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Limit Theorems for Weighted Sums of I.I.D. Random Variables with Applications to Self-Similar Processes

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1. Introduction.

Let $Z_{(\alpha)}(t)$ be a strictly stable process with index α $(0<\alpha\leqq 2)$ and let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent, identically distributed (i.i.d.) random variables such that $\{1/\phi(n)\} \sum_{j=1}^{n} X_j \xrightarrow{L} Z_{(\alpha)}(1)$ as $n \to \infty$ for some $\phi(n)$ ($\to \infty$ as $n \to \infty$). (Throughout this paper $\xrightarrow{L}$ denotes the convergence in distribution.) It is well known that

$$(1.1) \qquad Z_n(t) \equiv \{1/\phi(n)\} \sum_{j=1}^{[nt]} X_j \xrightarrow{L} Z_{(\alpha)}(t) \quad \text{as} \quad n \to \infty$$

over the function space $D([0, \infty) \to \mathbb{R})$ (see [1] and [7] for the definition).

In our previous paper [5], we studied the convergence of

$$(1.2) \qquad X_n \equiv \int_0^{\infty} f_n(u)\, dZ_n(u)$$

$$\left[= \sum_j f_n(j/n)\, X_j/\phi(n) \right], \quad n = 1,2, \cdots ,$$

to

$$(1.3) \qquad X \equiv \int_0^{\infty} f(u)\, dZ_{(\alpha)}(u)$$

under suitable assumptions on $f_n(u)$ and $f(u)$. (The most restrictive condition is that $\{f_n(u)\}_n$ is uniformly bounded.) As we discussed in the same paper the results are useful in various kinds of limit theorems where i.i.d. random variables are involved, and the purpose of this paper is to extend the above result to give an application to <u>self-similar processes</u>. A process $\{Y(t)\}_{t\geqq 0}$ is said to be self-similar with parameter H $(\in \mathbb{R})$ if $\{Y(ct)\}_{t\geqq 0}$ is identical in law to $\{c^H Y(t)\}_{t\geqq 0}$ for every $c > 0$. $Z_{(\alpha)}(t)$ is of course self-similar with $H = 1/\alpha$ and among others another example of self-similar processes which we shall be concerned with in this paper (section 6) is the fractional stable process defined by

$$(1.4) \qquad Y(t) = \int_{-\infty}^{\infty} (|t-u|^{\gamma} - |u|^{\gamma})\, dZ_{(\alpha)}(u) , \quad t \geqq 0 ,$$

where $0 < \alpha < 2$, $-1/\alpha < \gamma < 1 - 1/\alpha$, $\gamma \neq 0$ (see [8]), or by

$$(1.5) \qquad Y(t) = \int_{-\infty}^{\infty} (\{(t-u)^{+}\}^{\gamma} - \{(-u)^{+}\}^{\gamma})\, dZ_{(\alpha)}(u) , \quad t \geqq 0 ,$$

with the notation $a^{+} = \max(0, a)$, where $0 < \alpha \leqq 2$, $-1/\alpha < \gamma < 1 - 1/\alpha$. (See [3],[10] for the case $\alpha = 2$, and [11] for $0 < \alpha < 2$.) $Y(t)$ in (1.5) is called the fractional Brownian motion or the fractional Lévy motion according as $\alpha = 2$ or $0 < \alpha < 2$.

Since self-similar processes are always obtained as the limiting processes of time-space scaled processes, it would be of interest to find their domain of attraction. Here we say that a sequence of stationary random variables $\{Y_j\}$ belongs to the domain of attraction of $\{Y(t)\}$ if all finite-dimensional distributions of $\sum_{j=1}^{[nt]} Y_j$, with a suitable normalization $d(n)$, converges to those of $\{Y(t)\}$. It is known that, if $\{Y(t)\}$ is non-degenerate, continuous in probability and has a non-empty domain of attraction, then $\{Y(t)\}$ is self-similar with a parameter $H > 0$ and $d(n) = n^{H} L(n)$, where $L(\cdot)$ is a slowly varying function at infinity (see Lamperti [6]). A domain of attraction of (1.4) was found by one of the authors ([8]) and of the fractional Brownian motion was studied in [3].

Since the proof of [8] heavily depends on the computation of the Fourier transforms, we did not have an intuitive explanation to the result. However, we obtain a very natural heuristic proof if we formally apply the above argument of [5]. The only problem of this approach is that the integrand of (1.4) is not bounded in general and consequently we cannot apply the result of [5]. Hence, in this paper, we first relax the conditions of [5] so that the unbounded integrands can be treated and then we apply them to the above problems on the fractional stable process. As a byproduct of our new approach, we can extend the result of [8] in a few directions: Firstly, in (1.1), $\phi(n)$ is generally of the form $n^{1/\alpha}L(n)$, where $L(n)$ is a slowly varying function, while in [8], mainly due to technical difficulties, $L(n)$ is confined to be constant, which condition will be removed now. Secondly, the case of $\alpha = 2$ (the fractional Brownian motion) can be treated in parallel even in the case of infinite variance. Thirdly the class of fractional stable processes will be extended a little.

We shall start with non-Gaussian case ($0 < \alpha < 2$). In section 2, the definition of stochastic integrals based on stable processes is given and in section 3 we shall prove a basic inequality, which plays the crucial role to extend the results of [5]. In section 4 we shall study the general theory for the convergence of (1.2) to (1.3). The case of $\alpha = 2$ will be examined in section 5, and the results in sections 4 and 5 will be applied to the fractional stable process and the fractional Brownian motion in section 6.

2. Preliminaries.

Let $Z(t) = Z_{(\alpha)}(t)$, $t \geqq 0$, be a strictly stable Lévy process of index α $(0<\alpha<2)$ with the Lévy measure

$$(2.1) \qquad \nu(dx) = \alpha\{C_+ I[x > 0] + C_- I[x < 0]\} \, |x|^{-\alpha-1} dx$$

on $\mathbb{R} \setminus \{0\}$. Here $I[\cdot]$ is the indicator function and C_+, $C_- \geqq 0$, $C_+ + C_- > 0$. Namely,

$$\log E[\, e^{i\theta Z(t)}] = \begin{cases} t \displaystyle\int (e^{i\theta x} - 1) \, \nu(dx) & , \ 0 < \alpha < 1, \\ t \displaystyle\int (e^{i\theta x} - 1 - i\theta x I[\,|x| \leqq 1]) \nu(dx) & , \ \alpha = 1, \\ t \displaystyle\int (e^{i\theta x} - 1 - i\theta x) \, \nu(dx) & , \ 1 < \alpha < 2. \end{cases}$$

Note that $C_+ = C_-$ in case $\alpha = 1$ since we assumed that $Z(t)$ is strictly stable .

Let $N(dt\,dx)$ be the Poisson random measure defined by the discontinuities of $Z(t)$. As is well known the intensity measure is equal to $\nu(dx)$ (i.e., $E[\, N(dt\,dx)\,] = dt\, \nu(dx)$) and the Lévy-Itô representation theorem gives us

$$Z(t) = \begin{cases} \displaystyle\lim_{\delta \downarrow 0} \int_0^{t+} \int_{|x|>\delta} x \, N(dudx) & \text{a.s.,} \quad 0 < \alpha < 1, \\ \displaystyle\lim_{\delta \uparrow \infty} \int_0^{t+} \int_{|x|\leqq\delta} x \, \tilde{N}(dudx) & \text{a.s.,} \quad 1 \leqq \alpha < 2, \end{cases}$$

where $\tilde{N}(dudx) = N(dudx) - du\, \nu(dx)$.

Definition 2.1. Let $T \in (0, \infty]$. For a real-valued measurable function $f(u)$ on $[0, T] \cap [0, \infty)$, we define

$$(2.2) \qquad \int_0^{t+} f(u)\, dZ(u), \quad 0 \leqq t \leqq T,$$

by

$$(2.3) \qquad \int_0^{t+}\int_{|f(u)x| \leqq 1} f(u)x\, \tilde{N}(dudx) + \int_0^{t+}\int_{|f(u)x| > 1} f(u)x\, N(dudx) + a(t),$$

where

$$a(t) = \begin{cases} \displaystyle\int_0^t \int_{|f(u)x| \leqq 1} f(u)x\, du\, \nu(dx), & 0 < \alpha < 1, \\ 0, & \alpha = 1, \\ \displaystyle -\int_0^t \int_{|f(u)x| > 1} f(u)x\, du\, \nu(dx), & 1 < \alpha < 2, \end{cases}$$

provided that all terms of (2.3) are well-defined. (Throughout $\int_0^{t+}$ is understood to be $\int_{(0, \infty)}$ when $t = \infty$.)

Lemma 2.1. *If*

$$(2.4) \qquad \int_0^T |f(u)|^{\alpha}\, du < \infty,$$

then (2.2) *is well-defined and there exists a positive constant* C *depending only on* $\nu(dx)$ *such that for any* f *satisfying* (2.4) *and any* $\varepsilon > 0$ *we have that*

$$(2.5) \qquad P\{ \sup_{0\leq t\leq T} \left| \int_0^{t+} f(u)\, dZ(u) \right| > \varepsilon \}$$

$$\leq C(\varepsilon^{-2} + 1) \int_0^T |f(u)|^{\alpha}\, du \,.$$

<u>Proof</u>. Let $D = \{(u, x) : |f(u)x| \leq 1 \}$ and let I_D be its indicator function. Since

$$\int_0^T \int I_D\, (f(u)x)^2\, du\, \nu(dx) = C_1 \int_0^T |f(u)|^{\alpha}\, du < \infty ,$$

where $C_1 = (C_+ + C_-)\alpha/(2 - \alpha)$, the integral

$$X(t) \equiv \int_0^{t+} \int I_D\, f(u)x\, \tilde{N}(du\, dx) \quad , \qquad 0 \leq t \leq T ,$$

exists and is a square-integrable martingale. Therefore, we have that

$$(2.6) \qquad P\{ \sup_{0\leq t\leq T} |X(t)| \geq \varepsilon \} \leq \varepsilon^{-2} E[X(T)^2]$$

$$= C_1\varepsilon^{-2} \int_0^T |f(u)|^{\alpha}\, du \,.$$

On the other hand, since

$$\int_0^T \int I_{D^c}\, du\, \nu(dx) = C_2 \int_0^T |f(u)|^{\alpha}\, du < \infty ,$$

where $C_2 = C_+ + C_-$, we see that N restricted on D^c is almost surely a sum of finite numbers of Dirac measures. Hence,

$$Y(t) \equiv \int_0^{t+} \int I_{D^c} \, f(u)x \, N(du\,dx) \ , \quad 0 \leqq t \leqq T \ ,$$

is finite almost surely. Also we have that

$$(2.7) \qquad P\{ \sup_{0 \leqq t \leqq T} |Y(t)| > 0 \} = P\{ \int_0^T \int I_{D^c} \, N(du\,dx) \geqq 1 \}$$

$$\leqq \int_0^T \int I_{D^c} \, du \, \nu(dx) = C_2 \int_0^T |f(u)|^{\alpha} \, du$$

Furthermore, when $0 < \alpha < 1$,

$$(2.8) \qquad \sup_{0 \leqq t \leqq T} |a(t)| \leqq \int_0^T \int I_D \, |f(u)x| \, du \, \nu(dx)$$

$$= C_3 \int_0^T |f(u)|^{\alpha} \, du \ ,$$

where $C_3 = \alpha/(1 - \alpha)$, and when $1 < \alpha < 2$,

$$(2.9) \qquad \sup_{0 \leqq t \leqq T} |a(t)| \leqq \int_0^T \int I_{D^c} |f(u)x| \, du \, \nu(dx)$$

$$= C_4 \int_0^T |f(u)|^{\alpha} \, du \ ,$$

where $C_4 = \alpha/(\alpha - 1)$. (2.6)-(2.9) conclude (2.5). (q.e.d.)

Notice that $\int_0^{t+} f(u) \, dZ(u)$ is a functional of $\{Z(t)\}$ because so is $N(dt\,du)$. It should also be noticed that (2.3) may formally be rewritten as follows.

$$\int_0^{t+} f(u)\, dZ(u) = \begin{cases} \int_0^{t+}\int f(u)x\, N(dudx)\ , & 0 < \alpha < 1\ , \\ \\ \int_0^{t+}\int f(u)x\, \tilde{N}(dudx), & 1 \leq \alpha < 2. \end{cases}$$

Therefore, the integral (2.2) can be expressed in various ways. Indeed we have the following. The proof is easy and left to the reader.

Lemma 2.2. *Let* $f(u)$ *be a measurable function on* $[0, T] \cap [0, \infty)$ *satisfying (2.4). For any measurable set* D *in* $[0, T] \times (R\backslash\{0\})$ *such that*

$$\int\int_D (f(u)x)^2\, du\, \nu(dx) + \int\int_{D^c} du\, \nu(dx) < \infty\ ,$$

it holds that

$$\int_0^{t+} f(u)dZ(u) = \int_0^{t+}\int I_D f(u)x\, \tilde{N}(dudx) + \int_0^{t+}\int I_{D^c}\, f(u)x\, N(dudx) + a_D(t),$$

where

$$a_D(t) = \begin{cases} \int_0^t\int I_D\, f(u)x\, du\, \nu(dx), & 0 < \alpha < 1, \\ \\ 0\qquad\qquad\qquad , & \alpha = 1, \\ \\ -\int_0^t\int I_{D^c}\, f(u)x\, du\, \nu(dx), & 1 < \alpha < 2. \end{cases}$$

Remark. Once (2.5) is established we can prove the following properties by approximating the integrands by step functions.

(i) For any constants a and b,

$$\int_0^{t+} (a\, f(u) + b\, g(u))\, dZ(u)$$

$$= a \int_0^{t+} f(u)\, dZ(u) + b \int_0^{t+} g(u)\, dZ(u).$$

(ii) If $f(u)$ is continuous, then

$$\int_0^{t+} f(u)\, dZ(u) = \underset{n \to \infty}{\mathrm{l.i.p.}} \sum_j f(\tfrac{j}{n})\{Z(\tfrac{j+1}{n} \wedge t) - Z(\tfrac{j}{n} \wedge t)\},$$

where $a \wedge b = \min(a, b)$.

3. A basic inequality.

Throughout this paper the notation in the previous section is preserved. Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of independent random variables having common distribution function $F(x)$ and suppose that for some $\phi(n)$ ($\longrightarrow \infty$ as $n \to \infty$),

$$(3.1) \qquad \frac{1}{\phi(n)} \sum_{j=1}^{n} X_j \xrightarrow{L} Z_{(\alpha)}(1) \quad , \qquad \text{as} \quad n \to \infty ,$$

where $\xrightarrow{L}$ denotes the convergence in distribution. A necessary and sufficient condition for (3.1) is that $\nu_n(dx) \equiv n\, dF(\phi(n)x)$ converges to $\nu(dx)$ (in (2.1)) and in addition, when $1 < \alpha < 2$, $\int x\, \nu_n(dx) = 0$ and when $\alpha = 1$, $\int_{|x|<1} x\, \nu_n(dx) \longrightarrow 0$ as $n \to \infty$. (see section 2 of [5]).

Definition 3.1. Put

$$Z_n(t) = \frac{1}{\phi(n)} \sum_{j \leqq nt} X_j \quad , \ t \geqq 0 ,$$

and

$$\int_0^{t+} f(u)\, dZ_n(u) = \frac{1}{\phi(n)} \sum_{j \leqq nt} f\left(\frac{j}{n} \right) X_j \ . \qquad t \geqq 0 \ .$$

Furthermore, we define

$$(3.2) \qquad \int_0^{\infty} f(u)\, dZ_n(u) = \underset{T \to \infty}{\text{l.i.p.}} \int_0^{T+} f(u)\, dZ_n(u)$$

when the right-hand side exists.

If $\sum_{j \geqq 1} |f(j/n)|^b < \infty$ for some b with $0 < b < \alpha$, then we can show that the right-hand side of (3.2) exists (see Theorem 3.1).

The purpose of this section is to prove the following theorem which is an analogue of Lemma 2.1.

Theorem 3.1. *For any ε (>0), there exists a positive constant C depending only on ε and $F(x)$ such that*

$$(3.3) \qquad P\{ \sup_{0 \leqq t \leqq T} | \int_0^{t+} f(u)\, dZ_n(u) | \geqq a \}$$

$$\leqq C (a^{-2} + 1) \int_0^T \{|f(u)|^{\alpha+\varepsilon} + |f(u)|^{\alpha-\varepsilon}\}\, d\rho_n(u)$$

for every $a > 0$, every $T \in (0, \infty]$, every $n \geqq 1$ and for $f(u)$ such that the right-hand side of (3.3) is finite. Here, $\rho_n(u) = [nu]/n$.

For the proof we first prepare two lemmas.

Lemma 3.1. *For $\varepsilon > 0$, put $C_{\varepsilon,n} = \int \min \{|x|^{\alpha+\varepsilon}, |x|^{\alpha-\varepsilon}\}\, \nu_n(dx)$.*

(i) *When $0 \leqq \beta < \alpha$,*

$$(3.4) \qquad \int_0^{T+} \int_{|f(u)x| \geqq 1} |f(u)x|^{\beta}\, d\rho_n(u)\, \nu_n(dx)$$

$$\leqq C_{\varepsilon,n} \int_0^{T+} \{ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\}\, d\rho_n(u) .$$

(ii) <u>When</u> $\beta > \alpha$,

$$(3.5) \qquad \int_0^{T+} \int_{|f(u)x| \leqq 1} |f(u)x|^{\beta} \, d\rho_n(u) \ \nu_n(dx)$$

$$\leqq C_{\varepsilon,n} \int_0^{T+} \{ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon} \} \, d\rho_n(u) \ .$$

<u>Proof</u>. Since $a^{-\varepsilon} + a^{\varepsilon}$ is monotone increasing in $\varepsilon > 0$, so are the right-hand sides of (3.4) and (3.5). Hence it is enough to prove (3.4) and (3.5) for all sufficiently small $\varepsilon > 0$. Therefore, we shall assume that $0 < \varepsilon \leqq \alpha - \beta$ in (i) and $0 < \varepsilon \leqq \beta - \alpha$ in (ii).

(i) Since $\beta \leqq \alpha - \varepsilon < \alpha + \varepsilon$, we have

$$I[|f(u)x| \geqq 1] \ |f(u)x|^{\beta} \ \leqq \ |f(u)x|^{\alpha-\varepsilon} \wedge |f(u)x|^{\alpha+\varepsilon} \ .$$

Thus we obtain

$$\int_0^{T+} \int_{|f(u)x| \ \geqq \ 1} |f(u)x|^{\beta} \, d\rho_n(u) \ \nu_n(dx)$$

$$\leqq \int_0^{T+} \int \{ I[|x| \geqq 1] \ |f(u)x|^{\alpha-\varepsilon} + I[|x| \leqq 1] \ |f(u)x|^{\alpha+\varepsilon} \} \, d\rho_n(u) \nu_n(dx)$$

$$\leqq C_{\varepsilon,n} \int_0^{T+} \{ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon} \} \, d\rho_n(u) \ .$$

(ii) Similarly, since $\beta \geqq \alpha + \varepsilon > \alpha - \varepsilon$, we have

$$I[|f(u)x| \ \leqq \ 1] \ |f(u)x|^{\beta} \ \leqq \ |f(u)x|^{\alpha-\varepsilon} \wedge |f(u)x|^{\alpha+\varepsilon} \ .$$

Hence we obtain

$$\int_0^{T+} \int_{|f(u)x| \leqq 1} |f(u)x|^{\beta} \, d\rho_n(u) \; \nu_n(dx)$$

$$\leqq C_{\varepsilon,n} \int_0^{T+} \{|f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\} \, d\rho_n(u) \; . \qquad \text{(q.e.d.)}$$

Lemma 3.2. *We consider the case* $\alpha = 1$. *Put*

$$h_n(z) = \int_{|x| \leqq z} x \; \nu_n(dx) \; , \quad z > 0.$$

Then, for every $\varepsilon > 0$, *there exists a constant* $C(\varepsilon) > 0$ *depending only on* $\varepsilon > 0$ *and* $F(x)$ *such that*

$$\text{(3.6)} \qquad \sup_n |h_n(z)| \leqq C(\varepsilon) \, (z^{\varepsilon} + z^{-\varepsilon}) \quad , \quad z > 0 \; .$$

Proof. We have that for $z \geqq 1$,

$$|h_n(z)| \leqq |h_n(1)| + \int_{1 \leqq |x| \leqq z} |x| \; \nu_n(dx)$$

$$\leqq |h_n(1)| + z^{\varepsilon} \int_{1 \leqq |x|} |x|^{1-\varepsilon} \; \nu_n(dx) \quad ,$$

and for $z < 1$,

$$|h_n(z)| \leqq |h_n(1)| + \int_{z \leqq |x| \leqq 1} |x| \; \nu_n(dx)$$

$$\leqq |h_n(1)| + z^{-\varepsilon} \int_{|x|\leqq 1} |x|^{1+\varepsilon} \nu_n(dx) .$$

By our assuption for the case $\alpha = 1$, we have that $h_n(1) \longrightarrow 0$ as $n \to \infty$. Also note that, as $n \longrightarrow \infty$,

$$\int_{|x|\geqq 1} |x|^{1-\varepsilon} \nu_n(dx) \longrightarrow \int_{|x|\geqq 1} |x|^{1-\varepsilon} \nu(dx) < \infty$$

and

$$\int_{|x|\leqq 1} |x|^{1+\varepsilon} \nu_n(dx) \longrightarrow \int_{|x|\leqq 1} |x|^{1+\varepsilon} \nu(dx) < \infty \quad .$$

All together above conclude (3.6). (q.e.d)

Proof of Theorem 3.1. The idea of the proof is essentially the same as that of Lemma 2.1 if we adopt the following notation: Define

$$(3.7) \qquad N_n(dudx) = \sum_{j \geqq 1} \delta_{(j/n,\ X_j/\phi(n))}(dudx)$$

and

$$(3.8) \qquad \tilde{N}_n(dudx) = N_n(dudx) - E[N_n(dudx)]$$

$$= N_n(dudx) - d\rho_n(u)\, \nu_n(dx) .$$

Here, $\delta_{(u_0,x_0)}$ denotes the Dirac measure at (u_0, x_0).

With this notation, the analogue of (2.2)-(2.3) is as follows:

$$(3.9) \qquad \int_0^{t+} f(u)\, dZ_n(u) = \int_0^{t+} \int f(u)x\, N_n(dudx)$$

$$= X_n(t) + Y_n(t) + a_n(t) \quad , \quad t \geqq 0 \ ,$$

where

$$\text{(3.10)} \qquad X_n(t) = \int_0^{t+} \int_{|f(u)x| \leqq 1} f(u)x \ \tilde{N}_n(dudx) \ ,$$

$$\text{(3.11)} \qquad Y_n(t) = \int_0^{t+} \int_{|f(u)x| > 1} f(u)x \ N_n(dudx)$$

and

$$\text{(3.12)} \qquad a_n(t) = \int_0^{t+} \int_{|f(u)x| \leqq 1} f(u)x \ d\rho_n(u) \ \nu_n(dx) \ .$$

As to $X_n(t)$, we have

$$\text{(3.13)} \quad P\{ \sup_{0 \leqq t \leqq T} | X_n(t)| \geqq a/2 \}$$

$$\leqq 4 \ a^{-2} \int_0^{T+} \int_{|f(u)x| \leqq 1} (f(u)x)^2 \ d\rho_n(u) \quad \nu_n(dx)$$

(cf. (2.6)) Applying Lemma 3.1 (ii) with $\beta = 2$, we have from (3.13) that

$$\text{(3.14)} \quad P\{ \sup_{0 \leqq t \leqq T} | X_n(t)| \geqq a/2 \}$$

$$\leqq 4 \ a^{-2} \ C_{\varepsilon,n} \int_0^{T+} \{ \ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon} \} \ d\rho_n(u) \ .$$

As to $Y_n(t)$, we see in a similar way to (2.7) that

$$P\{ \sup_{0 \leqq t \leqq T} | Y_n(t)| > 0 \}$$

$$\leq \int_0^{T+} \int_{|f(u)x| > 1} d\rho_n(u)\ \nu_n(dx) \quad .$$

Applying Lemma 3.1(i) with $\beta = 0$, we obtain

$$(3.15) \quad P\{ \sup_{0\leq t\leq T} |\ Y_n(t)| > 0 \}$$

$$\leq C_{\varepsilon,n} \int_0^{T+} \{\ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\}\ d\rho_n(u) \ .$$

It remains to evaluate $a_n(t)$. When $0 < \alpha < 1$,

$$|a_n(t)| \leq \int_0^{T+} \int_{|f(u)x| \leq 1} |f(u)x|\ d\rho_n(u)\ \nu_n(dx)$$

$$\leq C_{\varepsilon,n} \int_0^{T+} \{\ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\}\ d\rho_n(u) \ .$$

by Lemma 3.1 (ii) with $\beta = 1$.

When $1 < \alpha < 2$, note that

$$a_n(t) = - \int_0^{t+} \int_{|f(u)x|>1} f(u)x\ d\rho_n(u)\ \nu_n(dx) \ ,$$

by $\int x\ \nu_n(dx) = 0$. Therefore, if we apply Lemma 3.1 (i) with $\beta = 1$,

$$|a_n(t)| \leq C_{\varepsilon,n} \int_0^{T+} \{\ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\}\ d\rho_n(u) \ .$$

Finally, when $\alpha = 1$, it follows from Lemma 3.2 that

$$|a_n(t)| = | \int_0^{t+} f(u)\ h_n(1/f(u))\ d\rho_n(u)\ |$$

$$\leq C(\varepsilon) \int_0^{T+} \{ |f(u)|^{1-\varepsilon} + |f(u)|^{1+\varepsilon}\} \, d\rho_n(u) \ .$$

Hence in any case, we have

$$(3.16) \qquad \sup_{0 \leq t \leq T} |a_n(t)| \leq \tilde{C}_\varepsilon \int_0^{T+} \{ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\} \, d\rho_n(u) \ .$$

where $\tilde{C}_\varepsilon = C(\varepsilon)$ or $C_{\varepsilon,n}$ according as $\alpha = 1$ or not.

Now combining (3.14), (3.15) and (3.16) , we see that

$$P\{ \sup_{0 \leq t \leq T} | X_n(t) + Y_n(t) + a_n(t)| \geq a \}$$

$$\leq P\{ \sup_{0 \leq t \leq T} | X_n(t)| \geq \frac{a}{2} \} + P\{ \sup_{0 \leq t \leq T} | Y_n(t)| > 0 \}$$

$$+ P\{ \sup_{0 \leq t \leq T} | a_n(t)| \geq \frac{a}{2} \}$$

$$\leq (4a^{-2} C_{\varepsilon,n} + C_{\varepsilon,n} + 2a^{-1} \tilde{C}_\varepsilon) \int_0^{T+} \{ |f(u)|^{\alpha-\varepsilon} + |f(u)|^{\alpha+\varepsilon}\} \, d\rho_n(u) \ .$$

Since

$$C_{\varepsilon,n} \longrightarrow \int \min \{|x|^{\alpha+\varepsilon}, |x|^{\alpha-\varepsilon}\} \, \nu(dx) \ , \text{ as } \quad n \longrightarrow \infty \ ,$$

we have the assertion.

4. Convergence to stochastic integrals.

In the previous sections we considered $\int_0^{t+} f(u)\, dZ_{(\alpha)}(u)$ and $\int_0^{t+} f(u)\, dZ_n(u)$. We now give a mild condition which assures the convergence of $X_n(t) = \int_0^{t+} f_n(u)\, dZ_n(u)$ to $X(t) = \int_0^{t+} f(u)\, dZ_{(\alpha)}(u)$.

Theorem 4.1. *Let* $f_n(u)$, $u \geqq 0$, $(n = 1, 2, \cdots)$ *be real-valued functions and let* $f(u)$, $u \geqq 0$, *be a measurable function satisfying*

(A1) $\quad f_n(u) \xrightarrow{\text{c.c.}} f(u)$, du-a.e.,

(*where* $f_n(u) \xrightarrow{\text{c.c.}} f(u)$ *at* $u = u_0$ *means that* $f_n(u_n) \longrightarrow f(u_0)$ *whenever* $u_n \longrightarrow u_0$), *and*

(A2) *for every* $T > 0$, *there exists* $\beta > \alpha$ *such that*

$$\sup_n \int_0^{T+} |f_n(u)|^\beta \, d\rho_n(u) < \infty , \quad (\rho_n(u) = [nu]/u).$$

Then

$$(4.1) \quad \left(\int_0^{t+} f_n(u)\, dZ_n(u),\ Z_n(t) \right) \xrightarrow{L} \left(\int_0^{t+} f(u)\, dZ(u),\ Z(t) \right)$$

as $n \to \infty$ *in* $D([0, \infty) \to \mathbb{R}^2)$.

Proof. It is enough to consider the convergence in every finite interval $[0, T]$. In our previous paper [5], we discussed the case where $\{f_n(u)\}$ is uniformly bounded with a slightly different formulation; we proved that if $|f_n(u)| \leqq A$ $(n \geqq 1,\ u \geqq 0)$ for some $A > 0$ and if (A1) is satisfied then

(4.2) $$\int_0^{t+}\int_{|x|\leqq\delta} f_n(u)x\ \tilde{N}_n(dudx) + \int_0^{t+}\int_{|x|>\delta} f_n(u)x\ N_n(dudx)$$

$$\xrightarrow{L} \int_0^{t+}\int_{|x|\leqq\delta} f(u)x\ \tilde{N}(dudx) + \int_0^{t+}\int_{|x|>\delta} f(u)x\ N(dudx)\ .$$

However, it is easy to see that

(4.3) $$\int_0^{t+}\int_{|x|\leqq\delta} f_n(u)x\ d\rho_n(u)\ \nu_n(dx) \longrightarrow b(t)\ ,\ \text{as}\ n \longrightarrow \infty\ ,$$

where

$$b(t) = \begin{cases} \displaystyle\int_0^{t+}\int_{|x|\leqq\delta} f(u)x\ du\ \nu(dx)\ , & 0 < \alpha < 1\ , \\ 0\ , & \alpha = 1\ , \\ \displaystyle -\int_0^{t+}\int_{|x|>\delta} f(u)x\ du\ \nu(dx), & 1 < \alpha < 2\ . \end{cases}$$

Hence (4.2) and (4.3) yield that $\int_0^{t+} f_n(u)\ dZ_n(u)$ converges in distribution to

$$\int_0^{t+}\int_{|x|\leqq\delta} f(u)x\ \tilde{N}(dudx) + \int_0^{t+}\int_{|x|>\delta} f(u)x\ N(dudx) + b(t)$$

which is equal to $\int_0^{t+} f(u)\ dZ(u)$ by Lemma 2.2. We thus have (4.1) for uniformly bounded $\{f_n(u)\}_n$. (The joint convergence can be proved by the Cramér-Wold method.)

We next consider the general case where $\{f_n(u)\}_n$ is not necessarily bounded. Put

$$f_n^M(u) = f_n(u)\ I[\ |f_n(u)| \leqq M\ ]$$

and

$$f^M(u) = f(u)\ I[\ |f(u)| \leqq M\ ]\ ,\ (\ M > 0\ ).$$

As we have seen in the above, (4.1) holds for $\{f_n^M(u)\}_n$ and $f^M(u)$ for every fixed $M > 0$. Therefore, it suffices to show that for any $a > 0$,

$$(4.4)\quad \lim_{M\to\infty} P\{ \sup_{0\leqq t\leqq T} \left| \int_0^{t+} (f_n^M(u) - f_n(u))\, dZ_n(u) \right| > a \} = 0\ ,\quad n \geqq 1,$$

$$(4.5)\quad \lim_{M\to\infty} P\{ \sup_{0\leqq t\leqq T} \left| \int_0^{t+} (f^M(u) - f(u))\, dZ(u) \right| > a \} = 0\ ,$$

and

$$(4.6)\quad \lim_{M\to\infty} \limsup_{n\to\infty} P\{ \sup_{0\leqq t\leqq T} \left| \int_0^{t+} (f_n^M(u) - f_n(u))\, dZ_n(u) \right| > a \} = 0$$

(see for example Theorem 4.2 of Billingsley [1]). (4.4) and (4.5) are clear from Theorem 3.1 and Lemma 2.1, respectively. As to (4.6), choose ε so that $0 < \alpha + \varepsilon < \beta$. Then by Theorem 3.1,

$$(4.7)\qquad P\{ \sup_{0\leqq t\leqq T} \left| \int_0^{t+} (f_n^M(u) - f_n(u))\, dZ_n(u) \right| > a \}$$

$$\leqq C\,(a^{-2} + 1) \int_0^{T+} \{ |f_n^M - f_n|^{\alpha-\varepsilon} + |f_n^M - f_n|^{\alpha+\varepsilon} \}\, d\rho_n(u)$$

$$= C\,(a^{-2} + 1) \int_0^{T+} \{ |f_n|^{\alpha-\varepsilon} + |f_n|^{\alpha+\varepsilon} \}\ I[\,|f_n| > M\ ]\, d\rho_n(u)$$

$$\leqq C\,(a^{-2} + 1)\ (\ M^{-\beta+\alpha-\varepsilon} + M^{-\beta+\alpha+\varepsilon}\ ) \int_0^{T+} |f_n|^{\beta}\, d\rho_n(u)\ .$$

Therefore, we obtain (4.6) from the assumption (A2). (q.e.d.)

Theorem 4.2. In addition to the assumptions (A1)-(A2) in Theorem 4.1 we assume

(A3) there exists an $\varepsilon > 0$ such that

$$(4.8) \qquad \lim_{T \to \infty} \limsup_{n \to \infty} \int_T^\infty \{ |f_n(u)|^{\alpha-\varepsilon} + |f_n(u)|^{\alpha+\varepsilon} \} \, d\rho_n(u) = 0.$$

Then we have that

$$\left(\int_0^\infty f_n(u) \, dZ_n(u), \, Z_n(t) \right) \xrightarrow{L} \left(\int_0^\infty f(u) \, dZ(u), \, Z(t) \right)$$

in $\mathbb{R} \times D([0, \infty) \to \mathbb{R})$. A sufficient condition for (A2) and (A3) is

(A4) there exist β and δ such that $\beta > \alpha$, $\delta > \beta/\alpha - 1$ and

$$(4.9) \qquad \sup_n \int_0^\infty |f_n(u)|^\beta \, (1 + u)^\delta \, d\rho_n(u) < \infty .$$

Proof. The story of the proof is the same as that of Theorem 4.1: Because of Theorem 4.1, it is enough to check

$$(4.10) \qquad \lim_{t \to \infty} \limsup_{n \to \infty} P\{ \, | \int_0^\infty f_n \, dZ_n - \int_0^{t+} f_n \, dZ_n \, | > a \, \} = 0$$

for every $a > 0$. However, as in (4.7) we can apply Theorem 3.1 to see

$$P\{ \, | \int_0^\infty f_n \, dZ_n - \int_0^{t+} f_n \, dZ_n \, | > a \, \}$$

$$\leqq C\,(a^{-2}+1)\int_t^{\infty}\{\,|f_n(u)|^{\alpha-\varepsilon}+|f_n(u)|^{\alpha+\varepsilon}\}\,d\rho_n(u)\ ,$$

which combined with (A3) proves (4.10). It remains to prove the latter half of the theorem. The sufficiency of (A4) for (A2) is clear and to see that for (A3) we need the next lemma the proof of which will be given later.

Lemma 4.1. *For any* $\beta, \beta', \delta, \delta' > 0$ *satisfying*

(4.11) $$\frac{\delta'+1}{\delta+1} < \frac{\beta'}{\beta} \leqq 1$$

and for any measurable function $f(u) \geqq 0$, *we have*

$$\int_0^{\infty} f(u)^{\beta'}(1+u)^{\delta'}\,d\rho_n(u)$$

$$\leqq C\,\{\int_0^{\infty} f(u)^{\beta}\,(1+u)^{\delta}\,d\rho_n(u)\,\}^{\beta'/\beta}\ ,$$

where $C > 0$ *is independent of* $f(u)$ *and* n.

We continue the proof of Theorem 4.2. Choose $\varepsilon > 0$ so that $0 < \alpha - \varepsilon < \alpha + \varepsilon < \beta$ and that

(4.12) $1 < ((\alpha-\varepsilon)/\beta)(\delta+1) < ((\alpha+\varepsilon)/\beta)(\delta+1)$.

This is possible because $0 < \alpha < \beta$ and $1 < (\alpha/\beta)(\delta+1)$ by the assumption. We apply Lemma 4.1 with $\beta' = \alpha+\varepsilon$ or $\alpha-\varepsilon$ and with

$\delta' = (1/2)\{((\alpha - \varepsilon)/\beta)(\delta + 1)-1\}$ (>0 by (4.12)). It is easy to check that (4.11) is satisfied. Hence,

$$\int_T^\infty \{ |f_n(u)|^{\alpha-\varepsilon} + |f_n(u)|^{\alpha+\varepsilon}\} \, d\rho_n(u)$$

$$\leqq T^{-\delta'} \int_T^\infty \{ |f_n(u)|^{\alpha-\varepsilon} + |f_n(u)|^{\alpha+\varepsilon}\} (1 + u)^{\delta'} \, d\rho_n(u)$$

$$\leqq C\, T^{-\delta'} \{ \left(\int_0^\infty |f_n(u)|^\beta (1 + u)^\delta \, d\rho_n(u)\right)^{(\alpha - \varepsilon)/\beta}$$

$$+ \left(\int_0^\infty |f_n(u)|^\beta (1 + u)^\delta \, d\rho_n(u)\right)^{(\alpha + \varepsilon)/\beta} \} .$$

Letting $T \longrightarrow \infty$, we get (4.8) from (4.9). (q.e.d.)

Proof of Lemma 4.1. It is enough to consider the case where $\beta'/\beta < 1$. Put $p = \beta/\beta'$ and let q be its conjugate (i.e., $1/p + 1/q = 1$). Then,

$$\int_0^\infty f(u)^{\beta'} (1 + u)^{\delta'} d\rho_n(u)$$

$$\leqq \int_0^\infty f(u)^{\beta/p} (1 + u)^{\delta/p} (1 + u)^{\delta' - \delta/p} \, d\rho_n(u)$$

$$\leqq \left(\int_0^\infty f(u)^\beta (1 + u)^\delta \, d\rho_n(u) \right)^{1/p} \left(\int_0^\infty (1 + u)^{(\delta' - \delta/p)q} \, d\rho_n(u) \right)^{1/q} .$$

Therefore, we get the conclusion with

$$C = \left(\int_0^\infty (1 + u)^{(\delta' - \delta/p)q} \, du \right)^{1/q}$$

which is finite because $(\delta' - \delta/p)q < -1$ by (4.11). (q.e.d.)

It is easy to extend the result of Theorem 4.2 to the case of $\mathbb{R}^d$-valued functions:

Theorem 4.3. *Suppose that, for every* $k = 1, 2, \cdots, d$ $(d \geqq 2)$, $\{f_n^k(u)\}_n$ *and* $f^k(u)$ *satisfy the assumptions (A1)-(A3) in Theorem 4.2. Then we have*

$$\left(\int_0^\infty f_n^1(u)\, dZ_n(u) , \cdots , \int_0^\infty f_n^d(u)\, dZ_n(u) \right)$$

$$\xrightarrow{L} \left(\int_0^\infty f^1(u)\, dZ(u) , \cdots , \int_0^\infty f^d(u)\, dZ(u) \right) \quad \text{as} \quad n \to \infty .$$

Proof. By the Cramér-Wold method we have the assertion immediately from Theorem 4.2. (q.e.d)

To conclude this section we consider the two-sided case. Let $\{Z_-(t)\}_{t \geqq 0}$ be an independent copy of $\{Z(t)\}_{t \geqq 0}$, and extend $Z(t)$ in such a way that

$$Z(t) = - Z_-(-t+0) \qquad \text{if} \quad t < 0 .$$

We call this $\{Z(t): -\infty < t < \infty\}$ a *two-sided* stable process with index α.

For a measurable function $f(u)$, $u \in \mathbb{R}$, we define

$$\int_{-\infty}^{\infty} f(u)\, dZ(u)$$

by

$$\int_0^{\infty} f(u)\, dZ_+(u) + \int_0^{\infty} f(-u)\, dZ_-(u) \tag{4.13}$$

provided that the both terms of (4.13) are well-defined.

So far we considered i.i.d. random variables $\{X_j\}_{j=1}^{\infty}$ with distribution function $F(x)$ satisfying (3.1) but in the rest of this section we shall consider the bilateral sequence $\{X_j\}_{j=-\infty}^{\infty}$ of i.i.d. random variables with the above distribution function. We define

$$\int_{-\infty}^{\infty} f(u)\, dZ_n(u) = \frac{1}{\phi(n)} \sum_{j \in \mathbb{Z}} f(\frac{j}{n})\, X_j ,$$

provided that the right-hand side exists. Then we have the next theorem immediately from Theorem 4.3.

<u>Theorem 4.4</u>. <u>Suppose that, for every</u> $k = 1, 2, \cdots , d$ $(d \geqq 1)$, <u>functions</u> $\{f_n^k(u)\}_{n=1}^{\infty}$ <u>and</u> $f^k(u)$ <u>on</u> $\mathbb{R}$ <u>satisfy that</u>

(A1)′ $\quad f_n^k(u) \xrightarrow{c.c.} f^k(u)$ du-a.e, <u>as</u> $n \to \infty$,

<u>and</u>

(A4)′ <u>there exist</u> β <u>and</u> δ <u>such that</u> $\beta > \alpha$, $\delta > \beta/\alpha - 1$ <u>and</u>

$$\sup_n \int_{-\infty}^{\infty} |f_n^k(u)|^{\beta} (1 + |u|)^{\delta}\, d\rho_n(u) < \infty , (\rho_n(u) = [nu]/n). \tag{4.14}$$

<u>Then we have that, as</u> $n \to \infty$,

$$\left(\int_{-\infty}^{\infty} f_n^1(u)\, dZ_n(u) , \cdots , \int_{-\infty}^{\infty} f_n^d(u)\, dZ_n(u) \right)$$

$$\xrightarrow{L} \left(\int_{-\infty}^{\infty} f^1(u)\, dZ(u) , \cdots , \int_{-\infty}^{\infty} f^d(u)\, dZ(u) \right)$$

as $n \to \infty$.

5. The case of $\alpha = 2$.

The purpose of this section is to establish the statements of Theorems 4.1 and 4.4 for the case $\alpha = 2$. (Theorems 4.2 and 4.3 can be similarly established, but will be omitted.) Let $Z(t)$ ($= Z_{(2)}(t)$) $= \sigma B(t)$ or $= -\sigma B_-(-t)$ according as $t \geqq 0$ or $t < 0$, where $\{B(t)\}_{t\geqq 0}$ and $\{B_-(t)\}_{t\geqq 0}$ are independent standard Brownian motions and $\sigma > 0$. Further, let $\{X_j\}_{j=-\infty}^{\infty}$ be a sequence of independent random variables with a common distribution function $F(x)$ such that for some $\phi(n)$ ($\longrightarrow \infty$ as $n \to \infty$), $(1/\phi(n)) \sum_{j=1}^{n} X_j \xrightarrow{L} Z(1)$. Put $Z_n(t) = (1/\phi(n)) \sum_{0<j\leqq nt} X_j$ or $-(1/\phi(n)) \sum_{nt<j\leqq 0} X_j$ according as $t > 0$ or not as before.

Theorem 5.1. *For any $\varepsilon > 0$, there exists a positive constant C depending only on ε and $F(x)$ such that, for every $T \leqq \infty$,*

$$(5.1) \qquad P\{ \sup_{0\leqq t\leqq T} \left| \int_0^{t+} f(u)\, dZ_n(u) \right| > a \}$$

$$\leqq C\, (a^{-2} + 1) \int_0^{T+} \{ |f(u)|^2 + |f(u)|^{2-\varepsilon} \}\, d\rho_n(u) .$$

Proof. When $E[X_1^2] < \infty$, the Kolmogorov inequality gives us

$$P\{ \sup_{0\leqq t\leqq T} \left| \int_0^{t+} f(u)\, dZ_n(u) \right| > a \}$$

$$\leqq a^{-2} E[X_1^2] \int_0^{T+} |f(u)|^2\, d\rho_n(u) ,$$

which proves (5.1). When $E[X_1^2] = \infty$, we define $X_n(t)$, $Y_n(t)$ and

$a_n(t)$ by (3.7)-(3.12). Then, (3.13) remains valid and hence by the next Lemma 5.1 we obtain

$$P\{ \sup_{0 \leq t \leq T} | X_n(t) | > a/2 \} \tag{5.2}$$

$$\leq 4 a^{-2} C_{\varepsilon,n} \int_0^{T+} \{ |f(u)|^2 + |f(u)|^{2-\varepsilon}\} d\rho_n(u) .$$

As to $Y_n(t)$, we have

$$P\{ \sup_{0 \leq t \leq T} | Y_n(t) | > 0 \}$$

$$\leq \int_0^{T+} \int I[|f(u)x| > 1] d\rho_n(u) \nu_n(dx)$$

$$\leq \int_0^{T+} \int \{(f(u)x)^2 I[|x| \leq 1] + |f(u)x|^{2-\varepsilon} I[|x|>1]\} d\rho_n(u) \nu_n(dx) .$$

Keeping in mind that, in the case where $E[X_j^2] = \infty$,

$$\int_{|x| \leq 1} x^2 \nu_n(dx) \longrightarrow \sigma^2 \quad \text{as} \quad n \to \infty , \tag{5.3}$$

and that

$$\int_{|x| > 1} |x|^{2-\varepsilon} \nu_n(dx) \longrightarrow 0 \quad \text{as} \quad n \to \infty , \tag{5.4}$$

we have

$$P\{ \sup_{0 \leq t \leq T} | Y_n(t) | > 0 \} \tag{5.5}$$

$$\leq \text{const.} \int_0^{T+} \{ |f(u)|^2 + |f(u)|^{2-\varepsilon}\} d\rho_n(u) .$$

As to $a_n(t)$, we similarly have

$$(5.6) \qquad |a_n(t)| \leqq \int_0^{T+} \int I[\ |f(u)x| > 1\ ]\ |f(u)x|\ d\rho_n(u)\ \nu_n(dx)$$

$$\leqq \text{ const. } \int_0^{T+} \{\ |f(u)|^2 + |f(u)|^{2-\varepsilon}\}\ d\rho_n(u)\ .$$

The assertion of the theorem follows from (5.2),(5.5) and (5.6).

Lemma 5.1. *If* $F(x)$ *belongs to the domain of attraction of a normal law and has infinite variance, then for every* $\varepsilon > 0$ *there exists* C_ε *depending only on* ε *and* $F(x)$ *such that*

$$(5.7) \qquad \int_{|x| \leqq z} x^2\ \nu_n(dx) \leqq C_\varepsilon\ (\ 1 + z^\varepsilon\ ), \quad \textit{for every} \quad z > 0\ ,$$

where $\nu_n(dx) = n\ dF(\phi(n)x)$ *as before*.

Proof. We have that for $z \geqq 1$

$$(5.8) \qquad \int_{|x| \leqq z} x^2\ \nu_n(dx)$$

$$\leqq \int_{|x| \leqq 1} x^2\ \nu_n(dx) + \int_{1 \leqq |x| \leqq z} x^2\ \nu_n(dx)$$

$$\leqq \int_{|x| \leqq 1} x^2\ \nu_n(dx) + z^\varepsilon \int_{1 \leqq |x|} |x|^{2-\varepsilon}\ \nu_n(dx)\ .$$

For z such that $0 < z < 1$,

$$(5.9) \qquad \int_{|x| \leqq z} x^2\ \nu_n(dx) \leqq \int_{|x| \leqq 1} |x|^{2-\varepsilon}\ \nu_n(dx)\ .$$

Thus (5.7) follows from (5.8) and (5.9) (see (5.3) and (5.4)). (q.e.d.)

Recall that we proved Theorems 4.1-4.4 using Theorem 3.1. Therefore, once we establish Theorem 5.1, which corresponds to Theorem 3.1, we obtain the following theorems in the same way.

Theorem 5.2. *Let* $f_n(u)$, $u \geqq 0$, $(n = 1, 2, \cdots)$ *be real-valued functions and let* $f(u)$, $u \geqq 0$, *be a measurable function satisfying (A1) in Theorem 4.1 and*

(A2) *for every* $T > 0$,

$$\sup_n \int_0^{T+} f_n(u)^2 \, d\rho_n(u) < \infty, \quad (\rho_n(u) = [nu]/u).$$

Then,

$$\left(\int_0^{t+} f_n(u) \, dZ_n(u), Z_n(t) \right) \xrightarrow{L} \left(\int_0^{t+} f(u) \, dZ(u), Z(t) \right)$$

as $n \to \infty$ *in* $D([0, \infty) \to \mathbb{R}^2)$.

Theorem 5.3. *The assertion of Theorem 4.4 is still valid even in the case of* $\alpha = 2$ *replacing the condition* (A4)′ *by*

(A4)′′ *there exists* $\delta > 0$ *such that*

(5.10) $$\sup_n \int_{-\infty}^{\infty} f_n(u)^2 (1 + |u|)^{\delta} \, d\rho_n(u) < \infty .$$

6. The fractional stable process.

In this section we shall apply the results in sections 4 and 5 to the fractional stable process ([8],[11],[12]) and the fractional Brownian motion ([3],[10]). Throughout this section, $Z(t) = Z_{(\alpha)}(t)$, ($t \in \mathbb{R}$) is the two-sided stable Lévy process of index α ($0<\alpha\leqq 2$) considered in the previous sections (i.e., for negative t, we define $Z(t) = -Z_{-}(-t+0)$ where $Z_{-}(t)$ is an independent copy of $\{Z_{(\alpha)}(t):t \geqq 0\}$).

Let $a_{+}, a_{-} \in \mathbb{R}$ and let γ be a real number such that

$$- 1/\alpha < \gamma < 1 - 1/\alpha \quad \text{and} \quad \gamma \neq 0 . \tag{6.1}$$

Define for $t\ (>0)$, $u\ (\neq 0, t)$,

$$f(t:u) = \begin{cases} -\displaystyle\int_{[-u,t-u]^{c}} \psi(x)\, dx , & \text{if } \gamma < 0 \text{ and } 0 < u < t, \\ \displaystyle\int_{[-u,t-u]} \psi(x)\, dx , & \text{otherwise ,} \end{cases}$$

where

$$\psi(x) = \psi(a_{+},a_{-};\gamma:x) = \{ a_{+}I[x>0] - a_{-}I[x<0]\}|x|^{\gamma-1}. \tag{6.2}$$

Notice that $f(t:u) = |t - u|^{\gamma} - |u|^{\gamma}$ when $a_{+} = a_{-} = \gamma$, while $f(t:u) = \{(t - u)^{+}\}^{\gamma} - \{(- u)^{+}\}^{\gamma}$ when $a_{+} = \gamma$ and $a_{-} = 0$.

Definition 6.1. For γ satisfying (6.1), we call the process $\{\Delta(t)\}_{t\geqq 0}$, defined by the following stochastic integral the <u>fractional stable</u>

process;

$$\Delta(t) = \Delta(a_+,a_-;\gamma:t) = \begin{cases} 0 & , \ t = 0 , \\ \int_{-\infty}^{\infty} f(t:u)\, dZ(u) & , \ t > 0. \end{cases}$$

Of course the stochastic integral is well-defined (see Lemma 2.1) and it is easy to see that $\Delta(t)$ is a self-similar process having parameter $H = 1/\alpha+\gamma$. We can also show that $\Delta(t)$ has stationary increments (i.e., for every $c > 0$, $\{\Delta(t+c)-\Delta(c)\}_t$ and $\{\Delta(t)\}_t$ have the same finite-dimensional distributions.). These facts can be proved directly but also follow from Theorems 6.1 and 6.2 below. The regularity of the sample paths of $\Delta(t)$ will be mentioned at the end of this section. (See [12] for some basic properties of self-similar processes having stationary increments.)

Examples. (1)

$$\Delta(\gamma,\gamma;\gamma:t) = \int_{-\infty}^{\infty} \{ |t - u|^{\gamma} - |u|^{\gamma}\}\, dZ(u)$$

is the fractional stable process considered in [8].

(2)

$$\Delta(\gamma,0;\gamma:t) = \int_{-\infty}^{0+}\{(t - u)^{\gamma} - (-u)^{\gamma}\}\, dZ(u) + \int_{0}^{t} (t - u)^{\gamma}\, dZ(u)$$

is the fractional Brownian in [3] and [10] or the fractional Lévy motion in [11] according as $\alpha = 2$ or $0 < \alpha < 2$. Also see [12] for fractional processes.

In this section we shall find some stationary sequences of random variables belonging to the domain of attraction of the process $\Delta(t)$. Our result will extend the limit theorems in [3] and [8]:

Let $\{c_j\}_{j=-\infty}^{\infty}$ be a sequence of real numbers such that

$$(6.3) \qquad c_n \sim \begin{cases} a_+ \, n^{\gamma-1} \, L(n) , & \text{as } n \to \infty , \\ \\ a_- \, |n|^{\gamma-1} \, L(|n|) , & \text{as } n \to -\infty , \end{cases}$$

where a_+, a_- and γ are the same as before and $L(n)$ is slowly varying at infinity. Here, $a_n \sim b_n$ means that $\lim a_n/b_n = 1$. Let $\{X_j\}_{j=-\infty}^{\infty}$ be as before; a sequence of i.i.d. random variables such that, for some $\phi(n)$, $(1/\phi(n)) \sum_{j=1}^{n} X_j \xrightarrow{L} Z_{(\alpha)}(1)$ as $n \to \infty$.

Define a stationay sequence $\{Y_k\}_{k=1}^{\infty}$ of random variables by

$$(6.4) \qquad Y_k = \sum_{j \in \mathbb{Z}} c_j X_{k-j} , \quad k = 1, 2, \cdots ,$$

and consider the partial sum

$$D_n(t) = \sum_{k=1}^{[nt]} Y_k , \qquad t > 0 , \quad n = 1, 2, \cdots .$$

Then we have

<u>Theorem 6.1</u>. <u>When</u> $0 < \gamma < 1 - 1/\alpha$,

$$\frac{1}{n^{\gamma}L(n)\phi(n)} D_n(t) \longrightarrow_{f.d.} \Delta(a_+,a_-;\gamma:t) \quad \underline{as} \quad n \longrightarrow \infty ,$$

<u>where</u> $\longrightarrow_{f.d.}$ <u>denotes the convergence of all finite-dimensional distributions</u>.

<u>Theorem 6.2</u>. <u>When</u> $-1/\alpha < \gamma < 0$,

$$\frac{1}{\phi(n)} D_n(t) \longrightarrow_{f.d.} A \cdot Z_{(\alpha)}(t) \quad , \quad \underline{as} \quad n \longrightarrow \infty , \tag{6.5}$$

<u>where</u> $A = \sum_j c_j$. <u>Furthermore, if in addition</u> $A = 0$, <u>we have</u>

$$\frac{1}{n^{\gamma}L(n)\phi(n)} D_n(t) \longrightarrow_{f.d.} \Delta(a_+,a_-;\gamma:t) , \quad \underline{as} \quad n \longrightarrow \infty .$$

<u>Proofs of Theorems 6.1 and 6.2</u>. We have

$$\frac{1}{\phi(n)} D_n(t) = \frac{1}{\phi(n)} \sum_{k=1}^{[nt]} Y_k$$

$$= \frac{1}{\phi(n)} \sum_j \Big(\sum_{k=1}^{[nt]} c_{k-j} \Big) X_j$$

$$= \sum_j \Big(\sum_{k=1-j}^{[nt]-j} c_k \Big) X_j/\phi(n) = \int_{-\infty}^{\infty} g_n(t:u)\, dZ_n(u) ,$$

where

$$g_n(t:u) = \sum_{k=1-[nu]}^{[nt]-[nu]} c_k \quad . \tag{6.6}$$

Hence, to apply Theorem 4.4 (5.3 when $\alpha = 2$) it is enough to check the continuous convergence of $f_n^k(u) = \{1/(n^\gamma L(n))\} g_n(t_k:u)$ (or $g_n(t_k:u)$ in the case of (6.5)) and the condition (4.14) ((5.10) when $\alpha = 2$) for every $0 < t_1 < t_2 < \cdots < t_d$. To this end we prepare two lemmas.
Since in Theorem 6.3 below the first half of Theorem 6.2 will be proved in a more general situation, we shall omit the proof of this part and assume that $A = 0$ in the proof of Theorem 6.2.

Lemma 6.1. *Let* $\gamma > 0$ *and* $t > 0$. *Then*

$$\frac{1}{n^\gamma L(n)} g_n(t:u) \xrightarrow{c.c.} f(t:u), \quad u \neq 0, t,$$

and for any $\varepsilon > 0$, *there exists a constant* $C = C(\varepsilon) > 0$ *such that*

$$\frac{1}{n^\gamma L(n)} |g_n(t:u)| \leq \begin{cases} C, & -t \leqq u \leqq 2t, \\ C\,|u|^{\gamma-1+\varepsilon}, & \text{elsewhere}. \end{cases} \tag{6.7}$$

Lemma 6.2. *Let* $\gamma < 0$ *and* $t > 0$. *If* $A = \sum_j c_j = 0$, *then*

$$\frac{1}{n^\gamma L(n)} g_n(t:u) \xrightarrow{c.c.} f(t:u), \quad u \neq 0, t, \tag{6.8}$$

and for any $\varepsilon > 0$, *there exists a constant* $C = C(\varepsilon) > 0$ *such that*

$$(6.9) \qquad \frac{1}{n^{\gamma}L(n)}\,|g_n(t:u)| \leqq \begin{cases} C\,\{|u|^{\gamma-\varepsilon} + |t-u|^{\gamma-\varepsilon}\} & , -t \leqq u \leqq 2t, \\ \\ C\,|u|^{\gamma-1+\varepsilon} & , \underline{\text{elsewhere}}. \end{cases}$$

We postpone the proof of above lemmas and continue the proof of Theorems 6.1 and 6.2. Since $-1/\alpha < \gamma < 1 - 1/\alpha$, we have that $\alpha(\gamma-1) < -1$ and $\alpha\gamma > -1$. Therefore, we can choose $\delta > 0$ so that $\alpha(\gamma-1)+\delta < -1$. Then if we take $\varepsilon > 0$ small enough, it holds that

$$(\alpha + \varepsilon)(\gamma - 1 + \varepsilon) + \delta < -1 \quad \text{and} \quad \varepsilon < \alpha\delta \quad ,$$

and that

$$(\alpha + \varepsilon)(\gamma - \varepsilon) > -1 \quad .$$

Namely, if we let $\beta = \alpha + \varepsilon$, then

$$(6.10) \qquad \beta(\gamma - 1 + \varepsilon) + \delta < -1 \ .$$

and

$$(6.11) \qquad \beta(\gamma - \varepsilon) > -1 \quad .$$

It also holds that $\beta > \alpha$ and $\delta > \beta/\alpha - 1$. Therefore, combining (6.10) and (6.11) with (6.7) or (6.9) we see that $f_n^k(u)$ satisfies (A4) of Theorem 4.3. Since (A1) is also proved in the above two lemmas, the assertions of the theorems follow from Theorem 4.3 (Theorem 5.3 when $\alpha = 2$). (q.e.d.)

It remains to prove Lemmas 6.1 and 6.2. Since the proof of these two are essentially the same, we shall prove the latter only.

Proof of Lemma 6.2. Put $\xi(x) = c_{[x]}$, $x \in \mathbb{R}$. Then,

$$(6.12) \qquad \frac{1}{n^{\gamma}L(n)}\, g_n(t:u) = \int_{-nu}^{nt-nu} \frac{\xi(x)}{n^{\gamma}L(n)}\, dx + \varepsilon_n(t:u)$$
$$= \int_{-u}^{t-u} \frac{\xi(nx)}{n^{\gamma-1}L(n)}\, dx + \varepsilon_n(t:u) \quad ,$$

where

$$|\varepsilon_n(t:u)| \leqq \frac{1}{n^{\gamma}L(n)} \{|\xi(1-nu)| + |\xi(nt-nu)|\} = O(\tfrac{1}{n}) \ .$$

If we note that $\xi(nx)/n^{\gamma-1}L(n)$ converges to $\psi(x)$ (in (6.2)) uniformly for x on every compact subset of $\mathbb{R}\setminus\{0\}$, we have from (6.12) that

$$\frac{1}{n^{\gamma}L(n)}\, g_n(t:u) \xrightarrow{\text{c.c.}} \int_{-u}^{t-u} \psi(x)\, dx \qquad \text{as } n \to \infty ,$$

when $u \in (-\infty, 0) \cup (t, \infty)$. When $u \in (0, t)$, using the assumption $A = 0$, we have from (6.12) that

$$\frac{1}{n^{\gamma}L(n)}\, g_n(t:u) = -\int_{[-u,t-u]^c} \frac{\xi(nx)}{n^{\gamma-1}L(n)}\, dx - \varepsilon_n(t:u)$$
$$\xrightarrow{\text{c.c.}} -\int_{[-u,t-u]^c} \psi(x)\, dx \quad , \qquad \text{as } n \to \infty \ .$$

Thus (6.8) is proved.

We finally show (6.9). To this end we note that for every $\varepsilon > 0$, there exist $K_1 > 0$ and $n_0 > 0$ such that

$$(6.13) \qquad \left| \frac{L(y)}{L(x)} \right| \leqq K_1 \left\{ \left(\frac{y}{x} \right)^{\varepsilon} + \left(\frac{y}{x} \right)^{-\varepsilon} \right\},$$

for any $x, y \geqq n_0$. (This can easily be seen by the canonical representation of slowly varying functions (see page 282 of [4]). Since $(n + n_0)^{\gamma-1}L(n + n_0) \sim n^{\gamma-1}L(n)$ as $n \to \infty$, we have from (6.3) that, for some K_2,

$$(6.14) \qquad |c_j| \leqq K_2 (|j| + n_0)^{\gamma-1} L(|j| + n_0), \text{ for all } j \in \mathbb{Z}.$$

Using (6.13), we have from (6.14) that

$$\left| \frac{c_j}{n^{\gamma-1}L(n)} \right| \leqq K_2 \left(\frac{|j|+n_0}{n} \right)^{\gamma-1} \frac{L(|j|+n_0)}{L(n)}$$

$$\leqq K_3 \left(\frac{|j|+n_0}{n} \right)^{\gamma-1\pm\varepsilon}, \text{ for all } j \in \mathbb{Z},$$

where $x^{\gamma-1\pm\varepsilon} = \max\{ x^{\gamma-1+\varepsilon}, x^{\gamma-1-\varepsilon} \}$ and $K_3 = K_1 \cdot K_2$.

Thus we have

$$(6.15) \qquad \frac{1}{n^{\gamma}L(n)} |g_n(t:u)| \leqq \frac{1}{n} \sum_{j=1-[nu]}^{[nt]-[nu]} \frac{|c_j|}{n^{\gamma-1}L(n)}$$

$$\leqq K_3 \frac{1}{n} \sum_{j=1-[nu]}^{[nt]-[nu]} \left(\frac{|j|+n_0}{n} \right)^{\gamma-1\pm\varepsilon}$$

$$\leqq K_3 \frac{1}{n} \int_{-nu-1}^{nt-nu+1} \left(\frac{|x|+n_0}{n} \right)^{\gamma-1\pm\varepsilon} dx$$

$$= K_3 \int_{-u-1/n}^{t-u+1/n} \left(|x| + \frac{n_0}{n} \right)^{\gamma-1\pm\varepsilon} dx$$

$$\leqq K_3 \int_{-(1/2)t-u}^{(3/2)t-u} |x|^{\gamma-1\pm\varepsilon}\, dx$$

for $n \geqq 2/t$. If $u \in (-\infty, -t) \cup (2t, \infty)$, then the last integral is dominated by const.$\times$ $|u|^{\gamma-1\pm\varepsilon}$, which is also bounded by another constant times $|u|^{\gamma-1+\varepsilon}$ on these intervals. This proves the second line of (6.9). If $u \in (-t, 2t)$, using the assumption $\sum_j c_j = 0$, we have

$$g_n(t:u) = -\left(\sum_{j<1-nu} c_j + \sum_{j>nt-nu} c_j \right) .$$

Similarly to (6.15), we obtain

$$\frac{1}{n^{\gamma}L(n)} |g_n(t:u)|$$

$$\leqq K_3 \left\{ \int_{-\infty}^{-u+1/n} + \int_{t-u-1/n}^{\infty} \right\} \left(|x| + \frac{n_0}{n}\right)^{\gamma-1\pm\varepsilon} dx \quad .$$

The first integral is dominated by

$$\text{const.}\times \int_{-\infty}^{-u} \left(\left|x - \frac{1}{n}\right| + \frac{n_0}{n} \right)^{\gamma-1\pm\varepsilon} dx$$

$$\leqq \text{const.}\times \int_{-\infty}^{-u} |x|^{\gamma-1\pm\varepsilon}\, dx$$

$$\leqq \text{const.}\times |u|^{\gamma-\varepsilon} ,$$

while the second one is similarly bounded by const.$\times$ $|t - u|^{\gamma-\varepsilon}$. This concludes the first line of (6.9) completeing the proof of Lemma 6.2.

Remark 1. As we mentioned before, the process $\Delta(t)$ generalizes the fractional stable process and the fractional Brownian motion studied in [3],[8],[10],[11] and [12]. Therefore, Theorems 6.1 and 6.2 are generalizations of the limit theorems related to those processes in the following sense. First, the limiting processes are generalized and secondly, even in the cases of the fractional stable process and the fractional Brownian motion, the domain of attraction is extended. Namely, for the fractional stable process, the case studied in [8] is that $c_j = j^{\gamma-1}$, $= 0$, or $= -|j|^{\gamma-1}$ according as $j > 0$, $= 0$ or < 0 and $\phi(n) = n^{1/\alpha}$. For the fractional Brownian motion, [3] studied the case of $E[X_1^2] < \infty$, although $\{c_j\}$ are more general, while our theorem includes the case where X_1 has infinite variance.

The first half of Theorem 6.2 can be generalized as follows:

Theorem 6.3. *Let* $\{X_j\}$ *and* $Z_{(\alpha)}(t)$ $(0<\alpha\leqq 2)$ *be as before and let* $\{c_j\}_{j=-\infty}^{\infty}$ *be a sequence of real numbers such that*

$$\sum_{j=-\infty}^{\infty} |c_j|^b < \infty \quad \textit{for some} \quad 0 < b < \alpha\ ,\ b \leqq 1\ .$$

Define $\{Y_k\}_{k=1}^{\infty}$ *by* (6.4). *Then*,

$$\frac{1}{\phi(n)} \sum_{k=1}^{[nt]} Y_k \xrightarrow{\ \ }_{f.d.} A\cdot Z_{(\alpha)}(t) \quad , \quad \textit{as} \quad n \longrightarrow \infty\ .$$

where $A = \sum_j c_j$.

Proof. The idea of the proof is the same as that of Theorems 6.1 and 6.2. It is enough to check the conditions of Theorem 4.3 (or Theorem 5. 3 when $\alpha = 2$) for $f_n^k(u) = g_n(t_k:u)$, which is defined in (6.6). Since (A1)$'$ is almost obvious, we need only to prove (A4)$'$ or (A4)$''$ according as $\alpha < 2$ or $= 2$. Since we can choose ε sufficiently small so that $b < \alpha-\varepsilon < \alpha+\varepsilon$, it is enough to show that , for every $p > b$,

$$\lim_{T \to \infty} \limsup_{n \to \infty} \int_{|u|>T} |g_n(t:u)|^p \, d\rho_n(u) = 0 , \tag{6.16}$$

When $p < 1$,

$$\int_{|u|>T} |g_n(t:u)|^p \, d\rho_n(u)$$

$$= \int_{|u|>T} \Big| \sum_{k=1-[nu]}^{[nt]-[nu]} c_k \Big|^p \, d\rho_n(u)$$

$$\leqq C \sum_{|j|>T} \sum_{k=1-[nj]}^{[nt]-[nj]} |c_k|^p$$

$$\leqq C \Big(\sum_{k>1-nT} + \sum_{k<nt-nT} \Big) |c_k|^p \sum_{j=[1/n-k/n]}^{[t-k/n]} 1$$

$$\leqq C\, t \Big(\sum_{k>1-nT} + \sum_{k<nt-nT} \Big) |c_k|^p \longrightarrow 0 ,$$

as $n \to \infty$ if $T > t$, which proves (6.16). When $p \geqq 1$,

$$\int_{|u|>T} |g_n(t:u)|^p \, d\rho_n(u)$$

$$\leqq C\, M^{p-1} \sum_{|j|>T} \sum_{k=1-[nj]}^{[nt]-[nj]} |c_k| \longrightarrow 0 ,$$

as $n \to \infty$ if $T > t$ because $\sum_j |c_j| < \infty$. (Here $M = \max_k |c_k|$.)

Thus (6.16) is proved. (q.e.d.)

Remark 2. Theorem 6.3 generalizes a result in [2], where only the one-dimendional distribution is considered with an additional restriction that $\alpha \neq 2$ and $c_j = 0$, $j \leqq 0$.

To conclude this section we remark the following simple result on self-similar processes with stationary increments.

Theorem 6.4. *Suppose that $X(t)$ is a self-similar process with parameter $H > 0$ having stationary increments. If $E|X(1)|^p < \infty$ for some $p > 1/H$, then $X(t)$ has a sample continuous version.*

Proof. Observe that

$$E[\,|X(t) - X(s)|^p\,] = E[\,|X(t-s)|^p\,]$$

$$= |t - s|^{pH}\, E[\,|X(1)|^p\,] = \text{const.}|t - s|^{pH} .$$

Since $pH > 1$ by asssumption, Kolmogorov's condition is satisfied.

(q.e.d.)

Notice that $E[|\Delta(1)|^q] < \infty$ if $0 < q < \alpha$ because the law of $\Delta(1)$ is the stable law of index α. Therefore, by the above theorem our $\Delta(t)$

has a sample continuous version when $\gamma > 0$ even though $Z_{(\alpha)}(t)$ does not unless $\alpha = 2$. This fact is reasonable in some sense because $\Delta(t)$ is some sort of the fractional integral or derivative of $Z_{(\alpha)}(t)$ according as $\gamma > 0$ or $\gamma < 0$. Hence, we can naturally expect that, when $\gamma > 0$, the process $\Delta(t)$ is more regular than $Z_{(\alpha)}(t)$. On the other hand, when $\gamma < 0$, there is an example where $\Delta(t)$ has completely irregular sample paths (see [9]).

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