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On generalized harmonic analysis in R^n

by

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I.Introduction

Let $\mathbf{M}^1(\mathbf{R}^1)$ be the class of functions which are locally integrable such that

$$\sup_{T>0}\frac{1}{2T}\int_{-T}^{T}|f(x)|dx<\infty. \tag{1.1}$$

Suppose K(x) is an absolutely continuous even function which satisfies the condition

$$(1+x^2)|K(x)| \le C, \quad x\in\mathbf{R}^1, \tag{1.2}$$

C being an constant. Then if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)\, dx = M(f) \tag{1.3}$$

exists, then we have the so called Wiener formula [4][5][6][7][15]

$$\lim_{\epsilon\to 0} \int_{\mathbf{R}^1} f(\frac{x}{\epsilon})\, K(x)\, dx = M(f) \int_{\mathbf{R}^1} K(x)\, dx, \tag{1.4}$$

provided $f\in\mathbf{M}^1(\mathbf{R}^1)$.

This formula was extended to the $\mathbf{R}^2$ case by the author [9] and also by Anzai-Koizumi-Matsuoka [1] with some restricted limit process. When $f(x)\ge 0$, the existence of either $M(f)$ or the left hand side of (1.4) implies the existence of the other. This was shown with $K(x)=\sin^2 x/(\pi x^2)$ by Wiener [14] [15], rather deep Tauberian theorem being used.

In **3** and **4** we shall give further extensions of above results to the $\mathbf{R}^n$ case.

Now let f(x) be a function satisfying $(1+|x|)^{-1}f(x)\in L^2(\mathbf{R}^1)$. Wiener [14] [15] defined

$$s(u) = \underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{(2\pi)^{1/2}} \left[\int_{|x|\le 1} \frac{e^{-iux}-1}{-ix} f(x)\, dx + \int_{1<|x|\le A} \frac{e^{-iux}}{-ix} f(x)\, dx \right], \tag{1.5}$$

where l.i.m. means limit in $L^2(\mathbf{R}^1)$. s(u) is called Wiener Transform. See also Bochner 2-transform for $f(x)/(1+|x|)\in L^1(\mathbf{R}^1)$ [2] . Wiener has shown the Wiener identity

$$M(|f(x)|^2) = \lim_{h\to 0}\frac{1}{2h}\int_{\mathbf{R}^1} |s(u+h) - s(u-h)|^2\, du, \tag{1.6}$$

in the sense that if either side exists, then the other side exists and the equality holds.

We shall give the $\mathbf{R}^n$ generalization of this identity in **7**. The $\mathbf{R}^2$ case was given by Matsuoka [13].

Let $f(x)$, $x \in \mathbf{R}^1$ be the characteristic function of a probability distribution function $F(u)$,

$$f(x) = \int_{\mathbf{R}^1} e^{ixu} \, dF(u). \tag{1.7}$$

$M(f)$ and $M(|f|^2)$ exist and the inversion formula gives

$$F(u) - F(0) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} \frac{e^{-iux} - 1}{-ix} f(x) \, dx, \tag{1.8}$$

where $F(u)$ is modified to be $\frac{1}{2}[F(u+0) - F(u-0)]$ for discontinuities u of F .

The function

$$C(h) = \frac{1}{2h} \int_{\mathbf{R}^1} [\, F(u+h) - F(u-h) \,]^2 \, du, \quad h > 0 \tag{1.9}$$

is called the mean concentration function of a distribution function $F(u)$ [8] [10].

Actually the right hand side of (1.9) exists and $C(h)$ is a nondecreasing function of $h \in \mathbf{R}^1_+$, $\mathbf{R}^1_+ = (0, \infty)$.

We have, from the Wiener formula,

$$\lim_{h \to 0+} C(h) = M(|f|^2). \tag{1.10}$$

In **8**, we define the mean concentration function in $\mathbf{R}^n$ and show some basic properties of it.

2.Lemmas.

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $x = (x_1, \ldots, x_n)$. The interval (a,b) in $\mathbf{R}^n$ is $\{x : a_k < x_k < b_k, 1 \le k \le n\}$. [a,b],(a,b],[a,b) are defined in a similar way. We write the integral of $f(x)$ in several ways like :

$$\int_{a<x<b} f(x) \, dx = \int_a^b f(x) \, dx = \int_{a_1}^{b_1} dx_1 \cdots \int_{a_n}^{b_n} dx_n \, f(x) = \left(\prod_{k=1}^{n} \int_{a_k}^{b_k} dx_k \right) f(x).$$

We first note the formula for integration by parts.

Lemma 1. *Suppose* $f(x) \in L^1(a,b)$, $a,b \in \mathbf{R}^n$ *being finite and* $\frac{\partial^j}{\partial x_{k_1} \cdots \partial x_{k_j}} g(x)$ *exists for every* $x \in [a,b]$ *and distinct* $(k_1, \ldots, k_j)$ $(1 \le k_m \le n, \; m = 1, .., j)$ *and continuous in* $[a,b]$. *Then for such* f,g *we have*

$$\int_a^b f(x)g(x)dx=\int_a^b f(x)dxg(b)$$

$$-\int_a^b dx\sum_{k=1}^n \frac{\partial}{\partial x_k} g(b_1,\ldots,b_{k-1},x_k,b_{k+1},\ldots,b_n)\cdot\int_{a_k}^{x_k} f(x_1,\ldots,x_{k-1},u_k,x_{k+1},\ldots,x_n)du_k$$

$$+\int_a^b dx\sum_{1\le k_1<k_2\le n}\frac{\partial_2}{\partial x_{k_1}\partial x_{k_2}} g(b_1,\ldots,b_{k_1-1},x_{k_1},b_{k_1+1},\ldots,b_{k_2-1},x_{k_2},b_{k_2+1},\ldots,b_n)$$

$$\cdot\int_{a_{k_1}}^{x_{k_1}} du_{k_1}\int_{a_{k_2}}^{x_{k_2}} du_{k_2} f(x_1,\ldots,x_{k_1-1},u_{k_1},x_{k_1+1},\ldots,x_{k_2-1},u_{k_2},x_{k_2+1},\ldots,x_n)$$

$$-+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^n\int_a^b dx\frac{\partial^n}{\partial x_1\cdots\partial x_n}g(x)\int_a^x f(u)du. \tag{2.1}$$

We frequently use the following lemma.

Lemma 2. *Suppse $f(x)$ is locally integrable in $\mathbf{R}_+^n$ $(=\{x\in\mathbf{R}^n : x_k>0,\ 1\le k\le n\})$ and*

$$\prod_{k=1}^n A_k^{-1}\int_0^A |f(x)|dx\le D \tag{2.2}$$

for $A=(A_1,\ldots,A_n)>0$, where D is a constant independent of A. Then, for every $\alpha>0$ $(\alpha=(\alpha_1,\ldots,\alpha_n))$, we have, for $0\le a<b\le\infty$, $a,b\in\mathbf{R}^n$, $0=(0,\ldots,0)$ (no confusion will be expected),

$$\int_a^b |f(x)|\prod_{k=1}^n(1+\alpha_k^2x_k^2)^{-1}dx\le C_nD\prod_{k=1}^n a_k^{-1}(1+\alpha_ka_k)^{-1} \tag{2.3}$$

and

$$\int_a^b |f(x)|\prod_{k=1}^n(1+\alpha_k^2x_k^2)^{-1}dx\le C_nD\prod_{k=1}^n b_k^{-1} \tag{2.4}$$

if $b<\infty$, where C_n is a constant depending only on n.

Proof. Take $|f(x)|$ in place of $f(x)$ and

$$g(x)=\prod_{k=1}^n(1+\alpha_k^2x_k^2)^{-1}$$

in Lemma 1. We then see, for $1\le k_1<\ldots<k_j\le n$,

$$\frac{\partial^j}{\partial x_{k_1}\ldots\partial x_{k_j}}g(x)=(-1)^j2^j\alpha_{k_1}^2\ldots\alpha_{k_j}^2x_{k_1}\ldots x_{k_j}\prod_{k=k_1\ldots k_j}(1+\alpha_k^2x_k^2)^{-2}\prod_{1\le k\le n,k\ne k_1,\ldots,k_j}(1+\alpha_k^2x_k^2)^{-1}. \tag{2.5}$$

The first term with $|f(x)|$ in place of $f(x)$ on the right hand side of (2.1), we have

$$\int_a^b |f(x)|dxg(b)\le D\prod_{k=1}^n b_k\prod_{k=1}^n(1+\alpha_k^2x_k^2)^{-1} \tag{2.6}$$

$$\le D\prod_{k=1}^n[\alpha_k^{-1}(1+\alpha_kx_k)^{-1}]. \tag{2.7}$$

We also see, for the general term, that it is not greater than

$$\int_a^b dx \sum_{1\le k_1<\ldots<k_j\le n} \left|\frac{\partial^j}{\partial x_{k_1}\cdots\partial x_{k_j}} g(b_1,\ldots,b_{k_1-1},x_{k_1},b_{k_1+1},\ldots,b_{k_j-1},x_{k_j},b_{k_j+1},\ldots,b_n)\right|$$

$$\cdot\left(\prod_{\mu=k_1,\ldots,k_j}\int_{a_\mu}^{b_\mu} du_\mu\right)|f(x_1,\ldots,x_{k_1-1},u_{k_1},x_{k_1+1},\ldots,x_{k_j-1},u_{k_j},x_{k_j+1},\ldots,x_n)|,$$

where $\sum$ runs over all $(k_1,\ldots,k_j)$ with $1\le k_1<\ldots<k_j\le n$ and the similar for $\prod_\mu$. The above is

$$=\left(\prod_{\mu=k_1,\ldots,k_j}\int_{a_\mu}^{b_\mu} dx_\mu\right)\sum_{1\le k_1<k_2<\ldots<k_j\le n}\left|\frac{\partial^j}{\partial x_{k_1}\cdots\partial x_{k_j}}\right.$$

$$g(b_1,\ldots,b_{k_1-1},x_{k_1},b_{k_1+1},\ldots,b_{k_j-1},x_{k_j},b_{k_j+1},\ldots,b_n)\cdot\left(\prod_{l\ne k_1,\ldots,k_j}\int_{a_l}^{b_l} dx_l\right)\left(\prod_{\nu=k_1,\ldots,k_j}\int_{a_\nu}^{b_\nu} dx_{k_\nu}\right)$$

$$\left.\cdot|f(x_1,\ldots,x_{k_1-1},u_{k_1},x_{k_1+1},\ldots,x_{k_j-1},u_{k_j},x_{k_j+1},\ldots,x_n)\right|.$$

Using the condition (2.2) for the integral inside Σ, we see that the last one does not exceed

$$\left(\prod_{\mu=k_1,\ldots,k_j}\int_{a_\mu}^{b_\mu} dx_\mu\right)\sum_{1\le k_1<\ldots<k_j\le n}\left|\frac{\partial^j}{\partial x_{k_1}\cdots\partial x_{k_j}} g(b_1,\ldots,x_{k_1},\ldots,b_n)\right| D2^n \prod_{l\ne k_1,\ldots,k_n} b_l \prod_{\nu=k_1,\ldots,k_j} x_\nu$$

which is, because of (2.5) and the fact that

$$\int_{a_k}^{b_k}(1+\alpha_k^2x_k^2)^{-1}dx_k\le 2\int_{a_k}^{b_k}(1+\alpha_kx_k)^{-2}dx_k\le 2\alpha_k^{-1}(1+\alpha_ka_k)^{-1}, \tag{2.8}$$

not greater than

$$\left(\prod_{\mu=k_1,\ldots,k_j}\int_{a_\mu}^{b_\mu} dx_\mu\right)2^j\prod_{k=k_1,\ldots,k_j}\alpha_k^2x_k(1+\alpha_k^2x_k^2)^{-2}\cdot\prod_{l\ne k_1,\ldots,k_j}(1+\alpha_l^2b_l^2)^{-1}D2^n\prod_{m\ne k_1,\ldots,k_n} b_m\prod_{h=k_1,\ldots,k_n} x_h$$

$$\le\left(\prod_{\mu=k_1,\ldots,k_j}\int_{a_\mu}^{b_\mu} dx_\mu\right)2^{n+j}D\prod_{k=k_1,\ldots,k_j}(1+\alpha_k^2x_k^2)^{-1}\prod_{l\ne k_1,\ldots,k_j}\alpha_l^{-1}(1+\alpha_lb_l)^{-1}. \tag{2.9}$$

Using (2.8), this is not greater than

$$2^{n+j}D\prod_{\mu=k_1,\ldots,k_j}2\alpha_\mu^{-1}(1+\alpha_\mu a_\mu)^{-1}\prod_{l\ne k_1,\ldots,k_j}\alpha_l^{-1}(1+\alpha_lb_l)^{-1}\le 2^{n+2j}D\prod_{k=1}^n\alpha_k^{-1}(1+\alpha_ka_k)^{-1}.$$

Inserting this estimate and (2.7) into the right hand side of(2.1) with $|f(x)|$ for $f(x)$

$$\int_a^b|f(x)|g(x)dx\le\left(1+\sum_{j=1}^n 2^{n+2j}\right)D\prod_{k=1}^n\alpha_k^{-1}(1+\alpha_ka_k)^{-1}$$

which is (2.3).

Since $\int_{a_k}^{b_k}(1+\alpha_k^2x_k^2)^{-1}dx_k\le b_k$, we immediately have (2.4) using (2.9).

Lemma 3. *Supposes, for any $\delta>0$, we may find $T_0\in\mathbf{R}^1$, $T_0>0$ such that for $T_k>T_0$ $k=1,2,\ldots,n$,*

$$\left|\left(\prod_{k=1}^n T_k^{-1}\right)\int_0^T f(x)dx\right|<\delta, \tag{2.10}$$

where $T=(T_1,\ldots,T_n)$. Let $0<a<1$ and $B\ge1$ be fixed numbers in $\mathbf{R}^1$. Take $\eta\in\mathbf{R}^1$ such that $a/\eta>T_0$.

Then for $0\le A_k\le B$, $0<\epsilon_k<\eta$, $k=1,2,\ldots,n$,

$$|\int_{\underline{a}}^{A} f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})dx|\le 2^n B^n\delta, \tag{2.11}$$

where $A=(A_1,\ldots,A_n)$, $\underline{a}=(a,\ldots,a)$

Proof. Denote the integral on the left side of (2.11) by J. We have

$$J=\prod_{k=1}^{n}(\int_0^{A_k}dx_k-\int_0^{a}dx_k)f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})$$
$$=[(\prod_{k=1}^{n}\int_0^{A_k}dx_k)+\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}(-1)^{n-j}(\prod_{k=k_1,\ldots,k_j}\int_0^{A_k}dx_k)$$
$$\cdot(\prod_{l\ne k_1,\ldots,k_j}\int_0^{a}dx_l)+(-1)^n(\prod_{k=1}^{n}\int_0^{a}dx_k)]f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n}). \tag{2.12}$$

For the middle on the right hand side, it is

$$\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}(-1)^{n-j}\prod_{m=1}^{n}\epsilon_m(\prod_{k=k_1,\ldots,k_j}\int_0^{A_k/\epsilon_k}dx_k)(\prod_{l\ne k_1,\ldots,k_j}\int_0^{a/\epsilon_l}du_l)f(u),$$

$u=(u_1,\ldots,u_n)$, which is, because of (20), not greater, in absolute value, than

$$\prod_{k=k_1,\ldots,k_j}A_k a^{n-j}\delta\le B^n\delta.$$

We have the similar estimates for the first and the third on the right hand side of (2.12). Hence we have altogether

$$|J|\le 2B^n\delta+\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}B^n\delta^n=2^nB^n\delta$$

which proves (2.11).

3. Wiener formula.

Write $\mathbf{R}_+^n=\{x\in\mathbf{R}^n: x_k>0,\ k=1,2,\ldots,n\}$, $x=(x_1,\ldots,x_n)$. Let $\mathbf{M}^1(\mathbf{R}_+^n)$ be the class of functions $f(x)$, $x\in\mathbf{R}_+^n$ which are locally integrable in $\mathbf{R}_+^n$ such that

$$\sup_{0<T}(\prod_{k=1}^{n}T_k^{-1})\int_0^T|f(x)|dx<\infty \tag{3.1}$$

with $T=(T_1,\ldots,T_n)$. Denote

$$M_+(f)=\lim_{T_1,\ldots,T_n\to\infty}(\prod_{k=1}^{n}T_k^{-1})\int_0^T f(x)dx \tag{3.2}$$

when the limit on the right hand side exists as $T_1,\ldots,T_n$ go to infinity independently.

Let $\mathbf{K}(\mathbf{R}_+^n)$ be the class of functions $K(x)$, $x\in\mathbf{R}_+^n$ such that $\frac{\partial^p}{\partial x_{k_1}\cdots\partial x_{k_p}}K(x)$ exists for any $1\le k_1<\ldots<k_p\le n$, in $\mathbf{R}_+^n$, $p=1,\ldots,n$ and is integrable in every compact domain in $\mathbf{R}_+^n$, and moreover

$$\prod_{k=1}^{n}(1+x_k^2)K(x)\leq C<\infty \tag{3.3}$$

holds for $x\in\mathbf{R}_+^n$, where C is a positive constant.

We then have the following Theorems 1 in which (3.4) is called the Wiener formula.

Theorem 1. *If $f(x)\in\mathbf{M}^1(\mathbf{R}_+^n)$, $K(x)\in\mathbf{K}(\mathbf{R}_+^n)$, and $M_+(f)$ exists, then*

$$\lim_{\epsilon_1,\ldots,\epsilon_n\to 0+}\int_{\mathbf{R}_+^n} f\left(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n}\right)K(x)dx=M_+(f)\int_{\mathbf{R}_+^n}K(x)dx. \tag{3.4}$$

Now write the class of locally integrable functions $f(x)$ in $\mathbf{R}_+^n$ for which (3.1) holds with $T_k=c_k u$, $u\in\mathbf{R}_+^1$ as $u\to\infty$, c_k, $k=1,2,\ldots,n$ being fixed constants, by $M_r^+(\mathbf{R}_+^n)$. $M_{r,+}(f)$ is similarly defined by (3.2) with $T_k=c_k u$, $u\to\infty$, when it exists. We then have

Theorem 2 . *If $f(x)\in\mathbf{M}_r^1(\mathbf{R}_+^n)$, $K(x)\in\mathbf{K}(\mathbf{R}_+^n)$ and $M_{r,+}(f)$ exists, then for $\epsilon\in\mathbf{R}_1^+$,*

$$\lim_{\epsilon\to 0+}\int_{\mathbf{R}^n} f\left(\frac{c_1x_1}{\epsilon},\ldots,\frac{c_nx_n}{\epsilon}\right)K(x)dx=M_{r,+}(f)\int_{\mathbf{R}_+^n}K(x)dx. \tag{3.5}$$

(3.5) is also called the Wiener formula.

Furthermore we define $\mathbf{M}^1(\mathbf{R}^n)$ to be the class of locally integrable functions $f(x)$ in $\mathbf{R}^n$, for which

$$\sup_{0<T}\prod_{k=1}^{n}(2T_k)^{-1}\int_{-T}^{T}|f(x)|dx<\infty. \tag{3.6}$$

Write

$$M(f)=\lim_{T_1,\ldots,T_n\to\infty}\left(\prod_{k=1}^{n}T_k^{-1}\right)\int_{-T}^{T}f(x)dx \tag{3.7}$$

when the limit on the right hand side exists.

Let $\mathbf{K}(\mathbf{R}^n)$ be the class of functions $K(x)$, $x\in\mathbf{R}^n$ defined in a way similar to $\mathbf{K}(\mathbf{R}_+^n)$ with $\mathbf{R}^n$ in place of $\mathbf{R}_+^n$. Namely for $K(x)\in\mathbf{K}(\mathbf{R}^n)$, $\frac{\partial^p}{\partial x_{k_1},\ldots,\partial x_{k_1}}K(x)$ exists, for any $1\leq k_1<\ldots<k_p\leq n$ in $\mathbf{R}^n$ and is integrable in every compact domain in $\mathbf{R}^n$ and satisties (3.3) for all $x\in\mathbf{R}^n$.

Then we have

Theorem 3. *If $f(x)\in\mathbf{M}^1(\mathbf{R}^n)$, $K(x)\in\mathbf{K}(\mathbf{R}^n)$ and $M(f)$ exists, then*

$$\lim_{\epsilon_1,\ldots,\epsilon_n\to 0+}\int_{\mathbf{R}^n} f\left(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n}\right)K(x)dx=M(f)\int_{\mathbf{R}^n}K(x)dx \tag{3.8}$$

Define $\mathbf{M}_r^1(\mathbf{R}_+^n)$ and $M_r(f)$ in ways just similar to those of $\mathbf{M}_r^1(\mathbf{R}_+^n)$ and $M_r^+(f)$ replacing $\mathbf{R}^n$ by $\mathbf{R}_+^n$.

We then have

Theorem 4. *If $f(x)\in M_r^1(\mathbf{R}^n)$, $K(x)\in K(\mathbf{R}_+^n)$ and $M_r(f)$ exists and moreover $K(x)$ is an even function for each variables $x_k, k=1,2,\ldots,n$, then*

$$\lim_{\epsilon\to 0+}\int_{\mathbf{R}^n} f(\frac{c_1x_1}{\epsilon},\ldots,\frac{c_nx_n}{\epsilon})K(x)dx=M_r(f)\int_{\mathbf{R}^n}K(x)dx. \tag{3.9}$$

4.Proofs of Theorems 1-4

Proof of Theorem 1. We may and do suppose $M_+(f)=0$. Let $A>1$, $A\in\mathbf{R}_+^1$. We write

$$\begin{aligned}I&=\int_{\mathbf{R}_+^n} f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})K(x)dx\\&=[\prod_{k=1}^n(\int_0^A dx_k+\int_A^\infty dx_k)]f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})K(x)\\&=[(\prod_{k=1}^n\int_0^A dx_k)+\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}(\prod_{k=1}^j\int_0^A dx_k\prod_{l=j+1}^n\int_A^\infty dx_l)+(\prod_{k=1}^n\int_A^\infty dx_k)]f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})K(x)\\&=I_1+I_2+I_3,\end{aligned}\tag{4.1}$$

say. We shall estimate I_2 first. From (3.3), we have

$$|I_2|\le C\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}(\prod_{k=k_1,\ldots,k_j}\epsilon_k\int_0^{A/\epsilon_k}du_k\cdot\prod_{l\ne k_1,\ldots,k_j}\epsilon_l\int_{A/\epsilon_l}^\infty du_l)|f(u)|\prod_{m=1}^n(1+\epsilon_m^2u_m^2)^{-1},$$

where $u=(u_1,\ldots,u_n)$. Appliying (2.3) of Lemma 2 and letting $0<\epsilon_k<1$, $k=1,2,\ldots,n$, we have

$$|I_2|\le C_nD\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}(1+A)^{-n+j}\le C_nD(1+A)^{-1},\tag{4.2}$$

where D is the quantity on the left hand side of (3.1) and C_n is a constant depending only on n and may be different on each occurrence.

In just the same way, we see that

$$|I_3|\le C_nD(1+A)^{-1}.\tag{4.3}$$

We now consider I_1. We split it in the following way. Take a positive number $a<A$ and write

$$\begin{aligned}I_1&=\prod_{k=1}^n(\int_0^a dx_k+\int_a^A dx_k)f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})K(x)\\&=[(\prod_{k=1}^n\int_0^a dx_k)+\sum_{j=1}^{n-1}\sum_{1\le k_1<\ldots<k_j\le n}(\prod_{k=k_1,\ldots,k_j}\int_0^a dx_k)(\prod_{l=k_1,\ldots,k_j}\int_a^A dx_l)+(\prod_{k=1}^n\int_a^A dx_k)]f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})K(x)\\&=I_{1,1}+I_{1,2}+I_{1,3},\end{aligned}$$

say. Because of (3.3).

$$|I_{1,1}|\le C(\prod_{k=1}^n\int_0^a dx_k)|f(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n})|\prod_{k=1}^n(1+x_l^2)^{-1}$$

$$\le C\prod_{m=1}^{n}\epsilon_m\Big(\prod_{k=1}^{n}\int_0^{a/\epsilon_k}du_k\Big)|f(u)|\prod_{l=1}^{n}(1+\epsilon_l^2u_l^2)^{-1}$$

$(u=(u_1,\ldots,u_n))$, which is, by (2.4) of Lemma 2

$$\le C_nDa^a, \tag{4.4}$$

where we suppose

$$1>a>\epsilon_k, \quad k=1,2,\ldots,n. \tag{4.5}$$

In a similar way, we easily have

$$|I_{1,2}|\le C_nDa^jA^{n-j}\le C_nDaA^n. \tag{4.6}$$

We use Lemma 1 to estimate $I_{1,3}$ and have

$$\begin{aligned}
&I_{1,3}\\
&=\Big(\prod_{k=1}^{n}\int_a^A dx_k\Big)f\Big(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n}\Big)K(A,\ldots,A)\\
&\quad -+\ldots\ldots\ldots\ldots\ldots\\
&\quad +(-1)^j\Big(\prod_{k=1}^{n}\int_a^A dx_k\Big)\sum_{1\le k_1<\ldots<k_j\le n}\frac{\partial^j}{\partial x_{k_1}\ldots\partial x_{k_j}}K(A,\ldots,A,x_{k_1},A,\ldots\ldots,A,x_{k_j},A,\ldots\ldots,A)\\
&\quad \Big(\prod_{l=1}^{j}\int_0^{x_{k_l}}du_l\Big)f\Big(\frac{x_1}{\epsilon_1},\ldots,\frac{x_{k_1-1}}{\epsilon_{k_1-1}},\frac{u_1}{\epsilon_{k_1}},\frac{x_{k_1+1}}{\epsilon_{k_1+1}},\ldots,\frac{x_{k_j-1}}{\epsilon_{k_j-1}},\frac{u_j}{\epsilon_{k_j}},\ldots,\frac{x_n}{\epsilon_n}\Big)\\
&\quad +\ldots\ldots\ldots\\
&\quad +(-1)^n\Big(\prod_{k=1}^{n}\int_a^A dx_k\Big)\frac{\partial^j}{\partial x_{k_1}\ldots\partial x_{k_1}}K(x)\Big(\prod_{k=1}^{n}\int_0^{x_k}du_k\Big)f\Big(\frac{u_1}{\epsilon_1},\ldots,\frac{u_n}{\epsilon_n}\Big).
\end{aligned} \tag{4.7}$$

The general term I_{13j} of this is , in absolute value, not greater than

$$\begin{aligned}
|I_{13j}|\le &\sum_{1\le k_1<\ldots<k_j\le n}\Big(\prod_{l=1}^{j}\int_a^A dx_{k_l}\Big)\Big|\frac{\partial^j}{\partial x_{k_1}\ldots\partial x_{k_j}}K(A,\ldots,A,x_{k_1},A,\ldots\ldots,A,x_{k_j},A,\ldots\ldots,A)\Big|\\
&\cdot\Big|\Big(\prod_{m=1}^{j}\int_a^{x_{k_m}}du_m\Big)\Big(\prod_{p\ne k_1,\ldots,k_j}\int_a^A dx_p\Big)\\
&\cdot f\Big(\frac{x_1}{\epsilon_1},\ldots,\frac{x_{k_1-1}}{\epsilon_{k_1-1}},\frac{u_1}{\epsilon_{k_1}},\frac{x_{k_1+1}}{\epsilon_{k_1+1}},\frac{x_{k_j-1}}{\epsilon_{k_j-1}},\frac{u_j}{\epsilon_{k_j}},\frac{x_{k_j+1}}{\epsilon_{k_j+1}},,\ldots,\frac{x_n}{\epsilon_n}\Big)\Big|.
\end{aligned} \tag{4.8}$$

Now take $T_0=T_0(\delta)>1$ such that (2.10) in Lemma 3 holds, which is possible by our assumption $M_+(f)=0$.

Take $\eta>0$, so small that

$$a/\eta>T_0 \tag{4.9}$$

Since $a\le x_{k_l}\le A$ in the integral in (4.8), we have from Lemma 3

$$\left|\left(\prod_{m=1}^{j}\int_a^{x_{k_m}}du_m\right)\left(\prod_{p\neq k_1,\ldots,k_j}\int_a^A dx_p\right)f\right|\le 2^nA^n\delta$$

and hence

$$|I_{13j}|\le 2^nA^n\delta C_{n,j}(a,A),$$

where

$$C_{n,j}(a,A)=\sum_{1\le k_1<\ldots<k_j\le n}\left(\prod_{l=1}^{j}\int_a^A dx_{k_l}\right)\left|\frac{\partial^j}{\partial x_{k_1}\ldots\partial x_{k_j}}K(A,\ldots,A,x_{k_1},A,\ldots\ldots,A,x_{k_j},A,\ldots.,A)\right|.$$

Writing $C_n(a,A)=\sum_{j=1}^{n}C_{n,j}(a,A)$, we have

$$|I_{13}|\le C_n(a,A)2^nA^n\delta. \tag{4.10}$$

(4.4), (4.6) and (4.10) give us

$$|I_1|\le C_n[D(a^n+aA^n+C_n(a,A)A^n\delta]\le C_n[Da(1+A^n)+C_n(a,A)A^n\delta]$$

Putting this together with (4.2) and (4.3) we finally have

$$|I|\le C_n[D(1+A)^{-1}+Da(1+A^n)+C_n(a,A)A^n\delta] \tag{4.11}$$

Now let $\epsilon>0$ be an arbitrary positive number. Choose A so large that $C_nD(1+A^{-1})<\epsilon/3$ and then choose a so small that $0<a<1$ and $C_nDa(1+A^n)<\epsilon/3$ For such A and a, we take η so small that (4.9) holds . Then for ϵ, with $0<\epsilon_k<\eta<a$, we have

$$|I|<\epsilon$$

which completes the proof of Theorem 1.

The proof of Theorem 2 will go on just in the similar way as in that of Theorem 1.

We shall now give the **ProofofTheorem3** . Write

$$\phi(x_1,\ldots,x_n)=\phi(x)=\frac{1}{2^n}\sum_{\substack{\eta_k=\pm1\\k=1,2,\ldots,n}}f(\eta_1x_1,\ldots,\eta_nx_n), \tag{4.12}$$

where $\sum$ means the summation for all $f(\eta_1x_1,\ldots,\eta_nx_n)$ each of $\eta_1,\ldots,\eta_n$ assuming $+1$ or -1 independently. Let $T=(T_1,\ldots,T_n)$. We have

$$\left(\prod_{k=1}^{n}T_k^{-1}\right)\int_0^T\phi(x)dx=2^{-n}\left(\prod_{k=1}^{n}T_k^{-1}\right)\int_{-T}^{T}f(x)dx \tag{4.13}$$

($x=(x_1,\ldots,x_n)$) which converges to $M(f)$ as $T_1,\ldots,T_n\to\infty$ by assumption. Then by Theorem 1,

$$\lim_{\epsilon_1,\ldots,\epsilon_n\to0+}\int_{\mathbf{R}_+^n}f\left(\frac{x_1}{\epsilon_1},\ldots,\frac{x_n}{\epsilon_n}\right)K(x)dx=M(f)\int_{\mathbf{R}_+^n}K(x)dx. \tag{4.14}$$

Since $\phi(x)$ and $K(x)$ are symmetric for each variable x_k , (4.14) is still true if we let $\epsilon_1,\ldots,\epsilon_n$ go to zero in any manner and the integral of the left hand side of (4.14) can be written by

$$\frac{1}{2^n}\sum_{\eta_k=\pm 1}\int_{\mathbf{R}_+^n} f(\frac{\eta_1 x_1}{\epsilon_1}, \ldots, \frac{\eta_n x_n}{\epsilon_n})K(x)dx = \frac{1}{2^n}\int_{\mathbf{R}^n} f(\frac{x_1}{\epsilon_1}, \ldots, \frac{x_n}{\epsilon_n})K(x)dx$$

and the right hand side of (4.14) is equal to

$$M(f)\frac{1}{2^n}\int_{\mathbf{R}^n} K(x)dx.$$

Thus (4.14) turns out to be (3.8), which proves Theorem 3 .

The proof of Theorem 4 is carried out in just the similar way as in that of Theorem 3.

5. Wiener formula and the general Tauberian theorems

Suppose $F(x)$, $x \in \mathbf{R}_+^n$ is bounded in $\mathbf{R}_+^n$. Then it belongs to $\mathbf{M}^1(\mathbf{R}_+^n)$ and Theorem 1 holds with $K(x) \in \mathbf{K}(\mathbf{R}_+^n)$. We shall show that when $f(x) \geq 0$ and is bounded, the existence of the left hand side of (3.4) is equivalent to the existence of $M_+(f)$ under a suitable condition on $K(x)$. Precisely we shall show the following theorem.

Theorem 5. *Suppose $f(x)$, $x \in \mathbf{R}_+^n$, is bounded and nonnegative in $\mathbf{R}_+^n$ and $K(x)$ satisfies the condition that (3.3) holds for $x \in \mathbf{R}_+^n$ and*

$$\int_{\mathbf{R}_+^n} K(x)dx = 1 \tag{5.1}$$

If the Melin transform of $K(x)$

$$\int_{\mathbf{R}_+^n} K(x)\prod_{k=1}^{n} x_k^{iu_k} dx \neq 0 \tag{5.2}$$

for all $u = (u_1, \ldots, u_n) \in \mathbf{R}^n$, then for some constant m the limit relations

$$\lim_{T_1, \ldots, T_n \to \infty} (\prod_{k=1}^{n} T_k^{-1})\int_0^T f(x)dx = m \tag{5.3}$$

$(T = (T_1, \ldots, T_n))$

and

$$\lim_{\epsilon_1, \ldots, \epsilon_n \to 0+} \int_{\mathbf{R}_+^n} f(\frac{x_1}{\epsilon_1}, \ldots, \frac{x_n}{\epsilon_n})K(x)dx = m \tag{5.4}$$

are equivalent to each other in the sense that the limit in (5.3) or in (5.4) exists, then the other limit exists and either relation (5.3) or (5.4) implies the other.

Theorem 6. *If $c_k > 0$, $k = 1,2,\ldots,n$ and all the conditions in Theorem 5 are satisfied, then the limit relations*

$$\lim_{T\to\infty} T^{-n}\left(\prod_{k=1}^{n} c_k^{-1}\int_0^{c_kT} dx_k\right)f(x)=m, \qquad T\in\mathbf{R}^1 \tag{5.5}$$

and

$$\lim_{\epsilon\to 0+}\int_{\mathbf{R}^n} f\left(\frac{c_1x_1}{\epsilon},\ldots,\frac{c_nx_n}{\epsilon}\right)K(x)dx=m \tag{5.6}$$

are equivalent to each other in the sense that if either of the limit in (5.5) or in (5.6) exists, then the other limit exists and either relation (5.5) or (5.6) imlpies the other.

In other to prove theses Theorems, we follow the lines of the proofs of Wiener [ ] and Anzai-Koizumi-Matsuoka [ ] in the following section and appeal to the general Tauberian theorems.

Denote by $U(\mathbf{R}^n)$ a subclass of $L^1(\mathbf{R}^n)$ of continuous functions $f(x)$ for which

$$\sum_{m=-\infty}^{\infty}\max_{m\le x\le m+1}|f(x)|<\infty, \tag{5.7}$$

where $m=(m_1,\ldots,m_n)$, a lattice point in $\mathbf{R}^n$, $1=(1,\ldots,1)$ and $\sum_{m=-\infty}^{\infty}$ means

$$\sum_{m_1=-\infty}^{\infty}\sum_{m_2=-\infty}^{\infty}\cdots\sum_{m_n=-\infty}^{\infty}.$$

Moreover let $U_0(\mathbf{R}^n)$ be a subspace of $U(\mathbf{R}^n)$ for which, for each $u\in\mathbf{R}^n$, there exists an $f(x)\in U_0(\mathbf{R}^n)$ with the property that

$$\hat{f}(u)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n} e^{-ixu}f(x)dx\neq 0. \tag{5.8}$$

The class of a single function $f(x)\in U(\mathbf{R}^n)$ with $\hat{f}(u)\neq 0$ for all $u\in\mathbf{R}^n$ is an $U_0(\mathbf{R}^n)$.

We now introduce a notation , for a function $g(x), x\in\mathbf{R}^n$ and $a,b\in\mathbf{R}^n$, $a<b$,

$$\begin{aligned}\Delta_{x=a}^{b}g(x)&=g(b_1,\ldots,b_n)-\sum_{k=1}^{n}g(b_1,\ldots,b_{k-1},a_k,b_{k+1},\ldots,b_n)\\&+\sum_{1\le k_1<k_2\le n}g(b_1,\ldots,b_{k_1-1},a_{k_1},b_{k_1+1},\ldots,b_{k_2-1},a_{k_2},b_{k_2+1},\ldots,b_n)\\&-+\cdots\cdots\cdots\\&+(-1)^n g(a_1,\ldots,a_n).\end{aligned} \tag{5.9}$$

We note that

$$\Delta_{x=a}^{b}g(x)=\Delta_{x_1=a_1}^{b_1}\cdots\Delta_{x_{n-1}=a_{n-1}}^{b_{n-1}}\Delta_{x_n=a_n}^{b_n}g(x).$$

The meaning of the right hand side will be obvious. Define as usual

$$\int_a^b |dg(x)| = \sup_D \sum_{\Delta_j} |\Delta_j g(x)|,$$

where $D:(\Delta_1, \ldots, \Delta_{2^n})$, $\bigcup_{j=1}^{2^n} \Delta_j = [a,b]$ is a decomposition of $[a,b]$ into closed nonoverlapping subintervals $\Delta_j (j=1,2,\ldots,2^n)$ and sup is taken for all such divisions. A function $f(x)$, $x \in \mathbf{R}^n$ with finite $\int_a^b |dg(x)|$ is of bounded variation over $[a,b]$ in $\mathbf{R}^n$.

Let $V(\mathbf{R}^n)$ be the class of functions of bounded variation in every compact interval in $\mathbf{R}^n$, for which

$$\sup_m \int_m^{m+1} |dg(x)| < \infty, \tag{5.10}$$

where m runs over all lattice points in $\mathbf{R}^n$. Then as is easily seen, the integral

$$\int_{\mathbf{R}^n} f(x-y)\,dg(y), \qquad x \in \mathbf{R}^n \tag{5.11}$$

exists for every pair of $f \in U(\mathbf{R}^n)$ and $g \in V(\mathbf{R}^n)$.

The Tauberian theorems we are going to appeal to are the followings.

Theorem A (Wiener) *Let $g(x) \in V(\mathbf{R}^n)$ and $K(x) \in U_0(\mathbf{R}^n)$. If*

$$\lim_{\zeta_1, \ldots, \zeta_n \to \infty} \int_{\mathbf{R}^n} K(\zeta - x)\,dg(x) = m \int_{\mathbf{R}^n} K(x)\,dx \tag{5.12}$$

for some constant m, then

$$\lim_{\zeta_1, \ldots, \zeta_n \to \infty} \int_{\mathbf{R}^n} L(\zeta - x)\,dg(x) = m \int_{\mathbf{R}^n} L(x)\,dx \tag{5.13}$$

holds for every $L(x) \in U(\mathbf{R}^n)$.

Theorem B (Anzai-Koizumi-Matsuoka) *Let $g(x) \in V(\mathbf{R}^n)$ and $K(x) \in U_0(\mathbf{R}^n)$. Let $\xi(t) = (\xi_1(t), \ldots, \xi_n(t)), t \in \mathbf{R}^1_+$, be a continuous function with*

$$\xi(0) = 0 \quad \text{and} \quad \xi_j(t) \to \infty, \; j = 1,2,\ldots,n, \; \textit{as } t \to \infty.$$

If

$$\lim_{t \to \infty} \int_{\mathbf{R}^n} K(\xi(t) - \zeta - x)\,dg(x) = m \int_{\mathbf{R}^n} K(x)\,dx \tag{5.14}$$

for some $\zeta \in \mathbf{R}^n$ and a constant m, then

$$\lim_{t \to \infty} \int_{\mathbf{R}^n} L(\xi(t) - \zeta - x)\,dg(x) = m \int_{\mathbf{R}^n} L(x)\,dx \tag{5.15}$$

holds for every $L(x) \in U(\mathbf{R}^n)$.

The $\mathbf{R}^2$ case of this theorem was given by Anzai-Koizumi-Matsuoka [1] and it is easy to generalize to the $\mathbf{R}^n$ case.

6. Proofs of Theorems 5 and 6.

Proof of Theorem 5. Let $x=(x_1, \ldots, x_n)\in\mathbf{R}^n_+$. We write $x_k=e^{\xi_k}$, $k=1,2,\ldots,n$, $\xi=(\xi_1, \ldots, \xi_n)$ and define $\psi(\xi)$ by

$$\psi(\xi)=f(e^{\xi_1}, \ldots, e^{\xi_n}). \tag{6.1}$$

$\psi(\xi)\geq 0$ and is bounded in $\mathbf{R}^n$. Also we write $T_k=e^{\zeta_k}, \epsilon_k=T_k^{-1}=e^{-\zeta_k}$, $k=1,2,\ldots,n$, $T=(T_1, \ldots, T_n)$, $\zeta=(\zeta_1, \ldots, \zeta_n)$, $\epsilon=(\epsilon_1, \ldots, \epsilon_n)$. Then

$$\prod_{k=1}^{n} T_k^{-1}\int_0^T f(x)dx=(\prod_{k=1}^{n} e^{-\zeta_k}\int_{-\infty}^{\zeta_k} e^{\xi_k} d\xi_k)\psi(\xi)$$

and (5.3) turns out to be

$$\lim_{\zeta_1, \ldots, \zeta_n\to\infty}(\prod_{k=1}^{n}\int_{-\infty}^{\zeta_k} e^{-(\zeta_k-\xi_k)}d\xi_k).\psi(\xi)=m. \tag{6.2}$$

On the other hand, the integral in (5.4) is

$$\prod_{k=1}^{n}\epsilon_k\int_{\mathbf{R}^n_+} f(x)K(\epsilon_1 x_1, \ldots, \epsilon_n x_n)dx=\prod_{k=1}^{n} e^{-\zeta_k}\int_{\mathbf{R}^n}\psi(\xi)K(e^{-(\zeta_1-\xi_1)},\ldots,e^{-(\zeta_n-\xi_n)})\prod_{l=1}^{n} e^{\xi_l}d\xi$$

and hence (5.4) turns out to be

$$\lim_{\xi_1, \ldots, \xi_n\to\infty}\int_{\mathbf{R}^n}\psi(\xi)\prod_{k=1}^{n} e^{-(\zeta_k-\xi_k)}K(e^{-(\zeta_1-\xi_1)},\ldots,e^{-(\zeta_n-\xi_n)})d\xi=m. \tag{6.3}$$

Therefore for our purpose, it is sufficient to prove the equivalence of (6.2) and (6.3).

Now write

$$g(\xi)=(\prod_{k=1}^{n}\int_0^{\xi_k} d\eta_k)\psi(\eta) \tag{6.4}$$

$(\eta=(\eta_1, \ldots, \eta_n))$. Note that $g(\zeta)\geq 0$, $\Delta_{\xi=a}^{b} g(\xi)\geq 0$ and then

$$\int_m^{m+1}|dg(\xi)|=\Delta_{\xi=m}^{m+1} g(\xi)=\int_m^{m+1}\psi(\xi)d\xi \ .$$

Since $\psi(\xi)$ is bounded

$$\sup_m\int_m^{m+1}|dg(\xi)|<\infty, \tag{6.5}$$

namely $g\in V(\mathbf{R}^n)$.

Write

$$K_1(\xi)=\prod_{k=1}^{n} e^{-\xi_k}, \quad \text{for } \xi\in\mathbf{R}^n_+ \ , \tag{6.6}$$

$$=0, \quad \textit{elsewhere in } \mathbf{R}^n$$

and

$$K_2(\xi)=K(e^{-\xi_1}, \ldots, e^{-\xi_n})\prod_{k=1}^{n} e^{-\xi_k}, \quad \xi\in\mathbf{R}^n \ . \tag{6.7}$$

We easily see that

$$\int_{\mathbf{R}^n} K_1(\xi)d\xi=\int_{\mathbf{R}^n} K_2(\xi)d\xi=1. \tag{6.8}$$

(6.2) and (6.3) then take the following forms respectively,

$$\lim_{\zeta_1,\ldots,\zeta_n\to\infty}\int K_1(\zeta-\xi)dg(\xi)=m, \tag{6.9}$$

$$\lim_{\zeta_1,\ldots,\zeta_n\to\infty}\int K_2(\zeta-\xi)dg(\xi)=m. \tag{6.10}$$

In order to show the equivalence of these relations, we appeal to Theorem A. For this, we first check that $K_2(\xi)$ satisfies the condition of K in Theorem A.

$K_2(\xi)$ is continuous and because of (3.3)

$$|K_2(\xi)|\le C\prod_{k=1}^{n}e^{-\xi_k}\prod_{l=1}^{n}(1+e^{-2\xi_l})^{-1}\le C\prod_{k=1}^{n}e^{-|\xi_k|}, \qquad \text{for } \xi\in\mathbf{R}^n$$

so that $K_2(\xi)\in U(\mathbf{R}^n)$. Moreover we see that

$$\hat{K}_2(u)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}K(e^{-\zeta_1},\ldots,e^{-\zeta_n})\prod_{k=1}^{n}e^{-\xi_k(1+iu_k)}d\xi\cdot\frac{(-1)^n}{(2\pi)^{n/2}}\int_{\mathbf{R}_+^n}K(x)\prod_{k=1}^{n}x_k^{iu_k}dx\neq 0, \quad (u\in\mathbf{R}^n)$$

by (5.2) and hence $K_2(u)\in U_0(\mathbf{R}^n)$. Therefore $K_2(x)$ satisfies the condition of K in Theorem A.

Next we consider $K_1(\xi)$. This does not satisfy the condition in Theorem A, since it is not continuous. We then take $\bar{K}_1(\xi,\epsilon)$, in place of $K_1(\xi)$, defined in the following way, ϵ being any positive number of $\mathbf{R}_+^1$.

Write $\boldsymbol{\epsilon}=(\epsilon,\ldots,\epsilon)$.

Denote, for $1\le k_1<\ldots<k_j\le n$, $1\le j\le n-1$,

$D_{k_1,\ldots,k_j}=\{\ \xi\ :\ \xi_k\ge 0\ \ (k=k_1,\ldots,k_j),\ 0>\xi_l\ge-\epsilon,\ (l\neq k_1,\ldots,k_j)\}$,
$D'_{k_1,\ldots,k_j}=\{\ \xi\ :\ \xi_k\ge 0\ \ (k=k_1,\ldots,k_j),\ 0>\xi_l,\ (l\neq k_1,\ldots,k_j)\}-D_{k_1,\ldots,k_j}$,
$E=\{\ \xi\ :0>\xi_k\ge-\epsilon,\ (1\le k\le n\ )\}$

and

$E'=\{\ \xi\ :0>\xi_k,\ (1\le k\le n\}-E$.

Obviously

$$\mathbf{R}^n=\mathbf{R}_+^n\cup\Big(\bigcup_{j=1}^{n-1}\ \bigcup_{1\le k_1<\ldots<k_j\le n}(D_{k_1,\ldots,k_j}\cup D'_{k_1,\ldots,k_j})\Big)\cup E\cup E'. \tag{6.11}$$

We now define

$$\bar{K}_1(\xi,\epsilon)=\Big(\prod_{k=1}^{n}\epsilon^{-1}\int_{\xi_k}^{\xi_k+\epsilon}d\eta_k\Big)K_1(\eta) \tag{6.12}$$

$\eta=(\eta_1,\ldots,\eta_k)$. We then have

$$\bar{K}_1(\xi,\epsilon)=\epsilon^{-n}(1-e^{-\epsilon})^n e^{-\xi_1-\dots-\xi_n}, \qquad \text{for } \xi\in\mathbf{R}^n_+,$$

$$\bar{K}_1(\xi,\epsilon)=\Big(\prod_{k=k_1,\dots,k_j}\epsilon^{-1}\int_{\xi_k}^{\xi_k+\epsilon}d\eta_k\Big)\Big(\prod_{l=k_1,\dots,k_j}\epsilon^{-1}\int_0^{\xi_l+\epsilon}d\eta_l\Big),$$

$$=\epsilon^{-n}(1-e^{-\epsilon})^j e^{-\xi_1-\dots-\xi_{k_j}}\prod_{l\neq k_1,\dots,k_j}(1-e^{-\xi_l-\epsilon}), \qquad \text{for } \xi\in D_{k_1,\dots,k_j},$$

$$=0, \qquad \text{for } \xi\in D'_{k_1,\dots,k_j},$$

$(1\le j\le n-1)$,

and

$$\bar{K}_1(\xi,\epsilon)=\Big(\prod_{k=1}^{n}\epsilon^{-1}\int_0^{\xi_k+\epsilon}d\eta_k\Big)K_1(\eta),$$

$$=\epsilon^{-n}\prod_{k=1}^{n}(1-e^{-\xi_k-\epsilon}), \qquad \text{for } \xi\in E,$$

$$=0, \qquad \text{for } \xi\in E'.$$

It is easy to see that $\bar{K}_1(\xi,\epsilon)$ is continuous in $\mathbf{R}^n$ and belongs to $U(\mathbf{R}^n)$.

We shall show that the class of $\bar{K}_1(\xi,\epsilon), 0<\epsilon$, is an $U_0(\mathbf{R}^n)$

$$\hat{\bar{K}}_1(u)=\hat{\bar{K}}_1(u,\epsilon)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}e^{-iu\cdot\xi}\bar{K}_1(\xi,\epsilon)d\xi$$

$$=\frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}e^{-iu\cdot\xi}\epsilon^{-n}\Big(\prod_{k=1}^{n}\int_{\xi_k}^{\xi_k+\epsilon}d\eta_k\Big)K_1(\eta,\epsilon)$$

$$=\frac{1}{(2\pi)^{n/2}}\int_{\mathbf{R}^n}e^{-iu\xi}\epsilon^{-n}\Big(\prod_{k=1}^{n}\int_0^{\epsilon}d\eta_k\Big)K_1(\xi+\eta,\epsilon)$$

$$=\frac{1}{\epsilon^n(2\pi)^{n/2}}\Big(\prod_{k=1}^{n}\int_0^{\epsilon}d\eta_k\Big)\int_{\mathbf{R}^n}K_1(\xi+\eta,\epsilon)e^{-iu\cdot\xi}d\xi$$

$$=\frac{1}{\epsilon^n(2\pi)^{n/2}}\prod_{k=1}^{n}\int_0^{\epsilon}e^{-\eta_k}d\eta_k\int_{-\eta_k}^{\infty}e^{-\xi_k(1+iu_k)}d\xi_k$$

$$=\frac{1}{\epsilon^n(2\pi)^{n/2}}\prod_{k=1}^{n}\frac{e^{iu_k\epsilon}-1}{iu_k(1+iu_k)}$$

$$=\Big(\frac{2}{\pi}\Big)^{n/2}\prod_{k=1}^{n}\frac{e^{iu_k\epsilon/2}}{1+iu_k}\frac{\sin(u_k\epsilon/2)}{u_k\epsilon}.$$

Hence $\hat{\bar{K}}_1(u,\epsilon)=0$ only for some $u_k=2m_k\pi/\epsilon$ $(m_k=0,\pm1,\dots)$. Since there is no u such that $\hat{\bar{K}}_1(u,\epsilon)=0$ for all $\epsilon>0$, $K_1(\xi,\epsilon)$ belongs to an $U_0(\mathbf{R}^n)$ and therefore from Theorem A

$$\lim_{\zeta_1,\ldots,\zeta_n\to\infty}\int_{\mathbf{R}^n}\bar{K}_1(\zeta-\xi)dg(\xi)=m \tag{6.13}$$

is equivalent to (6.10).

Thus what remains for the proof of Theorem 5 is to prove that (6.13) and (6.9) are equivalent to each other.

We prove this in what follows. First we shall show that (6.9) implies (6.13).

$$\int_{\mathbf{R}^n}\bar{K}_1(\zeta-\xi,\epsilon)dg(\xi)=\int_{\mathbf{R}^n}dg(\xi)\Big(\prod_{k=1}^{n}\epsilon^{-1}\int_{\zeta_k-\xi_k}^{\zeta_k-\xi_k+\epsilon}d\eta_k\Big)K_1(\zeta-\xi+\eta,\epsilon)$$
$$=\epsilon^{-n}\Big(\prod_{k=1}^{n}\int_0^{\epsilon}d\eta_k\Big)\int_{\mathbf{R}^n}K_1(\zeta-\xi+\eta)dg(\xi).$$

Since the inner integral is bounded as was mentioned before, the bounded convergence theorem and (6.8) give us that the last one converges, as $\zeta_1,\ldots,\zeta_n\to\infty$, to m and we have (6.13).

Next we suppose (6.13) holds good. Note that, for $\xi\in\mathbf{R}^n$,

$$\epsilon^{-n}(1-e^{-\epsilon})^nK_1(\xi,\epsilon)\le\bar{K}_1(\xi,\epsilon)\le\epsilon^{-n}(e^{\epsilon}-1)^nK_1(\xi+\underline{\epsilon},\epsilon) \tag{6.14}$$

$\underline{\epsilon}=(\epsilon,\ldots,\epsilon)$. Also note that $K_1(\xi,\epsilon)$, $\bar{K}_1(\xi,\epsilon)\ge0$ and $\Delta_{\xi=a}^{b}g(\xi)\ge0$ for any $a<b$. We then have

$$\begin{aligned}
&\epsilon^{-n}(1-e^{-\epsilon})^n\limsup_{\zeta_1,\ldots,\zeta_n\to\infty}\int_{\mathbf{R}^n}K_1(\zeta-\xi,\epsilon)dg(\xi)\\
&\le\limsup_{\zeta_1,\ldots,\zeta_n\to\infty}\int_{\mathbf{R}^n}\bar{K}_1(\zeta-\xi,\epsilon)dg(\xi)\\
&\le\epsilon^{-n}(e^{\epsilon}-1)^n\liminf_{\zeta_1,\ldots,\zeta_n\to\infty}\int_{\mathbf{R}^n}K_1(\zeta-\xi+\epsilon,\epsilon)dg(\xi)\\
&\le\epsilon^{-n}(e^{\epsilon}-1)^n\liminf_{\zeta_1,\ldots,\zeta_n\to\infty}\int_{\mathbf{R}^n}K_1(\zeta-\xi,\epsilon)dg(\xi).
\end{aligned} \tag{6.15}$$

Since by (6.13), the right hand side of (6.15) is equal to m, and we may take ϵ as close to zero as we want, the above inequality relation shows the validity of (6.9). This completes the proof of Theorem 5.

Theorem 6 is proved in a way quite similar to the proof of Theorem 5, if Theorem B is used. See Anzai-Koizumi-Matsuoka [1].

7. Wiener's identity in $\mathbf{R}^n$

We denote by $\mathbf{N}^2(\mathbf{R}^n)$ the class of functions $f(x)$, $x\in\mathbf{R}^n$, satisfying

$$\int_{\mathbf{R}^n}|f(x)|^2\prod_{k=1}^{n}(1+x_k^2)^{-1}dx<\infty \tag{7.1}$$

$x=(x_1,\ldots,x_n)$. We define by $\mathbf{W}^2(\mathbf{R}^n)$ the class of functions with the property that

$$\sup_{T>0}\prod_{k=1}^{n}(2T_k)^{-1}\int_{-T}^{T}|f(x)|^2dx<\infty \tag{7.2}$$

and

$$\lim_{T_1,\ldots,T_n\to\infty}\prod_{k=1}^{n}(2T_k)^{-1}\int_{-T}^{T}|f(x)|^2dx=M(|f|^2) \tag{7.3}$$

exists, $T=(T_1,..,T_n)$.

Now we define the Wiener transform of $f(x)\in\mathbf{N}^2(\mathbf{R}^n)$ $x\in\mathbf{R}^n$ by

$$s(u)=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{(2\pi)^{n/2}}\Big[\prod_{k=1}^{n}\Big(\int_{|x_k|<1}\frac{e^{-iu_kx_k}-1}{-ix_k}dx_k+\int_{1<|x_k|<A}\frac{e^{-iu_k\cdot x_k}}{-ix_k}dx_k\Big)\Big]f(x) \tag{7.4}$$

($A\in\mathbf{R}^1$, $A>1$). The l.i.m. means the limit in $L^2(\mathbf{R}^n)$. This is well defined. Note that $\mathbf{W}^2(\mathbf{R}^n)\subset\mathbf{N}^2(\mathbf{R}^n)$. This was shown in the course of proof of Theorem 3 (or Theorem 1) with $K(x)=\prod_{k=1}^{n}(1+x_k^2)^{-1}$.

$$s(u)=\frac{1}{(2\pi)^{n/2}}\sum_{j=0}^{n}J_j(u), \tag{7.5}$$

where

$$J_0(u)=\Big(\prod_{k=1}^{n}\int_{1<|x_k|<A}\frac{e^{-iu_kx_k}}{-ix_k}dx_k\Big)f(x), \tag{7.6}$$

$$J_n(u)=\Big(\prod_{k=1}^{n}\int_{|x_k|<1}\frac{e^{-iu_kx_k}-1}{-ix_k}dx_k\Big)f(x) \tag{7.7}$$

and for $1\le j\le n-1$,

$$J_j(u)=\Big(\prod_{k=k_1,\ldots,k_j,1\le k_1<\ldots<k_j\le n}\int_{|x_k|\le 1}\frac{e^{-iu_kx_k}-1}{-ix_k}dx_k\Big)\cdot\Big(\prod_{l\ne k_1,\ldots,k_j}\int_{1<|x_l|\le A}\frac{e^{-iu_lx_l}}{-ix_l}dx_l\Big)f(x). \tag{7.8}$$

Now for $v,h\in\mathbf{R}^n$, $v=(v_1,\ldots,v_n)$, $h=(h_1,\ldots,h_n)$, $h>0$, we have, noting that $\Delta_{u_k=a}^{b_k}(e^{-iu_kx_k}-1)=\Delta_{u_k=a_k}^{b_k}e^{-iu_kx_k}$, $\Delta_{u=a}^{b}=\Delta_{u_1=a_1}^{b_1}\ldots\Delta_{u_n=a_n}^{b_n}$,

$$\begin{aligned}\Delta_{u=v-h}^{v+h}s(u)&=\underset{A\to\infty}{\mathrm{l.i.m.}}\sum_{j=0}^{n}\Delta_{u=v-h}^{v+h}J_j(u)\\&=\underset{A\to\infty}{\mathrm{l.i.m.}}\Big(\prod_{k=1}^{n}\int_{1<|x_k|\le A}\frac{dx_k}{-ix_k}\Delta_{u_k=v_k-h_k}^{v_k+h_k}e^{-iu_kx_k}\Big)f(x)\\&+\sum_{j=1}^{n-1}\underset{A\to\infty}{\mathrm{l.i.m.}}\Big(\prod_{k=k_1,\ldots,k_j,1\le k_1<\ldots<k_j\le n}\int_{|x_k|\le 1}\frac{dx_k}{-ix_k}\Delta_{u_k=v_k-h_k}^{v_k+h_k}e^{-iu_kx_k}\Big)\\&\cdot\Big(\prod_{l\ne k_1,\ldots,k_j,1\le k_1<\ldots<k_j\le n}\int_{1<|x_l|\le A}\frac{dx_l}{-ix_l}\Delta_{u_l=v_l-h_l}^{v_l+h_l}e^{-iu_lx_l}\Big)f(x)\\&+\Big(\prod_{k=1}^{n}\int_{|x_k|\le 1}\frac{dx_k}{-ix_k}\Delta_{u_k=v_k-h_k}^{v_k+h_k}e^{-iu_kx_k}\Big)f(x).\end{aligned}$$

Since $\Delta_{u_k=v_k-h_k}^{v_k+h_k} e^{-iu_kx_k} = 2\frac{\sinh_k x_k}{x_k} e^{-ix_kv_k}$, we have

$$\Delta_{u=v-h}^{v+h} s(u) = \frac{1}{(2\pi)^{n/2}} [\prod_{k=1}^{n} \int_{1<|x_k|\leq A} dx_k + \sum_{j=1}^{n-1} (\prod_{k=k_1,\ldots,k_j, 1\leq k_1<\ldots<k_j\leq n} \int_{|x_k|\leq 1} dx_k)$$

$$\cdot(\prod_{l\neq k_1,\ldots,k_j} \int_{1<|x_l|\leq A} dx_l) + \prod_{k=1}^{n} \int_{|x_l|\leq 1} dx_k] 2^n \prod_{m=1}^{n} \frac{\sinh_m x_m}{x_m} e^{-ix\cdot v} f(x)$$

$$= (\frac{2}{\pi})^{n/2} \underset{A\to\infty}{\mathrm{l.i.m.}} \prod_{k=1}^{n} (\int_{|x_k|\leq 1} dx_k + \int_{1<|x_k|\leq A} dx_k) \prod_{m=1}^{n} \frac{\sinh_m x_m}{x_m} e^{-ix\cdot v} f(x)$$

$$= (\frac{2}{\pi})^{n/2} \underset{A\to\infty}{\mathrm{l.i.m.}} (\prod_{k=1}^{n} \int_{-A}^{A} \frac{\sinh_k x_k}{x_k} dx_k) e^{-ix\cdot v} f(x).$$

Hence we have shown that if $f(x)\in \mathbf{N}^2(\mathbf{R}^n)$ then the Fourier transform $\hat{\phi}(v,h)$ of

$$\phi(x,h) = \prod_{k=1}^{n} \frac{\sin^2 h_k x_k}{x_k} f(x) \qquad in\ L^2(\mathbf{R}^n)$$

is given by

$$\hat{\phi}(v,h) = \frac{1}{2^n} \Delta_{u=v-h}^{v+h} s(u) \tag{7.9}$$

$(h=(h_1,\ldots,h_n)>0)$.

Then Parseval relation gives us

$$(\frac{2}{\pi})^n \int_{\mathbf{R}^n} \prod_{k=1}^{n} \frac{\sin^2 h_k x_k}{x_k^2} |f(x)|^2 dx = \frac{1}{\pi^n} \prod_{k=1}^{n} h_k^{-1} \int_{\mathbf{R}^n} |\Delta_{u=v-h}^{v+h} s(u)|^2 du \tag{7.10}$$

$(h>0)$.

We now have the following theorems.

Theorem 7. *If $f(x)\in \mathbf{W}^2(\mathbf{R}^n)$, then*

$$M(|f|^2) = \lim_{h_1,\ldots,h_n\to 0+} \frac{1}{2^n} \prod_{k=1}^{n} h_k^{-1} \int_{\mathbf{R}^n} |\Delta_{u=v-h}^{v+h} s(u)|^2 du. \tag{7.11}$$

In (7.2) we suppose $T_k=c_kT$, $c_k>0$, $k=1,\ldots,n$, $T>0$, $T\in\mathbf{R}^1$ and let $T\to\infty$. Namely we consider the class of functions such that $M_r(|f|^2)$ exists. Such class is denoted by $\mathbf{W}_r^2(\mathbf{R}^n)$.

Theorem 8. *If $f(x)\in \mathbf{W}_r^2(\mathbf{R}^n)$, then*

$$M_r(|f|^2) = \lim_{\epsilon\to 0+} \frac{1}{2^n\epsilon^n} \prod_{k=1}^{n} c_k \cdot \int_{\mathbf{R}^n} |\Delta_{u=v-\eta}^{v+\eta} s(u)|^2 du,$$

where $\eta=(\epsilon c_1^{-1},\ldots,\epsilon c_n^{-1})$.

8. Mean concentration function in $\mathbf{R}^n$

Let $F(x)$ be a probability distribution function of a $\mathbf{R}^1$ -valued random variable $X(\omega)$ on a given probabilty space (Ω,F,P), $\omega\in\Omega$. The function of $h\in\mathbf{R}^1_+$,

$$C(h)=\frac{1}{2h}\int_{\mathbf{R}^1}[F(x+h)-F(x-h)]^2dx \tag{8.1}$$

always exists and is called the mean concentration function of $X(\omega)$ or of the distribution function $F(x)$. This was defined by the author [8] or see [10], as a counterpart of the Levy's concentration function.(Levy [11],[12] or see Doob [3]).

We shall make the generalization of $C(h)$ in $\mathbf{R}^n$ and give analogues of some basic properties of it.

Let $F(x)$, $x\in\mathbf{R}^n$, be the distribution function of an $\mathbf{R}^n$-valued random variable, $X(\omega)=(X_1(\omega), \ldots ,X_n(\omega))$. Write the characteristic functiuon of $X(\omega)$ by

$$f(t)=\int_{\mathbf{R}^n}e^{it\cdot x}dF(x), \tag{8.2}$$

$t=(t_1,\ldots,t_n)$. We write sometimes $dF(x)$ as $d_{x_1,\ldots,x_n}F(x)$ or $d_xF(x)$. We define the mean concentration function $C(h)$, $h\in\mathbf{R}^1_+$ of $X(\omega)$ by

$$C(h)=[\prod_{k=1}^{n}(2h_k)^{-1}]\cdot\int_{\mathbf{R}^n}[\Delta_{u=x-h}^{x+h}F(u)]^2dx. \tag{8.3}$$

From the inversion formula for a chacteristic function, it follows that, for a continuity interval $(x-h,x+h)$ of the distribution function $F(x)$.

$$\Delta_{u=x-h}^{x+h}F(u)=\frac{1}{(2\pi)^n}\lim_{T_1,\ldots,T_n\to\infty}\int_{-T}^{T}f(t)\prod_{k=1}^{n}\frac{e^{-it_k(x_k+h_k)}-e^{-it_k(x_k-h_k)}}{-it_k}dt$$

$(T=(T_1, \ldots ,T_n))$

$$=\lim_{T_1,\ldots,T_n\to\infty}\frac{1}{\pi^n}\int_{-T}^{T}f(t)\prod_{k=1}^{n}\frac{\sin h_kt_k}{t_k}e^{-it_kx_k}dt.$$

Since the integrand on the right hand side is a function of $L^2(\mathbf{R}^n)$, the Parseval relation gives us the following theorem.

Theorem 9. *The mean concentration function $C(h)$, $h\in\mathbf{R}^n$, $h=(h_1, \ldots ,h_n)>0$, satisfies*

$$C(h)=\frac{1}{\pi^n}\int_{\mathbf{R}^n}|f(t)|^2\prod_{k=1}^{n}\frac{\sin^2 h_kt_k}{h_kt_k^2}dt, \tag{8.4}$$

where $f(t)$ is the characteristic function.

We mention the following result which is an obvious generalization of the well known theorem for $\mathbf{R}^1$ case, and may be known.

Lemma 4. *Let* $a=(a_1,...a_n)\in\mathbf{R}^n$. *If* $f(t)$ *is the characteristic function of an* $\mathbf{R}^n$ *valued random variable X, then*

$$\lim_{T_1,\ldots,T_n\to\infty}\prod_{k=1}^{n}(2T_k)^{-1}\int_{-T}^{T}f(t)e^{-it\cdot a}dt=P(X=a) \tag{8.5}$$

$(T=(T_1,\ldots,T_n))$.

This is easily shown from

$$\prod_{k=1}^{n}(2T_k)^{-1}\cdot\int_{-T}^{T}f(t)e^{-it\cdot a}dt=\int_{\mathbf{R}^n}\prod_{k=1}^{n}\frac{\sin T_k(x_k-a_k)}{T_k(x_k-a_k)}dF(x)$$

and the fact that $\dfrac{\sin T_k(x_k-a_k)}{T_k(x_k-a_k)}$ boundedly converges to 0 if $x_k\neq a_k$ and is 1 if $x_k=a_k$.

We also note the following identity (8.6) which might be usefull in a further study of sequence of random variables.

Lemma 5. *For* $h=(h_1,\ldots,h_k)>0$

$$\prod_{k=1}^{n}(2h_k)^{-1}\int_0^{2h}\Delta_{v=x-u}^{x+u}F(v)du=\frac{1}{\pi^n}\int_{\mathbf{R}^n}f(t)\prod_{k=1}^{n}\frac{\sin^2 h_k t}{k_k t^2}e^{-itx}dt. \tag{8.6}$$

For $\mathbf{R}^1$ case of this Lemma, see [10] (Theorem 9.3.1)

The proof is quite analogous. The right hand side of (8.6) is equal to

$$\begin{aligned}
&\frac{1}{\pi^n}\int_{\mathbf{R}^n}dF(u)\int_{\mathbf{R}^n}e^{it\cdot(u-x)}\prod_{k=1}^{n}\frac{\sin^2 h_k t_k}{h_k t_k^2}dt\\
&=\int_{\mathbf{R}^n}dF(u)\Big(\prod_{k=1}^{n}\frac{1}{\pi}\int_{\mathbf{R}^1}\frac{\sin^2 h_k t_k}{h_k t_k^2}e^{it_k(u_k-x_k)}dt_k\Big)\\
&=\int_{\mathbf{R}^n}dF(u)\Big(\prod_{k=1}^{n}\frac{1}{h_k}U(u_k-x_k,h_k)\Big),
\end{aligned}$$

where

$$U(y,\alpha)=\begin{cases}0, & |y|\geq 2\alpha,\\ \alpha+y/2, & -2\alpha<y\leq 0,\\ \alpha-y/2, & 0<y\leq 2\alpha,\end{cases}$$

$\alpha,y\in\mathbf{R}^1$, $\alpha>0$. The last integral is

$$\begin{aligned}
&\int_{\mathbf{R}^n}\Big(\prod_{k=1}^{n}\frac{1}{k_k}U(v_k,h_k)\Big)d_vF(x+v)\\
&=\Big(\prod_{k=1}^{n}h_k^{-1}\Big)\int_{-2h_1}^{2h_1}\cdots\int_{-2h_{n-1}}^{2h_{n-1}}\prod_{k=1}^{n-1}U(v_k,h_k)
\end{aligned}$$

$$\cdot\int_0^{2h_n} U(v_n,h_n)d_v[F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n+v_n)-F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n-v_n)]$$

which is , by ordinary integration by parts applied to the integration regarding v_n, equal to

$$[\prod_{k=1}^{n-1}\frac{1}{h_k}\int_{-2h_k}^{2h_k}U(v_k,h_k)]d_{v_1,\ldots,v_{n-1}}h_n^{-1}[F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n+v_n)$$
$$-F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n-v_n)]_{v_n=0}^{2h_n}$$
$$+\frac{1}{2h_n}\int_0^{2h_n}[F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n+v_n)-F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},x_n-v_n)]dv_n\}$$
$$=[\prod_{k=1}^{n-1}\frac{1}{h_k}\int_{-2h_k}^{2h_k}U(v_k,h_k)]d_{v_1,\ldots,v_{n-1}}\{\frac{1}{2h_n}\int_0^{2h_n}\Delta_{w_n=x_n-v_n}^{x_n+v_n}F(x_1+v_1,\ldots,x_{n-1}+v_{n-1},w_n)dw_n\}.$$

Repeating this procedure, we have that the above is

$$\prod_{k=1}^{n-1}(2h_k)^{-1}\int_0^{2h}\Delta_{w=x-v}^{x+v}F(w)dv$$

$(w=(w_1,\ldots,w_n))$.

We now give the following basic properties of $C(h)$

Theorem 10. *The mean concentration function $C(h)$ of a distribution function $F(x)$, $h,x\in\mathbf{R}^n$, $h>0$, satisfies the following properties:*

(i) *$C(h)$ is nondecreasing, namely, if $0<h_1<h_2$ $(h_1,h_2\in\mathbf{R}_+^n)$ then*

$$C(h_1)\le C(h_2), \tag{8.7}$$

(ii)
$$0\le C(h)\le 1, \tag{8.8}$$

(iii) *$C(h)\to 0$, as $h_1,\ldots,h_n\to 0+$, if and only if $F(x)$ is contiuous in $\mathbf{R}^n$.*

Proof. Let $X=X(\omega)=(X_1,\ldots,X_n)$ be the $\mathbf{R}^n$ valued random variable whose distribution function is $F(x)$, $(x\in\mathbf{R}^n)$. Let $X'=(X'_1,\ldots,X'_n)$ be a random variable which has the same distribution as X and is independent of X. Then the characteristic function of $X-X'$ is given by $|f(t)|^2$. Writing the distribution function of $X-X'$ by $G(x)$, we have, from Lemma 5,(8.6) and Theorem 9,

$$C(h)=(\prod_{k=1}^{n}\frac{1}{2h_k}\int_0^{2h_k}dv_k)\Delta_{w=-v}^{v}G(w)$$

$(v=(v_1,\ldots,v_k))$. Since $\Delta_{w=-v}^{v}G(w)$ is nondecreasing, (i) is obvious.

(ii) also follows from Theorem 9, because $|f(t)|\le 1$ and $\frac{1}{\pi}\int_{\mathbf{R}^1}\frac{\sin^2\alpha t}{\alpha t^2}dt=1$, $\alpha\in\mathbf{R}_+^1$.

The proof of (iii) is done in the following way. From Theorem 3 with $K(x)=\prod_{k=1}^{n}\frac{\sin^2 x_k}{\pi x_k^2}$ and Lemma 3, we have

$$\lim_{h_1,\ldots,h_n\to 0+}C(h)=P(X-X'=0)$$

in which $P(X-X'=0)=0$ is equivalent to the fact that the distribution function $F(x)$ of X is continuous, and the proof of Theorem 10 is complete.

The fact of the last part of the proof will be probably known. But for completeness we give the proof of it.

First note that the continuity of $F(x)$ means by definition, that $F(s)$, the measure generated by $F(x)$ has no atom, which is, in turn, equivalent to the fact that all marginal distribution functions $F_k(x_k) = P(X_k<x_k)$ of X', $k=1,2,\ldots,n$, are continuous in $x_k \in \mathbf{R}^1$

Now we show that, for any $1 \le j \le n$,

$$\sum_{\nu=1}^{\infty} P^2(X_j=\nu^{(j)}) \le P(X-X'=0) \le \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty} P^2(X_k=\gamma_\nu^{(k)}) \tag{8.8}$$

where $\{\gamma_\nu^{(k)}, \nu=1,2,\ldots\}$ are the discontinuities of $F_k(x)$, $x \in \mathbf{R}^1$.

Proof of this is done in the following way. The distribution function $G(x)$ of $X-X'$ (X' is as in the proof of Theorem 10) is given by $\int_{\mathbf{R}^n} F_1(x-y)dF(y)$, where $F_1(x)=P(-X'<x) = \Delta_{u=-x}^{\infty}F(u)$ and hence for $h>0$, $h \in \mathbf{R}^n$,

$$P(-h \le X-X' < h)=\int_{\mathbf{R}^n}[\Delta_{u=y-h}^{\infty}F(u)-\Delta_{u=y+h}^{\infty}F(u)]dF(y). \tag{8.9}$$

Here we note

$$\begin{aligned}
&\Delta_{u=y-h}^{\infty}F(u)=P(X \ge y-h) \\
&=1-\sum_{k=1}^{n} F(\infty,\ldots,\infty,y_k-h_k,\infty,\ldots,\infty)-\sum_{1\le k_1<k_2\le n} F(\infty,\ldots,\infty,y_{k_1}-h_{k_1},\infty,\ldots,\infty,y_{k_2}-h_{k_2},\infty,\ldots,\infty) \\
&\qquad +-\ldots+(-1)^n F(y_1-h_1,\ldots,y_n-h_n)
\end{aligned}$$

and a similar relaton for $\Delta_{u=y+h}^{\infty}F(u)$. From theses the integrand of the integral on the right hand side of (8.9) can be written

$$\begin{aligned}
&\sum_{k=1}^{n} P(y_k-h_k \le X_k < y_k+h_k)-\sum_{1\le k_1<k_2\le n} P(y_{k_1}-h_{k_1} \le X_{k_1} < y_{k_1}+h_{k_1}, y_{k_2}-h_{k_2} \le X_{k_2} < y_{k_2}+h_{k_2}) \\
&\quad +-\ldots+(-1)^n P(y_1-h_1 \le X_1 < y_1+h_1,\ldots,y_n-h_n \le X_n < y_n+h_n).
\end{aligned}$$

Letting $h_1, \ldots, h_n \to 0+$, we have

$$\begin{aligned}
P(X-X'=0)=\int_{\mathbf{R}^n}dF(y)[P(X_k=y_k)-&\sum_{1\le k_1<k_2\le n} P(X_{k_1}=y_{k_1},X_{k_2}=y_{k_2}) \\
&+-\ldots\ldots\ldots\ldots+(-1)^n P(X_1=y_1,\ldots\ldots\ldots,X_n=y_n)].
\end{aligned}$$

Hence we have the relation

$$P(X-X'=0)=\int_{\mathbf{R}^n} P(\bigcup_{k=1}^{n}\{X_k=y_k\})dF(y) \tag{8.10}$$

The right hand side is not greater than

$$\sum_{k=1}^{n}\int_{\mathbf{R}^n}P(X_k=y_k)dF(y)=\sum_{k=1}^{n}\int_{\mathbf{R}^1}P(X_k=y_k)dF_k(y_k)=\sum_{k=1}^{n}\sum_{\nu=1}^{\infty}P^2(X_k=\gamma_\nu^{(k)})$$

and is not smaller than, for any $1\leq j\leq n$,

$$\int_{\mathbf{R}^n}P(X_j=y_j)dF(y)=\sum_{\nu=1}^{\infty}P^2(X_j=\gamma_\nu^{(j)}).$$

(8.8) is thus proved. With the remark given above, the fact we wanted to show is now obvious.

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