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Tauberian theorems and Hausdorff-Young's inequality

by

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ABSTRACT

A generalized Marcinkiewicz space $M_\alpha^p(\mathbf{R})$ $(1 \le p < \infty, 0 < \alpha < \infty)$ is defined as the class of locally p-integrable functions f on a real field $\mathbf{R}$ such that

$$\|f\|_{M_\alpha^p} = \limsup_{T \to \infty} \left[\frac{1}{(2T)^\alpha} \int_{-T}^{T} |f(x)|^p dx\right]^{1/p} < \infty.$$

Also we define the space $V_\alpha^p(\mathbf{R})$ $(1 \le p < \infty, 0 < \alpha < \infty)$ as the class of tempered distributions f for which

$$\|f\|_{V_\alpha^p} = \limsup_{h \to 0+} \; h^{\alpha(1-1/p)/2} \left[\int_{-\infty}^{\infty} |W(.,h) * f|^p dx\right]^{1/p} < \infty,$$

where $W(.,h)$ is a Gauss-Weierstrass kernel. We study on the relation between Marcinkiewicz space $M_\alpha^p(\mathbf{R})$ and Tauberian theorems, and show that Fourier transform $\hat{}$ is a bounded linear operator from $M_\alpha^p(\mathbf{R})$ into $V_\alpha^q(\mathbf{R})$ and from $V_\alpha^p(\mathbf{R})$ into $M_\alpha^q(\mathbf{R})$ $(1 < p \le 2, \frac{1}{p} + \frac{1}{q} = 1)$ which is seemed to be an extension of Hausdorff-Young's inequality.

I.Introduction

Let $\mathbf{R}$ be a real field. Let p be a real number such that $1 \le p < \infty$. A Marcinkiewicz space $M^p(\mathbf{R})$ is defined in [3] as the set of locally p-integrable functions f on $\mathbf{R}$ such that

$$\|f\|_{M^p} = \limsup_{T \to \infty} \left[\frac{1}{2T} \int_{-T}^{T} |f(x)|^p dx\right]^{1/p} < \infty.$$

The space had been studied by many mathematicians in the theory of almost periodic functions (see, for example, [1],[2] and [3]). It is known ([1],[3]) that $M^p(\mathbf{R})$ is a Banach space with norm $\|.\|_{M^p}$

under the identification that $f=g$ in $M^p(\mathbf{R})$ if and only if $||f-g||_{M^p}=0$. Recently, Lau and Lee have studied in [2] on the relation between Marcinkiewicz space $M^p(\mathbf{R})$ and Tauberian theorems, and shown that $M^p(\mathbf{R})$ was characterized as the set of locally p-integrable functions f on $\mathbf{R}$ such that

$$||f||_{M_h^p}=\limsup_{T\to\infty}[\int_{-\infty}^{\infty}|f(Tx)|^p h(x)dx]^{1/p}<\infty,$$

where h is a non-negative bounded continuous function on $\mathbf{R}$ such that $\tilde{h}(x)\in L^1(\mathbf{R})$, $|x|\tilde{h}(x)\to 0$ $(|x|\to\infty)$ where $\tilde{h}(x)=\sup_{|t|>x} h(t)$ and that there exist positive numbers b,q such that $h(x)\geq b\chi_{[-q,q]}(x)$, where $\chi_{[a,b]}$ is a characteristic function on a closed set $[a,b]$.

In this paper, we shall consider the class $M_\alpha^p(\mathbf{R})$ ($1\leq p<\infty$, $0<\alpha<\infty$) of locally p-integrable functions f on $\mathbf{R}$ such that

$$||f||_{M_\alpha^p}=\limsup_{T\to\infty}[\frac{1}{(2T)^\alpha}\int_{-T}^{T}|f(x)|^p dx]^{1/p}<\infty.$$

We call the class a generalized Marcinkiewicz space.

In the section II, we show that $M_\alpha^p(\mathbf{R})$ is a Banach space with norm $||.||_{M_\alpha^p}$ under the identification that $f=g$ in $M_\alpha^p(\mathbf{R})$ if and only if $||f-g||_{M_\alpha^p}=0$.

In the section III, we study on the relation between a generalized Marcinkiewicz space $M_\alpha^p(\mathbf{R})$ and Tauberian theorems, and show another characterization of $M_\alpha^p(\mathbf{R})$, which is similar to the results obtained by Lau and Lee ([2]) in the case $\alpha=1$.

In the section IV, we define the space $V_\alpha^p(\mathbf{R})$ $(1\leq p<\infty, 0<\alpha<\infty)$ as the class of tempered distributions f for which

$$||f||_{V_\alpha^p}=\limsup_{h\to 0+} h^{\alpha(1-1/p)/2}[\int_{-\infty}^{\infty}|W(.,h)*f|^p dx]^{1/p}<\infty$$

where $W(.,h)$ is a Gauss-Weierstrass kernel. And we show that Fourier transform $\hat{}$ is a bounded linear operator from $M_\alpha^p(\mathbf{R})$ into $V_\alpha^q(\mathbf{R})$ and from $V_\alpha^p(\mathbf{R})$ into $M_\alpha^q(\mathbf{R})$ $(1<p\leq 2,\ \frac{1}{p}+\frac{1}{q}=1)$ which is seemed to be an extension of Hausdorff-Young's inequality.

II. The completeness of a generalized Marcinkiewicz space.

Let $M_\alpha^p(\mathbf{R})$ be the generalized Marcinkiewicz space defined as in the introduction. In this section, we prove the following;

Theorem 2.1. *A generalized Marcinkiewicz space* $M_\alpha^p(\mathbf{R})$ $(1\leq p<\infty, 0<\alpha<\infty)$ *is a Banach space with norm* $||.||_{M_\alpha^p}$ *under the identification that* $f=g$ *in* $M_\alpha^p(\mathbf{R})$ *if and only if* $||f-g||_{M_\alpha^p}=0$.

Proof. It is clear that $||.||_{M^p_\alpha}: M^p_\alpha(\mathbf{R}) \to [0,\infty)$ satisfies the conditions of norm under the identification that $f=0$ if and only if $||f||_{M^p_\alpha}=0$.

It suffices to prove the completeness of $M^p_\alpha(\mathbf{R})$.

Let $\{f_n\}$ be a Cauchy sequence in $M^p_\alpha(\mathbf{R})$. Let $\{\epsilon_n\}_{n=1}^\infty$ be a positive non-increasing sequence with the limit 0 such that

$$||f_n - f_{n+q}||^p_{M^p_\alpha} = \limsup_{T\to\infty}[\frac{1}{(2T)^\alpha}\int_{-T}^{T}|f_{n+q}(x)-f_n(x)|^p dx] < \epsilon_n \qquad \text{for all } n \text{ and } q . \qquad (1)$$

We shall prove that a function $f(x)$ can be founded in $M^p_\alpha(\mathbf{R})$ such that

$$||f-f_n||^p_{M^p_\alpha} = \limsup_{T\to\infty}[\frac{1}{(2T)^\alpha}\int_{-T}^{T}|f(x)-f_n(x)|^p dx] < 2\epsilon_n \qquad \text{for all } n,$$

that is, for any n, there exists a $T_n > 0$ such that

$$\frac{1}{(2T)^\alpha}\int_{|x|<T}|f(x)-f_n(x)|^p dx < 2\epsilon_n \qquad \text{for all } T > T_n. \qquad (2)$$

We can take a positive number T_1 such that

$$\frac{1}{(2T)^\alpha}\int_{|x|<T}|f_2(x)-f_1(x)|^p dx < \epsilon_1 \qquad \text{for all } T>T_1 \quad \text{(by (1))}.$$

Suppose we have chosen T_{n-1}. Select $T_n (> T_{n-1})$ satisfying the following conditions (i),(ii) and (iii);

(i) $$\frac{1}{(2T)^\alpha}\int_{|x|<T}|f_{n+1}(x)-f_i(x)|^p dx < \epsilon_i \quad \text{for all } T>T_n \text{ and } i=1,2,\dots n \qquad \text{(by(1))},$$

(ii) $$\frac{1}{(2T_n)^\alpha-(2T_{n-1})^\alpha}\int_{T_{n-1}<|x|<T_n}|f_n(x)-f_i(x)|^p dx < \epsilon_i \quad \text{for all } i=1,2,\dots,n-1 \qquad \text{(by(1))} ,$$

and

(iii) $$\frac{1}{(2T_n)^\alpha}[\int_{0<|x|<T_1}|f_1(x)-f_n(x)|^p dx + \int_{T_1<|x|<T_2}|f_2(x)-f_n(x)|^p dx + \dots + \int_{T_{n-2}<|x|<T_{n-1}}|f_{n-1}(x)-f_n(x)|^p dx] < \epsilon_n .$$

Define

$$f(x) = \begin{cases} f_1(x) & 0 \le |x| < T_1 \\ f_2(x) & T_1 \le |x| < T_2 \\ \dots\dots\dots\dots & \\ f_n(x) & T_{n-1} \le |x| < T_n \\ \dots\dots\dots\dots & \end{cases}.$$

Let n be any non-negative integer. Let T be any real number such that $T>T_n$. Then there exists a non-negative integer $m (\ge n)$ such that

$$T_m \le T < T_{m+1}.$$

Hence we see, by the conditions (iii),(ii) and (i), that

$$\begin{aligned}
&\int_{-T}^{T}|f(x)-f_n(x)|^p dx \\
&= \int_{0<|x|<T_1} |f_1(x)-f_n(x)|^p dx + \ldots\ldots + \int_{T_{n-2}<|x|<T_{n-1}} |f_{n-1}(x)-f_n(x)|^p dx \\
&\qquad + \int_{T_{n-1}<|x|<T_n} |f_n(x)-f_n(x)|^p dx \\
&\qquad + \int_{T_n<|x|<T_{n+1}} |f_{n+1}(x)-f_n(x)|^p dx + \ldots + \int_{T_{m-1}<|x|<T_m} |f_m(x)-f_n(x)|^p dx \\
&\qquad + \int_{T_m<|x|<T} |f_{m+1}(x)-f_n(x)|^p dx \\
&\le \epsilon_n(2T_n)^\alpha + 0 + \epsilon_n[((2T_{n+1})^\alpha-(2T_n)^\alpha)+((2T_{n+2})^\alpha-(2T_{n+1})^\alpha)+\ldots+((2T_m)^\alpha-(2T_{m-1})^\alpha)] \\
&\qquad\qquad + \int_{|x|<T} |f_{m+1}(x)-f_n(x)|^p dx \qquad \text{(by (iii) and (ii))} \\
&\le \epsilon_n(2T_n)^\alpha + \epsilon_n[-(2T_n)^\alpha+(2T_m)^\alpha] + \epsilon_n(2T)^\alpha \qquad \text{(by (i))} \\
&\le 2\epsilon_n(2T)^\alpha \qquad \text{for all } T\ge T_n,
\end{aligned}$$

which implies (2). This completes the proof.

III. Another characterization of $M_\alpha^p(\mathbf{R})$ and Tauberian Theorems

The following Lemmas immediately follow (see [2]).

Lemma 3.1. *Let f,g be measurable functions on $[a,b]$ such that f is integrable and g is of bounded variation. Then,*

$$\int_a^b f(x)g(x)dx = \left(\int_a^b f(x)dx\right)g(b) - \int_a^b\left(\int_a^x f(t)dt\right)d\eta_g$$

where η_g is the corresponding measure of g (i.e. for any $x\in(a,b]$ $\eta_g(a,x]=g(x+)-g(a+)$).

Let α be a real number such that $0<\alpha<\infty$. Let S_α^+ denote the class of positive Borel measurable functions f on $[0,\infty)$ such that $\sup_{T>0}\frac{1}{T^\alpha}\int_0^T f(x)dx\le 1$.

Lemma 3.2. *For $T,t>0$ and $f\in S_\alpha^+$, then*

$$\int_0^t f(Tx)dx \le t^\alpha T^{\alpha-1}.$$

Proposition 3.3. *Let α be a positive real number. Let k be a non-negative non-increasing function on $[0,\infty)$ such that $\int_0^\infty x^{\alpha-1}k(x)dx<\infty$ and $x^\alpha k(x)\to 0$ $(x\to\infty)$.*

Then

(i) $$\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)k(x)dx \le \alpha\int_0^\infty x^{\alpha-1}k(x)dx \qquad \text{for all } f\in S_\alpha^+,\ T\ge 1$$

and

(ii) $$\lim_{u\to\infty}\frac{1}{T^{\alpha-1}}\int_u^\infty f(Tx)k(x)dx=0 \qquad \text{uniformly for all } f\in S_\alpha^+,\ T\ge 1.$$

Proof. Since k is a non-increasing function, the corresponding measure η_k is negative. Hence, we see, by Lemmas 3.1 and 3.2, that, for $0<u<v<\infty$, $f\in S_\alpha^+$,

$$\begin{aligned}
&\frac{1}{T^{\alpha-1}}\int_u^v f(Tx)k(x)dx\\
&=\frac{1}{T^{\alpha-1}}\int_u^v f(Tx)dx\,k(v)-\frac{1}{T^{\alpha-1}}\int_u^v\left(\int_u^x f(Tt)dt\right)d\eta_k\\
&\le\frac{1}{T^{\alpha-1}}\int_0^v f(Tx)dx\,k(v)-\frac{1}{T^{\alpha-1}}\int_u^v\left(\int_0^x f(Tt)dt\right)d\eta_k\\
&\le v^\alpha k(v)-\int_u^v x^\alpha d\eta_k\\
&\le u^\alpha k(u)+\alpha\int_u^v x^{\alpha-1}k(x)dx.
\end{aligned} \tag{1}$$

Letting $u\to 0+$ and $v\to\infty$ in (1), we obtain (i), and letting $u\to\infty$ and $v\to\infty$ $(u<v)$ in (1), we obtain (ii).

Lemma 3.4. *Let α be a positive real number. Let h be a non-negative bounded measurable function on $[0,\infty)$ such that $x^{\alpha-1}\tilde{h}(x)\in L^1[0,\infty)$ and $x^\alpha\tilde{h}(x)\to 0(x\to\infty)$, where $\tilde{h}(x)=\operatorname{ess.sup}_{t\ge x}h(t)$.*

Let $a_1,b_1,a_2,b_2,\ldots,a_k$ and b_k be real numbers such that $0\le a_1<b_1<a_2<\ldots<a_k<b_k<\infty$, and

$$\limsup_{\substack{\delta^+\to 0+\\ \delta^-\to 0+}}\frac{1}{\delta^-+\delta^+}\int_{b_i-\delta^-}^{b_i-\delta^+}h(t)dt\ge h(b_i) \quad \text{for all } i=1,2,\ldots,k,$$

and let

$$\eta=\sum_{i=1}^k\left(h(b_i)(b_i^\alpha-a_i^\alpha)-\alpha\int_{a_i}^{b_i}x^{\alpha-1}h(x)dx\right).$$

Then,

$$\sup_{f\in S_\alpha^+}\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx\ge\alpha\int_0^\infty x^{\alpha-1}h(x)dx+\eta.$$

Proof. It suffices to show that for any positive number ϵ, there exists an $f\in S_\alpha^+$ such that

$$\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx\ge\alpha\int_0^\infty x^{\alpha-1}h(x)dx+\eta-\epsilon. \tag{1}$$

When $a_1=0$, if we can take $a'_1>0$ such that

$$h(b_1)(a'_1)^\alpha<\frac{\epsilon}{4} \quad\text{and}\quad \alpha\int_0^{a'_1}x^{\alpha-1}\tilde{h}(x)dx<\frac{\epsilon}{4}$$

and if we can prove that there exists an $f \in S_\alpha^+$ such that

$$\limsup_{T\to\infty} \frac{1}{T^{\alpha-1}} \int_0^\infty f(Tx)h(x)dx \geq \alpha \int_0^\infty x^{\alpha-1}h(x)dx + \eta' - \epsilon/2$$

where

$$\eta' = (h(b_1)(b_1^\alpha - a'^\alpha_1) - \alpha\int_{a'_1}^{b_1} x^{\alpha-1}h(x)dx) + \sum_{i=2}^{k} (h(b_i)(b_i^\alpha - a_i^\alpha) - \alpha\int_{a_i}^{b_i} x^{\alpha-1}h(x)dx).$$

Then, (1) easily follows. Hence, without loss of generality, we can assume that $a_1 > 0$.

Let ϵ be any positive number. We can take positive numbers δ_i^- and δ_i^+ $(i=1,2,\ldots,k)$ such that

$$a_i < b_i - \delta_i^- < b_i + \delta_i^+ < a_{i+1} < b_{i+1} - \delta_{i+1}^- < b_{i+1} \quad (i=1,2,\ldots,k-1)$$

and

$$\sum_{i=1}^{k} \frac{(b_i+\delta_i^+)^\alpha - a_i^\alpha}{\delta_i^- + \delta_i^+} \int_{b_i-\delta_i^-}^{b_i+\delta_i^+} h(x)dx \geq \sum_{i=1}^{k} \alpha \int_{a_i}^{b_i+\delta_i^+} x^{\alpha-1}h(x)dx + \eta - \epsilon/3, \tag{2}$$

since

$$\limsup_{\substack{\delta_i^+\to 0+ \\ \delta_i^-\to 0+}} \frac{(b_i+\delta_i^+)^\alpha - a_i^\alpha}{\delta_i^- + \delta_i^+} \int_{b_i-\delta_i^-}^{b_i+\delta_i^+} h(x)dx \geq (b_i^\alpha - a_i^\alpha)h(b_i) \qquad (i=1,2,\ldots,k)$$

and

$$\lim_{\delta_i^+\to 0+} \alpha \int_{a_i}^{b_i+\delta_i^+} x^{\alpha-1}h(x)dx = \alpha\int_{a_i}^{b_i} x^{\alpha-1}h(x)dx \qquad (i=1,2,\ldots,k).$$

Since $\bar{h}$ satisfies the conditions of k in Proposition 3.3, we can take a large positive number C such that

$$\alpha\int_C^\infty x^{\alpha-1}h(x)dx < \frac{\epsilon}{6} \tag{3}$$

and

$$\frac{1}{T^{\alpha-1}}\int_C^\infty p(Tx)h(x)dx \leq \frac{1}{T^{\alpha-1}}\int_C^\infty p(Tx)\bar{h}(x)dx \leq \epsilon/6 \tag{4}$$

for all $p \in S_\alpha^+$, $T>0$.

Let $T_1 > 0$ and let

$$f_1(x) = \begin{cases} \alpha x^{\alpha-1} & 0 \leq x < CT_1 \\ 0 & CT_1 \leq x \end{cases}.$$

Suppose we have chosen T_{n-1}, f_{n-1}.

Select T_n such that

$$CT_{n-1} < a_1 T_n, \quad \frac{MC^\alpha}{T_n^\alpha} \sum_{i=1}^{n-1} T_i^\alpha < \epsilon/6 \quad \text{and} \quad \alpha\int_0^{\frac{CT_{n-1}}{T_n}} x^{\alpha-1}h(x)dx < \epsilon/6 \tag{5}$$

where $M=\mathrm{ess.}\sup_{t\geq 0}|h(t)|$.

And define

$$f_n(x)=\begin{cases} 0 & 0\leq x<CT_{n-1} \\ \alpha x^{\alpha-1} & CT_{n-1}\leq x<a_1T_n \\ 0 & a_1T_n\leq x<(b_1-\delta_1^-)T_n \\ \dfrac{(b_1+\delta_1^+)^\alpha-a_1^\alpha}{\delta_1^-+\delta_1^+} & (b_1-\delta_1^-)T_n\leq x<(b_1+\delta_1^+)T_n \\ \alpha x^{\alpha-1} & (b_1+\delta_1^+)T_n\leq x<a_2T_n \\ 0 & a_2T_n\leq x<(b_2-\delta_2^-)T_n \\ \cdots\cdots & \\ \cdots\cdots & \\ 0 & a_iT_n\leq x<(b_i-\delta_i^-)T_n \\ \dfrac{(b_i+\delta_i^+)^\alpha-a_i^\alpha}{\delta_i^-+\delta_i^+} & (b_i-\delta_i^-)T_n\leq x<(b_i+\delta_i^+)T_n \\ \alpha x^{\alpha-1} & (b_i+\delta_i^+)T_n\leq x<a_{i+1}T_n \\ 0 & a_{i+1}T_n\leq x<b_{i+1}T_n \\ \cdots\cdots & \\ \cdots\cdots & \\ 0 & a_kT_n\leq x<(b_k-\delta_k^-)T_n \\ \dfrac{(b_k+\delta_k^+)^\alpha-a_k^\alpha}{\delta_k^-+\delta_k^+} & (b_k-\delta_k^-)T_n\leq x<(b_k+\delta_k^+)T_n \\ \alpha x^{\alpha-1} & (b_k+\delta_k^+)T_n\leq x<CT_n \\ 0 & CT_n\leq x \ . \end{cases}$$

Note that the functions $\{f_n\}$ have disjoint supports and supp $[f_n]\subset[CT_{n-1},CT_n]$. Let $f=\sum_{n=1}^{\infty}f_n$.

It is easy to show that

$$\frac{1}{T^\alpha}\int_0^T f(x)dx=1 \quad \text{for } T\in(0,\infty)\ -\ \bigcup_{n=2}^{\infty}\ \bigcup_{i=1}^{k}(a_iT_n,(b_i+\delta_i^+)T_n)$$

and

$$\frac{1}{T^\alpha}\int_0^T f(x)dx\leq 1 \quad \text{for } T\in\bigcup_{n=2}^{\infty}\ \bigcup_{i=1}^{k}(a_iT_n,(b_i+\delta_i^+)T_n).$$

Hence $f\in S_\alpha^+$.

We see that, for $n>1$,

$$\begin{aligned} &\left|\frac{1}{T_n^{\alpha-1}}\int_0^\infty f(T_nx)h(x)dx-\frac{1}{T_n^{\alpha-1}}\int_0^C f_n(T_nx)h(x)dx\right| \\ &\leq\left|\frac{1}{T_n^{\alpha-1}}\int_0^C f(T_nx)h(x)dx-\frac{1}{T_n^{\alpha-1}}\int_0^C f_n(T_nx)h(x)dx\right|+\epsilon/6 \qquad \text{(by (4))} \\ &\leq\sum_{i=1}^{n-1}\frac{1}{T_n^{\alpha-1}}\int_0^C f_i(T_nx)h(x)dx+\sum_{i=n+1}^{\infty}\frac{1}{T_n^{\alpha-1}}\int_0^C f_i(T_nx)h(x)dx+\epsilon/6 \end{aligned}$$

$$=\sum_{i=1}^{n-1}\frac{1}{T_n^{\alpha-1}}\int_0^C f_i(T_n x)h(x)dx+\epsilon/6$$
$$=\sum_{i=1}^{n-1}\frac{1}{T_n^{\alpha}}\int_0^{CT_n} f_i(x)h(\frac{x}{T_n})dx+\epsilon/6$$
$$\le\sum_{i=1}^{n-1}\frac{M}{T_n^{\alpha}}\int_0^{CT_i} f_i(x)dx+\epsilon/6$$
$$\le\frac{MC^{\alpha}}{T_n^{\alpha}}\sum_{i=1}^{n-1}T_i^{\alpha}+\epsilon/6 \qquad \text{(by Lemma 3.2)}$$
$$\le\epsilon/3 \qquad \text{(by(5)).} \tag{6}$$

And we see that, for $n>1$,

$$\frac{1}{T_n^{\alpha-1}}\int_0^C f_n(T_n x)h(x)dx$$
$$=\frac{1}{T_n^{\alpha-1}}\int_{CT_{n-1}/T_n}^{a_1}\alpha(T_n x)^{\alpha-1}h(x)dx+\frac{1}{T_n^{\alpha-1}}\int_{b_1-\delta_1^-}^{b_1+\delta_1^+}\frac{(b_1+\delta_i^+)^{\alpha}-a_1^{\alpha}}{\delta_1^-+\delta_1^+}T_n^{\alpha-1}h(x)dx$$
$$+\frac{1}{T_n^{\alpha-1}}\int_{b_1+\delta_1^+}^{a_2}\alpha(T_n x)^{\alpha-1}h(x)dx+\frac{1}{T_n^{\alpha-1}}\int_{b_2-\delta_2^-}^{b_2+\delta_2^+}\frac{(b_2+\delta_i^+)^{\alpha}-a_2^{\alpha}}{\delta_2^-+\delta_2^+}T_n^{\alpha-1}h(x)dx$$

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$$+\frac{1}{T_n^{\alpha-1}}\int_{b_{i-1}+\delta_{i-1}^+}^{a_i}\alpha(T_n x)^{\alpha-1}h(x)dx+\frac{1}{T_n^{\alpha-1}}\int_{b_i-\delta_i^-}^{b_i+\delta_i^+}\frac{(b_i+\delta_i^+)^{\alpha}-a_i^{\alpha}}{\delta_i^-+\delta_i^+}T_n^{\alpha-1}h(x)dx$$

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$$+\frac{1}{T_n^{\alpha-1}}\int_{b_{k-1}+\delta_{k-1}^+}^{a_k}\alpha(T_n x)^{\alpha-1}h(x)dx+\frac{1}{T_n^{\alpha-1}}\int_{b_k-\delta_k^-}^{b_k+\delta_k^+}\frac{(b_k+\delta_i^+)^{\alpha}-a_i^{\alpha}}{\delta_k^-+\delta_k^+}T_n^{\alpha-1}h(x)dx$$
$$+\frac{1}{T_n^{\alpha-1}}\int_{b_k+\delta_k^+}^{C}\alpha(T_n x)^{\alpha-1}h(x)dx$$
$$\ge\int_{CT_{n-1}/T_n}^{a_1}\alpha x^{\alpha-1}h(x)dx$$
$$+\sum_{i=2}^{k}\int_{b_{i-1}+\delta_{i-1}^+}^{a_i}\alpha x^{\alpha-1}h(x)dx+[\sum_{i=1}^{k}\alpha\int_{a_i}^{b_i+\delta_i^+}x^{\alpha-1}h(x)dx+\eta-\epsilon/3]$$
$$+\int_{b_k+\delta_k^+}^{C}\alpha x^{\alpha-1}h(x)dx \qquad \text{(by (2))}$$
$$\ge\int_{CT_{n-1}/T_n}^{C}\alpha x^{\alpha-1}h(x)dx+\eta-\epsilon/3$$
$$\ge\int_{\infty}^{\infty}\alpha x^{\alpha-1}h(x)dx+\eta-2\epsilon/3 \qquad \text{(by(3)and(5)).}$$

From this and (6), theorem follows.

Lemma 3.5. *Let α be a positive real number. Let h be a non-negative bounded measurable function on $[0,\infty)$ such that $x^{\alpha-1}\bar{h}(x)\in L^1[0,\infty)$ and $x^{\alpha}\bar{h}(x)\to 0(x\to\infty)$, where $\bar{h}(x)=\operatorname{ess.sup}_{t\geq x} h(t)$.*

Then

$$\sup_{f\in S_\alpha^+}\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx=\alpha\int_0^\infty x^{\alpha-1}\bar{h}(x)dx.$$

Proof. Since $\bar{h}$ satisfies the conditions of k in Proposition 3.3, we see that

$$\sup_{f\in S_\alpha^+}\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx\leq \sup_{f\in S_\alpha^+}\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)\bar{h}(x)dx\leq\alpha\int_0^\infty x^{\alpha-1}\bar{h}(x)dx$$

We will prove the reverse inequality.

Let $A=\cup\{\ I\ :\ I$ *is an open interval in* $\mathbf{R}$ *such that* $\bar{h}(x)$ *is constant in* $I\}$.

Since A is an open set in $\mathbf{R}$, A is the union of a countable (or finite) sequence of disjoint open intervals $\{(a_i,c_i)\}$.

Since $\bar{h}$ is right continuous, we see that $h(x)=\bar{h}(x)$ a.e. in $\mathbf{R}-A$ and

$$\alpha\int_A x^{\alpha-1}\bar{h}(x)dx=\sum_{i=1}^{\infty}\bar{h}(c_i)(c_i^\alpha-a_i^\alpha).$$

Then we see that

$$\begin{aligned}\alpha\int_0^\infty x^{\alpha-1}\bar{h}(x)dx&=\alpha\int_{\mathbf{R}-A}+\alpha\int_A\\ &=\alpha\int_{\mathbf{R}-A}x^{\alpha-1}h(x)dx+\sum_{i=1}^{\infty}\bar{h}(c_i)(c_i^\alpha-a_i^\alpha)\\ &=\alpha\int_0^\infty x^{\alpha-1}h(x)dx+\sum_{i=1}^{\infty}(\bar{h}(c_i)(c_i^\alpha-a_i^\alpha)-\alpha\int_{a_i}^{c_i}x^{\alpha-1}h(x)dx).\end{aligned}$$

Let ϵ be any positive number. Since there exist Lebegue points in $\mathbf{R}$ almost everywhere, we see that

$$\limsup_{\substack{\delta^+\to 0+\\ \delta^-\to 0+}}\frac{1}{\delta^-+\delta^+}\int_{x-\delta^-}^{x-\delta^+}h(t)dt\geq h(x)\quad\text{for almost all } x\in(0,\infty).$$

Then, we can take a sequence $\{b_i\}_{i=1}^{\infty}$ such that

$$a_i<b_i<c_i\quad\text{and}\quad\limsup_{\substack{\delta^+\to 0+\\ \delta^-\to 0+}}\frac{1}{\delta^-+\delta^+}\int_{b_i-\delta^-}^{b_i-\delta^+}h(t)dt\geq h(b_i)\quad\text{for all } i=1,2,\ldots$$

and

$$\alpha\int_0^\infty x^{\alpha-1}h(x)dx+\sum_{i=1}^{\infty}(\bar{h}(b_i)(b_i^\alpha-a_i^\alpha)-\alpha\int_{a_i}^{b_i}x^{\alpha-1}h(x)dx)>\alpha\int_0^\infty x^{\alpha-1}\bar{h}(x)dx-\epsilon/2.$$

From this, we can choose a positive integer $k>0$ such that

$$\alpha\int_0^\infty x^{\alpha-1}h(x)dx+\sum_{i=1}^{k}(\bar{h}(b_i)(b_i^\alpha-a_i^\alpha)-\alpha\int_{a_i}^{b_i}x^{\alpha-1}h(x)dx)>\alpha\int_0^\infty x^{\alpha-1}\bar{h}dx-\epsilon$$

which implies, from Lemma 3.4, that

$$\sup_{f\in S_\alpha^+}\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx\geq\alpha\int_0^\infty x^{\alpha-1}\bar{h}(x)dx-\epsilon.$$

Since ϵ is arbitrary positive real, this completes the proof.

Let h be a non-negative bounded measurable function on $[0,\infty)$. $M_{h,\alpha}^+$ denotes the set of locally integrable functions f on $[0,\infty)$ such that

$$\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx<\infty.$$

$h(x)=1$ $(x\in[0,1))$, 0 $(x\in[1,\infty))$, $M_{h,\alpha}^+$ is also written by M_α^+ .

Note that $f\in M_\alpha^+$ if and only if

$$\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx<\infty.$$

Theorem 3.6. *Let α be a positive real number. Let h be a non-negative bounded measurable function on $[0,\infty)$ such that $x^{\alpha-1}\bar{h}(x)\in L^1[0,\infty)$ and $x^\alpha\bar{h}(x)\to0(x\to\infty)$, where $\bar{h}(x)=\operatorname{ess.sup}_{t\geq x}h(t)$.*

Then

$$\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx\leq C_1\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx \qquad \text{for all } f \text{ in } M_\alpha^+ ,$$

where $C_1=\alpha\int_0^\infty x^{\alpha-1}\bar{h}(x)dx$.

Moreover C_1 is the best estimation of the inequality for the class of function f in M_α^+.

Proof. Let f be any function in M_α^+ and let f_r be defined as

$$f_r(x)=\begin{cases}0 & 0\leq x<r\\ f(x) & r\leq x\end{cases}.$$

Then it is clear that

$$\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx=\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f_r(Tx)h(x)dx$$

and

$$(\sup_{T>r}\frac{1}{T^\alpha}\int_0^T f(x)dx)f_r\in S_\alpha^+.$$

Applying Lemma 3.5 with $(\sup_{T>r}\frac{1}{T^\alpha}\int_0^T f(x)dx)f_r$, we get

$$\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f_r(Tx)h(x)dx \le C_1(\sup_{T>r}\frac{1}{T^\alpha}\int_0^T f(x)dx)f_r$$

which implies that

$$\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx \le C_1\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx.$$

Also the last assertion is trivial by Lemma 3.5 and the fact that $f\in S_\alpha^+$ implies $\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx \le 1$.

This completes the proof.

Theorem 3.7. *Let α be a positive real number. Let h be a non-negative bounded measurable function on $[0,\infty)$. If $0<C_2=\sup\{q^\alpha b : h(x)\ge b\chi_{[0,q]}(x)$ a.e.$\}<\infty$, then*

$$\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx \le 1/C_2\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx \qquad \text{for all } f\in M_{h,\alpha}^+ .$$

Moreover, suppose $x^{\alpha-1}\bar{h}(x)\in L^1[0,\infty)$ and $x^\alpha\bar{h}(x)\to 0(x\to\infty)$. If $C_2=\sup_{x\in[0,\infty)}\{x^\alpha\bar{h}(x)\} = x_0^\alpha\bar{h}(x_0)$ for some $x_0\in[0,\infty)$, C_2 is the best estimation of the inequality for the class of function f in $M_{h,\alpha}^+$, where $\bar{h}(x)=\lim_{\delta\to 0+}\operatorname{ess.sup}_{|x-t|<\delta} h(t)$.

Proof. Let f be any function in $M_{h,\alpha}^+$.

We see that, for any $q,b>0$ such that $h(x)\ge b\chi_{[0,q]}(x)$a.e.,

$$\begin{aligned}\frac{1}{T^\alpha}\int_0^T f(x)dx &= \frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)\chi_{[0,1]}(x)dx\\ &= \frac{1}{bT^{\alpha-1}}\int_0^\infty f(\frac{T}{q}x)b\chi_{[0,q]}(x)dx\\ &\le \frac{1}{bq^\alpha}(\frac{q}{T})^{\alpha-1}\int_0^\infty f(\frac{T}{q}x)h(x)dx.\end{aligned}$$

By taking limit supremun on both sides, we see that

$$\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx \le \frac{1}{q^\alpha b}\limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx \qquad \text{for all } f\in M_{h,\alpha}^+ .$$

And, by taking supremun under the condition that $h(x)\ge b\chi_{[0,q]}(x)$ a.e., the first part of theorem follows.

Next we shall prove the second part. Let ϵ be any positive real number. It suffices to construct a non-negative function f such that

$$\limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx=1 \quad\text{and}\quad \limsup_{T\to\infty}\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx \le C_2+\epsilon.$$

By Proposition 3.3, we can choose $C>x_0$ such that

$$\frac{1}{T^{\alpha-1}}\int_C^\infty p(Tx)\bar{h}(x)dx \leq \epsilon/4 \qquad (p \in S_\alpha^+,\ T>0). \tag{1}$$

Let l be any positive number.

We shall show that there exists a small $\delta>0$ such that

$$\frac{1}{2\delta}\int_{\beta(x_0-\delta)}^{\beta(x_0+\delta)} h(x)dx \leq \beta\bar{h}(\beta x_0) + \frac{\epsilon}{4x_0^\alpha l^{\alpha-1}} \quad \text{for all } 0<\beta<l. \tag{2}$$

For any $x \in [0,\infty)$, there exists a $\delta_x>0$ such that

$$\operatorname*{ess.sup}_{|t-x|<\delta_x} h(t) \leq \bar{h}(x) + \frac{\epsilon}{4l^\alpha x_0^\alpha}.$$

Since $[0,x_0 l]$ is compact, there exist positive numbers $x_1,x_2,\dots,x_n$ such that

$$[0,x_0 l] \subset \bigcup_{k=1}^{n} \{y: |y-x_k|<\delta_{x_k}\}. \tag{3}$$

Define

$$\delta_0 = \frac{1}{2}\inf_{x\in[0,x_0 l]}\{\sup\{\eta; (x-\eta, x+\eta) \subset \{y: |y-x_k|<\delta_{x_k}\} \text{ for some } k\}\}.$$

From (3), it is clear that $\delta_0>0$.

Put $\delta = \frac{\delta_0}{l}$. Hence, we see that, for any $x \in [0,x_0 l]$,

$$\operatorname*{ess.sup}_{|t-x|<\delta_0} h(t) \leq \bar{h}(x) + \frac{\epsilon}{4l^\alpha x_0^\alpha}.$$

Then, from this, we see that

$$\begin{aligned}
\frac{1}{2\delta}\int_{\beta(x_0-\delta)}^{\beta(x_0+\delta)} h(x)dx &= \frac{l}{2\delta_0}\int_{\beta x_0-\frac{\beta}{l}\delta_0}^{\beta x_0+\frac{\beta}{l}\delta_0} h(x)dx \\
&\leq \frac{l}{2\delta_0}\frac{2\beta\delta_0}{l}\operatorname*{ess.sup}_{|\beta x_0-t|<\beta\delta_0/l} h(t) \\
&\leq \beta\operatorname*{ess.sup}_{|\beta x_0-t|<\delta_0} h(t) \\
&\leq \beta\bar{h}(\beta x_0) + \frac{\epsilon\beta}{4l^\alpha x_0^\alpha} \\
&\leq \beta\bar{h}(\beta x_0) + \frac{\epsilon}{4l^{\alpha-1}x_0^\alpha}.
\end{aligned}$$

Hence (2) holds. Put $l = \frac{C}{x_0}$. We can, by (2), take $0<\delta<\min\{x_0, C-x_0\}$ such that

$$\frac{\beta^{\alpha-1}(x_0+\delta)^\alpha}{2\delta}\int_{\beta(x_0-\delta)}^{\beta(x_0+\delta)} h(x)dx \leq \beta^\alpha(x_0+\delta)^\alpha\bar{h}(\beta x_0) + \frac{\epsilon\beta^{\alpha-1}(x_0+\delta)^\alpha}{4l^{\alpha-1}x_0^\alpha}$$

$$\leq \beta^\alpha x_0^\alpha \bar{h}(\beta x_0) + \beta^\alpha((x_0+\delta)^\alpha - x_0^\alpha)\dot{h}(\beta x_0) + \frac{\epsilon\beta^{\alpha-1}(x_0+\delta)^\alpha x_0^{\alpha-1}}{4C^{\alpha-1}x_0^\alpha}$$

$$\leq x_0{}^\alpha \bar{h}(x_0) + \frac{\epsilon}{2} \qquad (0 \leq \beta < C/x_0) \tag{4}$$

since $\lim_{\delta\to 0+} \beta^\alpha((x_0+\delta)^\alpha - x_0^\alpha)\dot{h}(\beta x_0) = 0$ uniformly for all $0 \leq \beta < C/x_0$.

Let $T_1 > 1$, and select $T_n \geq T_{n-1}$, so that

$$T_{n-1}/T_n < x_0/C, \qquad \frac{1}{T_n^\alpha}\sum_{i=1}^{n-1} T_i^\alpha < 1/n \tag{5}$$

and

$$\frac{(x_0+\delta)^\alpha}{2\delta}\frac{C^{\alpha-1}}{x_0^{\alpha-1}}\sum_{i=1}^{n-1}\int_0^{C(x_0+\delta)T_i/x_0T_n} h(x)dx < \frac{\epsilon}{4} \ . \tag{6}$$

et

$$f_n = \frac{(x_0+\delta)^\alpha T_n{}^{\alpha-1}}{2\delta}\chi_{[(x_0-\delta)T_n,(x_0+\delta)T_n]} \ .$$

Note that the function $\{f_n\}$ have disjoint supports. Let $f = \sum_{n=1}^{\infty} f_n$.

We see, for any n,

$$\frac{1}{(x_0+\delta)^\alpha T_n^\alpha}\int_0^{(x_0+\delta)T_n} f(x)dx$$

$$= \frac{1}{(x_0+\delta)^\alpha T_n^\alpha}\left[\frac{(x_0+\delta)^\alpha T_n{}^{\alpha-1}}{2\delta}2\delta T_n + \sum_{i=1}^{n-1}\frac{(x_0+\delta)^\alpha T_i{}^{\alpha-1}}{2\delta}2\delta T_i\right] \leq 1 + \frac{1}{T_n^\alpha}\sum_{i=1}^{n-1}T_i^\alpha \leq 1 + 1/n \quad \text{(by (5))}.$$

Since $\frac{1}{T^\alpha}\int_0^T f(x)dx$ has a local maximun at each $(x_0+\delta)T_n$, we see

$$\sup_{T>1}\frac{1}{T^\alpha}\int_0^T f(x)dx \leq 2 \quad \text{and} \quad \limsup_{T\to\infty}\frac{1}{T^\alpha}\int_0^T f(x)dx = 1.$$

Now, for any $T > 1$, there exists an n such that $x_0 T_n \leq CT < x_0 T_{n+1}$.

Since $0 < \frac{T_n}{T} < C/x_0$, we see that

$$\frac{1}{T^{\alpha-1}}\int_0^\infty f(Tx)h(x)dx \leq \frac{1}{T^{\alpha-1}}\int_0^C f(Tx)h(x)dx + \epsilon/4 \qquad \text{(by (1))}$$

$$= \sum_{i=1}^{n}\frac{1}{T^{\alpha-1}}\int_0^C f_i(Tx)h(x)dx + \epsilon/4$$

$$= \sum_{i=1}^{n}\frac{1}{T^{\alpha-1}}\int_{(x_0-\delta)T_i/T}^{(x_0+\delta)T_i/T}\frac{(x_0+\delta)^\alpha T_n{}^{\alpha-1}}{2\delta}h(x)dx + \epsilon/4$$

$$= \frac{(x_0+\delta)^\alpha T_n^{\alpha-1}}{2\delta T^{\alpha-1}}\int_{(x_0-\delta)T_n/T}^{(x_0+\delta)T_n/T} h(x)dx + \frac{(x_0+\delta)^\alpha T_n^{\alpha-1}}{2\delta T^{\alpha-1}}\sum_{i=1}^{n-1}\int_{(x_0-\delta)T_i/T}^{(x_0+\delta)T_i/T} h(x)dx + \epsilon/4$$

$$= x_0^\alpha \bar{h}(x_0) + \frac{(x_0+\delta)^\alpha}{2\delta}\frac{C^{\alpha-1}}{x_0^{\alpha-1}}\sum_{i=1}^{n-1}\int_0^{C(x_0+\delta)T_i/x_0T_n} h(x)dx + 3\epsilon/4 \qquad \text{(by(4))}.$$

$$\leq x_0^{\alpha}\bar{h}(x_0)+\epsilon=C_2+\epsilon \qquad \text{(by(6))}.$$

This completes the proof .

Let h be a non-negative bounded measurable function on **R** . $M^p_{h,\alpha}(\mathbf{R})$ denotes the set of locally p-integrable ($1\leq p<\infty$) functions f on **R** such that

$$\|f\|_{M^p_{h,\alpha}}=\limsup_{T\to\infty}\left[\frac{1}{(2T)^{\alpha-1}}\int_{-\infty}^{\infty}|f(Tx)|^p h(x)dx\right]^{1/p}<\infty.$$

Corollary 3.8. *Let α be a positive real number. Let h be a non-negative bounded measurable function on* **R** *such that $|x|^{\alpha-1}\bar{h}(x)\in L^1(\mathbf{R})$, $|x|^{\alpha}\bar{h}(x)\to 0$ ($|x|\to\infty$) where $\bar{h}(x)=\operatorname{ess.sup}_{|t|>|x|}h(t)$. And suppose that there exist positive numbers b,q such that $h(x)\geq b\chi_{[-q,q]}(x)$* a.e. .

hen, $M^p_{h,\alpha}(\mathbf{R})=M^p_{\alpha}(\mathbf{R})$ and there exist positive numbers C_1 and C_2 ($C_1\leq C_2$) such that

$$C_1\|f\|_{M^p_{h,\alpha}}\leq\|f\|_{M^p_{\alpha}}\leq C_2\|f\|_{M^p_{h,\alpha}} \qquad \text{for all } f\in M^p_{\alpha}(\mathbf{R})$$

that is, $M^p_{h,\alpha}(\mathbf{R})$ and $M^p_{\alpha}(\mathbf{R})$ are equivalent as the Banach space.

IV. Space $V^p_{\alpha}(\mathbf{R})$ and Hausdorff-Young's inequality

Definition 4.1. *Let α be a non-negative real and let p be $1\leq p<\infty$. A subspace $V^p_{\alpha}(\mathbf{R})$ of a space S' of tempered distributions is defined as the class of $f\in S'$ for which*

(i) *$W(.,h)*f$ is a measurable function, where* $W(t,h)=\dfrac{1}{(4\pi h)^{1/2}}e^{-t^2/4h}$,

furthermore

(ii) $\|f\|_{V^p_{\alpha}}=\limsup_{h\to 0+} h^{\alpha(1-1/p)/2}\left[\int_{-\infty}^{\infty}|W(.,h)*f|^p dx\right]^{1/p}<\infty.$

It is clear that $V^p_{\alpha}(\mathbf{R})$ is a normed space with norm $\|.\|_{V^p_{\alpha}}$ under the identification that $\|f-g\|_{V^p_{\alpha}}=0$ implies $f=g$ in $V^p_{\alpha}(\mathbf{R})$.

The Fourier transform $\hat{\phi}$ of a function ϕ in the space S of rapidly decreasing functions is defined by

$$\hat{\phi}(t)=\int_{-\infty}^{\infty}\phi(x)e^{-ixt}dx$$

Since the mapping $\phi\to\hat{\phi}$ of S onto S is linear continuous in the topology of S, the Fourier transform $\hat{u}$ of a tempered distribution u defined through

$$\langle\hat{u},\phi\rangle=\langle u,\hat{\phi}\rangle \qquad (\phi\in S).$$

Theorem 4.2. *Let* $1<p\le 2$ *and let* q *be the conjugate exponent, that is,* $\frac{1}{p}+\frac{1}{q}=1$. *Then the Fourier transform* $\hat{}$ *is a bounded linear operator from* $M_\alpha^p(\mathbf{R})$ *into* $V_\alpha^q(\mathbf{R})$.

Proof. Let f be any function in $M_\alpha^p(\mathbf{R})$. Since e^{-pt^2} satisfies the conditions of h in Corollary 3.8, we see, by Hausdorff-Young's inequality in $L^p(\mathbf{R})$ and Corollary 3.8, that

$$\begin{aligned}
\|\hat{f}\|_{V_\alpha^q} &= \limsup_{h\to 0+} h^{\alpha(1-1/q)/2}\Big[\int_{-\infty}^{\infty}|W(.,h)*\hat{f}|^q dx\Big]^{1/q} \\
&= \limsup_{h\to 0+} h^{\alpha(1-1/q)/2}\Big[\int_{-\infty}^{\infty}|\{e^{-ht^2}f(t)\}^{\wedge}|^q dx\Big]^{1/q} \\
&\le C \limsup_{h\to 0+} h^{\alpha/2p}\Big[\int_{-\infty}^{\infty}e^{-hpx^2}|f(x)|^p dx\Big]^{1/p} \\
&= C \limsup_{h\to 0+} \Big[h^{(\alpha-1)/2}\int_{-\infty}^{\infty}e^{-py^2}|f(y/h^{1/2})|^p dy\Big]^{1/p} \\
&\le C'\|f\|_{M_\alpha^p}.
\end{aligned}$$

This completes the proof.

Theorem 4.3. *Let* $1<p\le 2$ *and let* q *be the conjugate exponent, that is,* $\frac{1}{q}+\frac{1}{p}=1$. *Then the Fourier transform* $\hat{}$ *is a bounded linear operator from* $V_\alpha^p(\mathbf{R})$ *into* $M_\alpha^q(\mathbf{R})$.

Proof. Let f be any function in $M_\alpha^q(\mathbf{R})$.

We see, by Corollary 3.8 and Hausdorff-Young's inequality in $L^p(\mathbf{R})$, that that

$$\begin{aligned}
\|\hat{f}\|_{M_\alpha^q} &\le C \limsup_{h\to 0+} \Big[h^{(\alpha-1)/2}\int_{-\infty}^{\infty}e^{-qy^2}|\hat{f}(y/h^{1/2})|^q dy\Big]^{1/q} \\
&= C \limsup_{h\to 0+} h^{\alpha/2q}\Big[\int_{-\infty}^{\infty}e^{-hqx^2}|\hat{f}(x)|^q dx\Big]^{1/q} \\
&= C \limsup_{h\to 0+} h^{\alpha/2q}\Big[\int_{-\infty}^{\infty}|\hat{W}(.,h)(x)\hat{f}(x)|^q dx\Big]^{1/q} \\
&= C \limsup_{h\to 0+} h^{\alpha/2q}\Big[\int_{-\infty}^{\infty}|\{W(.,h)*f\}^{\wedge}|^q dx\Big]^{1/q} \\
&\le C' \limsup_{h\to 0+} h^{\alpha(1-1/p)/2}\Big[\int_{-\infty}^{\infty}|W(.,h)*f|^p dx\Big]^{1/p} \\
&\le C'\|f\|_{V_\alpha^p}.
\end{aligned}$$

This completes the proof.

Example

Let l^p denote the space of complex sequences $a=\{a_n\}_{n=-\infty}^{\infty}$ such that $\|a\|_{l_p} = [\sum_{n=-\infty}^{\infty}|a_n|^p]^{1/p}<\infty$

Correspond $a\in l^p$ to the tempered distribution u_a such that

$$u_a(x)=\sum_{n=-\infty}^{\infty}a_n\delta(x-2\pi n).$$

Since we can easily get that $\|u_a\|_{V_1^p} = \|a\|_p$, then l^p can be considered as the closed subspace of $V_1^p(\mathbf{R})$.

Also, let $L^p_{2\pi}$ denote the space of 2π periodic p-ordered integrable functions f, that is,

$$||f||_{L^p_{2\pi}}=[\frac{1}{2\pi}\int_0^{2\pi}|f(x)|^p dx]^{1/p}<\infty.$$

Since $||f||_{L^p_{2\pi}} = ||f||_{M^p_1}$, $L^p_{2\pi}$ is a closed subspace of $M^p_1(\mathbf{R})$.

This example and Theorem 4.2 and 4.3 show the following Corollary which is known as the Hausdorff-Young's theorem in Fourier series.

Corollary 4.4. *Let $1<p\leq 2$ and let q be the conjugate exponent. Then,*

(i) *If $f\in L^p_{2\pi}$ then $\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^q<\infty$. More precisely $[\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^q]^{1/q}\leq C||f||_{L^p_{2\pi}}$,*

and

(ii) *If $\{a_n\}_{n=-\infty}^{\infty}\in l^p$ then there exists a function $f\in L^q_{2\pi}$ such that $a_n=\hat{f}(n)$. Moreover $||f||_{L^q_{2\pi}}\leq C[\sum_{n=-\infty}^{\infty}|\hat{f}(n)|^p]^{1/p}$, where $\hat{f}(n)$ is an n-ordered Fourier coefficient of f.*

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