Research Report

KSTS/RR-86/004
17 Mar., 1986

Analytic Inequalities, and Rough Isometries Between Non-compact Riemannian Manifolds

by

Masahiko Kanai

Department of Mathematics
Faculty of Science and Technology
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-86/004
March 17, 1986

ANALYTIC INEQUALITIES, AND ROUGH ISOMETRIES BETWEEN NON-COMPACT RIEMANNIAN MANIFOLDS

Masahiko KANAI

Department of Mathematics
Faculty of Science and Technology
Keio University
Yokohama 223, Japan

1. Introduction

For a non-compact riemannian manifold, how it spreads at infinity is one of the most interesting problems we have to study, and, in this pont of view, its local geometry and topology are of no matter to us. The notion of rough isometry was introduced in [**K1**] in this spirit:

Definition. A map $\varphi : X \to Y$, *not necessarily continuous*, between metric spaces X and Y, is called a *rough isometry*, if the following two conditions are satisfied:

(i) for a sufficiently large $\epsilon > 0$, the ϵ-neighborhood of the image of φ in Y coincides with Y itself;

(ii) there are constants $a \geq 1$ and $b \geq 0$ such that

$$a^{-1}d(x_1, x_2) \; - \; b \; \leq d(\varphi(x_1), \varphi(x_2)) \; \leq a\, d(x_1, x_2) \; + \; b$$

for all $x_1, x_2 \in X$.

We say that X is *roughly isometric* to Y if there is a rough isometry of X into Y.

It is quite easy to see that being roughly isometric is an equivalence relation. In fact, (1) the composition $\psi \circ \varphi : X \to Z$ of two rough isometries $\varphi : X \to Y$ and $\psi : Y \to Z$ is again a rough isometry: (2) For a rough isometry $\varphi : X \to Y$, an "inverse" rough isometry $\varphi^- : Y \to X$ is constructed as follows; for $y \in Y$ take $x \in X$ so that $d(\varphi(x), y) < \epsilon$, where ϵ is the constant in the definition above, and set $\varphi^-(y) = x$. Here, we should note that the above construction of the inverse rough isometry φ^- is possible because we do not assume that a rough isometry is to be continuous: In general, φ^- is not continuous even if φ is continuous. This is a remarkable feature of rough isometries, and by virtue of it, we can identify some spaces of different topological types by rough isometries. For example, the inclusion map of the complete "periodic" surface in Fig.1 into the euclidean 3-space $\mathbf{R}^3$ is a rough isometry, and therefore the surface is roughly isometric to $\mathbf{R}^3$.

As we have just seen, a rough isometry does not, in general, preserve the topological structures of spaces, but in the preceding papers [**K1**] and [**K2**] we have exhibited that some geometric invariants and properties of non-compact riemannian manifolds

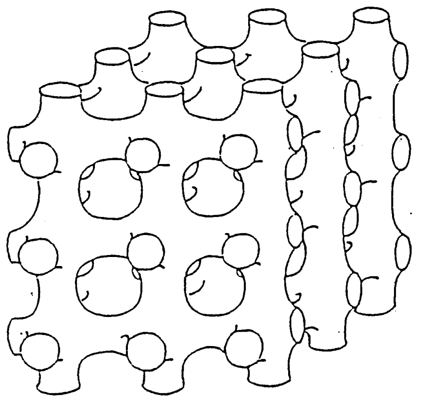

Fig.1

are inherited through rough isometries. One of them is the validity of isoperimetric inequalities. First of all, recall the classical isoperimetric inequality: It suggests that, for a bounded domain Ω in the euclidean space $\mathbf{R}^n$ with smooth boundary, the inequality

$$(\mathrm{vol}\,\Omega)^{1/n} \;\leq\; c_n \cdot (\mathrm{area}\,\partial\Omega)^{1/(n-1)}$$

alway holds with a constant c_n depending only on the dimension n. This leads us to the following definition of the isoperimetric constant $I_m(X)$ for a general complete riemannian manifold X with $\dim X \leq m \leq \infty$:

$$I_m(X) \;=\; \inf_{\Omega} \frac{\mathrm{area}\,\partial\Omega}{(\mathrm{vol}\,\Omega)^{(m-1)/m}}, \tag{1.1}$$

where Ω ranges through all the non-empty bounded domains in X with smooth boundaries, and, in the definition, we adopt the natural convention that $(m-1)/m = 1$ for $m = \infty$; in other words, $I_\infty(X)$ is, so-called, Cheeger's isoperimetric constant (for a non-compact riemannian manifold). Now the classical isoperimetric inequality is nothing but $I_n(\mathbf{R}^n) > 0$. And a theorem in the previous paper **[K1]** says that the validity of the isoperimetric inequality $I_m(X) > 0$ is preserved by rough isometries, under the additional condition that

($*$) the Ricci curvature is bounded below, and the injectivity radius is positive,

which ensures the uniformness of local geometry: More precisely we proved

Theorem 1.1. *Let X and Y be complete riemannian manifolds satisfying the condition ($*$) and roughly isometric to each other. Then, for $\max\{\dim X, \dim Y\} \leq m \leq \infty$, the inequality $I_m(X) > 0$ is equivalent to the inequality $I_m(Y) > 0$.*

A reason why isoperimetric inequalities have a lot of applications is that they are closely related to analytic inequalities. An application of this kind was, in fact, done in [**K1**], where we proved the Liouville theorem generalized in terms of rough isometries:

Theorem 1.2. *Let X be a complete riemannian manifold satisfying the condition $(*)$ and roughly isometric to the euclidean m-space with $m \geq \dim X$. Then any positive harmonic function on X is constant.*

One of the crucial steps of the proof of the above theorem is to translate Theorem 1.1 into an assertion concerned with a Sobolev inequality, and, to state it in a more concrete form, we should introduce the analytic constants $S_{l,m}(X)$ for a complete riemannian manifold X:

$$(1.2) \quad S_{l,m}(X) = \inf_{u \in C_0^\infty(X)} \frac{\left\{\int_X |\nabla u|^l \, dx\right\}^{1/l}}{\left\{\int_X |u|^{m/(m-1)} dx\right\}^{(m-1)/m}}, \qquad l \geq 1, \ 1 < m \leq \infty,$$

where $C_0^\infty(X)$ denotes the space of C^∞ functions on X with compact supports, and we again assume that $(m-1)/m = m/(m-1) = 1$ provided $m = \infty$. A fundamental fact relating isoperimetric inequalities to Sobolev inequalities is the identity

$$(1.3) \qquad I_m(X) = S_{1,m}(X)$$

due to Federer-Fleming [**FF**] and Maz'ya [**M**] (see also Osserman [**O**]), and, by virtue of it, Theorem 1.1 have

Corollary 1.3. *Let X and Y be as in Theorem 1.1. Then, for* $\max\{\dim X, \dim Y\} \leq m \leq \infty$, $S_{1,m}(X) > 0$ *if and only if* $S_{1,m}(Y) > 0$.

For the euclidean space $\mathbf{R}^m$, we have the Sobolev inequality $S_{1,m}(\mathbf{R}^m) > 0$, and, by the corollary above, we obtain $S_{1,m}(X) > 0$ for X as in Theorem 1.2, and this is one of the inequalities we need to prove Theorem 1.2.

Also, there are relations between an isoperimetric constant and an analytic constant other than the Federer-Fleming-Maz'ya identity (1.3). In fact, Cheeger [**Ch**] established the inequality

$$(1.4) \qquad \frac{1}{2} I_\infty(X) \leq S_{2,2}(X)$$

for an arbitrary complete riemannian manifold X (cf. Yau [**Y**]), while Buser proved in [**B**] that if X is a complete riemannian manifold with Ricci curvature bounded below then

$$(1.5) \qquad S_{2,2}(X)^2 \leq \text{const} \cdot I_\infty(X),$$

where the constant in the inequality depends only on the dimension of X and on the infimum of the Ricci curvature. Note that $S_{2,2}(X)^2$ appears as the best constant in the Poincaré inequality $\int_X |\nabla u|^2 dx \geq \text{const} \cdot \int_X u^2 dx$ for $u \in C_0^\infty(X)$, and therefore

$S_{2,2}(X)^2$ is equal to the infimum of the spectrum of $-\Delta$, the Laplace operator multiplied by -1, acting on L^2-functions on X. In particular, the inequalities of Cheeger (1.4) and of Buser (1.5) imply that

$$I_\infty(X) > 0 \quad \text{if and only if} \quad S_{2,2}(X) > 0 \tag{1.6}$$

for a complete riemannian manifold X with Ricci curvature bounded below, and consequently, together with Theorem 1.1, we have

Corollary 1.4. *Let X and Y be as in Theorem 1.1. Then $S_{2,2}(X) > 0$ is equivalent to $S_{2,2}(Y) > 0$.*

Now these two corollaries of Theorem 1.1 lead us to the natural question: To what extent are the analytic constants $S_{l,m}(X)$ preserved by rough isometries? In the present article, we will prove the following generalization of Corollary 1.4 without use of isoperimetric inequalities or Theorem 1.1.

Theorem 1.5. *Suppose that X and Y are complete riemannian manifolds satisfying the condition $(*)$ and roughly isometric to each other. Then, for $1 < m \leq \infty$ and $l \geq m/(m-1)$, $S_{l,m}(X) > 0$ if and only if $S_{l,m}(Y) > 0$.*

This theorem is also motivated by author's previous work [**K2**], in which he showed that the parabolicity is preserved by rough isometries: By definition, a riemannian manifold X is said to be parabolic if there is no positive superharmonic function on X other than constants, and the main result obtained in [**K2**] is

Theorem 1.6. *Let X and Y be complete riemannian manifolds which satisfy the condition $(*)$ and are roughly isometric to each other. Then X is parabolic if and only if so is Y.*

To prove the theorem, we first showed that a complete riemannian manifold X is non-parabolic if and only if $\operatorname{cap}\Omega > 0$ for a non-empty bounded domain Ω in X with smooth boundary, where the capacity $\operatorname{cap}\Omega$ of Ω is defined by

$$\operatorname{cap}\Omega \;= \inf\left\{\int_X |\nabla u|^2 dx \;:\; u \in C_0^\infty(X),\; u_{|\Omega} = 1\right\}.$$

Then Theorem 1.6 is reduced to the problem of showing that the non-vanishing of the capacity is preserved by rough isometries, and the proof of this fact is almost the same with that of Theorem 1.5, because the behavior of the capacity under rough isometries is quite similar to that of the analytic constant $S_{2,2}(X)$, as is expected from their similarity in the definitions.

The construction of this article is as follows. §§2 and 3 are devoted to the proof of Theorem 1.5, which will be done, as in the preceding works [**K1**] and [**K2**], by approximating "continuous" geometry of a riemannian manifold, say X, by "combinatorial" geometry of a certain discrete subset P of X endowed with a suitable combinatorial structure. We will call P a net in X, and its "intrinsic" aspects are considered in §2. In the next section, we will show that P actually approximates X, and will complete the proof of Theorem 1.5. Finally, in §4, we will discuss another application of the discrete approximation method. In particular we will reveal relationship between the work of Kesten [**Ks2**] on the random walks and the Cheeger-Buser inequalities (1.4) and (1.5): With the aid of our discrete approximation theorems, the latter (in a weaker form) will be followed from the former.

2. Intrinsic Studies of Nets

We begin this section with the precise "intrinsic" definition of nets. A *net* is a countable set P equipped with a family $\{N_p\}_{p\in P}$ indexed by the elements of P itself such that

(i) each N_p is a finite subset of P, and that
(ii) for $p,q \in P$, $p \in N_q$ if and only if $q \in N_p$.

A net is nothing but a kind of 1-dimensional graphs: In fact, each element of P can be considered as a vertex of a graph, and two vertices p and q are considered to be combined by an edge if $p \in N_q$. Now let P be a net. P is said to be *uniform* if $\sup_{p\in P} \#N_p < \infty$, where, for a set S, $\#S$ denotes its cardinality. A sequence $\mathbf{p} = (p_0, \ldots, p_L)$ of elements of P is called a *path from* p_0 *to* p_L *of length* L if $p_k \in N_{p_{k-1}}$ for $k = 1, \ldots, L$, and the net P is said to be *connected* if any two points of P are combined by a path. In the case when P is connected, the *combinatorial metric* δ of P is defined by

$$\delta(p,q) = \min\{\text{the lengths of paths from } p \text{ to } q\}$$

for $p, q \in P$. We always consider a connected net as a metric space with the combinatorial metric δ.

Next we introduce the analytic constants for the nets. Again let P be a net. For real-valued functions u and v on P, put

$$\begin{aligned} \langle Du, Dv\rangle(p) &= \sum_{q\in N_p} \{u(q)-u(p)\}\{v(q)-v(p)\}, \\ |Du|(p) &= \sqrt{\langle Du, Du\rangle(p)}, \qquad p \in P: \end{aligned} \tag{2.1}$$

The former is a combinatorial analogue of the inner product of the gradients of the functions u and v, and the latter the norm of the gradient of u. Now, for each $l \geq 1$ and $1 < m \leq \infty$, the analytic constant of the net P is defined by

$$S_{l,m}(P) = \inf_u \frac{\left\{\sum_{p\in P} |Du|^l(p)\right\}^{1/l}}{\left\{\sum_{p\in P} |u|^{m/(m-1)}(p)\right\}^{(m-1)/m}}, \tag{2.2}$$

where u ranges over all finitely supported functions on P. Now we have a combinatorial version of Theorem 1.5:

Proposition 2.1. *Let P and Q be uniform connected nets. If P is roughly isometric to Q, then $S_{l,m}(P) > 0$ is equivalent to $S_{l,m}(Q) > 0$ for any $l \geq 1$ and $1 < m \leq \infty$.*

Proof. In the proof, denote by $B_\rho(q)$ the "closed" ρ-ball in Q around $q \in Q$; i.e., $B_\rho(q) = \{r \in Q : \delta(r,q) \leq \rho\}$. Also, let $\varphi : P \to Q$ be a rough isometry such that $Q = \bigcup_{p \in P} B_\tau(\varphi(p))$ for some constant $\tau > 0$.

To begin with, suppose that v is an arbitrary non-negative function on Q with finite support, and define another finitely supported function $\overline{v}$ on Q by

$$\overline{v}(q) = \frac{1}{\# B_\tau(q)} \sum_{r \in B_\tau(q)} v(r).$$

Our first purpose is to derive the inequality (2.3) below. Let q and q' be points of Q with $\delta(q,q') = 1$. Then we have

$$\begin{aligned} |\overline{v}(q') - \overline{v}(q)| &= \left| \frac{1}{\# B_\tau(q')} \sum_{r' \in B_\tau(q')} v(r') - \frac{1}{\# B_\tau(q)} \sum_{r \in B_\tau(q)} v(r) \right| \\ &= \left| \frac{1}{\# B_\tau(q) \cdot \# B_\tau(q')} \sum_{r \in B_\tau(q), r' \in B_\tau(q')} \{v(r') - v(r)\} \right| \\ &\leq \frac{1}{\# B_\tau(q) \cdot \# B_\tau(q')} \sum_{r \in B_\tau(q), r' \in B_\tau(q')} |v(r') - v(r)|. \end{aligned}$$

Moreover, for $r \in B_\tau(q)$ and $r' \in B_\tau(q')$, combining them by a length-minimizing path $\mathbf{q} = (q_0, \ldots, q_L)$ with $q_0 = r$, $q_L = r'$, and of length $L \leq 2\tau + 1$, we obtain

$$\begin{aligned} |v(r') - v(r)| &\leq |v(q_0) - v(q_1)| + \cdots + |v(q_{L-1}) - v(q_L)| \\ &\leq |Dv|(q_0) + \cdots + |Dv|(q_{L-1}) \\ &\leq \sum_{q'' \in B_\tau(q)} |Dv|(q'') \end{aligned}$$

since $\delta(q_i, q) \leq \tau$ for $i = 0, \ldots, L-1$, and therefore we get

$$|\overline{v}(q') - \overline{v}(q)| \leq \sum_{q'' \in B_\tau(q)} |Dv|(q'').$$

Thus, for any $q \in Q$, we have

$$|D\overline{v}|(q) = \left\{ \sum_{q' \in N_q} \{\overline{v}(q') - \overline{v}(q)\}^2 \right\}^{1/2} \leq \nu_{Q,1}^{1/2} \sum_{r \in B_\tau(q)} |Du|(r),$$

and

$$|D\overline{v}|^l(q) \le \nu_{Q,1}^{l/2} \left\{ \sum_{r \in B_r(q)} |Dv|(r) \right\}^l \le \nu_{Q,1}^{l/2} \nu_{Q,r}^{l-1} \sum_{r \in B_r(q)} |Dv|^l(r)$$

by the Hölder inequality, where for any $\rho > 0$, we put $\nu_{Q,\rho} = \sup_{q \in Q} \# B_\rho(q)$ which has a finite value by the assumption of uniformness of Q. This yields

$$\sum_{q \in Q} |D\overline{v}|^l(q) \le \nu_{Q,1}^{l/2} \nu_{Q,r}^{l-1} \sum_{q \in Q} \sum_{r \in B_r(q)} |Dv|^l(r) \le \nu_{Q,1}^{l/2} \nu_{Q,r}^{l} \sum_{q \in Q} |Dv|^l(q),$$

i.e.,

$$\left\{ \sum_{q \in Q} |D\overline{v}|^l(q) \right\}^{1/l} \le c_1 \cdot \left\{ \sum_{q \in Q} |Dv|^l(q) \right\}^{1/l} \tag{2.3}$$

with a suitable constant c_1.

Now define a finitely supported non-negative function u on P by $u = \overline{v} \circ \varphi$. The next purpose of ours is to obtain the estimates (2.4) and (2.5) below. First note that, for $p, p' \in P$ with $\delta(p, p') = 1$, there is a constant L_0 such that $\delta(\varphi(p), \varphi(p')) \le L_0$ since φ is a rough isometry. Therefore, combining $\varphi(p)$ and $\varphi(p')$ by a path $\mathrm{q} = (q_0, \ldots, q_L)$ in Q with $q_0 = \varphi(p)$, $q_L = \varphi(p')$, and of length $L \le L_0$, we have

$$\begin{aligned} |u(p) - u(p')| &= |\overline{v}(q_0) - \overline{v}(q_L)| \\ &\le |\overline{v}(q_0) - \overline{v}(q_1)| + \cdots + |\overline{v}(q_{L-1}) - \overline{v}(q_L)| \\ &\le |D\overline{v}|(q_0) + \cdots + |D\overline{v}|(q_{L-1}) \\ &\le \sum_{q \in B_{L_0 - 1}(\varphi(p))} |D\overline{v}|(q), \end{aligned}$$

and this implies, as above,

$$\begin{aligned} |Du|^l(p) &\le \nu_{P,1}^{l/2} \left\{ \sum_{q \in B_{L_0-1}(\varphi(p))} |D\overline{v}|(q) \right\}^l \\ &\le \nu_{P,1}^{l/2} \nu_{Q,L_0-1}^{l-1} \sum_{q \in B_{L_0-1}(\varphi(p))} |D\overline{v}|^l(q) \end{aligned}$$

with $\nu_{P,\rho} = \sup_{p \in P} \#\{p' \in P : \delta(p', p) \le \rho\} < \infty$ for $\rho > 0$. Hence we obtain

$$\left\{ \sum_{p \in P} |Du|^l(p) \right\}^{1/l} \le c_2 \cdot \left\{ \sum_{q \in Q} |D\overline{v}|^l(q) \right\}^{1/l} \tag{2.4}$$

with a certain constant c_2. Finally, for each $p \in P$, we have

$$\begin{aligned} u^{m/(m-1)}(p) &= \left\{ \frac{1}{\#B_\tau(\varphi(p))} \sum_{q \in B_\tau(\varphi(p))} v(q) \right\}^{m/(m-1)} \\ &\geq \nu_{Q,\tau}^{-m/(m-1)} \sum_{q \in B_\tau(\varphi(p))} v^{m/(m-1)}(q), \end{aligned}$$

and consequently we get

$$\sum_{p \in P} u^{m/(m-1)}(p) \geq \nu_{Q,\tau}^{-m/(m-1)} \sum_{q \in Q} v^{m/(m-1)}(q)$$

since $Q = \bigcup_{p \in P} B_\tau(\varphi(p))$. This shows

$$\left\{ \sum_{p \in P} u^{m/(m-1)}(p) \right\}^{(m-1)/m} \geq c_3 \cdot \left\{ \sum_{q \in Q} v^{m/(m-1)}(q) \right\}^{(m-1)/m} \tag{2.5}$$

with a constant $c_3 > 0$.

By (2.3), (2.4) and (2.5) we conclude

$$\begin{aligned} &\frac{c_1 c_2}{c_3} \cdot \frac{\left\{ \sum_{q \in Q} |Dv|^l(q) \right\}^{1/l}}{\left\{ \sum_{q \in Q} v^{m/(m-1)}(q) \right\}^{(m-1)/m}} \\ &\qquad \geq \frac{\left\{ \sum_{p \in P} |Du|^l(p) \right\}^{1/l}}{\left\{ \sum_{p \in P} u^{m/(m-1)}(p) \right\}^{(m-1)/m}} \geq S_{l,m}(P) \end{aligned}$$

for an arbitrary non-negative function v on Q with finite support. Moreover because $|Dv| \geq |D|v||$ for any function v on Q, we obtain $(c_1 c_2 / c_3) \cdot S_{l,m}(Q) \geq S_{l,m}(P)$. This completes the proof of the proposition. ▯

3. Discrete Approximation Theorem

In this section, we construct a net P in a complete riemannian manifold X, and show that P indeed approximates X combinatorially. Then Theorem 1.5 will follow immediately.

Now let X be a complete riemannian manifold. A subset P of X is said to be *ϵ-separated* if $d(p,q) \geq \epsilon$ whenever p and q are distinct points of P, and for a maximal ϵ-separated subset of X, a structure of net on it is canonically defined by

$$N_p = \{ q \in P : 0 < d(p,q) \leq 3\epsilon \}, \qquad p \in P.$$

We call a maximal ϵ-separated subset P of X with this structure of net, an *ϵ-net in X*. Suppose that P is an ϵ-net in X. Then we have the following facts since P is maximally ϵ-separated in X:

(3.1) The open geodesic balls $B_{\epsilon/2}(p)$ in X of radius $\epsilon/2$ and with centers at $p \in P$ are disjoint;

(3.2) The geodesic balls $B_\epsilon(p)$, $p \in P$, cover X.

Moreover we can easily show that

(3.3) P is connected if X is connected.

So, in the rest of this paper, we assume that all manifolds and nets are connected.

To relate geometry of P to that of X, the following lemma proved in [**K1**] is fundamental.

Lemma 3.1. *Suppose that the Ricci curvature of X is bounded below. Then*

(1) *for $r > 0$ and $x \in X$, we have*

$$\#(P \cap B_r(x)) \leq \nu(r), \tag{3.4}$$

where $\nu(r)$ is a constant independent of x. In particular, P is uniform.

(2) *P is roughly isometric to X: In fact, there are constants $a > 1$ and $b > 0$ such that*

$$a^{-1}d(p,q) \leq \delta(p,q) \leq a\,d(p,q) + b \qquad \text{for } p,q \in P. \tag{3.5}$$

Consequently any two nets in X are roughly isometric to each other.

The most crucial part of the proof of Theorem 1.5 is the following discrete approximation theorem.

Theorem 3.2. *Let P be an ϵ-net in a complete riemannian manifold X satisfying the condition $(*)$. Then, for any $1 < m \leq \infty$ and $l \geq m/(m-1)$, $S_{l,m}(X) > 0$ if and only if $S_{l,m}(P) > 0$.*

We begin the proof of this theorem with referring to volume estimates of geodesic balls. For a complete riemannian manifold X with Ricci curvature bounded below, a standard comparison theorem gives

$$\operatorname{vol} B_r(x) \leq V_+(r) \qquad \text{for } x \in X \text{ and } r > 0. \tag{3.6}$$

On the other hand, for a complete riemannian manifold X with injectivity radius $\operatorname{inj} X > 0$, Croke [Cr] showed the inequality

$$\operatorname{vol} B_r(x) \geq V_-(r) \qquad \text{for } x \in X \text{ and } 0 < r \leq \frac{1}{2}\operatorname{inj} X. \tag{3.7}$$

Note that both $V_+(r)$ and $V_-(r)$ above are constants independent of $x \in X$.

Another necessity for the proof of Theorem 3.2 is

Lemma 3.3 (see, for the proof, [**K2**]). *Let X be a complete riemannian manifold with Ricci curvature bounded below. Then for the geodesic ball $B = B_r(p)$ in X of radius r and with the center p, there is a constant $\beta = \beta(r) > 0$ independent of p for which*

$$\int_B |\nabla u|\, dx \;\geq\; \beta \cdot \int_B |u - u^*|\, dx$$

holds for all $u \in C^\infty(\overline{B})$, where u^* denotes the integral mean of u over B; $u^* = (\mathrm{vol}\, B)^{-1} \int_B u\, dx$.

And this lemma yields

Corollary 3.4. *Let X and B be as in the lemma. Then for each $1 < m \leq \infty$ and $l \geq m/(m-1)$, there is a constant $\gamma = \gamma(r) > 0$ independent of p such that*

$$u^{*1/(m-1)} \left\{ \int_B |\nabla u|^l dx \right\}^{1/l} \geq \gamma \cdot \int_B \left| |u|^{m/(m-1)} - u^{*m/(m-1)} \right| dx \tag{3.8}$$

for all $u \in C^\infty(\bar{B})$ with

$$u^* \;=\; \left\{ \frac{1}{\mathrm{vol}\, B} \int_B |u|^{m/(m-1)} dx \right\}^{(m-1)/m} .$$

Proof. Apply Lemma 3.3 to $|u|^{m/(m-1)}$: Then with the Hölder inequality and (3.6), we have

$$\begin{aligned}
&\beta \cdot \int_B \left| |u|^{m/(m-1)} - u^{*m/(m-1)} \right| dx \\
&\leq \int_B \left| \nabla |u|^{m/(m-1)} \right| dx \\
&\leq \frac{m}{m-1} \int_B |u|^{1/(m-1)} |\nabla u| dx \\
&\leq \frac{m}{m-1} \left\{ \int_B |u|^{m/(m-1)} dx \right\}^{1/m} \left\{ \int_B |\nabla u|^{m/(m-1)} dx \right\}^{(m-1)/m} \\
&= \frac{m}{m-1} (\mathrm{vol}\, B)^{1/m} u^{*1/(m-1)} \left\{ \int_B |\nabla u|^{m/(m-1)} dx \right\}^{(m-1)/m} \\
&\leq \frac{m}{m-1} (\mathrm{vol}\, B)^{(l-1)/l} u^{*1/(m-1)} \left\{ \int_B |\nabla u|^l dx \right\}^{1/l} \\
&\leq \frac{m}{m-1} V_+(r)^{(l-1)/l} u^{*1/(m-1)} \left\{ \int_B |\nabla u|^l dx \right\}^{1/l} .
\end{aligned}$$

▯

Proof of Theorem 3.2. We may prove the theorem only in the case when $\epsilon \leq \mathrm{inj}\, X/2$, because any two nets in X are uniform and roughly isometric to each other by Lemma 3.1, and consequently, by Proposition 2.1, whether $S_{l,m}(P) > 0$ or not is independent of the choice of the net P in X.

First we prove the "if" part of the theorem, and to do this it is sufficient to show that

$$\frac{\left\{\int_X |\nabla u|^l dx\right\}^{1/l}}{\left\{\int_X |u|^{m/(m-1)} dx\right\}^{(m-1)/m}} \geq c_1 \cdot S_{l,m}(P) \tag{3.9}$$

for all $u \in C_0^\infty(X)$ with a suitable constant $c_1 > 0$. Now take $u \in C_0^\infty(X)$ arbitrarily, and define a finitely supported non-negative function u^* on P by

$$u^*(p) = \left\{\frac{1}{\operatorname{vol} B_{4\epsilon}(p)} \int_{B_{4\epsilon}(p)} |u|^{m/(m-1)} dx\right\}^{(m-1)/m}.$$

Then we immediately have

$$\int_X |u|^{m/(m-1)} dx \leq \sum_{p\in P} \int_{B_{4\epsilon}(p)} |u|^{m/(m-1)} dx \leq V_+(4\epsilon) \sum_{p\in P} u^{*m/(m-1)}(p), \tag{3.10}$$

because $\{B_{4\epsilon}(p) : p \in P\}$ covers X and because of (3.6). On the other hand, for $p, q \in P$ with $\delta(p,q) = 1$, we have

$$\begin{aligned}
&\{u^{*1/(m-1)}(p) + u^{*1/(m-1)}(q)\} \left\{\int_{B_{7\epsilon}(p)} |\nabla u|^l dx\right\}^{1/l} \\
&\geq u^{*1/(m-1)}(p) \left\{\int_{B_{4\epsilon}(p)} |\nabla u|^l dx\right\}^{1/l} \\
&\qquad + u^{*1/(m-1)}(q) \left\{\int_{B_{4\epsilon}(q)} |\nabla u|^l dx\right\}^{1/l} \\
&\geq \gamma(4\epsilon) \cdot \int_{B_{4\epsilon}(p)} \left||u|^{m/(m-1)}(x) - u^{*m/(m-1)}(p)\right| dx \\
&\qquad + \gamma(4\epsilon) \cdot \int_{B_{4\epsilon}(q)} \left||u|^{m/(m-1)}(x) - u^{*m/(m-1)}(q)\right| dx \\
&\geq \gamma(4\epsilon) \cdot \int_{B_{4\epsilon}(p)\cap B_{4\epsilon}(q)} \left\{\left||u|^{m/(m-1)}(x) - u^{*m/(m-1)}(p)\right|\right. \\
&\qquad\qquad \left. + \left||u|^{m/(m-1)}(x) - u^{*m/(m-1)}(q)\right|\right\} dx \\
&\geq \gamma(4\epsilon) \cdot \int_{B_{4\epsilon}(p)\cap B_{4\epsilon}(q)} \left|u^{*m/(m-1)}(p) - u^{*m/(m-1)}(q)\right| dx \\
&= \gamma(4\epsilon) \operatorname{vol}(B_{4\epsilon}(p) \cap B_{4\epsilon}(q)) \cdot \left|u^{*m/(m-1)}(p) - u^{*m/(m-1)}(q)\right| \\
&\geq \gamma(4\epsilon) \operatorname{vol} B_\epsilon(p) \cdot \left|u^{*m/(m-1)}(p) - u^{*m/(m-1)}(q)\right| \\
&\geq \gamma(4\epsilon) V_-(\epsilon) \cdot \left|u^{*m/(m-1)}(p) - u^{*m/(m-1)}(q)\right| :
\end{aligned} \tag{3.11}$$

The first inequality follows from the fact that $B_{4\epsilon}(p)$, $B_{4\epsilon}(q) \subset B_{7\epsilon}(p)$ because $d(p,q) \leq 3\epsilon$; the second inequality just follows from (3.8); the inequality before last is a consequence of the fact that $B_{\epsilon}(p) \subset B_{4\epsilon}(p) \cap B_{4\epsilon}(q)$, and the last is by Croke's inequality (3.7). Here note that for any real numbers $\xi, \eta \geq 0$ and $\alpha > 0$ we always have

$$2\,|\xi^{1+\alpha} - \eta^{1+\alpha}| \ \geq |\xi - \eta|(\xi^{\alpha} + \eta^{\alpha}).$$

Applying this to $\xi = u^*(p)$, $\eta = u^*(q)$, $\alpha = 1/(m-1)$ in (3.11), we obtain

$$\left\{ \int_{B_{7\epsilon}(p)} |\nabla u|^l dx \right\}^{1/l} \ \geq \frac{1}{2}\gamma(4\epsilon)V_-(\epsilon) \cdot |u^*(p) - u^*(q)|$$

for $p, q \in P$ with $\delta(p,q) = 1$, and therefore, by (3.4), we get

$$\left\{ \int_{B_{7\epsilon}(p)} |\nabla u|^l dx \right\}^{1/l} \ \geq \frac{1}{2}\gamma(4\epsilon)V_-(\epsilon)\nu(3\epsilon)^{-1/2} \cdot |Du^*|(p),$$

with $\nu(3\epsilon) \geq \sup_{p\in P} \#N_p$. This yields, again by (3.4),

$$\text{(3.12)} \qquad \int_X |\nabla u|^l dx \ \geq \nu(7\epsilon)^{-1} \sum_{p\in P} \int_{B_{7\epsilon}(p)} |\nabla u|^l dx \ \geq \ c_2 \sum_{p\in P} |Du^*|^l(p)$$

with a suitable constant $c_2 > 0$. Now the inequality (3.9) immediately follows from (3.10) and (3.12).

Next we give the "only if" part of the theorem; i.e., we will show that for any function u^* on P with finite support, the inequality

$$\text{(3.13)} \qquad \frac{\left\{ \sum_{p\in P} |Du^*|^l(p) \right\}^{1/l}}{\left\{ \sum_{p\in P} |u^*|^{m/(m-1)}(p) \right\}^{(m-1)/m}} \geq c_3 \cdot S_{l,m}(X)$$

always holds with a certain constant $c_3 > 0$. The proof of this inequality is rather easier than that of (3.9), and is done by "smoothing" u^* by use of a partition of unity of X. So, first of all, we construct a partition of unity associated to a covering of X by geodesic balls around $p \in P$. For each $p \in P$, define a function $\widehat{\eta}_p$ on X with finite support by

$$\widehat{\eta}_p(x) \ = \ \begin{cases} 1 \ - \ \frac{1}{2\epsilon} d(x,p) & \text{if} \quad x \in B_{2\epsilon}(p) \\ 0 & \text{otherwise,} \end{cases}$$

and then define a partition of unity, $\{\eta_p : p \in P\}$, by

$$\eta_p(x) \ = \ \frac{1}{\sum_{q\in P} \widehat{\eta}_q(x)} \, \widehat{\eta}_p(x).$$

It is easy to see that there are constants $c_4 > 0$ and c_5 independent of $p \in P$ such that $\eta_p \geq c_4$ on $B_{\epsilon/2}(p)$ and $|\nabla \eta_p| \leq c_5$. Now let u^* be an arbitrary finitely supported function on P. We may consider only non-negative u^*, because for general u^* we have $|Du^*| \geq |D|u^*||$. Define a non-negative function u on X by

$$u(x) \;=\; \sum_{p \in P} \eta_p(x) u^*(p).$$

This function u on X is not smooth but is Lipschitz continuous, and therefore differentiable almost everywhere. So we can treat this function u as a smooth function. Then, with Croke's inequality (3.7), we immediately have

$$\begin{aligned}\int_{B_{\epsilon/2}(p)} u^{m/(m-1)} dx &\geq \int_{B_{\epsilon/2}(p)} \eta_p^{m/(m-1)}(x)\, u^{*m/(m-1)}(p)\, dx \\ &\geq c_4^{m/(m-1)} \operatorname{vol} B_{\epsilon/2}(p) \cdot u^{*m/(m-1)}(p) \\ &\geq c_4^{m/(m-1)} V_-(\epsilon/2) \cdot u^{*m/(m-1)}(p)\end{aligned}$$

and this implies from (3.1) that

$$\text{(3.14)} \qquad \int_X u^{m/(m-1)} dx \geq \sum_{p \in P} \int_{B_{\epsilon/2}(p)} u^{m/(m-1)} dx \;\geq\; c_6 \sum_{p \in P} u^{*m/(m-1)}(p)$$

with some constant $c_6 > 0$. On the other hand, at a point $x \in B_\epsilon(p)$,

$$u(x) \;=\; \sum_{q \in N_p \cup \{p\}} u^*(q) \eta_q(x)$$

since each η_q is supported on $\overline{B_{2\epsilon}(q)}$, and

$$\nabla u(x) \;=\; \sum_{q \in N_p \cup \{p\}} u^*(q) \nabla \eta_q(x) \;=\; \sum_{q \in N_p} \{u^*(q) - u^*(p)\} \nabla \eta_q(x)$$

because $\sum_{q \in N_p \cup \{p\}} \nabla \eta_q(x) = 0$ (recall that $\sum_{q \in P} \eta_q = 1$). Thus by the Schwarz inequality, (3.4), and the fact that $|\nabla \eta_q| \leq c_5$, we obtain

$$|\nabla u|(x) \;\leq\; c_5 \nu(3\epsilon)^{1/2} \cdot |Du^*|(p)$$

for $x \in B_\epsilon(p)$. This implies with (3.6) that

$$\int_{B_\epsilon(p)} |\nabla u|^l dx \;\leq c_5^l \nu(3\epsilon)^{l/2} V_+(\epsilon) \cdot |Du^*|^l(p),$$

and therefore we have

$$\text{(3.15)} \qquad \int_X |\nabla u|^l dx \;\leq \sum_{p \in P} \int_{B_\epsilon(p)} |\nabla u|^l dx \;\leq c_7 \cdot \sum_{p \in P} |Du^*|^l(p).$$

Finally (3.13) follows from (3.14) and (3.15). ∎

Now we can complete

Proof of Theorem 1.5. Let X and Y be complete riemannian manifolds satisfying the condition $(*)$ which are roughly isometric to each other, and take nets P and Q in X and Y, respectively. Then, by Lemma 3.1 (2) and the assumption that X is roughly isometric to Y, P and Q, both of which are also uniform by Lemma 3.1 (1), are roughly isometric to each other, and consequently $S_{l,m}(P) > 0$ if and only if $S_{l,m}(Q) > 0$ by Proposition 2.1. On the other hand, Theorem 3.2 says that the inequalities $S_{l,m}(P) > 0$ and $S_{l,m}(Q) > 0$ are, respectively, equivalent to $S_{l,m}(X) > 0$ and $S_{l,m}(Y) > 0$. Thus we conclude that $S_{l,m}(X) > 0$ if and only if $S_{l,m}(Y) > 0$. This completes the proof of the theorem. ∎

4. Inequalities of Kesten and of Cheeger-Buser

In the preceding sections as well as our earlier works [**K1**] and [**K2**], we have seen that nets are enriched with combinatorial geometry: They have a lot of geometric notions corresponding to those for riemannian manifolds, such as volume growth rate, isoperimetric and analytic inequalities, and potential-theoretic and probability-theoretic notions. Also nets relate to riemannian manifolds through discrete approximation theorems, such as Theorem 3.2, which suggest that a net in a riemannian manifold is similar to the manifold. Furthermore we often find that problems are much easier in combinatorial category than riemannian category. So we may expect that by the aid of discrete approximation theorems we can utilize combinatorial geometry of nets to obtain results in riemannian geometry. In this section, we revisit the work of Kesten, which can be considered as a combinatorial version of the Cheeger-Buser inequalities, and from it, we will refind a weaker version of the Cheeger-Buser inequalities applying our discrete approximation theorems.

For this purpose, we should first recall the notion of isoperimetric constants for nets. Now let P be a net. For a subset S of P, its boundary is defined by

$$\partial S = \{p \in S : N_p \not\subset S\},$$

and, for each $1 < m \leq \infty$, the isoperimetric constant $I_m(P)$ of P is introduced by

$$I_m(P) = \inf_S \frac{\#\partial S}{(\#S)^{(m-1)/m}},$$

where S runs over all finite subsets of P. Then we have the following discrete approximation theorem which was the most essential in the proof of Theorem 1.1 (see [**K1**]):

Theorem 4.1. *Suppose that X is a complete riemannian manifold satisfying the condition $(*)$ and P is an ϵ-net in X with arbitrary $\epsilon > 0$. Then, for any $\dim X \leq m \leq \infty$, $I_m(P) > 0$ is equivalent to $I_m(X) > 0$.*

We will utilize this theorem later.

Now we refer to the work of Kesten [**Ks2**], which is stated in the following form in our language. (This work of Kesten was motivated by the study of random walks on discrete groups: For the probabilistic aspects, see Kesten's original papers [**Ks1**] and [**Ks2**].)

Proposition 4.2. *For a net P, we always have*

$$S_{2,2}(P) \geq \frac{1}{2} I_\infty(P). \tag{4.1}$$

Moreover if P is uniform, then

$$S_{2,2}(P)^2 \leq c \cdot I_\infty(P), \tag{4.2}$$

where c is a constant depending only on $\sup_{p\in P} \# N_p < \infty$.

Proof. First we prove the second inequality (4.2). Let S be an arbitrary non-empty finite subset of a uniform net P, and u the characteristic function of S. Then it is immediate to see that $\sum_{p\in P} u^2(p) = \# S$ and $\sum_{p\in P} |Du|^2(p) \leq c \cdot \# \partial S$, and consequently we have $c \cdot (\# \partial S / \# S) \geq \sum_{p\in P} |Du|^2(p) / \sum_{p\in P} u^2(p) \geq S_{2,2}(P)^2$, which implies (4.2).

Next we prove the first inequality (4.1). Let u be any non-negative finitely supported function on P. We will prove the inequality

$$\frac{\left\{\sum_{p\in P} |Du|^2(p)\right\}^{1/2}}{\left\{\sum_{p\in P} u^2(p)\right\}^{1/2}} \geq \frac{1}{2} I_\infty(P), \tag{4.3}$$

which implies (4.1) because $|Dv| \geq |D|v||$ for any function v on P. Put $S_t = \{p \in P : u^2(p) \geq t\}$ for $t > 0$. Then, by the definition of $I_\infty(P)$, we get

$$\int_0^\infty \# \partial S_t \, dt \geq I_\infty(P) \cdot \int_0^\infty \# S_t \, dt = I_\infty(P) \cdot \sum_{p\in P} u^2(p). \tag{4.4}$$

On the other hand, put $P_0 = \{p \in P : \min_{q\in N_p} u(q) < u(p)\}$. Then $p \in \partial S_t$ if and only if $p \in P_0$ and $\min_{q\in N_p} u^2(q) < t \leq u^2(p)$, and therefore we have

$$\int_0^\infty \# \partial S_t \, dt = \sum_{p\in P_0} \{u^2(p) - \min_{q\in N_p} u^2(q)\}.$$

Now, for each $p \in P_0$, take $q_0 \in N_p$ which minimizes $u(q)$ among $q \in N_p$. Then

$$\begin{aligned} u^2(p) - u^2(q_0) &= \{u(p) + u(q_0)\}\{u(p) - u(q_0)\} \\ &\leq 2u(p)\{u(p) - u(q_0)\} \\ &\leq 2u(p)|Du|(p), \end{aligned}$$

and this implies, by Schwarz's inequality, that

$$(4.5)\quad \int_0^{\infty} \#\partial S_t\, dt \;\le\; 2\sum_{p\in P} u(p)|Du|(p) \;\le\; 2\left\{\sum_{p\in P} u^2(p)\right\}^{1/2}\left\{\sum_{p\in P} |Du|^2(p)\right\}^{1/2}.$$

Now combining (4.4) and (4.5) we conclude (4.3). ∎

Note that, as was seen in the proof given above, the combinatorial version of Buser inequality, (4.2), is almost trivial, while Buser's inequality (1.5) needs more works, and using the combinatorial inequalities (4.1) and (4.2) proved just now, we can obtain the following assertion which is just (1.6) under a superfluous condition on the injectivity radius.

Corollary 4.3. *Let X be a complete riemannian manifold satisfying the condition $(*)$. Then $I_\infty(X) > 0$ if and only if $S_{2,2}(X) > 0$.*

This follows immediately from our discrete approximation theorems and Kesten's inequalities. In fact, for X in the corollary, take an ϵ-net P in it: Then $I_\infty(X) > 0$ iff $I_\infty(P) > 0$ by Theorem 4.1, $I_\infty(P) > 0$ iff $S_{2,2}(P) > 0$ by Proposition 4.2, and $S_{2,2}(P) > 0$ iff $S_{2,2}(X) > 0$ by Theorem 3.2.

So Corollary 4.3 can be considered as a typical example of applications of combinatorial geometry of nets to riemannian geometry.

REFERENCES

[B] P. Buser, *A note on the isoperimetric constant*, Ann. Sci. École Norm. Sup. **15** (1982), 213–230.

[Ch] J. Cheeger, *A lower bound for the smallest eigenvalue of the laplacian*, in "Rroblems in Analysis (A symposium in Honor of S. Bochner)", Princeton Univ. Press, Princeton, 1970, pp. 195–199.

[Cr] C. B. Croke, *Some isoperimetric inequalities and eigenvalue estimates*, Ann. Sci. École Norm. Sup. **13** (1980), 419–435.

[FF] H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. **72** (1960), 458–520.

[K1] M. Kanai, *Rough isometries, and combinatorial approximations of geometries of non- compact riemannian maniolds*, J. Math. Soc. Japan **37** (1985), 391–413.

[K2] ———, *Rough isometries and the parabolicity of riemannian maniolds.*

[K3] ———, *Rough isometries and isoperimetic inequalities for non-compact riemannian manifolds*, to appear in the proceedings of "The 6th Symposium on Differential Equations and Differential Geometry", held at Fudan Univ., Shanghai, 1985.

[Ks1] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.

[Ks2] ———, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156.

[M] V. G. Maz'ya, *Classes of domains and imbedding theorems for function spaces*, Dokl. Akad. Nauk SSSR **133** (1960), 527–530; English transl., Soviet Math. Dokl. **1** (1960), 882–885.

[O] R. Osserman, *The isoperimetric inequality*, Bull. Amer. Math. Soc. **84** (1978), 1182–1238.

[Y] S.-T. Yau, *Isoperimetric constants and the first eigenvalue of a compact riemannian manifold*, Ann. Sci. École Norm. Sup. **8** (1975), 487–507.