X \text{ is admissible} \\ 0 & \text{otherwise} \end{cases}$$

where $|X| = \sum_{\Gamma \in \mathcal{Y}} X(\Gamma)|\Gamma|$.

Then the partition function for Ising model in V with (+)-boundary condition is given by

$$Z_{V,+} = \sum_{X \subset V} \Psi_\beta(X), \tag{2-5}$$

where X ⊂ V means supp X ⊂ V .

Denote $X \in \mathcal{X}$ by $X = \sum_{i=1}^{k} m_i \gamma_i$, where $m_i \in \mathrm{N}$, $\gamma_i \in \mathcal{Y}$ (i=1,...,k) , and

$$X(\Gamma) = \begin{cases} m_i & \Gamma = \gamma_i \quad (i=1,\ldots,k) \\ 0 & \text{otherwise} \end{cases}$$

We say X is a polymer if for each pair of γ_p and γ_q there exists a chain $\gamma_{i_1}, \cdots \gamma_{i_s}$, $1 \le i_1, \ldots, i_s \le k$, such that $\gamma_{i_1} = \gamma_p$, $\gamma_{i_s} = \gamma_q$ and $\gamma_{i_n} \cap \gamma_{i_{n+1}} \neq \emptyset$ for each n . Let $\{\gamma_1, \ldots, \gamma_u\}$, $u = \sum_{i=1}^{k} m_i$, be a possible rearragement of X such that $\sum_{i=1}^{u} \gamma_i = \sum_{i=1}^{k} m_i \gamma_i$ .We make a graph G(X) from $X = \{\gamma_1, \ldots, \gamma_u\}$ as follows ;as vertices of G(X) we take all $\gamma_i \in$ X and draw a bond between γ_i and γ_j if $\gamma_i \cap \gamma_j \neq \emptyset$. A subgraph C $\subset$ G(X) is called connected if $\gamma_1 ,..., \gamma_u \subset$ C and for any γ_i and γ_j there exists a chain of bonds of C which connects γ_i and γ_j . Then α^T = Log α is rewritten in the form ,

$$\alpha^T(X) = \sum_{C \subset G(X):connected} (-1)^{\#\ of\ bonds\ in\ C} \tag{2-6}$$

We give the proof of (2-6) in the Appendix A .

It follows from (2-6) that

(*1) $\alpha^T(\cdot)$ is translation invariant ,i.e.

$\alpha^T(X_1) = \alpha^T(X_2)$ if supp X_1 is superimposed on supp X_2 by translation τ

and $X_1(\Gamma) = X_2(\tau\Gamma)$ for all $\Gamma \in$ supp X_1 ,

(*2) $\alpha^T(X) = 0$ unless X is a polymer .

From lemma 2-1 we have ,

$$Z_{V,+} = \exp\{ \textstyle\sum_{X \subset V} \Psi^T_\beta(X) \} \tag{2-7}$$

$$\Psi^T_\beta(X) = \exp(-\beta|X|)\alpha^T(X). \tag{2-8}$$

We remark that $\Psi^T_\beta(\cdot)$ also satisfies the above properties (*1) and (*2) .

Lemma 2-3 For sufficiently large β the following estimates are valid ,

(i)

$$\sum_{X \ni O} \frac{|\Psi^T_\beta(X)|}{X!} \leq C(\beta), \tag{2-9}$$

where X $\ni$ O means supp X $\ni$ O and $C(\beta) \rightarrow 0$ exponentially as $\beta \rightarrow \infty$,

(ii)

$$\sum_{X \ni O, |X| \geq k} \frac{|\Psi^T_\beta(X)|}{X!} \leq exp(-\beta_0 k) C(\beta - \beta_0),$$

if $\beta-\beta_0$ is also sufficiently large.

From (2-3) and (2-7) we get ,

$$P_V(\lambda) = \frac{1}{Z_V} \exp\{-\beta|\lambda| - \sum_{X\subset V,\ X\ i\ \lambda} \frac{\Psi_\beta^T(X)}{X!}\}, \tag{2-10}$$

where X i λ means some contour of supp X intersects λ .

2.3. Geometrical description of the interface

For an interface λ we shall introduce the conception of *wall* and *ceilling*. We call $S = \{ x = (x_1, x_2, x_3) \in \mathbf{R}^3 ; x_3 = \frac{1}{2} \}$ standard plane and define the projection $p(x)$ of a point $x = (x_1, x_2, x_3)$ on the standard plane by $p(x) = (x_1, x_2, \frac{1}{2})$.

We decompose all unit squares in the interface into two kinds of squares. We call a square q on λ a *wall square*, or *w−square*, if the vertical line $l(q)$ passing through a center of q intersects at more than one point of λ. ($l(q)$ is the line which is parallel to the third axis.) If $l(q)$ intersects λ at exactly one point we call a unit square q *a ceilling square*, or *c−square*. A set of W -squares and a set of C-squares are called wall part of λ, W(λ) , and ceiling part of λ, C(λ), respectively.

We decompose W(λ) into a finite number of connected components $\{ W_1, W_2, \ldots, W_n\}$, and C($\lambda$), $\{ C_1, C_2, \ldots, C_k \}$. We call each W_i *a wall'* and C_j *a ceilling*. If the set of walls $\{W_1, \ldots, W_n\}$ is given then the set of ceillings $\{ C_1, \ldots, C_k \}$ is uniquely determined.

Now we introduce the notion of *standard wall*. For a wall W of λ the complement of $p(W)$ contains exactly one infinite connected component , say $A_0(W)$. The *base* $C_0(W)$

of the wall W is defined as the ceilling which contacts with W and satisfies $p(\ C_0(W)) \subset A_0(W)$. We call the wall W of λ *standard wall* if $C_0(W)$ is contained in the standard plane . When $C_0(W)$ is contained in the plane $\{\ t \in \mathbf{R}^3\ ;\ t^3 = h_0\}$, we say the height of the base of W is h_0. When the height of the base of W_i is given by $s \neq 0$, we translate W_i by $(0,0,-s)$ and get the standard wall w_i.

In such a way we get a family of standard walls $\{w_1, \ldots, w_n\}$ from the interface λ. ⌒ʳe call a family $\mathbf{w} = \{\ w_1, \ldots, w_n\ \}$ of standard walls on S_V *admissible* if $p(w_i) \cap p(w_j) = \emptyset\ (\ i \neq j\ )$. If an admissible family of standard walls is given then we can construct an interface in a unique way. There is a *one−to−one* correspondence of an interface λ and an admissible family of standard walls.

2.4. Interacting System of Standard Walls

We first introduce the notion of the excess area of the standard wall w by,

$$|w| = \text{(the area of } w\text{)} - \text{(the area of } p(w))$$

When an interface λ is described by an admissible family $\mathbf{w} = \{w_1, \ldots, w_n\}$ of standard walls ,we get

$$|\lambda| = |\mathbf{w}| + |S_V|,$$

where $|\mathbf{w}| = \sum_{i=1}^{n} |w_i|$ and $S_V = S \cup V$. Then the probability distribution (2-10) is rewritten in the form ,

$$P_{L,M}(\lambda) = \frac{1}{Z_V} \exp\{-\beta|\mathbf{w}| - U_V(\mathbf{w})\}, \tag{2-11}$$

where

$$U_V(\mathbf{w}) = \sum_{X \subset V, X\, i\, \lambda} \frac{\Psi_\beta^T(X)}{X!} - \sum_{X \subset V, X\, i\, S} \frac{\Psi_\beta^T(X)}{X!}, \tag{2-12}$$

and Z_V is the normalized constant but it is not the same as in (2-10) . In (2-12) we subtracted the constant term from the first one in order that $U_V(\emptyset) = 0$.

Letting M $\to\infty$ in $V_{L,M}$ we consider the cylindrical region V* ,

$$V^* = V^*_L = \{t \in \mathbf{Z}^3 \; ; 0 \le t_1, t_2 \le L\},$$

and consider the probability distribution of the interface λ in V* . To do this we modify the definition of $U_V(\mathbf{w})$ and introduce the potential $U_{V^*}(\mathbf{w})$ for an admissible family of standard walls on S_V ,

$$U_{V^*}(\mathbf{w}) = \sum_{X \subset V^*,\, X\, i\lambda(\mathbf{w})} \frac{\Psi_\beta^T(X)}{X!} - \sum_{X \subset V^*,\, X\, i\, S} \frac{\Psi_\beta^T(X)}{X!}, \tag{2-13}$$

where $\lambda(\mathbf{w})$ is the interface in V* constructed from $\mathbf{w}$. Using the translation invariance of $\Psi_\beta^T(\cdot)$ we cancel the terms with p(X) $\cap$ p(w) = $\emptyset$ in (2-13) and get ,

$$U_{V^*}(\mathbf{w}) = \sum_{\substack{X \subset V^*, X\, i\, \lambda(\mathbf{w}) \\ p(X) \cap p(\mathbf{w}) \neq \emptyset}} \frac{\Psi_\beta^T(X)}{X!} - \sum_{\substack{X \subset V^*,\, X\, i\, S \\ p(X) \cap p(\mathbf{w}) \neq \emptyset}} \frac{\Psi_\beta^T(X)}{X!}, \tag{2-14}$$

where $p(\mathbf{w}) = \cup_{w \in \mathbf{w}}$ p(w) and p(X) = p(supp X) .

From the estimates in lemma 2-3 we get the following estimate for $U_{V^*}(\mathbf{w})$.

Lemma 2-4 For sufficiently large $\beta > 0$ the following estimate is valid ,

$$|U_{V^*}(\mathbf{w})| \le k_0(\beta) |\mathbf{w}|$$

where $k_0(\beta) \to 0$ exponentially as $\beta \to \infty$.

The proof of this lemma is given in Appendix A .

Now we regard the interface λ in V* as a configuration of "wall" particles $\mathbf{w} = \{w_1, \ldots, w_n\}$ on S_V, and define the probability distribution $P_L(\mathbf{w})$ for w by ,

$$P_L(\mathbf{w}) = \frac{1}{Z_{V*}} \exp\{-\beta|\mathbf{w}| - U_{V*}(\mathbf{w})\} \tag{2-15}$$

for sufficiently large β .

Lemma 2-4 shows the convergence of Z_{V*} ,

$$Z_{V*} = \sum_{\mathbf{w} \subset V*} \exp\{-\beta|\mathbf{w}| - U_{V*}(\mathbf{w})\} < \infty,$$

for sufficiently large β ,and this ensures that (2-15) is well defined .

It follows from the standrd argument that $P_{L,M}(\cdot) \to P_L(\cdot)$ weakly as $M \to \infty$.

We rewrite $U_{V*}(\mathbf{w})$ as the sum of local potentials $\Phi_{V*}(\mathbf{v})$, $\mathbf{v} \subset \mathbf{w}$,

$$U_{V*}(\mathbf{w}) = \sum_{\mathbf{v} \subset \mathbf{w}} \Phi_{V*}(\mathbf{v}).$$

From the Mobius inversion formula we get,

$$\Phi_{V*}(\mathbf{v}) = (-1)^{||\mathbf{v}||} \sum_{\alpha \subset \mathbf{v}} (-1)^{||\alpha||} U_{V*}(\alpha), \tag{2-16}$$

where $||\mathbf{v}||$ is the number of standard walls in $\mathbf{v}$.

It follows from (2-14) and (2-16) that

$$\Phi_{V*}(\mathbf{v}) = \Phi^1_{V*}(\mathbf{v}) + \Phi^2_{V*}(\mathbf{v}),$$

where

$$\Phi^1_{V*}(\mathbf{v}) = (-1)^{||\mathbf{v}||} \sum_{\emptyset \neq \alpha \subset \mathbf{v}} (-1)^{||\alpha||} \sum_{\substack{X\ i\ \lambda(\alpha), X \subset V* \\ p(X) \cap p(\alpha) \neq \emptyset}} \frac{\Psi^T_\beta(X)}{X!}$$

and

$$\Phi^2_{V*}(\mathbf{v}) = (-1)^{||\mathbf{v}||} \sum_{\emptyset \neq \alpha \subset \mathbf{v}} (-1)^{||\alpha||} \sum_{\substack{X\ i\ S, X \subset V* \\ p(X) \cap p(\alpha) \neq \emptyset}} \frac{\Psi^T_\beta(X)}{X!} .$$

We treat $\Phi^1_{V*}(\mathbf{v})$ and $\Phi^2_{V*}(\mathbf{v})$ separately. Exchanging the order of sums in $\Phi^1_{V*}(\mathbf{v})$ we get ,

$$\Phi^1_{V*}(\mathbf{v}) = (-1)^{||\mathbf{v}||} \sum_{X \subset V*} \frac{\Psi^T_\beta(X)}{X!} \sum_{\alpha \in \mathbf{v}(X)} (-1)^{||\alpha||}, \tag{2-17}$$

where

$$\mathbf{v}(X) = \{ \boldsymbol{\alpha} \,;\, \emptyset \neq \boldsymbol{\alpha} \subset \mathbf{v},\, X \ i\ \lambda(\boldsymbol{\alpha}) \,, p(X) \cap p(\boldsymbol{\alpha}) \neq \emptyset \ \}$$

For a given standard wall w on $\mathbf{Z}^2$ the complement of $p(w)$ has just one infinite connected component and several finite connected components. The set of these finite connected components is called the interior part of $p(w)$ and is denoted by Int $p(w)$. We say $w_0 \in \mathbf{v}$ is the inner most unless there is a wall $w \in \mathbf{v}$ such that $p(w) \subset$ Int $p(w_0)$. Denote by $\mathbf{v}_0$ the set of all inner most standard walls of $\mathbf{v}$.

For $v \in \mathbf{v}_0$ and $\boldsymbol{\alpha} \subset \mathbf{v}$ we denote by $v(\boldsymbol{\alpha})$ the part of $\lambda(\boldsymbol{\alpha})$ which projects on $f(v)$, where $f(v) = p(v) \cup$ Int $p(v)$. If $v \notin \boldsymbol{\alpha}$, $v(\boldsymbol{\alpha})$ is a part of ceilling of $\lambda(\boldsymbol{\alpha})$.

Let us suppose that $X \subset V^*$ satisfy the condition (*) ,

(*) there exists $v^* \in \mathbf{v}_0$ such that X doesn't intersect $v^*(\boldsymbol{\alpha})$ for all $\boldsymbol{\alpha} \subset \mathbf{v}$.

We shall prove that ,

$$\sum_{\boldsymbol{\alpha} \in \mathbf{v}(X)} (-1)^{\|\boldsymbol{\alpha}\|} = 0 \quad \text{if } X \ \textit{satisfies} \ \ (*).$$

To construct an interface which intersects X we only need standard walls of $\mathbf{v}$ v^*. If X i $\lambda(\boldsymbol{\alpha})$ and $p(X) \cap p(\boldsymbol{\alpha}) \neq \emptyset$ for $\boldsymbol{\alpha} \subset \mathbf{v} \setminus v^*$, X i $\lambda(\boldsymbol{\alpha} \cup v^*)$, since v^* is the inner most and X doesn't intersect $v^*(\boldsymbol{\alpha})$.

Hence $\mathbf{v}(X)$ is decomposed into two parts ,

$$\mathbf{v}(X) = \{ \boldsymbol{\alpha} \,;\, \boldsymbol{\alpha} \in (\mathbf{v} \setminus v^*)(X)\} + \{ \boldsymbol{\alpha} \cup v^* \,;\, \boldsymbol{\alpha} \in (\mathbf{v} \setminus v^*)(X)\},$$

and using this relation we get ,

$$\sum_{\boldsymbol{\alpha} \in \mathbf{v}(X)} (-1)^{\|\boldsymbol{\alpha}\|} = 0$$

Therefore $\Phi^1_{V^*}(\mathbf{v})$ is rewritten in the form,

$$\Phi^1_{V^*}(\mathbf{v}) = (-1)^{\|\mathbf{v}\|} \sum_{X \in J^1_{V^*}(\mathbf{v})} \frac{\Psi^T_\beta(X)}{X!} \sum_{\boldsymbol{\alpha} \in \mathbf{v}(X)} (-1)^{\|\boldsymbol{\alpha}\|}, \tag{2-18}$$

where

$J^1_{V^*}(\mathbf{v}) = \{ \ X \subset V^* \,;\, \textit{For any } v \in \mathbf{v}_0 \ X \ i \ v(\boldsymbol{\alpha}) \text{ for } \textit{some } \boldsymbol{\alpha} \subset \mathbf{v} \ \}$

We say w is external in $\mathbf{v}$ unless there is a wall v such that p $(w) \subset$ Int p (v) . When we construct an interface $\lambda(w)$ in $\mathbf{Z}^3$ from a single standard wall w , the ceilling part of $\lambda(w)$ has exactly one infinite connected component and several finite connected components. These finite connected components are called inner ceillings of w .

Take any X $\in I^1_{V*}(\mathbf{v})$. If p(X) $\cap\, p(w_0) = \varnothing$ for some external standard wall w_0 $\in \mathbf{v}$, then there exists an inner ceilling C of w_0 such that p(X) $\subset$ p(C) and p($\mathbf{v}_0$) $\subset$ p(C). If C is contained in the plane $\{ t=(t_1,t_2,t_3) \; ; t_3=d_0 \}$, we say the height of C is d_0 . We denote by X_k , $k \in \mathbf{Z}$, the translation of X by ($0,0,kd_0$)

When $d_0 > 0$ there is the positive integer $k_0 > 0$ satisfying

(i) $\mathbf{v}(X_k) = \varnothing$ for *all* $k>k_0$,

(ii) $\{\boldsymbol{\alpha} \in \mathbf{v}(X_{k_0}) \; ;\boldsymbol{\alpha} \ni w_0\} \neq \varnothing$, and

(iii) $\{\boldsymbol{\alpha} \in \mathbf{v}(X_{k_0}) \; ;\boldsymbol{\alpha} \not\ni w_0\} = \varnothing$,

and the negative integer $k_1 < 0$ satisfying

(i) $\mathbf{v}(X_k) = \varnothing$ for *any* $k < k_1$,

(ii) $\{\boldsymbol{\alpha} \in \mathbf{v}(X_{k_1}) \; ; \boldsymbol{\alpha} \not\ni w_0\} \neq \varnothing$. and

(iii) $\{\boldsymbol{\alpha} \in \mathbf{v}(X_{k_1}) \; ;\boldsymbol{\alpha} \ni w_0\} = \varnothing$.

For any k with $k_1 \leq k \leq k_0$ we put ,

$$\mathbf{v}(X_k) = \mathbf{v}^1(X_k) + \mathbf{v}^2(X_k),$$

where $\mathbf{v}^1(X_k) = \{\boldsymbol{\alpha} \in \mathbf{v}(X_k) \; ; \; w_0 \in \boldsymbol{\alpha}\}$ and $\mathbf{v}^2(X_k) = \{\boldsymbol{\alpha} \in \mathbf{v}(X_k) \; ; w_0 \quad \boldsymbol{\alpha}\}$.

Then $\mathbf{v}^1(X_k) = \{\boldsymbol{\alpha} \cup w_0 \; ; \boldsymbol{\alpha} \in \mathbf{v}^2(X_{k-1})\}$ for any $k_1 < k \leq k_0$.

Taking the translation invariance, $\Psi^T_\beta(X_k) = \Psi^T_\beta(X_{k-1})$, into account we get ,

$$\Psi^T_\beta(X_k) \sum_{\boldsymbol{\alpha} \in \mathbf{v}^1(X_k)} (-1)^{||\boldsymbol{\alpha}||} + \Psi^T_\beta(X_{k-1}) \sum_{\boldsymbol{\alpha} \in \mathbf{v}^2(X_{k-1})} (-1)^{||\boldsymbol{\alpha}||} = 0,$$

and moreover

$$\sum_{X\in I_{V*}(\mathbf{v})\,,\,p(X)\cap p(w_0)=\varnothing} \frac{\Psi_\beta^T(X)}{X!} \sum_{\boldsymbol{\alpha}\in\mathbf{v}(X)} (-1)^{\|\boldsymbol{\alpha}\|} = 0\,,$$

since $\mathbf{v}^2(X_{k_0}) = \varnothing$ and $\mathbf{v}^1(X_{k_1}) = \varnothing$.

Hence we have the final formula ,

$$\Phi^1_{V*}(\mathbf{v}) = (-1)^{\|\mathbf{v}\|} \sum_{X\in I_{V*}(\mathbf{v})} \frac{\Psi_\beta^T(X)}{X!} \sum_{\boldsymbol{\alpha}\in\mathbf{v}(X)} (-1)^{\|\boldsymbol{\alpha}\|}, \tag{2-19}$$

where

$I_{V*}(\mathbf{v}) = \{\ X\subset V*\ ;(i)\ \ p(X)\cap p(X)\neq\varnothing$ for *any* $w\in\mathbf{w}$ and (ii) for *any* $w\in\mathbf{v}_0$ *.nere is some* $\boldsymbol{\alpha}\subset\mathbf{v}$ *such that* $X\ i\ w(\boldsymbol{\alpha})\}$.

When $d_0 < 0$ the same formula is obtained .

In the same way as in (2-17) we get

$$\Phi^2_{V*}(\mathbf{v}) = (-1)^{\|\mathbf{V}\|} \sum_{X\subset V*\,,\,X\ i\ S} \frac{\Psi_\beta^T(X)}{X!} \sum_{\boldsymbol{\alpha}\in\mathbf{v}[X]} (-1)^{\|\boldsymbol{\alpha}\|}, \tag{2-20}$$

where

$\mathbf{v}[X] = \{\ \boldsymbol{\alpha}\ ;\varnothing\neq\boldsymbol{\alpha}\subset\mathbf{v}\ ,\ p(X)\cap p(\boldsymbol{\alpha})\neq\varnothing\}$

We put

$A(\mathbf{v}) = \{\ v\in\mathbf{v}\ ;p(X)\cap p(v)\neq\varnothing\ \}$

and

$B(\mathbf{v}) = \{\ v\in\mathbf{v}\ ;p(X)\cap p(v) = \varnothing\ \}$

then

$\mathbf{v}[X] = \{\ \boldsymbol{\alpha}_1\cup\boldsymbol{\alpha}_2\ ;\varnothing\neq\boldsymbol{\alpha}_1\subset A(\mathbf{v})\ ,\ \boldsymbol{\alpha}_2\subset B(\mathbf{v})\ \}$

If $B(\mathbf{v})\neq\varnothing$ then

$$\sum_{\boldsymbol{\alpha}\in\mathbf{v}[X]} (-1)^{\|\boldsymbol{\alpha}\|} = \sum_{\varnothing\neq\boldsymbol{\alpha}_1\subset A(\mathbf{v})} (-1)^{\|\boldsymbol{\alpha}_1\|} \sum_{\boldsymbol{\alpha}_2\subset B(\mathbf{v})} (-1)^{\|\boldsymbol{\alpha}_2\|} = 0\,.$$

Then we have

$$\Phi^2_{V*}(\mathbf{v}) = (-1)^{\|\mathbf{v}\|} \sum_{X \in II_{V*}(\mathbf{v})} \frac{\Psi^T_\beta(X)}{X!} \sum_{\alpha \in \mathbf{v}[X]} (-1)^{\|\alpha\|},$$

where

$II_{V*}(\mathbf{v}) = \{ X \subset V^* \; ; X \; i \; S \; , p(X) \cap p(v) \neq \emptyset$ for *all* $v \in \mathbf{v}\}$

We summarize the above results as Lemma 2-5 .

Lemma 2-5 For sufficiently large $\beta > 0$ we have ,

$$\Phi_{V*}(\mathbf{v}) = (-1)^{\|\mathbf{v}\|} \sum_{X \in I_{V*}(\mathbf{v})} \frac{\Psi^T_\beta(X)}{X!} \sum_{\alpha \in \mathbf{v}(X)} (-1)^{\|\alpha\|}$$

$$+ (-1)^{\|\mathbf{v}\|} \sum_{X \in II_{V*(\mathbf{v})}} \frac{\Psi^T_\beta(X)}{X!} \sum_{\alpha \in \mathbf{v}[X]} (-1)^{\|\alpha\|},$$

where

$I_{V*}(\mathbf{v}) = \{ X \subset V^* \; ; (i) \; p(X) \cap p(X) \neq \emptyset$ for *any* $w \in \mathbf{w}$ and (ii) for *any* $w \in \mathbf{v}_0$ *there is some* $\alpha \subset \mathbf{v}$ *such that* $X \; i \; w(\alpha)\}$.

and

$II_{V*}(\mathbf{v}) = \{X \subset V^* \; ; X \; i \; S \; , p(X) \cap p(v) \neq \emptyset$ for *all* $v \in \mathbf{v} \}$

Lemma 2-6 If β, β_0 and $\beta - \beta_0$ are sufficiently large we have the following upper bound of $|\Phi_{V*}(\mathbf{v})|$,

$$|\Phi_{V*}(\mathbf{v})| \leq 4^{\|\mathbf{v}\|+1} \cdot C(\beta - \beta_0) \cdot Min_{w \in \mathbf{v}_0} |w| \cdot \exp\{-2\beta_0 d(\mathbf{v})\}$$

$$\leq C(\beta - \beta_0) \cdot Min_{w \in \mathbf{v}_0} |w| \cdot \exp\{-\beta_0 d(\mathbf{v})\},$$

where $\mathbf{v}_0$ is the set of all inner most standard walls in $\mathbf{v}$ and $d(\mathbf{v})$ is the shortest length of the path connecting all $w \in \mathbf{v}$.

Proof.

We estimate the first term $\Phi^1_{V*}(\mathbf{v})$ and the second term $\Phi^2_{V*}(\mathbf{v})$ of $\Phi_{V*}(\mathbf{v})$ separately . For a given X and $w_0 \in \mathbf{v}_0$ we define the subset $\mathbf{v}(w_0;X)$ of $\mathbf{v}$ by ,

$$\mathbf{v}(w_0;X) = \{\boldsymbol{\alpha} \subset \mathbf{v} \, ; X \; i \; w_0(\boldsymbol{\alpha})\}.$$

Moreover we denote the part of the interface $\lambda(\boldsymbol{\alpha})$ which projects on $p(w_0)$ by $w_0{}^*(\boldsymbol{\alpha})$. Evidently $w_0{}^*(\boldsymbol{\alpha}) \subset w_0(\boldsymbol{\alpha})$.

If $X \in I_{V*}(\mathbf{v})$ then $\mathbf{v}(w_0;X) \neq \emptyset$ and X intersects $w_0(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha} \in \mathbf{v}(w_0;X)$. But X may not intersect $w_0{}^*(\boldsymbol{\alpha})$. We define the vertical distance $\rho(w_0{}^*(\boldsymbol{\alpha}) \, , X)$ between $w_0{}^*(\boldsymbol{\alpha})$ and X by ,

$$\rho(w_0{}^*(\boldsymbol{\alpha}),X) \;=\; Min\{\, |t-s| \, ; t \in suppX, \; s \in w_0{}^*(\boldsymbol{\alpha}) \; s.t. \; p(t) = p(s) \, \},$$

and put

$$d(w_0,X) = Min_{\boldsymbol{\alpha} \in \mathbf{v}(w_0;X)} \rho(w_0{}^*(\boldsymbol{\alpha}),X).$$

For any fixed $w_0 \in \mathbf{v}_0$ and $X \in I_{V*}(\mathbf{v})$ with $d(w_0,X) = k \geq 0$,suppX must contain the point q of the dual lattice $(\mathbf{Z}^3)^*$ such that the vertical distance between q and $w_0{}^*(\boldsymbol{\alpha})$is given by k . The number of such points q is bounded by $2^{\|\mathbf{v}\|} \cdot (|w_0| \vee 2|p(w_0)|)$ for any $w_0 \in \mathbf{v}_0$. We can also observe that $|X| \geq 2(d(\mathbf{v})+k)$ if $d(w_0,X) = k$

Hence we have ,

$$|\Phi^1_{V*}(\mathbf{v})| \leq 2^{\|\mathbf{v}\|} \cdot \sum_{X \in I_{V*}(\mathbf{v})} \frac{|\Psi^T_\beta(X)|}{X!} \tag{2-21}$$

$$\leq \; 4^{\|\mathbf{v}\|} \cdot Max\{|w_0|, 2|p(w_0)|\} \sum_{k=0}^{\infty} \sum_{X \; O, \; |X| \geq 2(k+d(\mathbf{v}))} \frac{|\Psi^T_\beta(X)|}{X!}$$

$$\leq \; 2 \cdot 4^{\|\mathbf{v}\|} \cdot |w_0| \cdot C(\beta-\beta_0) \cdot \sum_{k=0}^{\infty} \exp\{-2\beta_0(k+d(\mathbf{v}))\}$$

$$\leq 4^{\|\mathbf{v}\|+1} \cdot |w_0| \cdot C(\beta-\beta_0) \exp\{-2\beta_0 d(\mathbf{v})\}.$$

For any X $\in II_{V*}(\mathbf{v})$ we put ,

$$d^*(w_0,X) = d(p(w_0),suppX).$$

When $d^*(w_0,X) = k$, $|X| \geq 2(k+d(\mathbf{v}))$. Hence we can get the estimate of $\Phi^2_{V*}(\mathbf{v})$ in a similar way to $\Phi^1_{V*}(\mathbf{v})$,

$$|\Phi^2_{V*}(\mathbf{v})| \leq 3\cdot 2^{\|\mathbf{v}\|}\cdot |w_0|\cdot C(\beta-\beta_0)\cdot \exp\{-2\beta_0 d(\mathbf{v})\}. \qquad (2\text{-}22)$$

From (2-21) and (2-22) we get the proof of lemma .

3. Statement of Results

In the previous section we regarded the interface λ as the configuration $\mathbf{w}=\{w_1, \ldots ,w_n\}$ of standard walls on S_V , and expressed the probability distribution of λ as the Gibbsian distribution of $\mathbf{w}$ with self energy $|\mathbf{w}|$ and potential $U_{V*}(\mathbf{w})$. In (2-14) the sum of X was restricted in V* and this caused the V*-dependence of $U_{V*}(\mathbf{w})$. To avoid the difficulty in V*-dependence of the potential we introduce the potential $U(\mathbf{w})$ for an admissible family of standard walls on $\mathbf{Z}^2$,

$$U(\mathbf{w}) = \sum_{X\ i\ \lambda(\mathbf{w})\ ,p(X)\cap p(\mathbf{w})\neq\emptyset} \frac{\Psi^T_\beta(X)}{X!} - \sum_{X\ i\ S,p(X)\cap p(\mathbf{w})\neq\emptyset} \frac{\Psi^T_\beta(X)}{X!}, \qquad (3\text{-}1)$$

where $\lambda(\mathbf{w})$ is the interface in $\mathbf{Z}^3$ constructed from $\mathbf{w}$. Using Lemma 2-3 we get the same bound for $U_{V*}(\mathbf{w})$

$$|U(\mathbf{w})| \leq k_0(\beta)|\mathbf{w}| \qquad (3\text{-}2)$$

where $k_0(\beta)$ is the same function as Lemma 2-4.

The local potential $\Phi(\mathbf{v})$ induced from $U(\cdot)$ is expressed in a similar way to $\Phi_{V*}(\mathbf{v})$,

$$\Phi(\mathbf{v}) = (-1)^{\|\mathbf{v}\|} \sum_{X\ \in\ I(\mathbf{v})} \frac{\Psi^T_\beta(X)}{X!} \sum_{\alpha\in\mathbf{v}(X)} (-1)^{\|\alpha\|}$$

$$+ \ (-1)^{\|\mathbf{v}\|} \sum_{X \in II(\mathbf{v})} \frac{\Psi_\beta^T(X)}{X!} \sum_{\boldsymbol{\alpha} \in \mathbf{v}[X]} (-1)^{\|\boldsymbol{\alpha}\|} \tag{3-4}$$

where

$I(\mathbf{v}) = \{X \subset \mathbf{Z}^3; (i)\ p(X) \cap p(w) \neq \emptyset$ for *any* $w \in \mathbf{v}$ and

(ii) for *any* $w \in \mathbf{v}_0$ *there is some* $\boldsymbol{\alpha} \subset \mathbf{v}$ *such that* $X\ i\ w(\boldsymbol{\alpha})\}$

and

$II(\mathbf{v}) = \{X \subset \mathbf{Z}^3; X\ i\ S$ and $p(X) \cap p(v) \neq \emptyset$ for *all* $v \in \mathbf{v}\}$

Now we define the probability distribution $P^*_L(\mathbf{w})$ of $\mathbf{w}$ on S_V by,

$$P^*_L(\mathbf{w}) = \frac{1}{Z^*_L} \exp\{-\beta|\mathbf{w}| - U(\mathbf{w})\} \tag{3-4}$$

for sufficiently large β .

Let $\mathcal{P}$ be the set of standard walls on S and T : $\mathcal{P} \to \mathcal{P}$ be the mapping which maps every point of w $\in \mathcal{P}$ to its mirror image with respect to the standard plane S . (See Fig.)

We consider the functional F(w) defined for w $\in$ and assume the following two conditions on F ;

(i) $F(Tw) = -F(w)$

(ii) $|F(w)| \leq c_0|w|$,

where $c_0 > 0$.

For a given $\mathbf{w}$ on S_V we define $X^L_{(t,s)}(\mathbf{w})$, $0 \leq t,s \leq 1$, by

$$X^L_{(t,s)}(\mathbf{w}) = \frac{1}{\sigma(\beta)L} \sum_{w \in \mathbf{w};\, w \subset [0,tL] \times [0,sL]} F(w),$$

where the sum runs over all w $\in$w contained in

$$\{z=(z_1,z_2,\frac{1}{2}) \in S_V \ ;0 \leq z_1 \leq tL \text{ and } 0 \leq z_2 \leq sL\}$$

Now we can state our first result .

Theorem 1

For sufficiently large β there exists a function $\sigma(\beta) > 0$ and

$$P*_L(X^L_{(t_i,s_i)}(w) \in [T_i,S_i] \quad (i=1,\dots,k)) \rightarrow P(W(t_i,s_i) \in [T_i,S_i] \quad (i=1,\dots,k))$$

as $L\rightarrow\infty$ for any $0 \leq t_i , s_i \leq 1$, $T_i < S_i$ (i=1,...,k) , where (W(s,t) , P) is a Brownian sheet .

Next we define the continuous process $Y^L_{(t,s)}(w)$, 0 <= t,s <= 1 , as follows ,

(i) $Y^L(t,s) = X^L(\frac{k_1}{L},\frac{k_2}{L})$

if $(t,s) = (\frac{k_1}{L} , \frac{k_2}{L})$, $k_1,k_2 \in \mathbf{N}$,

(ii) $Y^L(t,s) = e_1X^L(\frac{k_1}{L},\frac{k_2+1}{L}) + e_2X^L(\frac{k_1+1}{L},\frac{k_2}{L}) + (1-e_1-e_2)X^L(\frac{k_1}{L},\frac{k_2}{L})$

if $(t,s) = (\frac{k_1}{L} + e_1,\frac{k_2}{L} + e_2)$, $e_1 + e_2 \leq 1$,($0 \leq e_1 , e_2 ,= 1$),

and

(iii) $Y^L(t,s) = (e_1 + e_2 - 1)X^L(\frac{k_1+1}{L} ,\frac{k_2+1}{L}) + (1-e_1)X^L(\frac{k_1}{L} ,\frac{k_2+1}{L})$

$+ (1-e_2)X^L(\frac{k_1+1}{L} ,\frac{k_2}{L})$,

if $(t,s) = (\frac{k_1}{L} + e_1 ,\frac{k_2}{L} + e_2)$, $e_1 + e_2 > 1$,($0 \leq e_1 , e_2 \leq 1$) .

On the triangular segment ,

$\{ (t,s) = (\frac{k_1}{L} + e_1 ,\frac{k_2}{L} + e_2) \in [0,1]\times[0,1] \ ; \ 0 \leq e_1 , e_2 \leq 1,$ and $e_1 + e_2 \leq 1\}$

,

$Y^L(\cdot)$ coincides with the plane passing through the three points $(\frac{k_1}{L}, \frac{k_1+1}{L}, X^L(\frac{k_1}{L}, \frac{k_1+1}{L}))$, $(\frac{k_1+1}{L}, \frac{k_2}{L}, X^L(\frac{k_1+1}{L}, k_2))$ and $(\frac{k_1}{L}, \frac{k_2}{L}, X^L(\frac{k_1}{L}, \frac{k_2}{L}))$. (See Fig.) It is obvious that $Y^L(t,s)$ is continuous in (t,s) $\in [0,1]\times[0,1]$. The finite dimensional distribution for $Y^L(t,s)$ also converges to the corresponding distribution of the Brownian Sheet .

Theorem 2

For sufficiently large β there exists a function $\sigma(\beta) > 0$ and

$$P_L^*(Y_{(t_i,s_i)}(w) \in [T_i,S_i] \ (i = 1,\ldots,k))$$

$$\rightarrow P(W(t_i,s_i) \in [T_i,S_i] \ (i = 1,\ldots,k))$$

as $L \rightarrow \infty$ for any $0 \leq t_i, s_i \leq 1$, $T_i < S_i$ $(i=1,\ldots,k)$, where (W(t,s),P) is a Brownian Sheet .

Finally we state our final result . Let C_0 be the space of continuous functions on $[0,1]\times[0,1]$ with supremum norm , and μ_L^* be the distribution of $\{Y^L_{(t,s)}\ ;\ 0\leq t, s \leq 1\}$ on C_0 derived from P_L^* .

Theorem 3

If β is sufficiently large , then μ_L^* converges weakly to the distribution of a Brownian sheet as $L \rightarrow \infty$.

4. Polymer expansion for the wall system

In§ 2.2 we developed the algebraic method for the contour expansion. In this section we apply the polymer expansion to the standard wall system similarly.

Let $\mathcal{P}$ be the set of all standard walls on the standard plane S, and $\mathfrak{F}(\mathcal{P})$ be the space of all mappings A from to the set of non-negative integers N satisfying

$$||A|| \equiv \sum_{w \in} A(w) < \infty$$

The functional space $\mathcal{F} = \mathcal{F}(\mathcal{P})$, a convolution product $*$, subspaces $\mathcal{F}_0$ and $\mathcal{F}_1$ the exponetial mapping Exp, the logalism mapping Log and the mapping D_A are defined in the same way as in § 2.2.

Lemmas 2-1 and 2-2 hold true in this case,too.

We say A is admissible if $A! = 1$ and $p(w_i) \cap p(w_j) = \emptyset$ for any w_i and $w_j \in supp\ A (i \neq j)$.

For an admissible $A \in \mathcal{F}(\mathcal{P})$ we put

$$U_\beta(A) \equiv U_\beta(\mathbf{w}),$$

where $\mathbf{w} = suppA$ and $U_\beta(\mathbf{w})$ is the same functions given by (3.1) in § 3 .

Now we introduce the function $\phi_\beta \in \mathcal{F}_1(\mathcal{P})$,

$$\phi_\beta(A) = \begin{cases} \exp\{-\beta|A| - U_\beta(A)\} & \text{if } A \text{ is admissible,} \\ 0 & \text{otherwise} \end{cases}$$

where $|A| = \sum_{w \in} |w| A(w)$. Then the partition function Z_{V^*} is rewritten in the form,

$$Z^*_{V^*} = \sum_{A \subset V^*} \phi_\beta(A),$$

where $A \subset V^*$ means $p(w) \subset S_V$ for all $w \in supp\ A$.

We shall introduce some terminologies which will be used in the sequel. For sufficiently large β_1 and β_2 we consider the generalised function $\phi_{\beta_1,\beta_2}(A)$ from $\phi_\beta(A)$;

$$\phi_{\beta_1,\beta_2}(A) \equiv \begin{cases} \exp\{-\beta_1|A| - U_{\beta_2}(A)\} & \text{if } A \text{ is admissible} \\ 0 & \text{otherwise} \end{cases}$$

If $\beta_1 = \beta_2$, then $\phi_{\beta_1,\beta_2} = \phi_\beta$. In § 2.2 the weight function $\phi_\beta(\cdot)$ of the contour expansion was, in a sense, multiplicative. But now we have the non-multiolicative factor $U_{\beta_2}(\cdot)$ and this cause the difficulty for the analysis of the polymer expansion.

First, we introduce the lexicographic order in $\mathbf{Z}^2$ in order to introduce the order in . We say $w \in \mathbf{v}$ is the first element in $\mathbf{v}$ if $w \in \mathbf{v}_0$ and i(w) < i(u) for all $u \in \mathbf{v}_0$, $u \neq w$, ,where i(w) is the first point of $p(w)$ in the lexicographic order in $\mathbf{Z}^2$. Denote by w_1 the first element of *supp A*.

For an admissible A we put

$$U_1^{\beta}(A) \equiv \sum_{w_1 \in T \subset A} \Phi_{\beta}(T),$$

where

$$\Phi_{\beta}(T) = \Phi_{\beta}(\ supp\ T\ ).$$

Furthermore,for an admissible A and B with $A \cap B = \emptyset$ and $B \neq \emptyset$ we put

$$W^{\beta}(A,B) \equiv \sum_{w_1 \in T \subset A} \Phi_{\beta}(T \cup B),$$

and

$$K^{\beta}(A,B) \equiv \begin{cases} \sum_{n \geq 1} \sum^{*}_{\{P_1,\ldots,P_n\}} \prod_{i=1}^{n} (\ exp(-W^{\beta}(A,P_i)\ -1) & \text{if} B \neq \emptyset \\ 1 & \text{if} B = \emptyset \end{cases}$$

where the sum $\sum^{*}_{P_1,\ldots,P_n}$ runs over all non-ordered sets $P_1,\ldots,P_n$ such that $\bigcup_{i=1}^{n} P_i = B$ and $P_i \neq P_j$ ($i \neq j$).

We define $\Delta_A(\cdot) \in \mathfrak{F}(\mathcal{P})$ for an admissible A by

$$\Delta_A(B) \equiv (\phi_{\beta_1,\beta_2}{}^{-1} * D_A \phi_{\beta_1,\beta_2})(B).$$

Then they satisfy the following recursion relation.

Lemma 4-1 (Recursion Formula for $\Delta_A(B)$)

For any $w \in$ and A such that $w \cup A$ is *N.O.* and $w < A$, we have

$$\frac{\Delta_{w \cup A}(B)}{B!} = \exp\{-\beta_1 |w| - U_1^{\beta_2}(A \cup w)\} \sum_{R \subset B;\, w \cup A \cup R:\, N.O.} K^{\beta_2}(w \cup A, R) \cdot$$

$$\sum_{\substack{Q \subset B \setminus R,\ Q \not\perp w, \\ Q \cup A \cup R: N.O.}} (-1)^{\|Q\|} \frac{\Delta_{A \cup R \cup Q}(B \setminus (Q \cup R))}{(B \setminus (Q \cup R))!}.$$

where A is N.O. means that A is admissible , and Q I w means that all elements of supp Q intersect w .

Put

$$I_m = I_m(\beta_1, \beta_2)$$
$$= \sup_{A:1 \le \|A\| \le m} \sum_{B:\|B\|=m-\|A\|} \frac{|\Delta_A(B)|}{B!} e^{\frac{\beta_1}{2}|A|}$$

From the recursion formula for $\Delta_A(B)$ we get the estimate for I_m which will play an important role.

Lemma 4-2

For sufficiently large β_1 and β_2, we have

$$I_{m+1} \le I_m k(\beta_1),$$

where $k(\beta_1) \to 0$ as $\beta_1 \to \infty$.

Cor.1 to Lemma 4-2

$$I_m \le k(\beta_1^m),$$

where $k(\beta_1) \to 0$ *as* $\beta_1 \to \infty$.

Cor.2 to Lemma 4-2

For sufficiently large β_1 and β_2 ,

$$\sum_{c \in B} \frac{|\phi^T_{\beta_1,\beta_2}(B)|}{B!} \le c_1 \exp\{-c_2\beta_1\}$$

where c_1 and c_2 are constants .

Proof of Cor.1

By induction we have

$$I_m \leq I_1 \{k(\beta_1)\}^{m+1} .$$

Here

$$I_1 = \sup_{w \in} |\Delta_w(\emptyset)| e^{\frac{\beta_1}{2}|w|} = \sup_{w \in} \exp\{-\frac{\beta_1}{2}|w| - \Phi^{\beta_2}(w)\}$$

Using the estimate of Φ^{β_2}, we get

$$I_1(\beta_1, \beta_2) \leq \exp(-c_1 \beta)$$

for sufficiently large β_1 and β_2 , where c_1 is the constant . Thus we get the estimate of I_m as above.

Proof of Cor.2

It follows from Lemma 2-2 that

$$D_w \phi^T_{\beta_1,\beta_2} = \phi^{-1}_{\beta_1,\beta_2} * D_w \phi_{\beta_1,\beta_2} = \Delta_w .$$

Hence we have,

$$\sum_{B \ni O} \frac{|\phi^T_{\beta_1,\beta_2}(B)|}{B!} = \sum_{w \ni O} \sum_{B} \frac{|\phi^T_{\beta_1,\beta_2}(w \cup B)|}{B!}$$

$$\leq \sum_{w \ni O} \sum_{m=0}^{\infty} I_m \exp\{-\frac{\beta_1}{2}|w|\}$$

$$\leq c_1 \exp\{-c_2 \beta_1\} ,$$

where c_1 and c_2 are constants.

5. Proof of Theorem 1

We define functions χ_j^L , $j=1,2,\ldots,k$ by

$$\chi_j^L \equiv \begin{cases} 1 & \text{if } w \subset [\,0, t_j L\,] \times [\,0, s_j L\,], \\ 0 & \textit{otherwise} \end{cases} .$$

For any $y_1,y_2,\dots,y_k \in \mathbf{R}$, we define the function $f_L = f_L(y_1,y_2,\dots,y_k)$, of $\mathbf{w}$, by

$$f_L(\mathbf{w}) = \sum_{i=1}^{k} y_i \sum_{w\in\mathbf{w}} F(w)\,\chi_i^L(w)$$

Consider the characteristic function $\theta_k^L(\mathbf{y}) = \theta_k^L(y_1,y_2,\cdots,y_k)$ of random vectors

$(\frac{1}{\sigma L}\sum F\cdot\chi_1^L,\ \dots,\ \frac{1}{\sigma L}\sum F\cdot\chi_k^L)$ defined by

$$\theta_k^L(\mathbf{y}) \equiv \langle e^{if_L/\sigma L} \rangle_{P_L^*} ,$$

where $\sigma > 0$.

To prove Thorem 1 it is sufficient to find some function $\sigma = \sigma(\beta) > 0$ defined for sufficiently large β and prove

$$\lim_{L\to\infty}\theta_k^L(\mathbf{y}) = \exp\{-\frac{1}{2}\sum_{m,n}(t_m \wedge t_n)\cdot(s_m \wedge s_n)\,y_m\,y_n\}$$

where the right hand side is the characteristic function of the random vectors $(W(t_1,s_1),\dots,W(t_k,s_k))$ with respect to the Brownian sheet.

First we rewrite θ_k^L in terms of the polymer functional $\phi_\beta(A)$, $A \in \mathfrak{A}(\mathcal{P})$,

$$\theta_k^L(\mathbf{y}) = \frac{\sum_{A\subset V^*}\exp\{if_L(A)/\sigma L\}\phi_\beta(A)}{\sum_{A\subset V^*}\phi_\beta(A)}.$$

where

$$f_L(A) \equiv \sum_{i=1}^{k} y_i \sum_{w\in A} A(w)\,F(w)\,\chi_i^L(w) \quad . \tag{5-1}$$

Since $\exp\{if_L(A)/\sigma L\}$ is a multiplicative function of A and

$$\sum_{A\subset V^*}|\exp\{\,if_L(A)/\sigma L\}\phi_\beta^T(A)| < \infty \qquad \text{for } \textit{sufficiently large}\ \beta > 0 \quad ,$$

we can apply Lemma 2-1 to $\theta_k^L(\mathbf{y})$ and get ,

$$\theta_k^L(\mathrm{y}) \ = \ \exp\{ \sum_{A \subset V^*} \frac{\phi_\beta^T(A)}{A!} (\exp\{\mathrm{i} f_L(A)/\sigma L\} - 1) \ \}. \qquad (5\text{-}2)$$

Using the Taylar expansion we get

$$\theta_k^L(\mathrm{y}) \ = \ \exp\{ \sum_{A \subset V^*} \frac{\phi_\beta^T(A)}{A!} (- \frac{f_L^2(A)}{2\sigma^2 L^2} + \frac{f_L^4(A)}{4!\sigma^4 L^4} \theta(\frac{f_L}{\sigma L}) \ \} \quad , \ |\theta(\cdot)| \le 1 \ , (5\text{-}3)$$

where the first and third terms in the Taylar expansion were cancelled from the fact that

$$i) \ \ f_L(TA) \ = \ -f_L(A) \ \text{ and } \ ii) \ \ \phi_\beta^T(TA) \ = \phi_\beta(A) \ .$$

We use the translation invariance of $\phi_\beta(A)$ for the analysis of $\theta_k^L(\mathrm{y})$. In the contour system the polymer function $\Psi_\beta(\cdot)$ satisfies the useful property that $\Psi_\beta^T(X) = 0$ unless X is a polymer ,but this property does not hold true for $\phi_\beta(\cdot)$.

With each $A \subset V^*$ we associate the minimal path in S_V connecting all standard walls of A in a unique way. We denote by $\bar{A}$ the union of such a minimal path and supp A .

We decompose the two-dimensional square $[0,L] \times [0,L]$ into the set of rectangulars $\{ G_{i\,j} \}$ as in the Fig. by using the line segments passing though the points , $(t_1 L,0),\ldots.,(t_k L,0)$ and $(0,s_1 L),\ldots,(0,s_k L)$.

From (5-3) we have ,

$$\theta_k^L(\mathrm{y}) \ = \ \exp\{ \frac{-1}{2\sigma^2 L^2} \sum_{i,j=1}^{k} \ \sum_{\bar{A} \subset G_{i\,j}} \frac{\phi_\beta^T(A)}{A!} f_L^2(A)$$

$$- \frac{1}{2\sigma^2 L^2} \sum_{\substack{\bar{A}\, i\, (\cup_{i,j} \partial G_{i,j}) \\ A \subset V^*}} \frac{\phi_\beta^T(A)\, f_L^2(A)}{A!} + \frac{1}{4!\sigma^4 L^4} \sum_{A \subset V^*} \frac{\phi_\beta^T(A)}{A!} f_L^4 \, \theta(\frac{f_L}{\sigma L}) \} \quad (5\text{-}4)$$

where in the second terms A runs over all A such that $\bar{A}$ intersects the boundary of some $G_{i,j}$.

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We prove the following two lemmas which will play a dominant role for the proof of the Theorem .

Let s(L) be the set of line segments in $[0,L]^2 \subset \mathbf{Z}^2$,and the total length $|s(L)|$ of all line segments is bounded from above by cL .

Lemma 5-1 For sufficiently large β we have ,

$$\lim_{L\to\infty} \frac{1}{L^2} \sum_{\substack{A\ i\ s(L) \\ A \subset V^*}} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k} = 0 ,$$

where $F(A) = \sum_w F(w)A(w)$.

Lemma 5-2 For sufficiently large β the following limits exist,

$$\lim_{L\to\infty} \frac{1}{L^2} \sum_{A \subset [0,tL] \times [0,sL]} \frac{\phi_\beta^T(A) F(A)^{2k}}{A!} = B_k(\beta) ts$$

and $B_k(\beta) > 0$ (k= 1,2) .

For the proof of lemma 5-1 we introduce two functions

$$\gamma(d) = \sum_{\substack{\underline{A}\ O \\ |\overline{A}|=d}} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k}$$

and

$$\delta(d) = \sum_{\substack{i(A)=0 \\ |\overline{A}|=d}} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k} ,$$

where $|\overline{A}| = |p(A)| +$ total length of the minimal path connecting all $w \in A$, and i(A) is the initial point in $\mathbf{Z}^2$ of p(A) in lexicographic order .

It is easy to see that $0 \le \gamma(d) = d\delta(d)$ and

$$\sum_d \frac{\gamma(d)}{d} = \sum_d \delta(d) = \sum_{i(A)=0} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k} < \infty .$$

Hence we have ,

$$\lim_{L\to\infty} \frac{1}{L} \sum_{d=4}^{L} \gamma(d) = 0 \quad .$$

Using these two facts we have ,

$$\frac{1}{L^2} \sum_{\substack{\bar{A}\ i\ s(L) \\ A \subset V^*}} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k}$$

$$\leq \frac{cL}{L^2} \sum_{\substack{\bar{A} \ni o \\ |A| \leq L}} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k} + \sum_{\substack{i(A)=0 \\ |A|>L}} \frac{|\phi_\beta^T(A)|}{A!} F(A)^{2k}$$

$$= c\frac{1}{L} \sum_{d=4}^{L} \gamma(d) + \sum_{d=L}^{\infty} \delta(d) \to 0 \ as \ L\to\infty.$$

This proves lemma 5-1. In the same way we can prove lemma 5-2.

Now we shall use these two lemmas for the proof of Theorem 1 .

Let us remark that $|\bigcup_{i,j} \partial G_{ij}| \leq KL$ for some constant $K > 0$. From lemma 5-1 we get,

$$\frac{1}{2\sigma^2 L^2} \sum_{\substack{A\ i\ (\bigcup_{i,j} \partial G_{ij}) \\ A \subset V^*}} \frac{|\phi_\beta^T(A)|\, f_L^2(A)}{A!} \to 0 \ as \ L \to \infty \tag{5-5}$$

The third term in (5-4) is estimated as follows ,

$$\frac{1}{4!\sigma^4 L^4} \sum_{A \subset V^*} \frac{|\phi_\beta^T(A)\, f_L^4(A)|}{A!} \ |\theta(\frac{f_L}{\sigma L})|$$

$$\leq \frac{L^2}{4!\sigma^4 L^4} \sum_{i(A)=0} \frac{|\phi_\beta^T(A)|\, F^4(A)}{A!} \left(\sum_{i=1}^{k} y_i\right)^4 \to 0 \ as \ L \to \infty \ . \tag{5-6}$$

The first summation I in (5-4) is rewritten in the form,

$$I = -\frac{1}{2\sigma^2 L^2} \sum_{i,j} \sum_{\bar{A} \subset G_{ij}} \frac{\phi_\beta^T(A)}{A!} \sum_{m,n}^{k} y_m y_n$$

$$\cdot\ (\sum_w A(w)\ F(w)\ \chi_m^L(w)\ )\cdot\ (\sum_w A(w)\ F(w)\ \chi_n^L(w)\ ).$$

Here it is easily seen that

$$(\sum_w A(w)\ F(w)\ \chi_m^L(w)\ )\cdot(\sum_w A(w)\ F(w)\ \chi_n^L(w)\ )\ =\ 0$$

unless $\bar{A} \subset [0,\ (t_m \wedge t_n)\ L\ ] \times [0,\ (s_m \wedge s_n)\ L\ ]$.

Therefore, we get the following convergence from lemma 5-2

$$I\ \to\ \frac{1}{2}\sum_{m,n=1}^{k} y_m y_n\ \cdot(t_m \wedge t_n)\cdot(s_m \wedge s_n)\quad as\ \ L \to \infty$$

Putting $\sigma\ \equiv\ B_1(\beta\ )^{\frac{1}{2}}$, and using (5-5) , (5-6), (5-7) , we get

$$\theta_k^L(\ \mathbf{y}\ )\quad \to\quad \exp\{\ -\frac{1}{2}\sum_{m,n} y_m\ y_n\ \cdot(\ t_m \wedge t_n\ )\ \cdot(\ s_m \wedge s_n\ )\ \}\quad as\ \ L\ \to\ \infty\quad .$$

This proves Theorem 1 .

6. Proof of Theorem 2 and Theorem 3

6.1. Proof of Theorem 2

The proof is very similar to the proof of Theorem 1 . For simplicity we only prove in the case of k=1. Fix (t,s) $\in [0,1]^2$ and put ,

$$k_1=k_1(L)=[tL] \quad , \quad k_2=k_2(L)=[sL] \quad \text{, and } V_L(k_1,k_2) = [0,k_1]\times[0,k_2] \ .$$

$Y^L(t,s)$ is described in the following form ,

$$Y^L(t,s) = e_1(L)X^L(k_1+1,k_2+1) + e_2(L)X^L(k_1,k_2+1) + e_3(L)X^L(k_1+1,k_2) + e_4(L)X^L(k_1,k_2) \ ,$$

where $e_1(L)\cdot e_4(L) = 0$ and $e_1(L) + e_2(L) + e_3(L) + e_4(L) = 1$.

Rewrite $Y^L(t,s)$ as follows ,

$$\begin{aligned} Y^L(t,s) = {} & X^L(k_1,k_2) \\ & + e_1(L)\cdot\frac{1}{\sigma L}\sum_{w\subset V_L(k_1+1,k_2+1)\setminus V_L(k_1,k_2)} F(w) \\ & + e_2(L)\cdot\frac{1}{\sigma L}\sum_{w\subset V_L(k_1,k_2+1)\setminus V_L(k_1,k_2)} F(w) \\ & + e_3(L)\cdot\frac{1}{\sigma L}\sum_{w\subset V_L(k_1+1,k_2)\setminus V_L(k_1,k_2)} F(w) \ . \end{aligned}$$

The first term is the volume term and remainder terms are boundary terms . In the same way as the proof of Th 1 , the following convergence is obtained ,

$$<\exp\{iy\cdot Y^L(t,s)\}>_{P*_L} \ \to \ \exp\{-\frac{1}{2}tsy^2\} \ .$$

6.2. Proof of Theorem 3

We have proved the convergence of the finite-dimensional distribution for $Y^L(t,s)$ in 6.1 . Now we shall prove the convergence as the distribution on $\mathbf{C}([0,1]^2\to\mathbf{R})$. From the definition of $Y^L(t,s)$ we know that

$$Y^L(t,s) = 0 \quad \text{if } t\times s = 0$$

with probability one .

We have only to prove the following condition (*) to prove Theorem 3 ;

$$(*) \quad \textit{there exist constants } c>0, \epsilon>0, \delta>0 \textit{ such that}$$
$$< |Y^L(\tau_1)-Y^L(\tau_2)|^{\epsilon} >_{P_L^*} \leq c\,|\tau_1-\tau_2|^{2+\delta}$$

for every τ_1 and $\tau_2 \in [0,1]^2$, if L is large enough .

For fixed $\tau_1=(t_1,s_1)$ and $\tau_2=(t_2,s_2) \in [0,1]^2$ we first show that ,

$$< |X^L(\tau_1)-X^L(\tau_2)|^6 > \leq \delta(\beta)\ |\tau_1-\tau_2|^3 \qquad (6-1)$$

for sufficiently large β , where $\delta(\beta) > 0$. Put

$$G^L(\mathrm{w}) = X^{L_{\tau_1}}(\mathrm{w}) - X^{L_{\tau_2}}(\mathrm{w})$$
$$= \frac{1}{\sigma(\beta)L}\{ \sum_{\substack{w\in \mathrm{w} \\ w\subset[0,t_1L]\times[0,s_1L]}} F(w) - \sum_{\substack{w\in \mathrm{w} \\ w\subset[0,t_2L]\times[0,s_2L]}} F(w) \}.$$

Applying the method of polymer expansion , we get

$$< \exp\{iG^L(\mathrm{w})y\} >_{P_L^*} = \frac{\sum_{A\subset S_V} \exp\{iG^L(A)y\}\ \phi_\beta(A)}{\sum_{A\subset S_V} \phi_\beta(A)}$$

$$= \exp\{ \sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} (\exp\{iG^L(A)y\}-1) \} ,$$

where

$$G^L(A) = \frac{1}{\sigma(\beta)L}\{ \sum_{w\subset V_L(\tau_1)} F(w)A(w) - \sum_{w\subset V_L(\tau_2)} F(w)A(w) \}$$

and $V_L(\tau_i) = [0,t_iL]\times[0,s_iL]\ (i=1,2)$.

From the standard argument ,

$$< G^L(.)^6 >_{P^*_L}$$
$$= \sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} G^L(A)^6 + 15 \sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} G^L(A)^4 \cdot \sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} G^L(A)^2$$
$$+ 15 \left(\sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} G^L(A)^2 \right)^3 \qquad (6-2)$$

Employing the properties of polymer expansion for $\phi_\beta(X)$, we shall show that

,

$$|\sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} G^L(A)^{2k}| \le \frac{L^2}{(\sigma(\beta)L)^{2k}}\cdot\delta_{2k}(\beta)\cdot|\tau_1-\tau_2| \quad (k=1,2,3) \quad (6-3).$$

for sufficiently large β , where $\delta_{2k}(\beta)\to 0$ *as* $\beta\to\infty$.

Let us first consider the case that $S_2 \le S_1$ and $t_1 \le t_2$. (See Fig.)

Let $h(w) = (t_h, s_h)$ and $r(w) = (t_r, s_r)$ be the points in $p(w)$ characterized by the condition (h) and (r) respectively ;

(h) $S_h \ge s$ for *all* $(t,s) \in p(w)$ and $t_h \ge t$ for *all* $(t,S_h) \in p(w)$

(r) $t_r \ge t$ for *all* $(t,s) \in p(w)$ and $s_r \le$ for *all* $(t_r,s) \in p(w)$.

(See Fig.)

In this case $G^L(A)$ is rewritten in the following form,

$$G^L(A) = \frac{1}{\sigma(\beta)L}\{\sum_{w\in A^{L}{}_1} F(w)A(w) - \sum_{w\in A^{L_2}} F(w)A(w)\} ,$$

where

$$A^{L}{}_1 = \{w : \textit{standard wall in } V_L(\tau_1) \textit{ such that } h(w) \in V_L(\tau_1)\backslash V_L(\tau_2)\}$$

and

$$A^{L_2} = \{w : \textit{standard wall in } V_L(\tau_2) \textit{ such that } r(w) \in V_L(\tau_2)\backslash V_L(\tau_1)\} .$$

Using the inequality (**) ;

$$(**) \quad (\frac{1}{n}\sum_{i=1}^n a_i)^m \le \frac{1}{n}\sum_{i=1}^n a_i^m \quad \text{for } a_i>0,\ m>1 ,$$

and the properties of polymer functionals , we get

$$|\sum_{A\subset S_V} \frac{\phi_\beta^T(A)}{A!} G^L(A)^{2k}|$$

$$\le \frac{1}{(\sigma(\beta)L)^{2k}} \sum_{A\subset S_V} \frac{|\phi_\beta^T(A)|}{A!}\{\sum_{w\in A^{L_1}\cup A^{L_2}} |F(w)|A(w)\}^{2k}$$

$$\le \frac{1}{(\sigma(\beta)L)^{2k}} \sum_{A\subset S_V} \frac{|\phi_\beta^T(A)|}{A!}(\#A^{L_1}\cup A^{L_2})^{2k-1} \sum_{w\in A^{L_1}\cup A^{L_2}} (|F(w)|A(w))^{2k}$$

$$\leq \frac{C^{2k}}{(\sigma(\beta)L)^{2k}} \sum_{A\subset S_V} \frac{|\phi_\beta^T(A)|}{A!}|A|^{2k-1} \sum_{w\in A^{L_1}\cup A^{L_2}} (\ |w|\,A(w)\ )^{2k}$$

$$\leq \frac{C^{2k}}{(\sigma(\beta)L)^{2k}} \sum_{w\in A^{L_1}\cup A^{L_2}} \sum_{A\subset S_V} \frac{|\phi_\beta^T(A)|}{A!}\ |A|^{4k}$$

$$\leq \frac{C^{2k}}{(\sigma(\beta)L)^{2k}} \cdot |\ V_L(\tau_1) \ominus V_L(\tau_2)\ | \cdot \sum_{o\in A} \frac{|\phi_\beta^T(A)|}{A!}\ |A|^{4k}$$

$$\leq \frac{L^2}{(\sigma(\beta)L)^{2k}}\ \delta_{2k}(\beta)\ |\ \tau_1-\tau_2\ | \ ,$$

where $\delta_{2k}(\beta) \to 0$ as $\beta \to \infty$.

When $s_1<s_2$ and $t_1\leq t_2$ (See Fig. Case 2) , we get the same estimate. Hence we could prove the estimate (6-3).

From (6-1) and (6-3) we get ,

$$< G^L(.)^6 >_{P_L^*}$$
$$\leq \frac{\delta_6(\beta)}{\sigma(\beta)^6}\cdot\frac{|\mathbf{k}_1-\mathbf{k}_2|}{L^5} + 15\ \frac{\delta_4(\beta)\delta_2(\beta)}{\sigma(\beta)^6}\cdot\frac{|\mathbf{k}_1-\mathbf{k}_2|^2}{L^4} + 15\ \frac{\delta_2(\beta)^3}{\sigma(\beta)^6}\cdot\frac{|\mathbf{k}_1-\mathbf{k}_2|^3}{L^3}$$
$$\leq 16\ \frac{\delta_2(\beta)^3}{\sigma(\beta)^6}\ \frac{|\mathbf{k}_1-\mathbf{k}_2|^3}{L^3}$$
$$\equiv \delta(\beta)\ |\tau_1-\tau_2|^3$$

for sufficiently large L , where $\tau_1 = \frac{1}{L}\mathbf{k}_1 = (\frac{n_1}{L},\frac{m_1}{L})$ and $\tau_2 = \frac{1}{L}\mathbf{k}_2 = (\frac{n_2}{L},\frac{m_2}{L})$

Hence we get the proof of (6-1).

The estimate (6-1) implies that the moment condition (*) of $Y^L(\tau)$ is satisfied with $\epsilon=6$ and $\delta=1$ for any discrete points $(\tau_1\ ,\ \tau_2)$ given by $\tau_k = \frac{1}{L}(n_k\ ,\ m_k)$, $n_k,m_k \in \mathbf{N}$ $(k=1,2)$.

Next, let us consider the case that $\tau_1=(t_1,s_1)$ and $\tau_2=(t_2,s_2)$ are contained in a single block enclosed by line segments passing through three points $A_0=\frac{1}{L}(n_1,m_1)$, $A_1=\frac{1}{L}(n_1,m_1+1)$, and $A_2=\frac{1}{L}(n_1+1,m_1)$, $(n_1,m_1 \in \mathrm{N})$.

It is easily seen that ,

$$| Y^L(\tau_1) - Y^L(\tau_2) | \leq |t_1-t_2|\cdot|Y^L(A_2)-Y^L(A_0)| = |s_1-s_2|\cdot|Y^L(A_1)-Y^L(A_0)| .$$

From the above estimate and the inequality (**) , we get

$$< |Y^L(\tau_1)-Y^L(\tau_2)|^6 >_{P^*_L} \leq |\tau_1-\tau_2|^6\cdot\frac{const.}{L^3}$$

$$\leq const.\cdot|\tau_1-\tau_2|^3$$

for sufficiently large L .

For arbitrarily given τ_1 and τ_2 $\in [0,1]^2$, we get the following estimate by using (6-1) and (**),

$$< |Y^L(\tau_1)-Y^L(\tau_2)| >_{P^*_L} \leq \delta(\beta)\cdot|\tau_1-\tau 2|^3$$

for sufficiently large β and L . This proves Theorem 3 .

7. Appendix A

7.1. Proof of (2-6)

Let $X = \sum_{i=1}^{k} \gamma_i$ be a possible rearrangement of $X = \sum_{i=1}^{m} n_i\gamma_i$. We define the function $f(\gamma_i,\gamma_j)$ by

$$f(\gamma_i,\gamma_j) = \begin{cases} -1 \text{ if } \gamma_i\cap\gamma_j\neq \emptyset \\ 0 \quad otherwise \end{cases}$$

Then $\alpha(X)$ is rewritten in the form,

$$\alpha(X)= \prod_{(i,j)\in g(X)}\{ f(\gamma_i,\gamma_j)+1\},$$

where $g(X) = \{ (i,j) \ ; \ 1\leq i<j\leq k \}$.

We proceed in the same way as usual,

$$\alpha(X) = 1 + \sum_{B\subset g(X)} \prod_{(i,j)\in B} f(\gamma_i,\gamma_j)$$

$$= 1 + \sum_{B\subset G_b(X)} (-1)^{\#(B)},$$

where $G_b(X) = \{(i,j)\in g(X)\ ;\ \gamma_i \cap \gamma_j \neq \varnothing\}$.

Let $w(.)$ be the right hand side of (2-4) , i.e.

$$\omega(X) = \sum_{C\subset G(X):conn.} (-1)^{\#\ of\ bonds\ in\ C}.$$

For any $B\subset G_b(X)$ we put

$$v(B) = \bigcup_{(i,j)\in B} \{\ i,j\ \}.$$

We rewrite $\omega(X)$ in the form,

$$\omega(X) = \sum_{B\subset G_b(X);conn.v(B)=\{1,2,\ldots,k\}}, \qquad \text{(A-1)}$$

where we say $B\subset G_b(X)$ is connected if for any i and $j\in v(B)$ there exist a path $i=i_0,\ i_1,\ .\ .\ .\ ,i_m=j$ in $v(B)$ such that $\gamma_{i_p}\cap\gamma_{i_{p+1}}\neq\varnothing$ for all p.

We shall prove that $Exp\ \omega\ = \alpha$ From (A-1) we have

$$Exp\ \omega(X) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{\{Y_1,\ldots,Y_n\} non\ ordered, \\ Y_1\cup\ldots\cup Y_n=\{1,\ldots,k\} \\ Y_i\cap Y_j=\varnothing\ (i\neq j)}} \prod_{j=1}^{n} \omega(\sum_{i\in Y_j} \gamma_i)$$

$$= 1 + \sum_{\substack{Y\subset\{1,2,\ldots,k\} \\ \#Y\geq 2}} \sum_{n=1}^{\infty} \sum_{\substack{\{Y_1,\ldots,Y_n\}:non\ ordered \\ Y_1\cup\ldots\cup Y_n=Y \\ Y_i\cap Y_j=\varnothing(i\neq j)}} \prod_{j=1}^{n} \omega(\sum_{i\in Y_j} \gamma_i)$$

$$= 1 + \sum_{Y\subset\{1,2,\ldots,k\}} \sum_{n=1}^{\infty} \sum_{\substack{B_1,\ .\ .\ .\ ,B_n\subset G_b(X)\ :\ conn. \\ v(B_1)\cup\ \cdots\ \cup v(B_n)=Y \\ v(B_i)\cap v(B_j)=\varnothing\ (i\neq j)}} \prod_{j=1}^{n} (-1)^{\#(B_j)}$$

$$= 1 + \sum_{Y\subset\{1,2,\ldots,k\}} \sum_{B\subset G_b(X),\nu(B)=Y} (-1)^{\#(B)}$$

$$= 1 + \sum_{B\subset G_b(X)} (-1)^{\#(B)}$$

$$= \alpha(X)$$

7.2. The proof of Lemma 2-3 (ii)

First of all, recall that $\Psi_\beta^T(X)$ is rewritten in the form,

$$\Psi_\beta^T(X) = \exp(-\beta|X|)\alpha^T(X).$$

Hence we have

$$\exp(\beta_0 k)\cdot \sum_{o\in X,|X|\geq k} \frac{|\Psi_\beta^T(X)|}{X!} \leq \sum_{X\ni O,|X|\geq k} \frac{\exp(-(\beta-\beta_0)|X|)\alpha^T(X)}{X!}$$

$$\leq \sum_{X\ni O} \frac{|\Psi_{\beta-\beta_0}^T(X)|}{X!} \leq c(\beta-\beta_0)$$

This prove lemma 2-3 (ii).

7.3. The proof of Lemma 2-4

First we put

$$U_{V^*}(w) = I_1 - I_2$$

where

$$I_1 = \sum_{\substack{X\ i\ \lambda(w) \\ p(X)\cap p(w)\neq\emptyset}} \frac{\Psi_\beta^T(X)}{X!}$$

and

$$I_2 = \sum_{\substack{X\ i\ S \\ p(X)\cap p(w)\neq\emptyset}} \frac{\Psi_\beta^T(X)}{X!}.$$

Moreover we decompose I_1 into the sum of I_{11} and I_{12},

$$I_{11} = \sum_{\substack{X\ i\ W(\lambda(\mathbf{w})) \\ p(X)\cap p(\mathbf{w})\neq\emptyset}} \frac{\Psi_\beta^T(X)}{X!}$$

$$I_{12} = \sum_{\substack{X\ i\ \lambda(\mathbf{w}) \\ X\ not\ i\ W(\lambda(\mathbf{w})) \\ p(X)\cap p(w)\neq\emptyset}} \frac{\Psi_\beta^T(X)}{X!}$$

where $W(\lambda(\mathbf{w}))$ means the wall part of the interface $\lambda(\mathbf{w})$, and X *not* i $W(\lambda(\mathbf{w}))$ means that any contour in *supp* X doesn't intersect $W(\lambda(\mathbf{w}))$.

Since $X\ i\ W(\lambda(\mathbf{w}))$, *supp* X contains at least one point of $W(\lambda(\mathbf{w}))$ and we get the following estimate for I_{11} from Lemma 2-3,(i),

$$|I_{11}| \leq |\textit{the total area of } W(\lambda(\mathbf{w}))| \sum_{X \ni O} \frac{|\Psi_\beta^T(X)|}{X!}$$

$$\leq 2|w|\, c(\beta).$$

Next we shall estimate I_{12} . Take any X from the summand in I_{12} . We define the vertical distance between X and $W(\lambda(w))$ by

$$\rho(X, W(\lambda(\mathbf{w}))) \equiv Min\{|t-s| : t\in W(\lambda(\mathbf{w})),\ s\in supp\ X,\ p(t) = p(s)\}.$$

For X in the summand of I_{12}, we get $\rho(X, W(\lambda(\mathbf{w}))) > 0$. When $\rho(X, W(\lambda(\mathbf{w}))) = k$ *supp* X must contain a point $q \in (\mathbf{Z}^3)^*$ satisfying $Min\{|t-q| : t\in W(\lambda(\mathbf{w})), p(t) = p(q)\} = k$. The number of such points q is bounded by $2|\mathbf{w}|$, so that we have,

$$|I_{12}| \leq \sum_{k=1}^{\infty} \sum_{d(X,W(\lambda(\mathbf{w}))=k} \frac{|\Psi_\beta^T(X)|}{X!}$$

$$\leq \sum_{k=1}^{\infty} 2|\mathrm{w}| \sum_{\substack{X \quad O \\ |X| \geq 2k}} \frac{|\Psi_\beta^T(X)|}{X!}$$

$$\leq const\ |w|\ exp(-2\beta_0)\ C(\beta-\beta_0)$$

if $\beta-\beta_0$ is sufficiently large .

Therefore we get

$$|I_1| \leq |\mathrm{w}| \{ 2C(\beta) + C_1 \exp(-2\beta_0) C(\beta-\beta_0) \}.$$

In a similar way to I_1 we have

$$|I_2| \leq \sum_{k=0}^{\infty} \sum_{\substack{X\ i\ S \\ p(X)\cap p(\mathrm{w}) \neq \emptyset \\ \rho(X, p(\mathrm{w}))=k}} \frac{|\Psi_\beta^T(X)|}{X!}$$

$$\leq 2|p(\mathrm{w})| \cdot \sum_{k=0}^{\infty} \sum_{\substack{X \quad O \\ |X| \geq 2k}} \frac{|\Psi_\beta^T(X)|}{X!}$$

$$\leq 2|\mathrm{w}| C(\beta-\beta_0) \cdot \sum_{k=0}^{\infty} \exp\{-2\beta_0 k\}$$

$$\leq const.\ |\mathrm{w}| C(\beta-\beta_0) \quad ,$$

if $\beta-\beta_0$ is sufficiently large .

Hence we get the estimate for $U_{V*}(\mathrm{w})$,

$$|U_{V*}(\mathrm{w})| \leq k(\beta)|\mathrm{w}|,$$

where $k(\beta) \to 0$ as $\beta \to \infty$.

7.4. The proof of the recursion formula for $\Delta_{A(B)}$,

First we write $\Delta_{A\cup w}/B!$ explicitely ;

$$\frac{\Delta_A \cup w(B)}{B!} = \sum_{\substack{(B_1,B_2) \\ B_1+B_2=B \\ A\cup w\cup B_2 : N.O.}} \frac{\phi_{\beta_1,\beta_2}^{-1}(B_1)}{B_1! B_2!} \exp\{-\beta_1|A\cup w\cup B_2| - U^{\beta_2}(A\cup w\cup B_2)\}.$$

Moreover it is easily seen that ,

$$U^{\beta_2}(A\cup w\cup B_2) = U_1^{\beta_2}(A\cup w) + U^{\beta_2}(A\cup B_2) + \sum_{\emptyset\neq S\subset B_2} W^{\beta_2}(A\cup w,S).$$

In the usual way we proceed ;

$$\prod_{\substack{S\subset B_2\\ S\neq\emptyset}} \exp(\,-W^{\beta_2}(\,A\cup w,S)\,)$$

$$= \prod_{\substack{S\subset B_2\\ S\neq\emptyset}} \{(exp(\,-W^{\beta_2}(\,A\cup w,S\,)\,)-1) + 1\}$$

$$= 1+\sum_{n\geq 1}\ \sum_{\substack{\{P_1,\ldots,P_n\}\\ P_i\subset Y_2 \text{ for any } i\\ P_i\neq\emptyset}}\ \prod_{i=1}^{n} \{\exp\,(\,W^{\beta_2}(A\cup w,P_i))-1\,\}$$

$$= 1+\sum_{n\geq 1}\ \sum_{\substack{R\subset B_2\\ R\neq\emptyset}}\ \sum_{\substack{\{P_1,\ldots,P_n\}\\ \bigcup_{i=1}^{n} P_i=R\\ P_i\neq\emptyset}}\ \prod_{i=1}^{n} \{\exp(\,-W^{\beta_2}(\,A\cup w,P_i\,)\,)\,-1\,\}$$

$$=1 + \sum_{\substack{R\subset B_2\\ R\neq\emptyset}} K^{\beta_2}(A\cup w,R)$$

$$= \sum_{R\subset B_2} K^{\beta_2}(\,A\cup w,R\,)$$

Using these results we get ,

$$\frac{\Delta_{A\cup w}(B)}{B!} = \exp\{-\beta_1|w|-U_1^{\beta_2}(A\cup w,R)\} \sum_{\substack{R\subset B\\ R;N.O.}} K^{\beta_2}(\,A\cup w,R\,)$$

$$\sum_{\substack{Q\subset B\setminus R\\ A\cup w\cup Q\cup R\,:N.O.}} \frac{\phi^{-1}_{\beta_1,\beta_2}(B\setminus(R\cup Q))\ \ \phi_{\beta_1,\beta_2}(A\cup R\cup Q)}{(\,B\setminus(R\cup Q)\,)!}$$

Here we put ,

$$I = \sum_{\substack{Q \subset B \setminus R \\ A \cup w \cup Q \cup R : N.O.}} \frac{\phi^{-1}_{\beta_1,\beta_2}(B\setminus(R \cup Q)\ )\ \phi_{\beta_1,\beta_2}(A \cup R \cup Q)}{(B\setminus(R \cup Q)\ )!},$$

and remark that Q in the summand deos not intersect w , because $A \cup w \cup Q \cup R$ is N.O. .

Thus we get ,

$$I = \{ \sum_{Q \subset B\setminus R} - \sum_{\substack{Q \subset B\setminus R \\ Q\ i\ w}} \} \frac{\phi^{-1}_{\beta_1,\beta_2}(B\setminus(R \cup Q))\ \ (D_{A\cup R}\phi_{\beta_1,\beta_2})(Q)}{(B\setminus(R \cup Q)\ )\ !}$$

$$= \frac{\Delta_{A\cup R}(Q)}{(B\setminus R)!} - \sum_{\substack{Q \subset B\setminus R \\ Q\ i\ w}} \frac{\phi^{-1}_{\beta_1,\beta_2}(\ B\setminus(R \cup Q))\ \ (D_{A\cup R}\phi_{\beta_1,\beta_2})(Q)}{(\ B\setminus(\ R \cup Q)\ )!}$$

where Q i w means that some element in suppQ intersects w .

Let Θ, Θ_1, ..., Θ_n, ... be as follow ,

$$\Theta = \{ Q \subset B\setminus R\ ; Q\ i\ w\ \}$$

$$\Theta_n = \{ T \subset B\setminus R\ ; \|T\|=1,\ \ T\ I\ w\ \}\ \ ,n \geq 1,$$

where $T\ I\ w$ means that all $w^1 \in T$ intersect w .

Then we get

$$\Theta = \sum_{n \geq 1} \Theta_n,$$

$$\sum_{\substack{Q \subset B\setminus R \\ Q\ i\ w}} = \sum_{Q \in \Theta}$$

$$= \sum_{T \in \Theta_1} \sum_{\substack{S \subset B\setminus R \\ T \subset S}} - \sum_{T \in \Theta_2} \sum_{\substack{S \subset B\setminus R \\ T \subset S}} + \ldots + (-1)^{n-1} \sum_{T \in \Theta_n} \sum_{\substack{S \subset B\setminus R \\ T \subset S}} + \cdots$$

Therefore,

$$I = \sum_{n \geq 1} (-1)^{n-1} \sum_{T \in \Theta_n} \sum_{\substack{S \subset B\setminus R \\ T \subset T}} \frac{\phi^{-1}_{\beta_1,\beta_2}(\ B\setminus(R \cup S))\ \ D_{A\cup R}\phi_{\beta_1,\beta_2}(S)}{(\ B\setminus(R \cup S))!}$$

$$= \sum_{n\geq 1}(-1)^{n-1} \sum_{T\in\Theta_n} \frac{\Delta_{A\cup R\cup T}(\,B\backslash(R\cup T)\,)}{(\,B\backslash(R\cup T)\,)!}$$

$$= -\sum_{\substack{\emptyset\neq T\subset B\backslash R \\ B\ I\ w}} (-1)^{\|T\|} \frac{\Delta_{A\cup R\cup T}(\,B\backslash(\,R\cup T\,)\,)}{(\,B\backslash(\,R\cup T)\,)!}$$

The result follows from this form .

7.5. Proof of lemma 4-2

For any fixed A with $0 \leq \|A\| \leq$ m , it follows from the recursion formula for $\Delta_{A\cup w}$ (B) that

$$\sum_{B:\|B\| = m-\|A\|} \frac{|\Delta_{A\cup w}(B)|}{B!} \exp\{\frac{1}{2}\beta_1|X\cup w|\}$$

$$\leq \exp\{-\frac{1}{2}\beta_1|w|-U_1^{\beta_2}(A\cup w)\} \cdot \sum_{B:\|B\|=m-\|A\|} \sum_{R\subset B\ A\cup R\cup w:N.O.} |K^{\beta_2}(A\cup w,R)|$$

$$\cdot \sum_{\substack{Q\subset B\backslash R\ ,\ Q\ I\ w \\ Q\ :N.O.}} \frac{|\Delta_{A\cup R\cup Q}(\,B\backslash(R\cup Q))|}{(\,B\backslash(R\cup Q))!} \cdot\exp\{\frac{1}{2}\beta_1|A\cup R\cup Q|\}\cdot\exp\{-\frac{1}{2}\beta_1|R\cup Q|\}$$

$$= \exp\{-\frac{1}{2}\beta_1|w|-U_1^{\beta_2}(A\cup w)\}$$

$$\cdot \sum_{\substack{R:A\cup w\cup R;\,N.O. \\ \|R\|\leq m-\|A\|}} |K^{\beta_2}(A\cup w,R)| \cdot \sum_{\substack{Q:Q\cup R\cup A;\,N.O.,Q\ I\ w \\ \|Q\|+\|R\|\leq m-\|A\|}} \exp\{-\frac{1}{2}\beta_1|R\cup Q|\}$$

$$\cdot \{ \sum_{\substack{W:\|W\|+\|Q\|+\|R\|=m-\|A\| \\ W\cup Q\cup R;\,N.O.}} \frac{|\Delta_{A\cup R\cup Q}(W)|}{W!} \cdot \exp(\frac{1}{2}\beta_1|A\cup R\cup Q|) \ \}$$

$$\leq I_m \cdot \exp\{-\frac{1}{2}\beta_1|w|-U_1^{\beta_2}(A\cup w)\}$$

$$\cdot \sum_{\substack{R:A\cup w\cup R:\,N.O.\\ \|R\|\le m-\|A\|}} |K^{\beta_2}(A\cup w,R)| \sum_{\substack{Q:Q\cup R\cup A;N.O.,Q\ I\ w\\ \|Q\|+\|R\|\le m-\|A\|}} \exp\{-\frac{1}{2}\beta_1|R\cup Q|\},$$

where w is the first element in $w\cup A$.

Therefore ,we have only to estimate J given by

$$J = \exp\{-\frac{1}{2}\beta_1|w| - U_1^{\beta_2}(A\cup w)\}$$

$$\cdot \sum_{R:A\cup w\cup R;\,N.O.} |K^{\beta_2}(A\cup w,R)| \cdot \sum_{\substack{Q:Q\cup R\cup A;\,N.O.\\ Q\ I\ w}} \exp\{-\frac{1}{2}\beta_1|R\cup Q|\}.$$

We prepare several lemmas for the estimate of J .

Lemma A-1

Let w be the first element in $A\cup w$. When $\beta_2,\beta_2-\beta_0,\beta_0$ are sufficiently large the following estimate holds ,

$$|U_1^{\beta_2}(A\cup w)| \le k_1(\beta_2)|w| \quad .$$

Proof

Put suppA $=$ **v**.

We rewrite $U_1^{\beta_2}(\mathbf{v}\cup w)$ in the form ,

$$U_1^{\beta_2}(\mathbf{v}\cup w) = U^{\beta_2}(\mathbf{v}\cup w) - U^{\beta_2}(\mathbf{v})$$

$$= \Big(\sum_{\substack{X\ i\ \lambda(\mathbf{v}\cup w)\\ p(X)\cap p(\mathbf{v}\cup w)\neq\emptyset}} \frac{\Psi_{\beta_2}^T(X)}{X!} - \sum_{\substack{X\ i\ \lambda(\mathbf{v})\\ p(X)\cap p(\mathbf{v})\neq\emptyset}} \frac{\Psi_{\beta_2}^T(X)}{X!} \Big)$$

$$+\Big(\sum_{\substack{X\ i\ S\\ p(X)\cap P(\mathbf{v}\cup w)\neq\emptyset}} \frac{\Psi_{\beta_2}^T(X)}{X!} - \sum_{\substack{X\ i\ S\\ p(X)\cap p(\mathbf{v})\neq\emptyset}} \frac{\Psi_{\beta_2}^2(X)}{X!} \Big)$$

$$= I_1 + I_2.$$

Bearing in mind that w is the inner most element in $w \cup v$ we get ,

$$I_1 = \sum_{\substack{X\, i\, \lambda(v\cup w)\\ p(w)\cap p(X)\neq\emptyset}} - \sum_{\substack{X\, i\, \lambda(v)\\ p(w)\cap p(X)\neq\emptyset\\ p(w)\cap p(v)\neq\emptyset}} ,$$

where the terms with $p(w)\cup p(X)=\emptyset$ were cancelled by using the translation invariance of $\Psi^T_{\beta_2}(X)$.

From the above formula and using the method developped in the proof of lemma 2-4 we get ,

$$|I_1| \leq k(\beta_2)|w|,$$

where $k(\beta_2)\to 0$ exponentially fast as $\beta_2\to\infty$.

In a similar way to I_1 we get the estimate for I_2 ,

$$|I_2| \leq k'(\beta_2)|w|,$$

where $k'(\beta_2)\to 0$ exponentially fast as $\beta_2\to\infty$.

This proves the lemma .

Lemma A-2

For sufficiently large β we have

$$\sum_{\substack{Q\, I\, w\\ Q\, :\, N.O.}} \exp\{-\frac{1}{2}\beta\, |Q|\} \leq (1+ g(\beta))^{|w|} ,$$

where $g(\beta)\to 0$ as $\beta\to\infty$.

Proof Since all element of $supp Q$ intersect w , we have ,

$$\sum_{\substack{Q\, I\, w\\ Q\, :\, N.O.}} \exp\{\frac{-1}{2}\beta|Q|\} \leq \sum_{k=0}^{|w|+|p(w)|} \binom{|w|+|p(w)|}{k} \Big(\sum_{o\in v} \exp\{\frac{-1}{2}\beta|v|\}\Big)^k .$$

The number of elements of the set

$\{v\ ;\ standard\ wall\ s.t.\ o\in p(v)\ \text{and}\ |v| + m\ \}$ is bounded by c^m for some $c > 0$, so that we have

$$\sum_{o\in v} \exp\{-\frac{1}{2}\beta|v|\} \le \sum_{k=4}^{\infty} c^k \exp\{-\frac{1}{2}\beta k\} .$$

Combining these facts we get the proof .

Using lemma A-1 and lemma A-2 we have ,

$$J \le \exp\{-\frac{1}{2}\beta_1|w| + k_1(\beta_2)|w|\}(1+g(\beta_1))^{|w|}$$

$$\cdot\{ 1 + \sum_{n=1}^{\infty} \sum_{\substack{\{P_1,\ldots,P_n\} \\ P_1\cup\ldots\cup P_n \cup A\cup w: N.O.}} \prod_{i=1}^{n} |\exp\{-W(A\cup w,P_i)\}-1| \cdot \exp\{-\frac{1}{2}\beta_1|P_1\cup\ldots\cup P_n|\} \}$$

$$\le \exp\{-\frac{1}{2}\beta_1|w| + k_1(\beta_2)|w|\}(1+g(\beta_1))^{|w|}$$

$$\{ 1 + \sum_{n=1}^{\infty} \sum_{\substack{\{P_1,\ldots,P_n\} \\ P_1\cup\ldots\cup P_n \cup A\cup w: N.O.}} \prod_{i=1}^{n} |W(A\cup w,P_i)| \cdot\exp(\sum_{i=1}^{n} |W(A\cup w,P_i)| - \frac{1}{2}\beta_1|P_1\cup\ldots\cup P_n|) \}$$

$$= \exp\{-\frac{1}{2}\beta_1|w| + k_1(\beta_2)|w|\}(1+g(\beta_1)) J^* .$$

To obtain estimates for $W(A\cup w,P_i)$ we prepare the following lemma .

Lemma A-3

If $\beta-\beta_0$ is sufficiently large , then the following estimate holds for any $\mathbf{w}$ and $w\in\mathbf{w}$,

$$\sum_{\substack{\mathbf{v}\subset\mathbf{w} \\ ;w\in\mathbf{v}_0}} |\Phi_\beta(\mathbf{v})| \exp\{\beta_0 d(\mathbf{v})\} \le k_2(\beta)|w| .$$

Proof

It follows from the explicit formula of $\Phi_\beta(\mathbf{v})$ that

$$\sum_{\substack{\mathbf{v}\subset\mathbf{w}\\ ;w\in\mathbf{v}_0}} |\Phi_\beta(\mathbf{v})|\ \{\beta_0 d(\mathbf{v})\} \leq \sum_{\mathbf{v}\subset\mathbf{w}} \sum_{X\in I(\mathbf{v})} \frac{|\Psi_\beta^T(X)|}{X!} \exp\{\beta_0 d(\mathbf{v})\}\cdot\#\mathbf{v}(X)$$

$$+ \sum_{\substack{\mathbf{v}\subset\mathbf{w}\\ w\in\mathbf{v}_0}} \sum_{X\in II(\mathbf{v})} \frac{|\Psi_\beta^T(X)|}{X!} \exp\{\beta_0 d(\mathbf{v})\}\cdot\#\mathbf{v}[X]$$

$$\equiv I_1 + I_2 \quad .$$

Exchanging the order of sums in I_1 we have ,

$$I_1 = \sum_{X\in J(\mathbf{w},w)} \sum_{\mathbf{v}\in(\mathbf{w},w)(X)} \frac{|\Psi_\beta^T(X)|}{X!} \exp\{\beta_0 d(\mathbf{v})\} \quad ,$$

where

$$J(\mathbf{w},w) = \{X;X\in I(\mathbf{v}) \text{ for } \textit{some } \mathbf{v}\subset\mathbf{w}\ \textit{s.t. } w\in\mathbf{v}_0\} \ ,$$

and

$$(\mathbf{w},\mathbf{w})(\mathrm{X}) = \{\mathbf{v}\subset\mathbf{w}\ ; X\in I(\mathbf{v}) \text{ and } w\in\mathbf{v}_0\ \} \ .$$

For a given $X\in J(\mathbf{w},w)$ we put ,

$$H(X,\mathbf{w}) = \{v\in\mathbf{w}\ ; p(v)\cap p(X)\neq\emptyset\} \quad .$$

If $|p(X)| = k$, then $\#H(X,\mathbf{w}) \leq k$ and $d(\mathbf{v})\leq k$ for any $\mathbf{v}\in(\mathbf{w},w)(X)$.

We introduce the following distance between w and X ,

$$\rho^*(w,X) = \underset{\substack{\mathbf{v}\subset H(X,\mathbf{w})\\ X\in I(\mathbf{v})\\ w\in\mathbf{v}_0}}{Min}\ \underset{\substack{\alpha\subset\mathbf{v}\\ X\ i\ w(\alpha)}}{Min}\ \rho(w^*(\alpha),suppX) \quad .$$

Suppose that $\rho^*(w,X) = h$ and $|p(X)| = k$. There exists $\alpha\subset H(X,\mathbf{w})$ such that $X\ i\ w(\alpha)$ for such X . Since $\rho^*(w,X) = h$, $\rho(w^*(\alpha)\ ,\ suppX) \geq h$. Hence we have $|X| >= 2h + k$.

Therefore we have ,

$$I_1 \le \sum_{k=4}^{\infty} \sum_{h=0}^{\infty} \sum_{\substack{X \in J(w\ ,\ w) \\ |p(X)| = k \\ \rho^*(w,X) = h}} \frac{|\Psi_\beta^T(X)|}{X!} \exp\{\beta_0 k\} \cdot 2^k$$

$$\le 2|w| \sum_{k=4}^{\infty} \sum_{h=0}^{\infty} \sum_{\substack{X\ \ O \\ |X| \ge 2h+k}} 2^k \exp\{\beta_0 k\} \frac{|\Psi_\beta^T(X)|}{X!}$$

$$\equiv c(\beta)|w| \quad ,$$

where $c(\beta) \to 0$ as $\beta \to \infty$.

In a similar way to I_1 , we have ,

$$I_2 \le c'(\beta)|w| \quad ,$$

where $c'(\beta) \to \infty$ as $\beta \to \infty$.

Now we shall estimate J^* . Let $w_1, .., w_k$ be the standrd walls of $\bigcup_{i=1}^{n} P_i$ contained in f(w) . Using lemma A-3 we have ,

$$\sum_{i=1}^{n} |W^{\beta_2}(A \cup w, P_i)| \le \sum_{i=1}^{n} \sum_{B \subset A} |\Phi^{\beta_2}(B \cup w \cup P_i)|$$

$$\le \sum_{i=1}^{n} \sum_{\substack{B \subset A \cup P_1 \cup \cdots \cup P_n \\ w_i \in B_0}} |\Phi^{\beta_2}(B)| + \sum_{\substack{B \subset A \cup P_1 \cup \ldots \cup P_n \\ w \in B_0}} |\Phi^{\beta_2}(B)|$$

$$\le k_2(\beta_2)(|w| + |w_1| + \cdots + |w_k|)$$

$$\le k_2(\beta_2)(|w| + |P_1 \cup \cdots \cup P_n|) \quad .$$

Hence we have the following estimate for J* for sufficiently large β_1 and β_2 ,

$$\frac{J^*}{\exp\{k_2(\beta_2)|w|\}} \leq \sum_{n=0}^{\infty} \sum_{\substack{\{P_1,\ldots,P_n\} \\ A\cup w\cup P_1\cdots\cup P_n : N.O.}} \prod_{i=1}^{n} |W(A\cup w,P_i)| \exp\{-\frac{1}{3}\beta_1|P_1\cup\cdots\cup P_n|\}$$

$$\leq \sum_{T\cup A\cup w : N.O.} \exp\{-\frac{1}{3}\beta_1|T|-\frac{1}{3}\beta_1 d(w\cup T)\}$$

$$\cdot \sum_{n=0}^{\infty} \frac{1}{n!}\{\sum_{P\subset T} |W(A\cup w,P)| \exp(\frac{1}{3}\beta_1 d(w\cup P))\}^n \quad .$$

In the same way as before we get ,

$$\sum_{P\subset T} |W(A\cup w,P)| \exp\{\frac{1}{3}\beta_1 d(w\cup P)\}$$
$$\leq k_2(\beta_2)(|w| + |T|) \quad .$$

Therefore , we have ,

$$J^* \leq \exp\{k_2(\beta_2)|w|\} \sum_{T\cup A\cup w : N.O.} \exp\{-\frac{1}{4}\beta_1|T|-\frac{1}{3}\beta_1 d(w\cup T)\}$$

$$\leq \exp\{k_2(\beta_2)|w|\}\sum_{l=1}^{\infty} \exp\{-\frac{1}{3}\beta_1 l\}$$

$$\sum_{k=1}^{l}\binom{l}{k}(\sum_{w \quad O} \exp(-\beta_0|w|))^k\cdot|w|$$

$$= |w|\exp\{k_2(\beta_2)|w|\}c(\beta_1) \quad ,$$

where $c(\beta_1)\to 0$ as $\beta_1\to\infty$.

Putting above estimates together we have ,

$$J \leq \exp\{-\frac{1}{3}\beta_1|w| + (k_1(\beta_2) + k_2(\beta_2))|w|\}(1+g(\beta_1))^{|w|}\cdot|w|c(\beta_1)$$

$$\leq c(\beta_1)\sup_{k\geq 4} k\cdot\exp(-\frac{1}{4}\beta_1) \quad .$$

This proves lemma 4-2 .

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Fig. 1

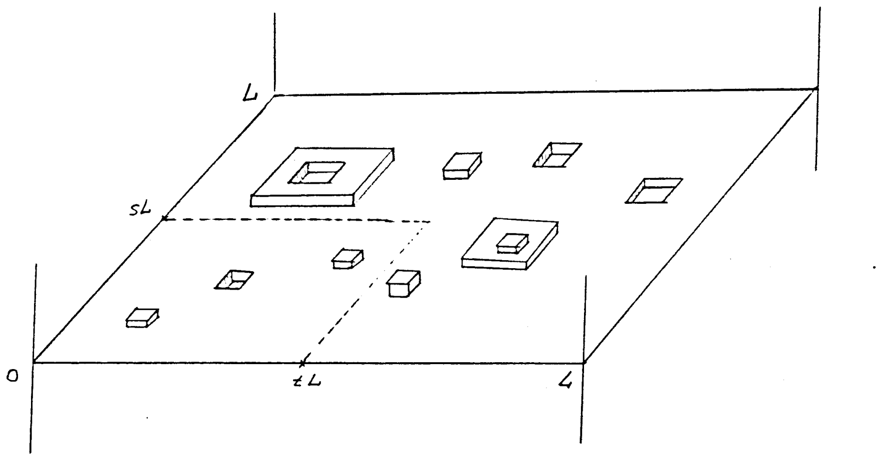

Fig.2

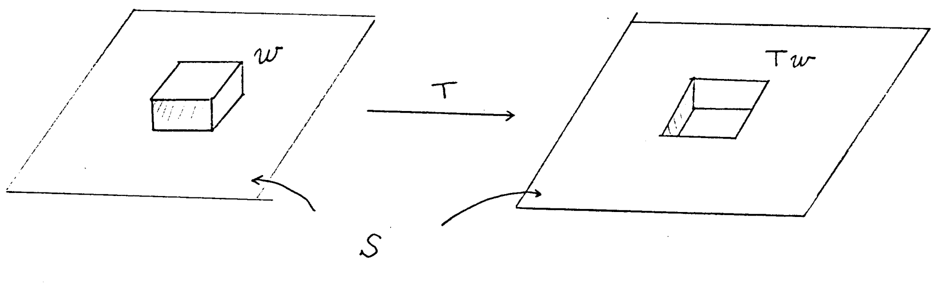

Fig.3

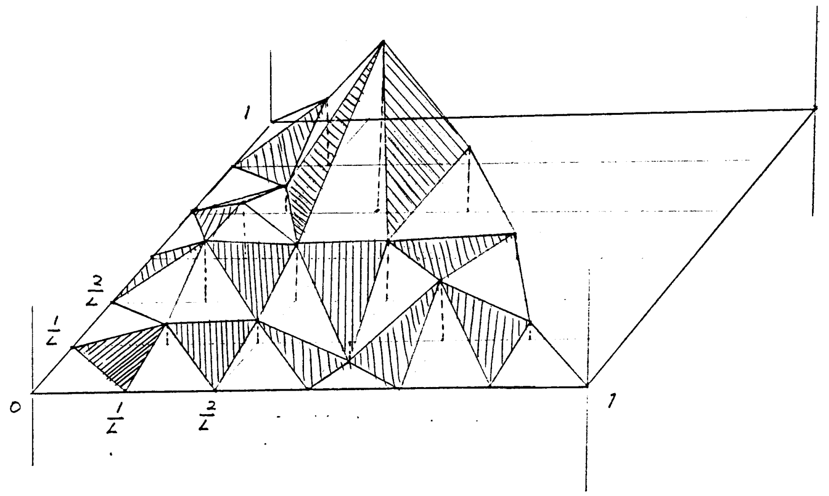
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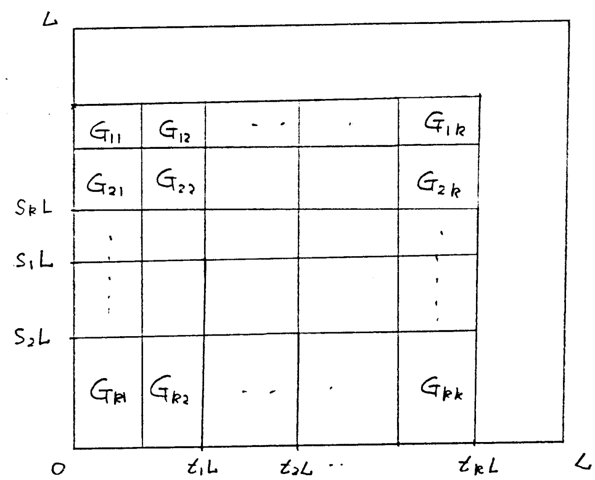
Fig.4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t2L
tkL
L