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DIFFUSION WITH INTERACTIONS AND COLLISIONS BETWEEN COLOURED PARTICLES AND THE PROPAGATION OF CHAOS

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Introduction.

Motivated by the works [7], [8], and [9] on a statistical problem of segregation, we introduce the following model : Consider $m+n$ coloured particles diffusing on the line. The blue ones, which are denoted by $X_1,\cdots,X_m$, are distributed to the left of the red ones, denoted by $Y_1,\cdots,Y_n$. When the rightmost particle among the X_i and the leftmost one among the Y_j collide, they can not pass one another, but are reflected. However, particles with the same colour can pass one another when they meet. Therefore, the system which we are considering consists of two segregated groups of blue and red particles.

Interactions between particles in the system are prescribed in terms of drift coefficients: A blue particle X_i is a Brownian motion B_i^- with a drift given by

$$\frac{1}{m}\sum_{k=1}^{m} b_{11}(X_i,X_k) + \frac{1}{n}\sum_{k=1}^{n} b_{12}(X_i,Y_k) ,$$

as long as it stays away from red particles, and a red particle Y_j is a Brownian motion B_j^+ with a drift

$$\frac{1}{m}\sum_{k=1}^{m} b_{21}(Y_j,X_k) + \frac{1}{n}\sum_{k=1}^{n} b_{22}(Y_j,Y_k) ,$$

as long as it stays away from blue particles. Here we assume $\{B_i^-, B_j^+ : i=1,\cdots,m,\ j=1,\cdots,n\}$ is a family of mutually independent Brownian motions on the same line. Since the X_i and Y_j collide and are reflected, they satisfy a system of stochastic differential equations (SDE) with moving reflecting boundary conditions (see (4.2)). One can show the existence and uniqueness of solutions for the SDE , applying a result on reflecting processes on a convex domain in $\mathbb{R}^d$ given in Tanaka [15], or in the way which will be explained in §1.

We are interested in what happens when the number of particles m (resp. n) tends to infinity under the constraint $m/(m+n) \to \theta$, where $0 < \theta < 1$ is the fixed proportion of blue particles in the system, that is, the propagation of chaos of two segregated groups of interacting particles (see McKean [5],[6] for propagation of chaos). Since segregation occurs through collisions between blues and reds, there is no fixed segregating front between the two groups (whereas the segregating front is assumed to be fixed in Nagasawa-Tanaka [11]). As will be seen, the front $\gamma^{(m,n)}(t) = \max_{1\le i\le m} X_i(t)$ of blue particles converges to a non-random segregating front $\gamma(t)$ which is determined by the limit distribution of the sequence of empirical distributions $U^{(m,n)}(t)$ of all particles in the observed system, when the number of particles in the system tends to infinity under the prescribed constraint. The system $(X_1(t),\cdots,X_m(t), Y_1(t),\cdots,Y_n(t))$ itself converges in law to an infinite number of independent copies of a non-linear diffusion process $(X(t),Y(t))$ (see SDE (55)), each component of which is reflected at the segregating front $\gamma(t)$ (Theorem 3 and Theorem 3').

We first discuss, in §1, a way of constructing reflected paths segregated into two groups, following the idea of Harris [2]. In §2 we analyse a relation between a chaotic family and a chaotic family of two groups (of probability distributions on $\mathbb{R}^1$). The results in §2 will be applied in §3 to show a propagation of chaos result for non-interacting diffusion processes of two groups, constructed as in §1. A system of interacting particles of two groups with a moving reflecting boundary is constructed in §5 (Theorem 1) under a Lipschitz condition on the interactions $b_{ij}(x,y)$. §4 is a short preparation for §5, the material in which is taken from [15].

We explain the propagation of chaos for the system of interacting coloured particles in §6. As we shall see there, solutions of SDE (55) cannot be uniquely specified without knowing the segregating front $\gamma(t)$ defined by $u(t,(-\infty,\gamma(t)]) = \theta$ in advance, and hence we have to obtain beforehand the limit distribution $u(t)$ of the empirical distribution $U^{(m,n)}(t)$ of the system (see Remark in §6). For this purpose we will solve an associated non-linear SDE on the line (without reflection) using the Maruyama formula [4] for drift transformations in §7 (Theorem 2). Finally, the propagation of chaos for interacting diffusion processes of two types will be proved in §8, where boundedness of the interactions $b_{ij}(x,y)$ is assumed. This boundedness assumption is too restrictive to cover some applications to the statistical model treated in [7], [8], [9] and [10].

1. Reflected paths segregated into two groups

Let $\{w_1, w_2, \cdots, w_{m+n}\} \subset W = C(\mathbb{R}^+, \mathbb{R}^1)$ be(*) a family of paths satisfying the following two conditions:

(1) $$\max_{1 \le i \le m} w_i(0) \le \min_{1 \le j \le n} w_{m+j}(0) ,$$

(2) any three of $w_i(t)$, $1 \le i \le m+n$, do not meet at the same time.

We will define a family $\{\tilde{w}_i : i = 1,2,\cdots,m+n\}$ of reflected paths segregated into two groups. This will be one of the main tools in this paper to handle the propagation of chaos.

First of all we rearrange $w_1(t),\cdots,w_{m+n}(t)$ in order from least to greatest and denote the ordered ones by

(3) $$\hat{w}_1(t) \le \hat{w}_2(t) \le \cdots \le \hat{w}_{m+n}(t) .$$

Then, in terms of the ordered paths, the <u>front</u> of the left m-particles (resp. of the right n-particles) can be defined by

(4) $$\gamma_\ell(t) = \hat{w}_m(t) ,$$

(5) $$\gamma_r(t) = \hat{w}_{m+1}(t) .$$

We then construct <u>reflected paths</u> $\tilde{w}_1,\cdots,\tilde{w}_{m+n}$ segregated into two groups. Roughly speaking, we paint $w_1(0),\cdots,w_m(0)$ with blue and $w_{m+1}(0),\cdots,w_{m+n}(0)$ with red. If the rightmost particle among the blues and the leftmost particle among the reds meet and exchange their positions, they must exchange their colours. We interprete this exchange of colours as if the two particles are reflected.

(*) $C(\mathbb{R}^+,\mathbb{R}^1)$ denotes the space of continuous paths $w: \mathbb{R}^+=[0,\infty) \to \mathbb{R}^1$.

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The paths so interpreted are the reflected paths segregated into two groups. A precise definition of trajectories of blue (resp. red) particles is given as follows.

1°) For each i, $1 \leq i \leq m$ (fixed henceforth) set $s_0 = 0$,

$$t_0 = \inf\{ t \geq 0 : w_i(t) = \gamma_r(t)\} ,$$

and define

(6.a) $$\tilde{w}_i(t) = w_i(t) , \quad \text{for} \quad 0 \leq t \leq t_0 .$$

2°) Reading the suffix j_0 different from i such that

$$\gamma_r(t_0) = w_{j_0}(t_0) ,$$

(such a suffix j_0 being unique by the assumption (2)), we set

$$s_1 = \inf\{t \geq t_0 : w_i(t) \wedge w_{j_0}(t) = \hat{w}_{m-1}(t) \text{ or } w_i(t) \vee w_{j_0}(t) = \hat{w}_{m+2}(t)\}$$

and define

(6.b) $$\tilde{w}_i(t) = w_i(t) \wedge w_{j_0}(t) , \quad \text{for} \quad t_0 \leq t \leq s_1 .$$

Thus by (6.a) and (6.b)

(6.c) $$\tilde{w}_i(t) , \quad \text{for} \quad 0 \leq t \leq s_1$$

is well-defined.

3°) Assume that $\tilde{w}_i(t)$ is defined up to $0 \leq t \leq s_k$, $(k \geq 1)$. In order to extend the defining set of $\tilde{w}_i$ we take a sufficiently small $\varepsilon > 0$ and find a suffix i_k such that

$$\tilde{w}_i(t) = w_{i_k}(t) , \quad \text{for} \quad t \in (s_k - \varepsilon, s_k] .$$

Defining t_k by

$$t_k = \inf\{t \geq s_k : w_{i_k}(t) = \gamma_r(t)\} ,$$

we read the suffix j_k different from i_k such that

$$\gamma_r(t_k) = w_{j_k}(t_k) ,$$

and define s_{k+1} by

$$s_{k+1} = \inf\{t \geqq s_k : w_{i_k}(t) \wedge w_{j_k}(t) = \hat{w}_{m-1}(t) \text{ or } w_{i_k}(t) \vee w_{j_k}(t) = \hat{w}_{m+2}(t)\}.$$

We now put

(6.d) $$\tilde{w}_i(t) = w_{i_k}(t) , \text{ for } s_k \leqq t \leqq t_k ,$$
$$= w_{i_k}(t) \wedge w_{j_k}(t) , \text{ for } t_k \leqq t \leqq s_{k+1} .$$

Thus we have constructed

$$\tilde{w}_i(t) \quad \text{for} \quad 0 \leqq t \leqq s_{k+1} ,$$

for arbitrary k. Since the property (2) implies

$$\lim_{k\to\infty} s_k = \infty ,$$

the reflected path $\tilde{w}_i(t)$ (staying on the left hand side) is well-defined for $\forall t \in \mathbb{R}^+$ and $1 \leqq i \leqq m$.

4°) For $m+1 \leqq j \leqq m+n$ we define the reflected paths $\tilde{w}_j(t)$ (staying on the right hand side) in the same way with necessary alteration ($\gamma_r \to \gamma_\ell$, $w_i \wedge w_j \to w_i \vee w_j$, etc.). The reflected paths $\tilde{w}_j(t)$, $m+1 \leqq j \leqq m+n$, are defined so that

$$\gamma_\ell(t) \leqq \tilde{w}_j(t) ,$$

whereas

$$\tilde{w}_i(t) \leqq \gamma_r(t) , \quad \text{for } 1 \leqq i \leqq m.$$

Thus the family $\{w_i : 1 \le i \le m+n\}$ is segregated into two groups so that

$$\max_{1\le i\le m} \tilde{w}_i(t) = \gamma_\ell(t) \le \gamma_r(t) = \min_{1\le j\le n} \tilde{w}_{m+j}(t) .$$

If $m = n = 1$, then $\tilde{w}_1(t) = w_1(t) \wedge w_2(t)$ and $\tilde{w}_2(t) = w_1(t) \vee w_2(t)$.

Definition 1. The paths $\{\tilde{w}_1, \cdots, \tilde{w}_m, \tilde{w}_{m+1}, \cdots, \tilde{w}_{m+n}\}$ constructed above from a given family $\{w_1, \cdots, w_{m+n}\}$ satisfying the conditions (1) and (2) will be called reflected paths (processes) of two segregated groups (for short, reflected processes).

Let us deduce stochastic differential equations (SDE) which are satisfied by the reflected processes of two segregated groups, when the given paths (processes) satisfy the following SDE

$$(7) \qquad x(t) = x(0) + B(t) + \int_0^t b(s, x(s))ds ,$$

where $B(t)$ is a 1-dimensional Brownian motion starting at 0 which is independent of the initial value $x(0)$ and $b(s,x)$ is a locally bounded measurable function defined on $\mathbb{R}^+ \times \mathbb{R}^1$.

When we speak of a family $\{x_1(t), \cdots, x_n(t)\}$ of copies of (solutions) of the SDE (7), we always assume that they are those of (7) with independent $\{B_1(t), \cdots, B_n(t)\}$ which are independent of the initial values $(x_1(0), \cdots, x_n(0))$. However, $(x_1(t), \cdots, x_n(t))$ are not necessarily independent of each other. We assume in the following that the SDE has a unique solution in the law sense.(*)

Let us consider a system of SDE's with moving reflecting boundary condition

(*) For a special SDE such as (7) the uniqueness in the law sense and the pathwise uniqueness are equivalent.

$$(8)\quad \begin{cases} X(t) = X(0) + B^{-}(t) + \int_0^t b(s,X(s))ds - \Phi(t) , \\ Y(t) = Y(0) + B^{+}(t) + \int_0^t b(s,Y(s))ds + \Phi(t) , \end{cases}$$

where $X(0) \leqq Y(0)$, $\{B^{-}(t),B^{+}(t)\}$ are independent 1-dimensional Brownian motions starting from 0 which are independent of $\{X(0), Y(0)\}$, $X(t) \leqq Y(t)$ for all $t \geqq 0$, and

(9) $\Phi(t)$ is continuous, non-decreasing, $\Phi(0) = 0$, and $\mathrm{supp}(d\Phi) \subset \{t \geqq 0 : X(t) = Y(t)\}$.

Lemma 1. Let $x_1(t)$ (resp. $x_2(t)$) be the solution of the SDE (7) with $B_1(t)$ (resp. $B_2(t)$), where $B_1(t)$ and $B_2(t)$ are independent Brownian motions. Let $\tilde{x}_1(t) = x_1(t) \wedge x_2(t)$ and $\tilde{x}_2(t) = x_1(t) \vee x_2(t)$. Then the process $(\tilde{x}_1(t),\tilde{x}_2(t))$ satisfies the system of SDE's (8) with the initial value $(\tilde{x}_1(0),\tilde{x}_2(0))$, where $B^{-}(t)$, $B^{+}(t)$ and $\Phi(t)$ are explicitely given by

$$(10.1)\quad B^{-}(t) = \int_0^t 1_{\{\tilde{x}_1(s)=x_1(s)\}}dB_1(s) + \int_0^t 1_{\{\tilde{x}_1(s)=x_2(s)\}}dB_2(s) ,$$

$$(10.2)\quad B^{+}(t) = \int_0^t 1_{\{\tilde{x}_2(s)=x_1(s)\}}dB_1(s) + \int_0^t 1_{\{\tilde{x}_2(s)=x_2(s)\}}dB_2(s) ,$$

$$(10.3)\quad \Phi(t) = \frac{1}{2}\int_0^t \delta_{\{x_1(s)=x_2(s)\}}ds = \frac{1}{2}\int_0^t \delta_{\{\tilde{x}_1(s)=\tilde{x}_2(s)\}}ds ,$$

the meaning of which will be clarified in the proof.

Proof. Choose a nonnegative continuous even function g on $\mathbb{R}^1$ with a compact support such that $\int_{\mathbb{R}^1} g(x)dx = 1$, and put

$$g_N(x) = Ng(Nx) ,$$

$$f_N(x) = \int_0^x dy \int_0^y g_N(z)dz , \qquad x \in \mathbb{R}^1 .$$

Then, it is clear that $f_N \in C^2(\mathbb{R}^1)$ and $f_N(x) \to \frac{1}{2}|x|$ uniformly as $N \to \infty$, and hence

$$f_N(x_1-x_2) + \frac{1}{2}(x_1+x_2) \to x_1 \vee x_2 .$$

An application of Itô's formula yields

$$\begin{aligned}
&f_N(x_1(t)-x_2(t)) + \frac{1}{2}(x_1(t)+x_2(t)) - f_N(x_1(0)-x_2(0)) - \frac{1}{2}(x_1(0)+x_2(0)) \\
&= \int_0^t \{\frac{1}{2}+f_N'(x_1(s)-x_2(s))\}\{dB_1(s) + b(s,x_1(s))ds\} \\
&+\int_0^t \{\frac{1}{2}-f_N'(x_1(s)-x_2(s))\}\{dB_2(s) + b(s,x_2(s))ds\} \\
&+\frac{1}{2}\int_0^t g_N(x_1(s)-x_2(s))ds ,
\end{aligned}$$

from which we obtain, taking the limit $N \to \infty$,

$$\tilde{x}_2(t) - \tilde{x}_2(0) = B^+(t) + \int_0^t b(s,\tilde{x}_2(s))ds + \Phi(t) ,$$

where

$$\begin{aligned}
\Phi(t) &= \frac{1}{2}\int_0^t \delta_{\{x_1(s)=x_2(s)\}}ds \\
&= \lim_{N\to\infty} \frac{1}{2}\int_0^t g_N(x_1(s)-x_2(s))ds \\
&= \lim_{N\to\infty} \frac{1}{2}\int_0^t g_N(\tilde{x}_1(s)-\tilde{x}_2(s))ds .
\end{aligned}$$

The formula for $\tilde{x}_1(t)$ can be obtained similarly.

Lemma 1 can be extended to a system of m+n particles. Let $\{x_1(t),\cdots,x_{m+n}(t)\}$ be a family of copies of solutions of the SDE (7). We assume that $\max_{1\le i\le m} x_i(0) \le \min_{1\le j\le n} x_{m+j}(0)$ and that any three of $x_i(0)$, $1 \le i \le m+n$, do not coincide. Since the paths $x_1(\cdot)$, $\cdots,x_{m+n}(\cdot)$ satisfy the condition (2) with probability 1, we can define the reflected processes $\tilde{x}_1(t),\cdots,\tilde{x}_m(t),\tilde{x}_{m+1}(t),\cdots,\tilde{x}_{m+n}(t)$ of two segregated groups from $x_1(t),\cdots,x_{m+n}(t)$. We will show that the reflected processes satisfy the following system of SDE's

$$(11)\begin{cases} X_i(t) = X_i(t) + B_i^-(t) + \int_0^t b(s,X_i(t))ds - \sum_{k=1}^{n} \Phi_{ik}(t), & 1 \leq i \leq m , \\ Y_j(t) = Y_j(t) + B_j^+(t) + \int_0^t b(s,Y_j(s))ds + \sum_{k=1}^{m} \Phi_{kj}(t), & 1 \leq j \leq n , \end{cases}$$

where $X_i(0)$, $Y_j(0)$, $1 \leq i \leq m$, $1 \leq j \leq n$, satisfy the condition (1) i.e.

$$\max_{1\leq i\leq m} X_i(0) \leq \min_{1\leq j\leq n} Y_j(0)$$

and

$$(12.1) \qquad B_i^-(t) = \sum_{k=1}^{m+n} \int_0^t 1_{\{\tilde{x}_i(s)=x_k(s)\}} dB_k(s) \ , \qquad 1 \leq i \leq m \ ,$$

$$(12.2) \qquad B_j^+(t) = \sum_{k=1}^{m+n} \int_0^t 1_{\{\tilde{x}_{m+j}(s)=x_k(s)\}} dB_k(s) \ , \qquad 1 \leq j \leq n \ ,$$

$$(12.3) \qquad \Phi_{ij}(t) = \frac{1}{2}\int_0^t 1_{\{\tilde{x}_i(s)=\gamma_\ell(s),\tilde{x}_{m+j}(s)=\gamma_r(s)\}} \delta_{\{\tilde{x}_i(s)=\tilde{x}_{m+j}(s)\}} ds \ , \qquad 1 \leq i \leq m \ , \ 1 \leq j \leq n \ .$$

Notice that $B_i^-(t)$, $B_j^+(t)$, $1 \leq i \leq m$, $1 \leq j \leq n$, are 1-dimensional Brownian motions starting from 0, which are independent of each other and of $\{X_i(0),Y_j(0)\}$, $\Phi_{ij}(t)$ satisfies the condition (9) for $X_i(t)$ and $Y_j(t)$, and

$$\max_{1\leq i\leq m} X_i(t) \leq \min_{1\leq j\leq n} Y_j(t) \quad , \quad \text{for } \forall\, t \geq 0 \ .$$

<u>Lemma 2</u>. Let $\{x_1(t),\cdots,x_{m+n}(t)\}$ be a family of m+n copies of the solution of the SDE (7) satisfying the condition (1) and (2).

Then the reflected process of two segregated groups $(\tilde{x}_1(t),\cdots,\tilde{x}_m(t),\tilde{x}_{m+1}(t),\cdots,\tilde{x}_{m+n}(t))$ constructed from the given family satisfies the system of SDE's (11), where $X_i(t)=\tilde{x}_i(t)$, for $1\leq i\leq m$, and $Y_j(t)=\tilde{x}_{m+j}(t)$, for $1\leq j\leq n$.

<u>Proof</u>. Let $\underline{\underline{F}}_t$ be the smallest σ-field with respect to which $x_i(s)$, $0\leq \forall s\leq t$, $1\leq i\leq m+n$, are measurable. Fixing $1\leq i\leq m$, let us observe $x_i(t)$. The sequence of times s_k and t_k, $k=1,2,\cdots$, appearing in the definition of the reflected paths $\tilde{x}_i(t)$ are $\underline{\underline{F}}$-stopping times(*) and denoted by corresponding capital letters S_k and T_k, $k=1,2,\cdots$. The suffixes i_k and j_k appearing in the definition of $\tilde{x}_i(t)$ are random variables measurable with respect to $\underline{\underline{F}}_{S_k}$ and $\underline{\underline{F}}_{T_k}$ respectively.(**) We will denote them by capital letters I_k and J_k, respectively.

Let us denote for $k=1,2,\cdots,m+n$,

$$\hat{x}_k(t) = x_k(T_0+t)$$

$$\hat{B}_k(t) = B_k(T_0+t)-B_k(T_0), \quad t\geq 0 .$$

Then, $\hat{B}_i(t)$ and $\hat{B}_{J_0}(t)$ are independent and $\hat{\underline{\underline{F}}}_t(=\underline{\underline{F}}_{T_0+t})$-adapted Brownian motions which are also independent of $\hat{x}_i(0)$ and $\hat{x}_{J_0}(0)$. Therefore by Lemma 1 we have

$$(13)\quad \hat{x}_i(t)\wedge\hat{x}_{J_0}(t) = x_i(T_0)+\int_0^t 1_{\{\hat{x}_i(s)<\hat{x}_{J_0}(s)\}}d\hat{B}_i(s)$$
$$+\int_0^t 1_{\{\hat{x}_i(s)>\hat{x}_{J_0}(s)\}}d\hat{B}_{J_0}(s)+\int_0^t b(T_0+s,\hat{x}_i(s)\wedge\hat{x}_{J_0}(s))ds$$
$$-\lim_{N\to\infty}\frac{1}{2}\int_0^t g_N(\hat{x}_i(s)-\hat{x}_{J_0}(s))ds$$

(*) T is an $\underline{\underline{F}}$-stopping time $\Longleftrightarrow$ $\{T\leq t\}\in\underline{\underline{F}}_t$, for $\forall t\geq 0$.

(**) $A\in\underline{\underline{F}}_T$ $\Longleftrightarrow$ $A\cap\{T\leq t\}\in\underline{\underline{F}}_t$, for $\forall t\geq 0$.

Then the reflected process of two segregated groups $(\tilde{x}_1(t),\cdots,\tilde{x}_m(t),\tilde{x}_{m+1}(t),\cdots,\tilde{x}_{m+n}(t))$ constructed from the given family satisfies the system of SDE's (11), where $X_i(t)=\tilde{x}_i(t)$, for $1\le i\le m$, and $Y_j(t)=\tilde{x}_{m+j}(t)$, for $1\le j\le n$.

<u>Proof</u>. Let $\underline{\underline{F}}_t$ be the smallest σ-field with respect to which $x_i(s)$, $0\le \forall s\le t$, $1\le i\le m+n$, are measurable. Fixing $1\le i\le m$, let us observe $x_i(t)$. The sequence of times s_k and t_k, k=1,2,$\cdots$, appearing in the definition of the reflected paths $\tilde{x}_i(t)$ are $\underline{\underline{F}}$-stopping times(*) and denoted by corresponding capital letters S_k and T_k, k=1,2,$\cdots$. The suffixes i_k and j_k appearing in the definition of $\tilde{x}_i(t)$ are random variables measurable with respect to $\underline{\underline{F}}_{S_k}$ and $\underline{\underline{F}}_{T_k}$ respectively.(**) We will denote them by capital letters I_k and J_k, respectively.

Let us denote for $k=1,2,\cdots,m+n$,

$$\hat{x}_k(t) = x_k(T_0+t)$$

$$\hat{B}_k(t) = B_k(T_0+t)-B_k(T_0), \quad t\ge 0 .$$

Then, $\hat{B}_i(t)$ and $\hat{B}_{J_0}(t)$ are independent and $\hat{\underline{\underline{F}}}_t(=\underline{\underline{F}}_{T_0+t})$-adapted Brownian motions which are also independent of $\hat{x}_i(0)$ and $\hat{x}_{J_0}(0)$. Therefore by Lemma 1 we have

$$(13)\quad \hat{x}_i(t)\wedge\hat{x}_{J_0}(t) = x_i(T_0)+\int_0^t 1_{\{\hat{x}_i(s)<\hat{x}_{J_0}(s)\}}d\hat{B}_i(s)$$
$$+\int_0^t 1_{\{\hat{x}_i(s)>\hat{x}_{J_0}(s)\}}d\hat{B}_{J_0}(s)+\int_0^t b(T_0+s,\hat{x}_i(s)\wedge\hat{x}_{J_0}(s))ds$$
$$-\lim_{N\to\infty}\frac{1}{2}\int_0^t g_N(\hat{x}_i(s)-\hat{x}_{J_0}(s))ds$$

(*) T is an $\underline{\underline{F}}$-stopping time $\Longleftrightarrow \{T\le t\}\in\underline{\underline{F}}_t$, for $\forall t\ge 0$.

(**) $A\in\underline{\underline{F}}_T \Longleftrightarrow A\cap\{T\le t\}\in\underline{\underline{F}}_t$, for $\forall t\ge 0$.

(14.2) $1_{\{T_0<t\}}A_k^+(\hat{T}) = \int_0^t 1_{\{T_0<s\}}1_{\{x_i(s)>x_k(s)\}}dB_k(s)$,

(14.3) $1_{\{T_0<t\}}A_k^0(\hat{T}) = \int_0^t 1_{\{T_0<s\}}b(s,x_i(s)\wedge x_k(s))ds$

(14.4) $1_{\{T_0<t\}}A_k^N(\hat{T}) = \frac{1}{2}\int_0^t 1_{\{T_0<s\}}g_N(x_i(s)-x_k(s))ds$.

Therefore we have, combining (14) with (13) in which t is replaced by $\hat{T}$,

(15)
$$\begin{aligned}\tilde{x}_i(t) &= 1_{\{t\le T_0\}}x_i(t) + 1_{\{T_0<t\}}\hat{x}_i(\hat{T})\wedge\hat{x}_{J_0}(\hat{T}) \\ &= 1_{\{t\le T_0\}}x_i(t) + 1_{\{T_0<t\}}x_i(T_0) \\ &\quad + \sum_{\substack{k=1\\k\ne i}}^{m+n} 1_{\{J_0=k\}}\Big\{\int_0^t 1_{\{T_0<s\}}1_{\{x_i(s)<x_k(s)\}}dB_i(s) \\ &\qquad +\int_0^t 1_{\{T_0<s\}}1_{\{x_i(s)>x_k(s)\}}dB_k(s) \\ &\qquad +\int_0^t 1_{\{T_0<s\}}b(s,x_i(s)\wedge x_k(s))ds \\ &\qquad - \lim_{N\to\infty}\frac{1}{2}\int_0^t 1_{\{T_0<s\}}g_N(x_i(s)-x_k(s))ds\Big\}\end{aligned}$$

Since

$$\begin{aligned}&1_{\{t\le T_0\}}x_i(t) + 1_{\{T_0<t\}}x_i(T_0) \\ &= \tilde{x}_i(0) + \int_0^t 1_{\{s\le T_0\}}dB_i(s) + \int_0^t 1_{\{s\le T_0\}}b(s,\tilde{x}_i(s))ds ,\end{aligned}$$

the right hand side of (15) is equal to

$$\tilde{x}_i(0) + B_i^-(t) + \int_0^t b(s,\tilde{x}_i(s))ds - \sum_{k=1}^{n}\Phi_{ik}(t) , \qquad 0 \le t \le S_1 ,$$

where

$$\begin{aligned}B_i^-(t) &= \int_0^t 1_{\{s\le T_0\}}dB_i(s) + \sum_{\substack{k=1\\k\ne i}}^{m+n} 1_{\{J_0=k\}}\Big\{\int_0^t 1_{\{T_0<s\}}1_{\{x_i(s)<x_k(s)\}}dB_i(s) \\ &\qquad +\int_0^t 1_{\{T_0<s\}}1_{\{x_i(s)>x_k(s)\}}dB_k(s)\Big\} \\ &= \text{the right hand side of (12.1), for } 0 \le t \le S_1 ,\end{aligned}$$

and

$$\sum_{k=1}^{n} \Phi_{ik}(t) = \frac{1}{2}\lim_{N\to\infty} \sum_{\substack{k=1\\k\neq i}}^{m+n} 1_{\{J_0=k\}}\int_0^t 1_{\{T_0<s\}} g_N(x_i(s)-x_k(s))ds$$

$$= \lim_{N\to\infty} \frac{1}{2}\sum_{\substack{k=1\\k\neq i}}^{m+n} \sum_{j=1}^{n} 1_{\{J_0=k\}}\int_0^t 1_{\{T_0<s\}} 1_{\{\tilde{x}_{m+j}(s)=x_i(s)\vee x_k(s)\}} g_N(\tilde{x}_i(s)-\tilde{x}_{m+j}(s))ds$$

$$= \frac{1}{2}\sum_{j=1}^{n} \lim_{N\to\infty}\int_0^t 1_{\{\tilde{x}_i(s)=\gamma_\ell(s),\ \tilde{x}_{m+j}(s)=\gamma_r(s)\}} g_N(\tilde{x}_i(s)-\tilde{x}_{m+j}(s))ds$$

$=$ the sum of the right hand side of (12.3), for $0 \leq t \leq S_1$.

Therefore

$$\tilde{x}_i(t) = \tilde{x}_i(0) + B_i^-(t) + \int_0^t b(s,\tilde{x}_i(s))ds - \sum_{k=1}^{n} \Phi_{ik}(t) \ , \text{ for } \ 0 \leq t \leq S_1 \ .$$

In a similar way we can show that the above formula holds for $0 \leq t \leq S_k$, $\forall k \geq 1$. This completes the proof of Lemma 2.

2. Chaotic families of two groups of probability distributions on $\mathbb{R}^1$: Order statistics

We consider a subset $\Gamma_{m,n}$ of $\mathbb{R}^{m+n}$ defined by

$$\Gamma_{m,n} = \{(x_1,\cdots, x_m , y_1,\cdots, y_n) \in \mathbb{R}^{m+n}: \max_{1\leq i\leq m} x_i \leq \min_{1\leq j\leq n} y_j \text{ and any three of its coordinates do not coincide}\} .$$

$\underline{\underline{P}}_{m,n}$ denotes the family of all probability measures μ on $\Gamma_{m,n}$ which have the following permutation invariance of the left m-coordinates $(x_1,\cdots,x_m)$ and the right n-coordinates $(y_1,\cdots,y_n)$:

$$\langle\mu, f_{\sigma,\tau}\rangle = \langle\mu,f\rangle$$

for all bounded Borel functions f on $\Gamma_{m,n}$ and all permutations σ and τ of $(1,\cdots,m)$ and $(1,\cdots,n)$ respectively, where

$$f_{\sigma,\tau}(x_1,\cdots,x_m,y_1,\cdots,y_n) = f(x_{\sigma(1)},\cdots,x_{\sigma(m)},y_{\tau(1)},\cdots,y_{\tau(n)}) ,$$

and

$$\langle\mu,f\rangle = \int_{\Gamma_{m,n}} f(x_1,\cdots,x_m, y_1,\cdots,y_n)\mu(dx_1,\cdots,dx_m, dy_1,\cdots,dy_n) .$$

An example of $\mu \in \underline{\underline{P}}_{m,n}$ which will be given below will appear in the following sections. Let ν be a symmetric probability distribution on $\mathbb{R}^{m+n}$ and $(X_1,\cdots,X_{m+n})$ be an $\mathbb{R}^{m+n}$-valued random variable whose distribution is ν. Let $(\hat{X}_1,\cdots,\hat{X}_{m+n})$ be the order statistics of $(X_1,\cdots,X_{m+n})$, i.e. $\hat{X}_1 \leq \hat{X}_2 \leq \cdots\leq \hat{X}_{m+n}$. Let $\hat{\nu}$ be the distribution of the ordered random variable

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$(\hat{X}_1,\cdots,\hat{X}_{m+n})$. Let us assume furthermore that ν satisfies

$$(16)\quad \nu\Big(\{(x_1,\cdots,x_{m+n}) \in \mathbb{R}^{m+n}:\ x_i = x_j = x_k \text{ for some } 1 \leq i < j < k \leq m+n\}\Big) = 0 .$$

Then the distribution $\hat{\nu}$ of the ordered random variable is clearly a probability distribution on $\Gamma_{m,n}$. Finally we define $\mu=\pi_{m,n}\nu$ by

$$(17)\quad \mu = \pi_{m,n}\nu = \frac{1}{m!n!}\sum_{\sigma,\tau} \hat{\nu}_{\sigma,\tau} ,$$

where $\hat{\nu}_{\sigma,\tau}$ is defined by

$$<\hat{\nu}_{\sigma,\tau}, f> = <\hat{\nu}, f_{\sigma,\tau}> ,$$

for all bounded Borel functions f on $\Gamma_{m,n}$ and $\sum_{\sigma,\tau}$ denotes the sum over all permutations σ and τ of $(1,2,\cdots,m)$ and $(1,2,\cdots,n)$, respectively. It is clear that $\mu \in \underline{\underline{P}}_{m,n}$. Let S and T be independent random variables taking values in permutations $\{\sigma\}$ and $\{\tau\}$, respectively, which are uniformly distributed with

$$P[S = \sigma] = \frac{1}{m!},\quad P[T = \tau] = \frac{1}{n!}$$

and independent of $\{X_1,\cdots,X_{m+n}\}$. Then the measure $\mu \in \underline{\underline{P}}_{m,n}$ defined by (17) is the probability distribution of

$$(18)\quad (\hat{X}_{S(1)},\cdots,\hat{X}_{S(m)}, \hat{Y}_{T(1)},\cdots,\hat{Y}_{T(n)})$$

where $\hat{Y}_j = \hat{X}_{m+j}$, $1 \leq j \leq n$.

Definition 2. A family $\{\nu_p\}$ of probability distributions on $\mathbb{R}^p$, $p = 1,2,\cdots$, is called a (symmetric) u-chaotic family if

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(19.a) ν_p is a symmetric probability distribution on $\mathbb{R}^p$,

(19.b) $\nu_p\Big(\{(x_1,\cdots,x_p) \in \mathbb{R}^p : x_i = x_j = x_k$ for some $1 \leq i < j < k \leq p\}\Big) = 0$,

(19.c) $\{\nu_p\}$ is u-chaotic for a probability distribution u on $\mathbb{R}^1$, i.e. for any $k \geq 1$ and bounded continuous functions $f_1,\cdots,f_k$,

$$\lim_{p\to\infty} \langle \nu_p, f_1\otimes \cdots \otimes f_k \otimes 1 \underbrace{\otimes\cdots\otimes}_{p-k} 1\rangle = \prod_{i=1}^{k} \langle u, f_k\rangle .$$

Given a symmetric u-chaotic family $\{\nu_p, p=1,2,\cdots\}$ we define a family $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m, n = 1,2,\cdots\}$ by (17). In the following we investigate an asymptotic behaviour of the family $\{\mu_{m,n}\}$ as $m, n \to\infty$ under the constraint

$$\lim_{m,n\to\infty} \frac{m}{m+n} = \theta \ , \ 0 < \theta < 1 \ , \ (\theta \text{ fixed}).$$

We will abbriviate this limit as $m,n \to \infty(\theta)$.

We say that a probability distribution u on $\mathbb{R}^1$ satisfies the <u>positivity condition</u> for a given θ , if there exists a constant $\gamma = \gamma_\theta$ such that*)

(20) $u((-\infty,\gamma-\varepsilon)) < u((-\infty,\gamma)) = \theta = u((-\infty,\gamma]) < u((-\infty,\gamma+\varepsilon))$, for $\forall\, \varepsilon > 0$,

in particular, $u(\{\gamma\}) = 0$.

<u>Lemma 3</u>. Let $\{\nu_p : p=1,2,\cdots\}$ be a symmetric u-chaotic family, where u satisfies (20) for a given θ . Let $(X_1,\cdots,X_{m+n})$ be a random variable distributed by ν_{m+n} and $(\hat{X}_1,\cdots,\hat{X}_{m+n}) = (\hat{X}_1,\cdots,\hat{X}_m,\hat{Y}_1,\cdots,\hat{Y}_n)$ be its order statistics. Then for $\forall\ \varepsilon > 0$

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*) γ is the θ-quantile of the probability distribution u.

(21) $$P[\gamma - \varepsilon < \hat{X}_m \leq \hat{Y}_1 < \gamma + \varepsilon] \to 1 ,$$

as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$.

Proof. The assertion of the lemma is an implication of the law of large numbers for the order statistics $(\hat{X}_1,\cdots,\hat{X}_{m+n})$.*) Let us denote $\bar{g} = \langle u,g\rangle$ for a bounded continuous function g . Since $\{\nu_p\}$ is a symmetric u-chaotic family

$$E[|\frac{1}{m+n}\sum_{i=1}^{m+n} g(X_i) - \bar{g}|^2]$$

$$= \frac{1}{m+n}E[|g(X_1)-\bar{g}|^2] + \frac{(m+n)(m+n-1)}{(m+n)^2} E[(g(X_1) - \bar{g})(g(X_2) - \bar{g})]$$

$$\to 0 , \text{ as } m,n \to \infty .$$

Therefore, by Chebyschev inequality

(22) $$\frac{1}{m+n}\sum_{i=1}^{m+n} g(X_i) \to \bar{g} , \text{ in probability.}$$

If we take $g \in C(\mathbb{R}^1)$ such that $g = 1$ on $(-\infty,\gamma-\varepsilon)$, $0 \leq g \leq 1$, and $\mathrm{supp}(g) \subset (-\infty,\gamma)$, then

(23) $$\bar{g} = \langle u,g\rangle < \theta ,$$

by the positivity condition (20).

Therefore,

$$P[\hat{X}_m \leq \gamma-\varepsilon]$$

$$= P[\#\{X_i : X_i \in (-\infty,\gamma-\varepsilon], 1 \leq i \leq m+n\} \geq m]$$

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*) See e.g. Wilks [18] for the order statistics of i.i.d. random variables .

$$= P[\frac{1}{m+n} \sum_{i=1}^{m+n} 1_{(-\infty,\gamma-\varepsilon]}(X_i) \geq \frac{m}{m+n}]$$

$$\leq P[\frac{1}{m+n} \sum_{i=1}^{m+n} g(X_i) \geq \frac{m}{m+n}] \to 0 \ , \ \text{as} \quad m,n \to \infty$$

because of (22) and (23) combined with the fact that $\frac{m}{m+n} \to \theta (>\overline{g})$. In the same way we can show that $P[\hat{Y}_1 \geq \gamma+\varepsilon] \to 0$. This completes the proof of Lemma 3.

Definition 3. Let u_ℓ and u_r be probability distributions on $\mathbb{R}^1$ satisfying that $\text{supp}(u_\ell) \subset (-\infty,\gamma]$ and $\text{supp}(u_r) \subset [\gamma,\infty)$. A family $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m,n=1,2,\cdots\}$ is called (u_ℓ,u_r)-chaotic family of two groups for a given $0 < \theta < 1$, if it satisfies for any $M \geq 1$, $N \geq 1$ and $f_1,\cdots,f_M,g_1,\cdots,g_N \in C_b(\mathbb{R}^1)$,(*)

(24) $$\langle \mu_{m,n}, f_1 \otimes \cdots \otimes f_M \otimes \overbrace{1 \otimes \cdots \otimes 1}^{m-M} \otimes g_1 \otimes \cdots \otimes g_N \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-N} \rangle$$
$$\to \prod_{i=1}^{M} \langle u_\ell, f_i \rangle \prod_{j=1}^{N} \langle u_r, g_j \rangle \ ,$$

as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$.

Lemma 4. Let $\{\nu_p\}$ be a symmetric u-chaotic family, where u satisfies the positivity condition (20) for a given θ . Then the family $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m,n = 1,2,\cdots\}$ defined from $\{\nu_p\}$ by (17) is a (u_ℓ,u_r)-chaotic family of two groups, where u_ℓ and u_r are defined by

(25) $$\langle u_\ell, f \rangle = \int f(x) \frac{1}{\theta} 1_{(-\infty,\gamma]}(x) u(dx) \ ,$$
$$\langle u_r, f \rangle = \int f(x) \frac{1}{1-\theta} 1_{(\gamma,\infty)}(x) u(dx) \ , \quad \text{for } \forall f \in C_b(\mathbb{R}^1) \ .$$

Proof. In terms of random variables $(\hat{X}_{S(1)},\cdots,\hat{X}_{S(m)},\hat{Y}_{T(1)},\cdots,\hat{Y}_{T(n)})$ given in (18) the left hand side of (24), denoted

(*) $C_b(\mathbb{R}^1)$ is the space of bounded continuous functions on $\mathbb{R}^1$.

by (L), is

$$(L) = E[\prod_{i=1}^{M} f_i(\hat{X}_{S(i)}) \prod_{j=1}^{N} g_j(\hat{Y}_{T(j)})]$$
$$= \frac{1}{m!n!} \sum_{\sigma,\tau} E[\prod_{i=1}^{M} f_i(\hat{X}_{\sigma(i)}) \prod_{j=1}^{N} g_j(\hat{Y}_{\tau(j)})] ,$$

where we assume that $\mathrm{supp}(f_i) \subset (-\infty,\gamma)$, $1 \leq i \leq M$ and $\mathrm{supp}(g_j) \subset (\gamma,\infty)$, $1 \leq j \leq N$. We apply a formula on summation:

$$\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_M=1}^{m} a_{i_1 i_2 \cdots i_M}$$
$$= \sum{}^{*} a_{i_1 \cdots i_M} + \sum{}' = \frac{1}{(m-M)!} \sum_{\sigma} a_{\sigma(1)\cdots\sigma(M)} + \sum{}'$$

where $\sum^*$ denotes the sum over all $(i_1,\cdots,i_M)$, each element of which is different from others, and the number of terms in the summation $\sum'$ is $m^M - m(m-1) \cdots (m-M+1)$, and σ runs over all permutations of $(1,2,\cdots,m)$ in the summation $\sum_\sigma$. Then we have

$$(L) = \frac{1}{m!n!} \sum_{\tau} (m-M)! \{ \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_M=1}^{m} E[\prod_{p=1}^{M} f_p(X_{i_p}) \prod_{j=1}^{N} g_j(\hat{Y}_{\tau(j)})] - \sum{}' \} .$$

Because

$$| \frac{1}{m!n!} \sum_{\tau} (m-M)! \sum{}' |$$
$$\leq \frac{\text{const.}}{m!n!} \sum_{\tau} (m-M)! \{m^M - m(m-1) \cdots (m-M+1)\}$$
$$= \text{const.} \frac{m^M - m(m-1)\cdots(m-M+1)}{m(m-1)\cdots(m-M+1)} \to 0$$

as $m,n \to \infty$, we can write (L) as follows:

$$(L) = \frac{(m-M)!}{m!\,n!} \sum_{\tau} \sum_{i_1=1}^{m} \cdots \sum_{i_M=1}^{m} E[\prod_{p=1}^{M} f_p(\hat{X}_{i_p}) \prod_{j=1}^{N} g_j(\hat{Y}_{\tau(j)})] + o(1)$$

$$= \frac{(m-M)!}{m!} \frac{(n-N)!}{n!} \sum_{i_1=1}^{m} \cdots \sum_{i_M=1}^{m} \sum_{j_1=1}^{n} \cdots \sum_{j_N=1}^{n} E[\prod_{p=1}^{M} f_p(\hat{X}_{i_p}) \prod_{q=1}^{N} g_q(\hat{Y}_{j_q})] + o(1)$$

$$= \frac{(m-M)!}{m!} \frac{(n-N)!}{n!} E[\prod_{p=1}^{M} \sum_{i=1}^{m} f_p(\hat{X}_i) \prod_{q=1}^{N} \sum_{j=1}^{n} g_q(\hat{Y}_j)] + o(1) .$$

This can be written with the help of Lemma 3 ,

$$= \frac{(m-M)!}{m!} \frac{(n-N)!}{n!} E[\prod_{p=1}^{M} \sum_{i=1}^{m} f_p(\hat{X}_i) \prod_{q=1}^{N} \sum_{j=1}^{n} g_q(\hat{Y}_j) 1_{\{\gamma-\varepsilon<\hat{X}_m \leqq \hat{Y}_1<\gamma+\varepsilon\}}] + o(1) .$$

We choose $\varepsilon > 0$ such that

$$\mathrm{supp}(f_i) \subset (-\infty,\gamma-\varepsilon) , \ 1 \leqq i \leqq M , \quad \text{and}$$

$$\mathrm{supp}(g_j) \subset (\gamma+\varepsilon,\infty) , \ 1 \leqq j \leqq N .$$

Then, on the set $\{\gamma - \varepsilon < \hat{X}_m \leqq \hat{Y}_1 < \gamma + \varepsilon\}$, we can write

$$\sum_{i=1}^{m} f_p(\hat{X}_i) = \sum_{i=1}^{m+n} f_p(X_i) ,$$

$$\sum_{j=1}^{n} g_q(\hat{Y}_j) = \sum_{j=1}^{m+n} g_q(X_j) ,$$

where $(X_1,\cdots,X_{m+n})$ is distributed by ν_{m+n} . Therefore

$$(L) = \frac{(m-M)!}{m!} \frac{(n-N)!}{n!} E[\prod_{p=1}^{M} \sum_{i=1}^{m+n} f_p(X_i) \prod_{q=1}^{N} \sum_{j=1}^{m+n} g_q(X_j) 1_{\{\gamma-\varepsilon<\hat{X}_m \leqq \hat{Y}_1<\gamma+\varepsilon\}}] + o(1)$$

applying Lemma 3 again, we can take away $1_{\{\gamma-\varepsilon<\hat{X}_m \leq \hat{Y}_1<\gamma+\varepsilon\}}$ from the expectation above, and hence

$$= \frac{m^M}{m(m-1)\cdots(m-M+1)} \frac{n^N}{n(n-1)\cdots(n-N+1)} E[\prod_{p=1}^{M} \frac{1}{m}\sum_{i=1}^{m+n} f_p(X_i) \prod_{q=1}^{N} \frac{1}{n}\sum_{j=1}^{m+n} g_q(X_j)] + o(1) \to \prod_{p=1}^{M} \frac{\langle u, f_p\rangle}{\theta} \prod_{q=1}^{N} \frac{\langle u, g_q\rangle}{1-\theta} .$$

as $m,n\to\infty$ with the constraint $m/(m+n)\to\theta$, because $\{\nu_{m+n}\}$ is a symmetric u-chaotic family. On the other hand it is clear that

$$\frac{\langle u,f_p\rangle}{\theta} = \langle u_\ell,f_p\rangle , \qquad \frac{\langle u,g_q\rangle}{1-\theta} = \langle u_r,g_q\rangle .$$

This completes the proof under the restriction that $\mathrm{supp}(f_i)\subset(-\infty,\gamma)$ and $\mathrm{supp}(g_j)\subset(\gamma,\infty)$. Removing this restriction is easy.

Lemma 4 characterizes an asymptotic behaviour of a family $\{\mu_{m,n}\in \underline{\underline{P}}_{m,n}\}$ which is defined from a given symmetric u-chaotic family $\{\nu_p\}$ by (17). In the following we will prove a converse of Lemma 4.

For a given $\mu\in\underline{\underline{P}}_{m,n}$ we define a symmetric distribution ν on $\mathbb{R}^{m+n}$ by

$$(26) \qquad \nu = \frac{1}{(m+n)!}\sum_{\sigma}\mu_\sigma ,$$

where $\sum_\sigma$ is the sum over all permutations σ of $(1,\cdots,m+n)$ and μ_σ is defined by

$$\langle\mu_\sigma,f\rangle = \langle\mu,f_\sigma\rangle , \quad \text{for } \forall f\in C(\mathbb{R}^{m+n})$$

with $f_\sigma(x_1,\cdots,x_{m+n}) = f(x_{\sigma(1)},\cdots,x_{\sigma(m+n)})$. Then it is clear that

(27) $$\pi_{m,n}\nu = \mu \ .$$

Let $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m,n = 1,2,\cdots\}$ be a (u_ℓ,u_r)-chaotic family of two groups for a given $0 < \theta < 1$. A converse of Lemma 4 can be formulated as follows.

Lemma 5. Let $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m,n = 1,2,\cdots\}$ be a (u_ℓ,u_r)-chaotic family of two groups for a given $0 < \theta < 1$ and define symmetric distribution $\nu_{m,n}$ by (26) from $\mu_{m,n}$. Then $\{\nu_{m,n} : m,n = 1,2,\cdots\}$ is a symmetric u-chaotic family, where u is defined by $u = \theta u_\ell + (1-\theta)u_r$, i.e. $\nu_{m,n}$ converges weakly to $u \otimes u \otimes \cdots$, as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$.

Proof. Let $(\tilde{X}_1,\cdots,\tilde{X}_{m+n})$ be a $\mu_{m,n}$-distributed random variable and S be a random variable which is independent of $(\tilde{X}_1,\cdots,\tilde{X}_{m+n})$ and taking values in all permutations of $(1,2,\cdots,m+n)$ with

$$P[S = \sigma] = \frac{1}{(m+n)!} \ .$$

Then by the definition of $\nu_{m,n}$

(28) $$\langle \nu_{m,n}, f_1 \otimes \cdots \otimes f_M \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-M} \otimes g_1 \otimes \cdots \otimes g_N \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-N} \rangle$$
$$= E[\prod_{i=1}^{M} f_i(\tilde{X}_{S(i)}) \prod_{j=1}^{N} g_j(\tilde{X}_{S(m+j)})]$$

for $f_1,\cdots,f_M$, $g_1,\cdots,g_N \in C_b(\mathbb{P}^1)$ Define u_i by

$$u_i = \begin{cases} u_\ell, & \text{for } 1 \leq i \leq m , \\ u_r, & \text{for } m+1 \leq i \leq m+n . \end{cases}$$

Because of (24), the right hand side of (28) can be written as

$$E\Big[\prod_{i=1}^{M} \langle u_{S(i)}, f_i\rangle \prod_{j=1}^{N} \langle u_{S(m+j)}, g_j\rangle\Big] + o(1) \tag{29}$$

where $o(1)$ converges to zero when $m,n\to\infty$ under the constraint $m/(m+n)\to\theta$. Let us denote

$$C_i = \{1 \leq S(i) \leq m\}$$

$$D_j = \{n+1 \leq S(m+j) \leq m+n\}$$

where S depends on (m,n) and hence so do C_i and D_j, but it is not indicated for simplicity. Then, it is clear that the family of events $\{C_i, D_j : 1\leq i\leq M,\ 1\leq j\leq N\}$ is asymptotically independent as $m,n\to\infty$ under the constraint $m/(m+n)\to\theta$, and

$$P[C_i] \to \theta, \quad P[D_j] \to 1-\theta$$

for each fixed i and j , respectively. Therefore (29) converges to

$$\prod_{i=1}^{M} \langle\theta u_\ell+(1-\theta)u_r, f_i\rangle \prod_{j=1}^{N} \langle\theta u_\ell+(1-\theta)u_r, g_j\rangle$$

$$= \prod_{i=1}^{M} \langle u, f_i\rangle \prod_{j=1}^{N} \langle u, g_j\rangle ,$$

as $m,n\to\infty$ under the constraint $m/(m+n)\to\theta$. This completes the proof of Lemma 5.

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3. Propagation of chaos; the case of no interaction

Given a probability distribution u on $\mathbb{R}^1$ satisfying the positivity condition (20) we define the left θ-part u_ℓ and the right $(1-\theta)$-part u_r of the distribution u, respectively, by

(30.a) $$\langle u_\ell, f\rangle = \langle u, \theta^{-1} 1_{(-\infty,\gamma]} f\rangle ,$$

(30.b) $$\langle u_r, f\rangle = \langle u, (1-\theta)^{-1} 1_{(\gamma,\infty)} f\rangle ,$$

for $\forall f \in C_b(\mathbb{R}^1)$, where γ is the θ-quantile of u ; thus we have

$$u = \theta u_\ell + (1-\theta) u_r .$$

When we represent u decomposed as above, we always understand that u_ℓ and u_r are given by (30).

Suppose that we are given a (u_ℓ, u_r)-chaotic family $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m,n = 1,2,\cdots\}$ of two groups and define $\nu_{m,n}$, $m,n = 1,2,\cdots$, by (26). Then $\nu_{m,n}$ has the property (16) and $\pi_{m,n}\nu_{m,n} = \mu_{m,n}$ by (27). Moreover, $\{\nu_{m,n} : m,n = 1,2,\cdots\}$ is a symmetric u-chaotic family by Lemma 5. Consider a system of SDE's

(31) $$y_i(t) = y_i(0) + B_i(t) + \int_0^t b(s, y_i(s))ds , \quad 1 \leq i \leq m+n ,$$

with a bounded drift coefficient $b(s,x)$, where $\{B_i(t)\}$ is a family of independent 1-dimensional Brownian motions starting at 0. For each pair (m,n) let $\underline{\underline{y}}(t) = \underline{\underline{y}}^{(m,n)}(t) = (y_1(t), \cdots, y_{m+n}(t))$ be the solution of (31) with a $\nu_{m,n}$-distributed initial value which is independent of the Brownian motions. Let $\nu_{m,n}(t)$ be the probability distribution of $\underline{\underline{y}}^{(m,n)}(t)$. Then it is easy to see that for each $t \geq 0$

(32) $\{\nu_{m,n}(t) : m,n = 1,2,\cdots\}$ is a symmetric $u(t)$-chaotic family,

where $u(t)$ is the probability distribution, at time t, of the solution of the SDE (7) with a u-distributed initial value. We now introduce $\underline{\underline{x}}(t) = (x_1(t),\cdots,x_{m+n}(t))$ as follows: First we find a random variable S which takes values in the permutations of $\{1,2,\cdots,m+n\}$ and is independent of the Brownian motion $\underline{\underline{B}}(t) = (B_1(t),\cdots,B_{m+n}(t))$ such that $(y_{S(1)}(0),\cdots,y_{S(m+n)}(0))$ is distributed according to $\mu_{m,n}$. We then set

$$x_i(t) = y_{S(i)}(t) \quad \text{and} \quad B_i'(t) = B_{S(i)}(t) .$$

It is clear that $\underline{\underline{x}}(t) = (x_1(t),\cdots,x_{m+n}(t))$ satisfies the conditions (1) and (2) with probability one and is a solution of the SDE (31) with the initial distribution $\mu_{m,n}$ and with $B_i'(t)$ in place of $B_i(t)$. By Lemma 2 the reflected process of two segregated groups

(33) $\tilde{\underline{\underline{x}}}(t) = (\tilde{x}_1(t),\cdots,\tilde{x}_m(t),\tilde{x}_{m+1}(t),\cdots,\tilde{x}_{m+n}(t))$

constructed from the $\underline{\underline{x}}(t)$ satisfies the system (11) of SDE's, where $X_i(t) = \tilde{x}_i(t)$, $1 \leq i \leq m$ and $Y_j(t) = \tilde{x}_{m+j}(t)$, $1 \leq j \leq n$.

Lemma 6. Let $\gamma(t)$ be the segregating front (θ-quantile) of the probability distribution $u(t)$, and $\gamma_\ell^{(m,n)}(t) = \max_{1\leq i\leq m} \tilde{x}_i(t)$, $\gamma_r^{(m,n)}(t) = \min_{1\leq j\leq n} \tilde{x}_{m+j}(t)$. Then for $\varepsilon > 0$

$$P[\gamma(t) - \varepsilon < \gamma_\ell^{(m,n)}(t) \leq \gamma_r^{(m,n)}(t) < \gamma(t) + \varepsilon] \to 1,$$

as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$.

Proof. It is clear that

$$\gamma_\ell(t) = \hat{x}_m(t) = \hat{y}_m(t) \quad \text{and} \quad \gamma_r(t) = \hat{x}_{m+1}(t) = \hat{y}_{m+1}(t) .$$

Since the family $\{\nu_{m,n}(t)\}$ of the distribution of $\underline{\underline{y}}(t)$ is $u(t)$-chaotic, we have by Lemma 3

$$P[\gamma(t) - \varepsilon < \gamma_\ell(t) \leq \gamma_r(t) < \gamma(t) + \varepsilon] \to 1$$

as $m,n \to \infty$ under the constraint $m/(m,n) \to \theta$. We note furthermore that $(\gamma_\ell^{(m,n)}(t), \gamma_r^{(m,n)}(t))$ is identical in law to $(\gamma_\ell(t), \gamma_r(t))$. This completes the proof of Lemma 6.

Lemma 7. Let $\tilde{\mu}_{m,n}(t)$ be the probability distribution of $\underline{\underline{\tilde{x}}}(t)$ in (33). Then $\{\tilde{\mu}_{m,n}(t) : m,n = 1,2,\cdots\}$ is a $(u(t)_\ell, u(t)_r)$-chaotic family of two groups, where $u(t)=\theta u(t)_\ell+(1-\theta)u(t)_r$. That is, the propagation of chaos holds for the diffusion process described by (11) with the initial distribution $\mu_{m,n} = \pi_{m,n}\nu_{m,n}$ defined by (17).

Proof. The distribution $\tilde{\mu}_{m,n}(t)$ coincides with the one $\mu_{m,n}(t) \in \underline{\underline{P}}_{m,n}$ which is defined by (17) from $\nu_{m,n}(t)$, because the probability distribution of $(\tilde{x}_1(t),\cdots,\tilde{x}_m(t))$ (resp. $(\tilde{x}_{m+1}(t), \cdots,\tilde{x}_{m+n}(t))$) is symmetric. Therefore we can apply Lemma 4 to the family $\{\tilde{\mu}_{m,n}(t)\}$. This completes the proof of Lemma 7.

The propagation of chaos for diffusion processes with interraction will be discussed in § 6 and §8.

4. A system of Skorokhod equations with moving reflecting boundary conditions

Given $\{w_1,\cdots,w_m , z_1,\cdots,z_n\} \subset \underline{\underline{W}} = C(\mathbb{R}^+,\mathbb{R}^1)$ satisfying the initial condition

(34) $$\max_{1\le i\le m} w_i(0) \le \min_{1\le j\le n} z_j(0) ,$$

we consider a system of Skorokhod equations

(35) $$\begin{cases} \xi_i(t) = w_i(t) - \sum_{k=1}^{n} \varphi_{ik}(t) , & 1 \le i \le m , \\ \eta_j(t) = z_j(t) + \sum_{k=1}^{m} \varphi_{kj}(t) , & 1 \le j \le n . \end{cases}$$

The problem is to find out a solution $(\xi_i, \eta_j, \varphi_{ij})$ of (35) with the following properties:

(36.a) $\xi_i, \eta_j \in \underline{\underline{W}}$ and $\max_{1\le i\le m} \xi_i(t) \le \min_{1\le j\le n} \eta_j(t)$, for $\forall\ t \ge 0$,

(36.b) $\varphi_{ij}(t)$ $(1\le i\le m,\ 1\le j\le n)$ is continuous, monotone non-decreasing in $t \ge 0$, $\varphi_{ij}(0) = 0$, and

$$\mathrm{supp}(d\varphi_{ij}) \subset \{t \ge 0 : \xi_i(t) = \eta_j(t)\} .$$

The Skorokhod problem (35) of a system of equations can be viewed as a Skorokhod equation in a convex domain in $\mathbb{R}^{m+n}$ as follows: Let us denote

(37) $$D = \{(x_1,\cdots,x_m,y_1,\cdots,y_n) \in \mathbb{R}^{m+n} : \max_{1\le i\le m} x_i < \min_{1\le j\le n} y_j\} .$$

The domain D can be represented as

$$D = \bigcap_{\substack{1 \le i \le m \\ 1 \le j \le n}} D_{ij} .$$

where $D_{ij} = \{(x_1, \cdots, x_m, y_1, \cdots, y_n) \in \mathbb{R}^{m+n} : x_i < y_j\}$. Since the set D_{ij} is clearly a convex domain in $\mathbb{R}^{m+n}$, so is the domain D .

In a convex domain D one can consider a Shorokod problem (Tanaka [15]): Given $\underline{w} \in C(\mathbb{R}^+, \mathbb{R}^{m+n})$ with $\underline{w}(0) \in \overline{D}$, solve

$$(38) \qquad \underline{\xi}(t) = \underline{w}(t) + \int_0^t \underline{n}(s) d\varphi ,$$

where $\underline{\xi}$ and φ should satisfy the conditions;

(39.a) $\underline{\xi} \in C(\mathbb{R}^+, \overline{D})$,

(39.b) φ is continuous, nondecreasing, $\varphi(0) = 0$, and

$\operatorname{supp}(d\varphi) \subset \{t \ge 0 : \underline{\xi}(t) \in \partial D\}$,

(39.c) $\underline{n}(s) \in N_{\underline{\xi}(s)}$, if $\underline{\xi}(s) \in \partial D$,

where $N_{\underline{x}}$, $\underline{x} \in \partial D$, denotes the totality of inward normal unit vectors of D at $\underline{x}$.

Remark. An inward normal unit vector is a unit vector which is perpendicular to a supporting hyperplane and pointing inward . For the convex domain D given in (37) , $N_{\underline{x}}$ can be expressed as

$$N_{\underline{x}} = \{\underline{n} = \sum_{(i,j) \in L_{\underline{x}}} c_{ij} \underline{n}_{ij} : c_{ij} \ge 0 , \|\underline{n}\| = 1\} ,^{*)}$$

where for $(i,j) \in L_{\underline{x}} = \{(i,j) : x_i = y_j\}$

*) This is proved by Saisho. See also [12] for a similar problem.

$$\underline{\underline{n}}_{ij} = (0,\cdots,\frac{-1}{\sqrt{2}},0,\cdots,0,\frac{1}{\sqrt{2}},0,\cdots,0) .$$

If we view the Skorokhod equation (38) componentwise, it is nothing but the system (35) of Skorokhod equations. Therefore we have

Lemma 8. (i) For given $w_1,\cdots,w_m,z_1,\cdots,z_n \in \underline{\underline{W}}$ satisfying the condition (34) the equation (35) has a unique solution.

(ii) Let $(\xi_i,\eta_j,\varphi_{ij})$ satisfy the Skorokhod equation

$$(40)\quad \begin{cases} \xi_i(t) = w_i(t) + \int_0^t a_i(s)ds - \sum_{k=1}^n \varphi_{ik}(t) , & 1 \leq i \leq m , \\ \eta_j(t) = z_j(t) + \int_0^t b_j(s)ds + \sum_{k=1}^m \varphi_{kj}(t) , & 1 \leq j \leq n , \end{cases}$$

where (34) is assumed, and let $(\bar{\xi}_i,\bar{\eta}_j,\bar{\varphi}_{ij})$ satisfy the Eq. (40) with $\bar{w}_i,\bar{z}_j,\bar{a}_i$ and $\bar{b}_j$ in place of w_i,z_j,a_i and b_j. Moreover assume that $w_i(t)-w_i(0) = \bar{w}_i(t)-\bar{w}_i(0)$ and $z_j(t)-z_j(0) = \bar{z}_j(t)-\bar{z}_j(0)$, for $\forall t \geq 0$. Then for $t \in [0,T]$

$$(41)\quad \sum_{i=1}^m |\xi_i(t) - \bar{\xi}_i(t)|^2 + \sum_{j=1}^n |\eta_j(t) - \bar{\eta}_j(t)|^2$$

$$\leq e^T\{ \sum_{i=1}^m |w_i(0) - \bar{w}_i(0)|^2 + \sum_{j=1}^n |z_j(0) - \bar{z}_j(0)|^2 + \int_0^t (\sum_{i=1}^m |a_i(s) - \bar{a}_i(s)|^2 + \sum_{j=1}^n |b_j(s) - \bar{b}_j(s)|^2) ds\} .$$

Proof. For (i) see Tanaka [15].(*) To prove (ii) denote the left hand side of (41) by $f(t)$ and the right hand side by $e^T g(t)$. Then the first inequality in Remark 2.2 of [15] implies

$$f(t) \leq \int_0^t f(s)ds + g(t) ,$$

from which follows (41) by Gronwall's lemma.

(*) When w_i and z_j are Brownian motions with drifts, one can get ξ_i and η_j as the reflected processes of two groups constructed from w_i and z_j in §1 (see Lemma 2).

5. A system of interacting particles with moving reflecting boundary conditions

We consider a system of stochastic differential equations

$$(42)\begin{cases} X_i(t)=X_i(0)+B_i^-(t)+\frac{1}{m}\sum_{k=1}^{m}\int_0^t b_{11}(X_i(s),X_k(s))ds + \\ \qquad +\frac{1}{n}\sum_{k=1}^{n}\int_0^t b_{12}(X_i(s),Y_k(s))ds - \sum_{k=1}^{n}\Phi_{ik}(t), \quad 1\le i\le m, \\ Y_j(t)=Y_j(0)+B_j^+(t)+\frac{1}{m}\sum_{k=1}^{m}\int_0^t b_{21}(Y_j(s),X_k(s))ds + \\ \qquad +\frac{1}{n}\sum_{k=1}^{n}\int_0^t b_{22}(Y_j(s),Y_k(s))ds + \sum_{k=1}^{m}\Phi_{kj}(t), \quad 1\le j\le n, \end{cases}$$ (*)

where the initial values satisfy the conditions

(43.a) $\max_{1\le i\le m} X_i(0) \le \min_{1\le j\le n} Y_j(0)$,

(43.b) any three of $\{X_i(0),Y_j(0) : 1\le i\le m,\ 1\le j\le n\}$ do not meet,

and $\{B_i^-(t),B_j^+(t) : 1\le i\le m,\ 1\le j\le n\}$ are independent one dimensional Brownian motions staring at 0 which are independent of the initial values $\{X_1(0),Y_j(0) : 1\le i\le m,\ 1\le j\le n\}$.

Solutions of the SDE (42) should satisfy

(44.a) $\max_{1\le i\le m} X_i(t) \le \min_{1\le j\le n} Y_j(t)$, for $\forall\ t\ge 0$,

(44.b) $\Phi_{ij}(t)$ is continuous, monotone nondecreasing, $\Phi_{ij}(0)=0$ and

$$\mathrm{supp}\,(d\Phi_{ij}) \subset \{t\ge 0 : X_i(t) = Y_j(t)\} .$$

When we emphasize the dependence of a solution on (m,n) , we write

---

(*) We denote $\Phi_i(t)=\sum_{k=1}^{n}\Phi_{ik}(t)$ and $\Psi_j(t)=\sum_{k=1}^{m}\Phi_{kj}(t)$.

it as $(X_i^{(m,n)}(t),\ Y_j^{(m,n)}(t),\ \Phi_{ij}^{(m,n)}(t))$.

<u>Theorem 1</u>. If the interactions $b_{ij}(x,y)$ satisfy a Lipschitz condition

$$(45)\qquad \sum_{i=1}^{2}\sum_{j=1}^{2}|b_{ij}(x,y)-b_{ij}(x',y')|^2 \leqq c\{|x-x'|^2+|y-y'|^2\},$$

then there exists a (pathwise) unique solution of SDE's (42).

<u>Proof</u>. Let us consider a deterministic Skorokhod problem : For given $w_i, z_j \in W$ satisfying the condition (34), solve

$$(46)\quad \begin{cases} \xi_i(t)=w_i(t)+\frac{1}{m}\sum_{\ell=1}^{m}\int_0^t b_{11}(\xi_i(s),\xi_\ell(s))ds+\frac{1}{n}\sum_{\ell=1}^{n}\int_0^t b_{12}(\xi_i(s),\eta_\ell(s))ds \\ \qquad\qquad -\sum_{\ell=1}^{n}\varphi_{i\ell}(t),\quad 1\leqq i\leqq m, \\ \eta_j(t)=z_j(t)+\frac{1}{m}\sum_{\ell=1}^{m}\int_0^t b_{21}(\eta_j(s),\xi_\ell(s))ds+\frac{1}{n}\sum_{\ell=1}^{n}\int_0^t b_{22}(\eta_j(s),\eta_\ell(s))ds \\ \qquad\qquad +\sum_{\ell=1}^{m}\varphi_{\ell j}(t),\quad 1\leqq j\leqq n, \end{cases}$$

under the conditions (36.a) and (36.b) .

Let $(\xi_i^{(0)}, \eta_j^{(0)}, \varphi_{ij}^{(0)})$ be a solution of

$$\begin{cases} \xi_i^{(0)}(t) = w_i(t) - \sum_{\ell=1}^{n}\varphi_{i\ell}^{(0)}(t), \\ \eta_j^{(0)}(t) = z_j(t) + \sum_{\ell=1}^{m}\varphi_{\ell j}^{(0)}(t), \end{cases}$$

and for $k \geqq 1$ $(\xi_i^{(k)}, \eta_j^{(k)}, \varphi_{ij}^{(k)})$ be a solution of

$$
(47)\begin{cases}
\xi_i^{(k)}(t)=w_i(t)+\frac{1}{m}\sum_{\ell=1}^{m}\int_0^t b_{11}(\xi_i^{(k-1)}(s),\xi_\ell^{(k-1)}(s))ds \\
\qquad +\frac{1}{n}\sum_{\ell=1}^{n}\int_0^t b_{12}(\xi_i^{(k-1)}(s),\eta_\ell^{(k-1)}(s))ds-\sum_{\ell=1}^{n}\varphi_{i\ell}^{(k)}(t), \\
\eta_j^{(k)}(t)=z_i(t)+\frac{1}{m}\sum_{\ell=1}^{m}\int_0^t b_{21}(\eta_j^{(k-1)}(s),\xi_\ell^{(k-1)}(s))ds \\
\qquad +\frac{1}{n}\sum_{\ell=1}^{n}\int_0^t b_{22}(\eta_j^{(k-1)}(s),\eta_\ell^{(k-1)}(s))ds+\sum_{\ell=1}^{m}\varphi_{\ell j}^{(k)}(t).
\end{cases}
$$

The existence and uniqueness of $(\xi_i^{(k)}, \eta_j^{(k)}, \varphi_{ij}^{(k)})$, $k \geq 0$, are guaranteed by Lemma 8, (i) of § 4. Furthermore Lemma 8, (ii) implies that for $0 \leqq t \leqq T < \infty$

$$
\sum_{i=1}^{m}|\xi_i^{(k+1)}(t)-\xi_i^{(k)}(t)|^2+\sum_{j=1}^{n}|\eta_j^{(k+1)}(t)-\eta_j^{(k)}(t)|^2
$$

$$
\begin{aligned}
\leq e^T\{ & \sum_{i=1}^{m}\frac{1}{m}\sum_{\ell=1}^{m}\int_0^t |b_{11}(\xi_i^{(k)}(s),\ \xi_\ell^{(k)}(s)) \\
& \qquad - b_{11}(\xi_i^{(k-1)}(s),\xi_\ell^{(k-1)}(s))|^2 ds \\
& + \sum_{i=1}^{m}\frac{1}{n}\sum_{\ell=1}^{n}\int_0^t |b_{12}(\xi_j^{(k)}(s),\ \eta_\ell^{(k)}(s)) \\
& \qquad - b_{12}(\xi_i^{(k-1)}(s),\eta_\ell^{(k-1)}(s))|^2 ds \\
& + \sum_{j=1}^{n}\frac{1}{m}\sum_{\ell=1}^{m}\int_0^t |b_{21}(\eta_j^{(k)}(s),\ \xi_\ell^{(k)}(s)) \\
& \qquad - b_{21}(\eta_j^{(k-1)}(s),\xi_\ell^{(k-1)}(s))|^2 ds \\
& + \sum_{j=1}^{n}\frac{1}{n}\sum_{\ell=1}^{n}\int_0^t |b_{22}(\eta_j^{(k)}(s),\ \eta_\ell^{(k)}(s)) \\
& \qquad - b_{22}(\eta_j^{(k-1)}(s),\eta_\ell^{(k-1)}(s))|^2 ds\},
\end{aligned}
$$

then, because of the Lipschitz condition (45),

$$\le 4ce^T \int_0^t \{ \sum_{i=1}^m |\xi_i^{(k)}(s)-\xi_i^{(k-1)}(s)|^2 + \sum_{j=1}^n |\eta_j^{(k)}(s)-\eta_j^{(k-1)}(s)|^2 \}ds$$

and hence, for $0 \le t \le T$

$$\sum_{i=1}^m |\xi_i^{(k+1)}(t)-\xi_i^{(k)}(t)|^2 + \sum_{j=1}^n |\eta_j^{(k+1)}(t)-\eta_j^{(k)}(t)|^2 \le \frac{K(4ce^T)^k T^k}{k!} ,$$

where

$$K = \max_{0\le t\le T} \{ \sum_{i=1}^m |\xi_i^{(1)}(t)-\xi_i^{(0)}(t)|^2 + \sum_{j=1}^n |\eta_j^{(1)}(t)-\eta_j^{(0)}(t)|^2 \} .$$

Therefore $\xi_i^{(k)}(t)$ and $\eta_j^{(k)}(t)$ converge uniformly in $t \in [0,T]$ as $k \to \infty$. Put

$$\xi_i(t) = \lim_{k\to\infty} \xi_i^{(k)}(t) , \quad \eta_j(t) = \lim_{k\to\infty} \eta_j^{(k)}(t).$$

Taking the limit $k \to \infty$ in (47), we have

$$(48) \quad \begin{cases} \xi_i(t) = w_i(t) + \frac{1}{m} \sum_{\ell=1}^m \int_0^t b_{11}(\xi_i(s), \xi_\ell(s))ds \\ \qquad + \frac{1}{n} \sum_{\ell=1}^n \int_0^t b_{12}(\xi_i(s),\eta_\ell(s))ds - \varphi_i(t) , \\ \eta_j(t) = z_j(t) + \frac{1}{m} \sum_{\ell=1}^m \int_0^t b_{21}(\eta_j(s),\xi_\ell(s))ds \\ \qquad + \frac{1}{n} \sum_{\ell=1}^n \int_0^t b_{22}(\eta_j(s),\eta_\ell(s))ds + \psi_j(t) , \end{cases}$$

where

$$\varphi_i(t) = \lim_{k\to\infty} \sum_{\ell=1}^n \varphi_{i\ell}^{(k)}(t) , \quad \psi_j(t) = \lim_{k\to\infty} \sum_{\ell=1}^m \varphi_{\ell j}^{(k)}(t) .$$

It is clear that $\{\xi_i, \eta_j\}$ satisfy the condition

$$\max_{1\le i\le m} \xi_i(t) \le \min_{1\le j\le n} \eta_j(t) , \quad \text{for } 0 \le t \le T ,$$

because $\{\xi_i^{(k)}, \eta_j^{(k)}\}$ satisfy the above inequality. If $\xi_i(t) \neq \eta_j(t)$ for $0 \leq j \leq n$, then $\xi_i^{(k)}(t) \neq \eta_j^{(k)}(t)$ for $0 \leq j \leq n$ in a neighbourhood υ of t. Then $\sum_{\ell=1}^{n} \varphi_{i\ell}^{(k)}(t)$ is constant in υ and hence $\varphi_i(t)$ is constant in υ. This means that $\mathrm{supp}(d\varphi_i) \subset \{ t \geq 0 : \xi_i(t) = \eta_j(t)$ for some $1 \leq j \leq n\}$. The same property holds for ψ_j. It is clear that φ_i, ψ_j are continuous and monotone nondecreasing. Therefore $(\xi_i, \eta_j, \varphi_i, \psi_j)$ is a solution of (48) (i.e.(46) with φ_{ij} defined by $d\varphi_{ij} = d\varphi_i 1_{\{t:\xi_i(t)=\eta_j(t)\}}$). The uniqueness follows from Lemma 8.

To indicate the dependence of the solution on $\{w_i, z_j\}$, let us write

$$\xi_i(t) = \xi_i(t, w_1,\cdots,w_m, z_1,\cdots,z_n) ,$$

$$\eta_j(t) = \eta_j(t, w_1,\cdots,w_m, z_1,\cdots,z_n) .$$

If we define $X_i(t)$ and $Y_j(t)$ by

$$X_i(t) = \xi_i(t, w_1,\cdots,w_m, z_1,\cdots,z_n)$$

$$Y_j(t) = \eta_j(t, w_1,\cdots,w_m, z_1,\cdots,z_n)$$

evaluated at

$$w_i = X_i(0) + B_i^- , \quad 1 \leq i \leq m ,$$

$$z_j = Y_j(0) + B_j^+ , \quad 1 \leq j \leq n ,$$

then $(X_1(t),\cdots,X_m(t), Y_1(t),\cdots,Y_n(t))$ gives a unique solution of (42).

6. A system of nonlinear diffusion processes which is expected to be the limit as $m,n\to\infty$ under the constraint $m/(m+n)\to\theta$.

For $u \in \underline{\underline{P}}(\mathbb{R}^1) = \{$all probability distributions on $\mathbb{R}^1\}$ we define the segregating front $\gamma(u)$ (θ-quantile) of the probability distribution u by

$$(49)\qquad \gamma(u)=\gamma(u,\theta)=\min\{x : u((-\infty,x)) \leqq \theta \leqq u((-\infty,x])\} ,$$

and $b_{11}[x,u_\ell]$ and $b_{12}[x,u_r]$, $(x,u) \in \mathbb{R}^1 \times \underline{\underline{P}}(\mathbb{R}^1)$, by

$$(50)\qquad \begin{aligned} b_{11}[x,u_\ell] &= \int b_{11}(x,y)\frac{1}{\theta}1_{(-\infty,\gamma(u)]}(y)u(dy) , \\ b_{12}[x,u_r] &= \int b_{12}(x,y)\frac{1}{1-\theta}1_{(\gamma(u),\infty)}(y)u(dy) . \end{aligned}$$

$b_{21}[x,u_\ell]$ and $b_{22}[x,u_r]$ are defined in the same way.

Let $\{X_i(t), Y_j(t) : 1 \leqq i \leqq m , 1 \leqq j \leqq n\}$ be the solution of the system of SDE's (42) and $U(t)$ be the empirical distribution of the solution, i.e

$$(51)\qquad U(t) = \frac{1}{m+n}\{\sum_{i=1}^{m}\delta_{X_i(t)} + \sum_{j=1}^{n}\delta_{Y_j(t)}\} .$$

Then the equations (42) can be written in terms of $b_{ij}[\cdot,\cdot]$ defined by (50) with $\theta = m/(m+n)$ and of $U(t)$, as follows

$$(52)\quad \begin{cases} X_i(t)=X_i(0)+B_i^-(t)+\int_0^t b_{11}[X_i(s),U(s)_\ell]ds+\int_0^t b_{12}[X_i(s),U(s)_r]ds \\ \qquad\qquad - \sum_{k=1}^{n}\Phi_{ik}(t) , \quad 1 \leqq i \leqq m , \\ Y_j(t)=Y_j(0)+B_j^+(t)+\int_0^t b_{21}[Y_j(s),U(s)_\ell]ds+\int_0^t b_{22}[Y_j(s),U(s)_r]ds \\ \qquad\qquad + \sum_{k=1}^{m}\Phi_{kj}(t) , \quad 1 \leqq j \leqq n . \end{cases}$$

In fact, for example,

$$(53)\quad \frac{1}{m}\sum_{k=1}^{m} b_{11}(x,X_k(t)) = \frac{1}{m}\int b_{11}(x,y)1_{(-\infty,\gamma^{\cdot}(t)]}(y)\sum_{k=1}^{m}\delta_{X_k(t)}(dy)$$

where $\gamma^{\cdot}(t) = \gamma^{(m,n)}(t)$ denotes the front of X-particles:

$$(54)\qquad \gamma^{\cdot}(t) = \max_{1\le i\le m} X_i(t) .$$

Because of (44.a)*) we can add $\sum_{k=1}^{n}\delta_{Y_k(t)}(dy)$ formally, and hence the right hand side of (53) can be written as

$$\int b_{11}(x,y)\,\frac{m+n}{m}\,1_{(-\infty,\gamma^{\cdot}(t)]}(y)U(t,dy)$$

$$= b_{11}[x,U(t)_{\ell}] \text{ , almost everywhere in } t \text{ ,}$$

since $\gamma^{\cdot}(t)$ in (54) coincides with the one defined by (49) for $u = U(t)$ almost everywhere in t.

Assuming that the empirical distribution $U(t) = U^{(m,n)}(t)$ defined by (51) converges to a probability distribution $u(t)$, as $m,n\to\infty$ under the constraint $m/(m+n)\to\theta$, we take the limit formally in (52) (namely for i=1, j=1) . Then we obtain a system of stochastic differential equations

$$(55)\begin{cases} X(t)=X(0)+B^{-}(t)+\int_0^t b_{11}[X(s),u(s)_{\ell}]ds + \int_0^t b_{12}[X(s),u(s)_r]ds-\Phi(t), \\ Y(t)=Y(0)+B^{+}(t)+\int_0^t b_{21}[Y(s),u(s)_{\ell}]ds + \int_0^t b_{22}[Y(s),u(s)_r]ds+\Psi(t), \end{cases}$$

where it should be remarked that $u(t)$ is a fixed probability distribution $u(t) = \lim_{m,n\to\infty(\theta)} U^{(m,n)}(t)$.

---

*) $X_m(t)=Y_1(t)$ may happen at some $t\ge 0$, but the set of such t has the Lebesgue measure zero.

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In (55) the influence of the cloud $u(t)$ of particles onto $X(t)$ (resp. $Y(t)$) is divided into two parts according to the decomposition $u(t) = \theta u(t)_\ell + (1-\theta)u(t)_r$, where $u(t)_\ell$ (resp. $u(t)_r$) is a probability distribution on $(-\infty,\gamma(t)]$ (resp. on $(\gamma(t),\infty)$) . To get (55) we are also assuming that the segregating front $\gamma(t) = \gamma(u(t))$ of the $u(t)$ defined by (49) is the limit of the front $\gamma^{\bullet}(t) = \gamma^{(m,n)}(t)$ of X-particles defined by (54).

The limit $u(t)$ of the empirical distribution $U^{(m,n)}(t)$ will be characterized in §7 as the probability distribution of a solution of a nonlinear SDE on $\mathbb{R}^1$. Since we ignore the difference of $X_i(t)$ and $Y_j(t)$ when we observe them through the empirical distribution $U^{(m,n)}(t)$, the reflecting boundary condition does not appear explicitly in the equation which characterizes the limit distribution $u(t)$.

The propagation of chaos of the solution of (52) is a limit theorem stronger than the convergence $U^{(m,n)}(t) \to u(t)$, and it can be formulated as follows :

(Propagation of chaos) Let $\mu_{m,n}(t)$ be the probability distribution of the solution $(X_1(t),\cdots,X_m(t),Y_1(t),\cdots,Y_n(t))$ of (52) with an initial distribution which is (u_ℓ,u_r)-chaotic. Then $\{\mu_{m,n}(t) : m,n = 1,2,\cdots\}$ is a $(u(t)_\ell,u(t)_r)$-chaotic family of two groups; i.e. for $\forall M \le m$, $N \le n$ and $\forall f_1,\cdots,f_M,g_1,\cdots g_N \in C_0(\mathbb{R}^1)$

$$(56)\quad E\Big[\prod_{i=1}^{M} f_i(X_i(t)) \prod_{j=1}^{N} g_j(Y_j(t))\Big] \to \prod_{i=1}^{M} \langle u(t)_\ell, f_i\rangle \prod_{j=1}^{N} \langle u(t)_r, g_j\rangle ,$$

as $m,n\to\infty$ with the constraint $m/(m+n) \to \theta$.

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The propagation of chaos as stated above will be shown in §8 with the help of Lemma 7 in §3.

Remark. Consider SDE's (55) under the conditions (i) $u(s)_\ell$ (resp. $u(s)_r$) is the probability distribution of $X(s)$ (resp. $Y(s)$), (ii) $X(t) \leq \gamma(t) \leq Y(t)$, for $\forall\, t \geq 0$, (iii) $\mathrm{supp}(d\Phi) \subset \{t \geq 0 : X(t) = \gamma(t)\}$ and $\mathrm{supp}(d\Psi) \subset \{t \geq 0 : Y(t) = \gamma(t)\}$. These conditions are not enough to specify a unique solution of (55). To do so we must prescribe $\gamma(t)$ in advance. In our case $\gamma(t)$ is specified by (iv) $u(t,(\infty,\gamma(t)]) = \theta$ in terms of the limit distribution $u(t)$ of the empirical distribution $U^{(m,n)}(t)$. This means that we must know the limit distribution $u(t)$ (or at least a candidate for it) beforehand, not by solving the SDE's (55) but by some other means. This will be done in the next section. If this is done, we can paraphrase the propagation of chaos in terms of the solution of (55) as follows.

(Propagation of chaos) The (M,N) components $(X_1(t),\cdots, X_M(t),Y_1(t),\cdots,Y_N(t))$ of the solution of (52) with an initial distribution which is (u_ℓ,u_r)-chaotic converge in law to (M,N)-independent copies $(X_{(1)}(t),\cdots,X_{(M)}(t),Y_{(1)}(t),\cdots,Y_{(N)}(t))$ of the solution $(X(t),Y(t))$ of (55) with the conditions (ii), (iii) and (iv) in the above remark, as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$.*)

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*) It will be shown in Lemma 17 of §8 that $(X(t),Y(t))$ is $u(t)_\ell \otimes u(t)_r$-distributed.

7. An associated nonlinear stochastic differential equation

Let $\{b_{ij}(x,y) : i,j = 1,2\}$ be bounded measurable functions on $\mathbb{R}^2$ with the bound

(57) $$\beta = \max_{i,j=1,2} \| b_{ij} \|_\infty < \infty ,$$

from which we define $b[x,u]$, $(x,u) \in \mathbb{R}^1 \times \underline{\underline{P}}(\mathbb{R}^1)$ as follows: Let $\gamma(u)$, $u \in \underline{\underline{P}}(\mathbb{R}^1)$, be the segregating front of u defined by (49) for a given $0 < \theta < 1$ (fixed throughout the discussion) and define

(58) $$b[x,u] = \int_{\mathbb{R}^1} b(x,y,u)u(dy) ,$$

where

(59) $$b(x,y,u)=\begin{cases} b_{11}(x,y)\frac{1}{\theta}1_{(-\infty,\gamma(u)]}(y) + b_{12}(x,y)\frac{1}{1-\theta}1_{(\gamma(u),\infty)}(y) , & x \leq \gamma(u), \\ b_{21}(x,y)\frac{1}{\theta}1_{(-\infty,\gamma(u)]}(y) + b_{22}(x,y)\frac{1}{1-\theta}1_{(\gamma(u),\infty)}(y) , & x > \gamma(u) . \end{cases}$$

It is clear that

(60) $$|b[x,u]| \leqq 2\beta .$$

We consider a nonlinear SDE on $\mathbb{R}^1$ with the drift coefficient $b[x,u]$ defined by (58) :

(61) $$X(t) = X(0) + B(t) + \int_0^t b[X(s),u(s)]ds ,$$

where $u(s)$ denotes the probability distribution of $X(s)$ and $B(t)$ is the 1-dimensional Brownian motion which is independent of the initial value $X(0)$ distributed according to $u(0) = u_0 \in \underline{\underline{P}}(\mathbb{R}^1)$.

The goal of this section is the following theorem.

Theorem 2. There exists a unique (pathwise) solution of the nonlinear SDE (61).

A proof of Theorem 2 will be given in a series of Lemmas.

Let P denote the Wiener measure on $W = C(\mathbb{R}^+,\mathbb{R}^1)$ with $P[w(0) \in dx] = u_0(dx)$ and $B(W)$ be the usual coordinate σ-field in W. In terms of (Cameron-Martin) Maruyama formula [4], one can rephrase the problem of solving the SDE (61) in the following equivalent form: Find a probability measure Q on W satisfying

(62) $$\frac{dQ}{dP} = M ,$$

with the Maruyama density

(63) $$M = \exp\{\int_0^T b[w(t),u^M(t)]dw(t) - \frac{1}{2}\int_0^T |b[w(t),u^M(t)]|^2 dt\} ,$$

where

$u^M(t) =$ the probability distribution of $w(t)$ with respect to the probability measure $Q = MP$.

That is, $\langle u^M(t),f\rangle = \int_W f(w(t))Q(dw) = \int_W f(w(t))M(w)P(dw)$. The existence of such a probability measure Q will be shown by virture of the fixed point theorem of Schauder-Tychonoff in the following.*)

In $L^2(P) = L^2(W,P)$ we define a subset K by

(64) $$K = \{M : \text{(i)} \ \|M\|_{L^2(P)} \leq c , \ \text{(ii)} \ \langle M,1\rangle_{L^2(P)} = 1 , \ \text{and} \ \text{(iii)} \int_A MdP \geq \frac{1}{4} c^{-2} \{P(A)\}^2, \text{ for } \forall A \in B(W)\}$$

(*) The idea of applying a fixed point theorm was suggested by Th. Brox. H.Föllmer has given a proof in $L^1(P)$ defining a convex set in terms of entropy.

where c is a constant defined by

(65) $$c = e^{2\beta^2 T} .$$

It is clear that the set K defined above is a convex and weakly compact subset of $L^2(P)$.

On the set K we define a mapping

(66) $$F : M \to \tilde{M}$$

where

(67) $$\tilde{M} = \exp\{\int_0^T b[w(t),u^M(t)]dw(t)-\frac{1}{2}\int_0^T |b[w(t),u^M(t)]|^2 dt\} ,$$

and $u^M(t)$ is the probability distribution of $w(t)$ with respect to the probability measure MdP on W. A fixed point $M \in K$ of the mapping F solves our problem of finding out a solution of (62).

<u>Lemma 9.</u> The mapping F defined by (66) maps K into itself.

<u>Proof</u>. For $\tilde{M}$ the first condition (i) of (64) is satisfied, because

$$\|\tilde{M}\|^2_{L^2(P)} = \int_W dP \exp\{ 2\int_0^T b[w(t),u^M(t)]dw(t) - \int_0^T |b[w(t),u^M(t)]|^2 dt\}$$
$$\leq e^{4\beta^2 T} = c^2 .$$

The second condition (ii) $\langle\tilde{M},1\rangle_{L^2(P)} = 1$ is obvious. For the third condition (iii) we have

$$\int_W \frac{1}{\tilde{M}} dP = \int_W dP \exp\{-\int_0^T b[w(t),u^M(t)]dw(t) + \frac{1}{2}\int_0^T |b[w(t),u^M(t)]|^2 dt\}$$
$$\leq e^{4\beta^2 T} = c^2 ,$$

and hence

$$P[\ \frac{1}{\tilde{M}} > a\ ] \leq c^2/a\ ,$$

that is,

(68) $$P[\tilde{M} \geq \frac{1}{a}] \geq 1 - \frac{c^2}{a}.$$

Therefore, with $C = \{w: \tilde{M}(w) \geq 1/a\}$

$$\int_A \tilde{M}dP \geq \int_{A\cap C} \frac{1}{a}dP = \frac{1}{a}P[\ A\cap C\ ] \geq \frac{1}{a}\{P[A] + P[C] - 1\}$$
$$\geq \frac{1}{a}\{P[A] - \frac{c^2}{a}\}$$

where made is use of the estimate (68). If we choose the constant a to be $2c^2/P[A]$, then we have

$$\int_A \tilde{M}dP \geq \frac{1}{4c^2}\{\ P[A]\ \}^2\ .$$

Thus we have $\tilde{M} \in K$, completing the proof.

<u>Lemma 10</u>. (i) If u_n converges weakly to u which satisfies the positivity condition (20), then the segregating front $\gamma(u_n)$ of u_n converges to the segregating front $\gamma(u)$ of u .
(ii) Let $M_n \in K$ converge weakly to $M \in K$ as $n \to \infty$, and $u_n(t)$ (resp. $u(t)$) be the probability distribution of $w(t)$ with respect to $M_n dP$ (resp. MdP). Then $u_n(t)$ converges weakly to $u(t)$ which satisfies the positivity condition (20), and hence $\gamma(u_n(t))$ converges to $\gamma(u(t))$ for each t .

<u>Proof</u>. (i) Writing $\gamma = \gamma(u)$, we have for each $\varepsilon > 0$

(69) $$u((-\infty,\gamma-\varepsilon)) < u((-\infty,\gamma)) = \theta = u((-\infty,\gamma]) < u((-\infty,\gamma+\varepsilon))\ ,$$

because u satisfies the positivity condition (20). Assume that u_n converges weakly to u. Then $\overline{\lim}_{n\to\infty} u_n((-\infty,\gamma-\varepsilon]) \leq u((-\infty,\gamma-\varepsilon])$ and $\underline{\lim}_{n\to\infty} u_n((-\infty,\gamma+\varepsilon)) \geq u((-\infty,\gamma+\varepsilon))$. Hence (69) implies that

(70) $$u_n((-\infty,\gamma-\varepsilon]) < \theta \quad \text{and} \quad \theta < u_n((-\infty,\gamma+\varepsilon))$$

for sufficiently large n. However, this means, in view of (49) defining $\gamma(u_n)$, that

(71) $$\gamma-\varepsilon \leqq \gamma(u_n) \leqq \gamma+\varepsilon \ .$$

Therefore $\gamma(u_n)$ converges to $\gamma(u)$ as $n \to \infty$.

(ii) If M_n converges to M weakly, then for $\forall\ f \in C_0(\mathbb{R}^1)$

$$\langle f, u_n(t)-u(t)\rangle = \langle M_n - M, f(w(t))\rangle_{L^2(P)} \to 0 \ .$$

Thus $u_n(t)$ converges to $u(t)$ weakly as $n \to \infty$. Because of (iii) in (64),

$$u(t,(\gamma-\varepsilon,\gamma]) = \int_A M dP \geqq \frac{1}{c^2}(P[A])^2 > 0 \ ,$$

where $A = \{w : \gamma(t)-\varepsilon < w(t) \leqq \gamma(t)\}$, i.e., $u(t)$ satisfies the positivity condition (20). Therefore $\gamma(u_n(t))$ converges to $\gamma(u(t))$ by (i), which completes the proof of the assertion (ii).

Lemma 11. Under the same notation as in (ii) of Lemma 10,

(72) $$\lim_{n\to\infty} b[x,u_n(t)] = b[x,u(t)] \ , \text{ for } \forall\, x \neq \gamma(t) = \gamma(u(t)) \text{ and } \forall t > 0.$$

Proof. Let us assume that $x < \gamma(t)$ (the same argument applies for $x > \gamma(t)$). We may, then, also assume $x < \gamma_n(t) = \gamma(u_n)$ by Lemma 10. Therefore, by the definition of $b[x,u]$ given in (58), we have

$$b[x,u_n(t)] = \int b_{11}(x,w(t))1_{\{w(t)\leqq\gamma_n(t)\}}(w)\ M_n(w)dP$$
$$+\int b_{12}(x,w(t))1_{\{w(t)>\gamma_n(t)\}}(w)\ M_n(w)dP \ ,$$

which converges to

$$\int b_{11}(x,w(t))1_{\{w(t)\leqq\gamma(t)\}}(w)\ M(w)dP$$
$$+\int b_{12}(x,w(t))1_{\{w(t)>\gamma(t)\}}(w)\ M(w)dP$$

$$= b[x,u(t)] ,$$

as $n\to\infty$, by Lemma 10 and the assumption that $M_n \to M$ weakly in $L^2(P)$.

Lemma 12. The mapping F from K into K defined by (66) is weakly continuous.

Proof. Let us write $\tilde{M}_n = F(M_n)$ and $\tilde{M} = F(M)$ and assume that $M_n \to M$ weakly. Using the same notation as in Lemma 10, we have

$$\begin{aligned}&\langle \tilde{M}_n, f\rangle_{L^2(P)} \\ &= \int dPf(w)\exp\{\int_0^T b[w(t),u_n(t)]dw(t) - \frac{1}{2}\int_0^T |b[w(t),u_n(t)]|^2 dt\} ,\end{aligned}$$

which converges to

$$\begin{aligned}&\int dPf(w)\exp\{\int_0^T b[w(t),u(t)]dw(t) - \frac{1}{2}\int_0^T |b[w(t),u(t)]|^2 dt\} \\ &= \langle \tilde{M}, f\rangle_{L^2(P)} , \quad \text{for } \forall f \in L^2(P) ,\end{aligned}$$

because

$$\lim_{n\to\infty} \int_W dP \int_0^T dt |b[w(t),u_n(t)] - b[w(t),u(t)]|^2 = 0$$

by Lemma 11. This completes the proof of Lemma 12.

Therefore, applying the fixed point theorem of Schauder-Tychonoff (cf.e.g. [1]) to the mapping F on K, we obtain $M \in K$ such that $dQ = MdP$ solves the problem (62), in other words, there exists a solution of SDE (61).

Lemma 13. The probability distribution $u(t)$ of a solution $X(t)$ of the SDE (61) is uniquely determined by the initial distribution $u_0 \in P(\mathbb{R}^1)$.

Proof. Under the assumption that u_0 satisfies the positivity condition (20), one can get this uniqueness result indirectly through the proof of the propagation of chaos. Taking a solution $X(t)$ from those of the SDE (61), we will prove in the next section that the empirical distribution $U^{(m,n)}(t)$ of the solutions of (42) (or (52)) converges weakly to the probability distribution $u(t)$ of $X(t)$. Therefore $u(t)$ is uniquely determined.

An analytic proof for Lemma 13 due to H.Amann, without assuming (20) for u_0, will be given in Appendix.

Let us denote $b(t,x) = b[x,u(t)]$ and consider an SDE on $\mathbb{R}^1$

$$X(t) = X(0) + B(t) + \int_0^t b(s,X(s))ds .$$

Since $b(t,x)$ is bounded and measurable in (t,x), we can conclude by Zvonkin's theorem [19] (cf. also Veretennikov [17]) that $X(t)$ is actually the strong solution and the pathwise uniqueness holds. This completes the proof of Theorem 2.

In the next section we need the following facts.

Lemma 14. Let $u(t)$ be the probability distribution of a solution of the SDE (61) and $\gamma(t) = \gamma(u(t))$. Then $\gamma(t)$ is continuous in $(0,\infty)$. If $u(0)$ satisfies the positivity condition (20), then $\gamma(t)$ is continuous on $[0,\infty)$.

8. Propagation of chaos for interacting diffusion processes of two types

Given a probability distribution u on $\mathbb{R}^1$ satisfying the positivity condition (20), we define u_ℓ and u_r by (30). Given a (u_ℓ,u_r)-chaotic family $\{\mu_{m,n} \in \underline{\underline{P}}_{m,n} : m,n = 1,2,\cdots\}$ of two groups, we consider the solution

$$(73) \qquad \underline{\underline{X}}^{(m,n)}(t) = (X_1(t),\cdots,X_m(t),Y_1(t),\cdots,Y_n(t))$$

of the SDE (42) with the initial distribution $\mu_{m,n}$, where the drift coefficients $b_{ij}(x,y)$, $i,j = 1,2$, are assumed to be bounded and with the Lipshitz condition (45).

Theorem 3. (Propagation of chaos) Let $\mu_{m,n}(t)$ be the probability distribution of $\underline{\underline{X}}^{(m,n)}(t)$ given in (73). Let $u(t)_\ell$ and $u(t)_r$ be defined through the decomposition $u(t) = \theta u(t)_\ell + (1-\theta)u(t)_r$, where $u(t)$ is the probability distribution, at time $t \geq 0$, of a unique solution of the SDE (61) with the initial distribution u. Then $\{\mu_{m,n}(t) : m,n = 1,2,\cdots\}$ is a $(u(t)_\ell,u(t)_r)$-chaotic family of two groups, i.e., for arbitrary but fixed $M \leq m$, $N \leq n$, $\forall f_1,\cdots,f_M,g_1,\cdots,g_N \in C_b(\mathbb{R}^1)$ and $t \geq 0$,

$$(74) \qquad E[\prod_{i=1}^{M} f_i(X_i(t)\prod_{j=1}^{N} g_j(Y_j(t))] \to \prod_{i=1}^{M} \langle u(t)_\ell, f_i\rangle \prod_{j=1}^{N} \langle u(t)_r, g_j\rangle ,$$

as $m,n \to \infty$ with the constraint $m/(m+n) \to \theta$. Moreover, the empirical distribution $U^{(m,n)}(t)$ of the $\underline{\underline{X}}^{(m,n)}(t)$ converges to $u(t)$ in the sense that for any $f \in C_0(\mathbb{R}^1)$

$$(75) \qquad \langle U^{(m,n)}(t),f\rangle \to \langle u(t),f\rangle \qquad \text{(in probability)}$$

as $m,n \to \infty$ (θ), and consequently $\gamma^{(m,n)}(t) = \max_{1\leq i\leq m} X_i(t)$ converges in probability to the segregating front $\gamma(t)$ of $u(t)$.

Proof. Pick up an arbitrary solution of the SDE (61) with the initial distribution u and denote the probability distribution of the solution at time $t \geq 0$ by $u(t)$. We will prove that (74) and (75) hold for this $u(t)$ without knowing that $u(t)$ is uniquely determined from the initial value u. The uniqueness of $u(t)$ will then follow from (75).

Let $b^u(t,x) = b[x,u(t)]$ and $\underline{\underline{y}}(t) = (y_1(t),\cdots,y_{m+n}(t))$ be the solution of (31) with $b(t,x) = b^u(t,x)$ having the initial distribution $\nu_{m,n} = \pi_{m,n}^{-1}\mu_{m,n}$. Let $\underline{\underline{\tilde{x}}}(t) = (\tilde{x}_1(t),\cdots,\tilde{x}_{m+n}(t))$ be the reflected process (33) of two segregated groups constructed in §3. Its initial distribution is $\mu_{m,n}$. We write

$$\text{(76)} \qquad \begin{cases} \tilde{X}_i(t) = \tilde{x}_i(t), & \text{for } 1 \leq i \leq m, \\ \tilde{Y}_j(t) = \tilde{x}_{m+j}(t), & \text{for } 1 \leq j \leq n, \end{cases}$$

and $\underline{\underline{\tilde{X}}}(t) = (\tilde{X}_1(t),\cdots,\tilde{X}_m(t),\tilde{Y}_1(t),\cdots,\tilde{Y}_n(t))$. Lemma 2 claims that

$$\text{(77)} \qquad \begin{cases} \tilde{X}_i(t) = \tilde{X}_i(0) + B_i^-(t) + \int_0^t b^u(s,\tilde{X}_i(s))ds - \sum_{k=1}^n \Phi_{ik}(t), & 1\leq i\leq m, \\ \tilde{Y}_j(t) = \tilde{Y}_j(0) + B_j^+(t) + \int_0^t b^u(s,\tilde{Y}_j(s))ds + \sum_{k=1}^m \Phi_{kj}(t), & 1\leq j\leq n, \end{cases}$$

hold. If necessary we denote $\underline{\underline{\tilde{X}}}^{(m,n)}(t)$ for $\underline{\underline{\tilde{X}}}(t)$ to indicate the dependence on (m,n).

Let $\tilde{U}^{(m,n)}(t)$ be the empirical distribution of $\underline{\underline{\tilde{X}}}^{(m,n)}(t)$. Then it is clear by the definition of $\underline{\underline{\tilde{X}}}^{(m,n)}(t)$ that

$$\text{(78)} \qquad \tilde{U}^{(m,n)}(t) = \frac{1}{m+n}\Big\{ \sum_{i=1}^m \delta_{\tilde{X}_i(t)} + \sum_{j=1}^n \delta_{\tilde{Y}_j(t)} \Big\} = \frac{1}{m+n}\sum_{i=1}^{m+n} \delta_{y_i(t)}.$$

What is important here is that the two processes $\tilde{\underline{\underline{X}}}(t)$ and $\underline{\underline{y}}(t)$ do not show up any difference if we observe them through their empirical distributions. Therefore, by (32) the empirical distribution $\tilde{U}^{(m,n)}(t)$ converges weakly to $u(t)$. Moreover, Lemma 7 claims that the family of distributions $\tilde{\mu}_{m,n}(t)$ of $\tilde{\underline{\underline{X}}}^{(m,n)}(t)$, $m,n = 1,2,\cdots$, forms a $(u(t)_\ell, u(t)_r)$-chaotic family of two groups. We state these facts as a lemma for later reference.

Lemma 15. (i) The empirical distribution $\tilde{U}^{(m,n)}(t)$ of the reflected process $\tilde{\underline{\underline{X}}}^{(m,n)}(t)$ of two segregated groups converges weakly to the probability distribution $u(t)$.
(ii) The family of probability distributions of $\tilde{\underline{\underline{X}}}^{(m,n)}(t)$, $m,n = 1,2,\cdots$, is a $(u(t)_\ell, u(t)_r)$-chaotic family of two groups, i.e., for arbitrary but fixed $M \leq m$, $N \leq n$ and $\forall f_1,\cdots,f_M$, $g_1,\cdots,g_N \in C_0(\mathbb{R}^1)$

$$E\Big[\prod_{i=1}^{M} f_i(\tilde{X}_i^{(m,n)}(t)) \prod_{j=1}^{N} g_j(\tilde{Y}_j^{(m,n)}(t))\Big] \to \prod_{i=1}^{M} \langle u(t)_\ell, f_i\rangle \prod_{j=1}^{N} \langle u(t)_r, g_j\rangle ,$$

as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$.

Next we note that the system (52) which is the same as (42) can be written as

$$(79)\begin{cases} X_i(t) = X_i(0) + B_i^-(t) + \int_0^t b^{(m,n)}[X_i(s), U^{(m,n)}(s)]ds - \sum_{k=1}^{n} \Phi_{ik}(t) , \\ \qquad\qquad 1 \leq i \leq m , \\ Y_j(t) = Y_j(0) + B_j^+(t) + \int_0^t b^{(m,n)}[Y_i(s), U^{(m,n)}(s)]ds + \sum_{k=1}^{m} \Phi_{kj}(t) , \\ \qquad\qquad 1 \leq j \leq n , \end{cases}$$

where $U^{(m,n)}(t)$ denotes the empirical distribution of $(X_1(t), \cdots, X_m(t), Y_1(t), \cdots, Y_n(t))$ and $b^{(m,n)}[x,u]$ is defined by

$$b^{(m,n)}[x,u] = \begin{cases} b_{11}[x,u_\ell] + b_{12}[x,u_r] , & \text{for } x > \gamma(u) , \\ b_{21}[x,u_\ell] + b_{22}[x,u_r] , & \text{for } x \leq \gamma(u) , \end{cases}$$

with $\theta = m/(m+n)$ in (58) and (59). Since we are interested only in the probability law of the solution $\underline{\underline{X}}^{(m,n)}(t)$ of (79), we may assume that

(80) $$\underline{\underline{X}}^{(m,n)}(0) = \tilde{\underline{\underline{X}}}^{(m,n)}(0) .$$

We will show in the following that the difference $\underline{\underline{X}}^{(m,n)}(t) - \tilde{\underline{\underline{X}}}^{(m,n)}(t)$ " converges to zero in L^2 " as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$, and hence $\underline{\underline{X}}^{(m,n)}(t)$ is $(u(t)_\ell, u(t)_r)$-chaotic, because so is $\tilde{\underline{\underline{X}}}^{(m,n)}(t)$ by Lemma 15.

<u>Lemma 16</u>. Under the assumption (80)

$$\lim_{m,n\to\infty(\theta)} E[|X_i^{(m,n)}(t)-\tilde{X}_i^{(m,n)}(t)|^2 + |Y_j^{(m,n)}(t)-\tilde{Y}_j^{(m,n)}(t)|^2] = 0 .$$

<u>Proof</u>. Applying (ii) of Lemma 8 to the SDE's (77) and (79), we have

(81) $$\sum_{i=1}^{m} |X_i(t) - \tilde{X}_i(t)|^2 + \sum_{j=1}^{n} |Y_j(t) - \tilde{Y}_j(t)|^2$$
$$\leq e^T\{ \sum_{i=1}^{m} \int_0^t ds |b^{(m,n)}[X_i(s),U(s)] - b[\tilde{X}_i(s),u(s)]|^2 + \sum_{j=1}^{n} \int_0^t ds |b^{(m,n)}[Y_j(s),U(s)] - b[\tilde{Y}_j(s),u(s)]|^2\} ,$$

for $t \in [0,T]$, where $U(t)$ stands for $U^{(m,n)}(t)$. Each term of the right hand side of (81) can be estimated as follows: Writing

$$I_1 = |b^{(m,n)}[X_i(s),U(s)] - b^{(m,n)}[\tilde{X}_i(s),\tilde{U}(s)]| ,$$

$$I_2 = |b^{(m,n)}[\tilde{X}_i(s),\tilde{U}(s)] - b[\tilde{X}_i(s),u(s)]| ,$$

where $\tilde{U}(t)$ stands for $\tilde{U}^{(m,n)}(t)$, we have

$$|b^{(m,n)}[X_i(s),U(s)] - b[\tilde{X}_i(s),u(s)]|^2 \leqq 2(I_1^2 + I_2^2) .$$

To estimate I_1 we notice that $b^{(m,n)}[\cdot,\cdot]$ in I_1 is defined in terms of $b_{11}[\cdot,\cdot]$ and $b_{12}[\cdot,\cdot]$ and hence

$$I_1^2 \leqq 2\{\frac{1}{m}\sum_{k=1}^{m}|b_{11}(X_i,X_k)-b_{11}(\tilde{X}_i,\tilde{X}_k)|^2+\frac{1}{n}\sum_{k=1}^{n}|b_{12}(X_i,Y_k)-b_{12}(\tilde{X}_i,\tilde{Y}_k)|^2\}$$

$$\leqq 2c\{2|X_i-\tilde{X}_i|^2 + \frac{1}{m}\sum_{k=1}^{m}|X_k-\tilde{X}_k|^2 + \frac{1}{n}\sum_{k=1}^{n}|Y_k-\tilde{Y}_k|^2\} ,$$

where the Lipschitz condition (45) is employed. One can assume that $(m+n)/m$, $(m+n)/n \leqq$ const., because $m/(m+n) \to \theta$, $0 < \theta < 1$. Hence

$$(82) \quad |I_1|^2 \leqq \text{const.}\{|X_i-\tilde{X}_i|^2 + \frac{1}{m+n}(\sum_{k=1}^{m}|X_k-\tilde{X}_k|^2 + \sum_{k=1}^{n}|Y_k-\tilde{Y}_k|^2)\}.$$

To estimate I_2 we notice that the front $\tilde{\gamma}^{(m,n)} = \gamma(\tilde{U}^{(m,n)}(t))$ converges to $\gamma = \gamma(u(t))$ by Lemma 10,(i), because $\tilde{U}^{(m,n)}(t)$ converges weakly to $u(t)$ by Lemma 15 and $u(t)$ satisfies the positivity condition (20). Therefore, for $x < \gamma$ (hence we may assume $x < \tilde{\gamma}^{(m,n)}$)

$$b^{(m,n)}[x,\tilde{U}^{(m,n)}] = \int b_{11}(x,y)\frac{m+n}{m}1_{(-\infty,\tilde{\gamma}^{(m,n)}]}(y)\tilde{U}^{(m,n)}(dy)$$
$$+\int b_{12}(x,y)\frac{m+n}{n}1_{(\tilde{\gamma}^{(m,n)},\infty)}(y)\tilde{U}^{(m,n)}(dy)$$

converges to

$$b[x,u] = \int b_{11}(x,y)\frac{1}{\theta}1_{(-\infty,\gamma]}(y)u(dy)+\int b_{12}(x,y)\frac{1}{1-\theta}1_{(\gamma,\infty)}(y)u(dy) ,$$

as $m,n \to \infty$ under the constraint $m/(m+n) \to \theta$, and hence

(83) $$E[\int_0^T |I_2|^2 ds] \to 0 .$$

Combining (81) with (82) and (83), we have

(84) $$\sum_{i=1}^m E|X_i(t)-\tilde{X}_i(t)|^2 + \sum_{j=1}^n E|Y_j(t)-\tilde{Y}_j(t)|^2$$

$$\leqq \text{const.} \sum_{i=1}^m \int_0^t ds\{E|X_i(s)-\tilde{X}_i(s)|^2 + \frac{1}{m+n}(\sum_{k=1}^m E|X_k(s)-\tilde{X}_k(s)|^2 + \sum_{k=1}^n E|Y_k(s)-\tilde{Y}_k(s)|^2)\}$$

$$+ \text{const.} \sum_{j=1}^n \int_0^t ds\{E|Y_j(s)-\tilde{Y}_j(s)|^2 + \frac{1}{m+n}(\sum_{k=1}^m E|X_k(s)-\tilde{X}_k(s)|^2 + \sum_{k=1}^n E|Y_k(s)-\tilde{Y}_k(s)|^2)\}$$

$$+ \sum_{i=1}^m o(1) + \sum_{j=1}^n o(1) .$$

Since expectations $E|X_i(t)-\tilde{X}_i(t)|^2$ (resp. $E|Y_j(t)-\tilde{Y}_j(t)|^2$) do not depend on i (resp. on j), it follows from (84) that

(85) $$\frac{m}{m+n}E|X_i(t)-\tilde{X}_i(t)|^2 + \frac{n}{m+n}E|Y_j(t)-\tilde{Y}_j(t)|^2$$

$$\leqq \text{const.} \int_0^t ds\{\frac{m}{m+n}E|X_i(s)-\tilde{X}_i(s)|^2 + \frac{n}{m+n}E|Y_j(s)-\tilde{Y}_j(s)|^2\} + o(1) ,$$

independently of (i,j). Since $b^{(m,n)}[\cdot,\cdot]$ and $b[\cdot,\cdot]$ are bounded, the integral of the right hand side of (85) is finite by (81). Therefore, by Gronwall's lemma, it holds that the left hand side of (85) is bounded by $o(1)$, that is, it converges to zero as $m,n \to \infty$ with the constraint $m/(m+n) \to \theta$, independently of (i,j). This completes the proof of Lemma 16.

Now we can complete the proof of the theorem as follows. Combining Lemma 15 with Lemma 16, we get (74). Since $\{\mu_{m,n}(t)\}$ is a $(u(t)_\ell, u(t)_r)$-chaotic family of two groups, the family $\{\nu_{m,n}(t)\}$ defined by (26) from $\{\mu_{m,n}(t)\}$ is a $u(t)$-chaotic family by Lemma 5. Therefore, the empirical distribution $U^{(m,n)}(t)$ converges to $u(t)$, because $U^{(m,n)}(t)$ does not depend on permutations of coordinates. The convergence of $\gamma^{(m,n)}(t)$ to $\gamma(t)$ follows from Lemma 10. This completes the proof of the theorem.

In order to characterize the probability distributions $u(t)_\ell$ and $u(t)_r$, we consider the system of SDE's (55) under the conditions

$$(86)\quad \begin{cases} u(t,(-\infty,\gamma(t)]) = \theta\ ,^{(*)} \\ X(t) \leqq \gamma(t) \leqq Y(t)\ , \text{ for } \forall\ t \geqq 0, \\ \operatorname{supp}(d\Phi) \subset \{t\geqq 0:\ X(t) = \gamma(t)\}\ , \\ \operatorname{supp}(d\Psi) \subset \{t\geqq 0:\ Y(t) = \gamma(t)\}\ . \end{cases}$$

Then (55) is equivalent to the following system of Skorokhod equations on $(-\infty,0]$ and $[0,\infty)$

$$(87)\quad \begin{cases} \bar{X}(t) = X(0) + B^-(t) - \gamma(t) + \int_0^t b[\bar{X}(s)+\gamma(s),u(s)]ds - \Phi(t) \\ \bar{Y}(t) = Y(0) + B^+(t) - \gamma(t) + \int_0^t b[\bar{Y}(s)+\gamma(s),u(s)]ds + \Psi(t) \end{cases}$$

where $\bar{X}(t) \leqq 0 \leqq \bar{Y}(0)$, $\operatorname{supp}(d\Phi) \subset \{t\geqq 0:\ \bar{X}(t) = 0\}$ and $\operatorname{supp}(d\Psi) \subset \{t\geqq 0:\ \bar{Y}(t) = 0\}$. We assume that $(X(0),Y(0))$ is distributed according to $u(0)_\ell \otimes u(0)_r$. The unique solution for (87) exists clearly and satisfies

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(*) It should be noticed that $u(t)$ here and in (87) is the fixed one, i.e., the probability distribution, at time t, of the unique solution of the SDE (61).

$$X(t) = \bar{X}(t) + \gamma(t)$$
$$Y(t) = \bar{Y}(t) + \gamma(t) .$$

Lemma 17. The solution $(X(t),Y(t))$ of the system of SDE's (55) under the conditions (86) is $u(t)_\ell \otimes u(t)_r$-distributed.

Proof. Let $\underset{=}{\tilde{X}}(t) = (\tilde{X}_1(t),\cdots,\tilde{X}_m(t),\tilde{Y}_1(t),\cdots,\tilde{Y}_n(t))$ be the reflected process of two segregated groups, which satisfies the system of SDE's (77). One can transform it into a system of Skorokhod equations on $(-\infty,0]$ and $[0,\infty)$:

$$(88)\quad \begin{cases} \tilde{X}_i(t)-\gamma_\ell(t) = \tilde{X}_i(0)+B_i^-(t)-\gamma_\ell(t)+\int_0^t b[\tilde{X}_i(s),u(s)]ds-\sum_{k=1}^{n}\Phi_{ik}(t) , \\ \tilde{Y}_j(t)-\gamma_r(t) = \tilde{Y}_j(0)+B_j^+(t)-\gamma_r(t)+\int_0^t b[\tilde{Y}_j(s),u(s)]ds+\sum_{k=1}^{m}\Phi_{kj}(t) , \end{cases}$$

where

$$(89)\qquad \gamma_\ell(t) = \max_{1\leq i\leq m} \tilde{X}_i(t) , \quad \gamma_r(t) = \min_{1\leq j\leq n} \tilde{Y}_j(t) .$$

The probability distribution of $(\tilde{X}_i(t),\tilde{Y}_j(t))$, for fixed (i,j), converges to $u(t)_\ell \otimes u(t)_r$ by Lemma 15. Therefore the fronts $\gamma_\ell(t)$ and $\gamma_r(t)$ converge to $\gamma(t)$ by Lemma 10. Therefore $(\tilde{X}_i(t),\tilde{Y}_j(t))$ converges in law to $(X(t),Y(t))$ as $m,n \to \infty$ with the constraint $m/(m+n) \to \theta$.(*) This means that $(X(t),Y(t))$ is distributed by $u(t)_\ell \otimes u(t)_r$, completing the proof.

(*) To be precise, we proceed through the Skorokhod's realization of almost sure convergence.

Therefore, the SDE (55) is actually a system of nonlinear stochastic differential equations under the conditions (86), since $u(t)_\ell$ (resp. $u(t)_r$) is the probability distribution of $X(t)$ (resp. $Y(t)$). Accordingly, we can state the propagation of chaos (Theorem 3) in terms of the solution of the SDE's (55) as follows:

Theorem 3'. (Propagation of chaos) Under the same assumptions as in Theorem 3, the (M,N)-components $(X_1(t),\cdots,X_M(t), Y_1(t),\cdots,Y_N(t))$ of the solution of (42) converge in law to the independent (M,N)-copies $(X_{(1)}(t),\cdots,X_{(M)}(t),Y_{(1)}(t),\cdots,Y_{(N)}(t))$ of the solution $(X(t),Y(t))$ of the nonlinear SDE's (55) with the conditions (86) as $m,n \to \infty$, where $(X(t),Y(t))$ is $u(t)_\ell \otimes u(t)_r$-distributed and $u(t) = \theta u(t)_\ell + (1-\theta)u(t)_r$ is the probability distribution of the unique solution of the nonlinear SDE (61).

Appendix

Thanks to H.Amann we can avoid the long detour through the proof of the propagation of chaos to show the uniqueness of solutions of the nonlinear SDE (61). Here is presented his proof.

First we consider an integral equation on C_0^* = the dual Banach space of $C_0 = C_0(\mathbb{R}^1)$: (*)

(90) $$u(t) = P_t^* u_0 + \int_0^t P_{t-s}^* G(u(s))ds \quad , \quad u(0) = u_0 \in C_0^* \quad ,$$

where P_t^* is the dual of the Brownian semigroup P_t (on C_0) and G is a Lipschitz continuous function on C_0^* taking values in $(C_0^1)^*$ = the dual Banach space of $C_0^1(\mathbb{R}^1)$, i.e. it satisfies

(91) $$\| G(u) - G(v) \|_{(C_0^1)^*} \leq \text{const.} \| u - v \|_{C_0^*} \quad , \quad \text{for } \forall u,v \in C_0^* \quad .$$

Lemma A-1.(**) The uniqueness holds for solutions of the equation (90) under the assumption (91) for G .

Proof. We notice first that $P_t \in L(C_0, C_0^1)$ and the operator norm $\| P_t \|_{L(C_0,C_0^1)}$ is bounded by const. $1/\sqrt{t}$ for $t > 0$, because

$$\| \frac{d}{dx} P_t f \| \leq \| f \| \frac{1}{\sqrt{t}} \int \frac{|x|}{\sqrt{t}} \frac{1}{\sqrt{2\pi t}} e^{-\frac{|x|^2}{2t}} dx \quad , \quad \text{for } \forall f \in C_0 \quad .$$

Therefore $P_t^* \in L((C_0^1)^*, C_0^*)$ and

(92) $$\| P_t^* \|_{L((C_0^1)^*, C_0^*)} \leq \text{const.} \frac{1}{\sqrt{t}} \quad , \quad \text{for } t > 0.$$

Let $u(t)$ and $v(t)$ be solutions of (90). By (91) and (92),

$$\| u(t) - v(t) \|_{C_0^*} \leq \text{const.} \int_0^t \frac{1}{\sqrt{t-s}} \| u(s) - v(s) \|_{C_0^*} ds \quad ,$$

from which follows $u(t) = v(t)$, completing the proof.

(*) $C_0(\mathbb{R}^1)$ denotes the space of continuous functions on $\mathbb{R}^1$ vanishing at infinity. $C_0^k(\mathbb{R}^1) = \{ f \in C_0(\mathbb{R}^1) : f^{(i)} \in C_0(\mathbb{R}^1) \text{ for } \forall i \leq k \}$.

(**) The lemma holds for higher dimensional Brownian semigroups.

Let $X(t)$ be a solution of the SDE (61) and $u(t)$ be the probability distribution of $X(t)$. Then, by Itô's formula

$$(93)\quad 0 = \langle u_0, g(0)\rangle + \int_0^T dt \langle u(t), \frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2} + b[\cdot,u(t)]\frac{\partial g}{\partial x}\rangle$$

for any $g(t,x)$ which is C^1 in t with $g(t,x)=0$ for $\forall\, t \geq T$ and C_0^2 as a function of x. We insert into (93) special g of the form

$$g(t,x) = \int_t^T P_{s-t}h(s,x)ds\ ,\quad h = af\quad ,\ a \in C^1([0,T])\ ,\ f \in C_0,$$

It is clear that g satisfies the condition stated above and

$$\frac{\partial g}{\partial t} = -\frac{1}{2}\frac{\partial^2 g}{\partial x^2} - h\ ,\ g(T,\cdot) = 0\ .$$

Therefore we have

$$(94)\quad 0 = \langle u_0, \int_0^T P_s h(s)ds\rangle + \int_0^T dt\langle u(t), -h(s) + b[\cdot,u(t)]\frac{\partial g}{\partial x}\rangle\ .$$

Let us define $G(u)$ by

$$(95)\qquad \langle G(u),f\rangle = \langle u, b[\cdot,u]\frac{df}{dx}\rangle \quad \text{for}\ \forall\, f \in C_0^1\ .$$

Then the third term of (94) can be written, by Fubini, as

$$\begin{aligned}
&\int_0^T dt\langle u(t),\ b[\cdot,u(t)]\frac{\partial g}{\partial x}(t)\rangle \\
&= \int_0^T dt\langle G(u(t)),g(t)\rangle = \int_0^T dt\langle G(u(t)),\int_t^T P_{s-t}h(s)ds\rangle\ , \\
&= \int_0^T ds\int_0^s dt\,\langle G(u(t)),P_{s-t}h(s)\rangle = \int_0^T ds\int_0^s dt\langle P_{s-t}^* G(u(t)),h(s)\rangle\ ,
\end{aligned}$$

and hence we have

$$\begin{aligned}
0 &= \int_0^T ds\langle P_s^* u_0 + \int_0^s dt\ P_{s-t}^* G(u(t)) - u(s), h(s)\rangle \\
&= \int_0^T ds\ a(s)\ \langle P_s^* u_0 + \int_0^s dt\ P_{s-t}^* G(u(t)) - u(s), f\rangle\ .
\end{aligned}$$

Since this holds for $\forall\, a \in C^1([0,T])$, we have for $\forall\, t \in [0,T]$

$$<u(t)-P_t^* u_0-\int_0^t ds\ P_{t-s}^* G(u(s)),f> = 0 \ , \text{ for } \forall f \in C_0 \ ,$$

i.e. u(t) satisfies the equation (90). As for the continuity (91) of the function G(u) defined by (95) it holds that

$$|<G(u)-G(v),f>| \leq \text{const.} \|f\|_{C_0^1} \|u-v\|_{C_0^*} \ , \quad \text{for } \forall f \in C_0^1 \ .$$

This can be verified, deviding the left hand side into three parts

$$<u-v,b[u]f'> + <v,f'\int b(\cdot,y,u)(u-v)(dy)> + <v,f'\int\{b(\cdot,y,u)-b(\cdot,y,v)\}v(dy)> \ .$$

In fact, it is enough to estimate the third term, and especially the ones which contain $b_{11}(x,y)$, because the same argument can be applied to other terms containing b_{ij}. Assuming $\gamma(u) \leq \gamma(v)$,

$$I_{11} = \int_{(-\infty,\gamma(u)]} v(dx)f'(x)\int_{(-\infty,\gamma(u)]} b_{11}(x,y)v(dy) - \int_{(-\infty,\gamma(v)]} v(dx)f'(x)\int_{(-\infty,\gamma(v)]} b_{11}(x,y)v(dy)$$

$$= -\int_{(-\infty,\gamma(u)]} v(dx)f'(x)\int_{(\gamma(u),\gamma(v)]} b_{11}(x,y)v(dy) - \int_{(\gamma(u),\gamma(v)]} v(dx)f'(x)\int_{(-\infty,\gamma(v)]} b_{11}(x,y)v(dy) \ ,$$

and then applying the equality

$$\int_{(\gamma(u),\gamma(v)]} v(dy) = \int_{(-\infty,\gamma(u)]} (u-v)(dy) \ ,$$

we get $|I_{11}| \leq \text{const.} \|f\|_{C_0^1} \|u-v\|_{C_0^*}$. Therefore, by Lemma A-1, we can complete the proof of the uniqueness, i.e.,

Lemma A-2. The probability distribution u(t) of a solution of the nonlinear SDE (61)

$$(61) \qquad X(t) = X(0) + B(t) + \int_0^t b[X(s),u(s)]ds$$

is uniquely determined by the initial distribution u_0 of X(0) .

References

[1] Dunford, N. and Schwartz, J., Linear Operators, Vol. 1, Interscience 1958.

[2] Harris, T.E., Diffusion with "collisions" between particles, J. Appl. Prob. 2, (1965), 323-338.

[3] Kusuoka, S. and Tamura, Y., Gibbs measures for mean field potentials, J. Fac. Sci. Univ. Tokyo Sect. 1A Math. 31 (1984), 223-245.

[4] Maruyama, G., On the transition probability functions of the Markov processes, Nat. Sci. Rep. Ochanomizu Univ. 5, (1954), 10-20.

[5] McKean, H.P., A class of Markov processes associated with non-linear parabolic equations, Proc. Nat. Acad. Sci. 56 (1966), 1907-1911.

[6] McKean, H.P., Propagation of chaos for a class of non-linear parabolic equations, Lecture Series in Differential Equations, Catholic Univ. (1967), 41-57.

[7] Nagasawa, M., Segregation of a population in an environment, J. Math. Biology 9 (1980), 213-235.

[8] Nagasawa, M., An application of segregation model for septation of Escherichia Coli, J. Th. Biology 90 (1981), 445-455.

[9] Nagasawa, M. and Yasue, K., A statistical model of mesons, Publication de l'Institut recherche Mathématique Avancée (CNRS) 33 (1982/83), Université Lois Pasteur (Strasbourg), 1-48.

[10] Nagasawa, M. and Tanaka, H., A diffusion process in a singular mean-drift-field, Z. Wahr. verw. Gebiete 68 (1985), 247-269.

[11] Nagasawa, M. and Tanaka, H., Propagation of chaos for diffusing particles of two types with singular mean field interaction, to appear in Z. Wahr. verw. Gebiete.

[12] Saisho,Y. and Tanaka,H., Stochastic differential equations for mutually reflecting Brownian balls (preprint).

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November 30, 1985

[13] Shiga,T. and Tanaka,H., Central limit theorem for a system of Markovian particles with mean field interaction, Z. Wahrsch. verw. Gebiete 69 (1985), 439-445.

[14] Sznitman, A.S., Non-linear reflecting diffusion processes, and propagation of chaos, and fluctuations associated, J. of Funct. Anal. 56 (1984), 311-336.

[15] Tanaka, H., Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. 9 (1979), 163-177.

[16] Tanaka, H., Limit theorems for certain diffusion processes with interaction, Stochastic Analysis, ed. by K. Itô, Kinokuniya Co. Ltd., Tokyo, North-Holland 1984, 469-488.

[17] Veretennikov, A.Ju., On strong solutions and explicit formulas for solutions of stochastic integral equations, Mat. Sb. 111 (1980), English transl. in Math USSR Sbornik 39 (1981), 387-403.

[18] Wilks, S.S., Mathematical statistics, John Wiley & Sons, Inc. 1962.

[19] Zvonkin, A.K. A transformation of the phase space of a diffusion process that removes the drift, Mat.Sb. 93 (1974), English trans. in Math.USSR Sb. 22 (1974), 129-149.