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Z_2 Equivariant Cohomology and its Applications to Orthogonal geodesic chords

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ABSTRACT

A geodesic joining two points of the boundary of a Riemannian manifold is called an *orthogonal geodesic chord* if it intersects with the boundary orthogonally.

Bos' theorem [B] asserts any convex n-disk has n orthogonal geodesic chords. In this case, the involution $\gamma \to \gamma^{-1} \equiv \gamma(1-\cdot)$ can be considered and equivariant (co)homology of a path space plays essential role. This note gives n such chords for compact contractible manifold instead of the n-disk.

These chords are expected to give periodic solutions on energy surfaces of Hamiltonians of the type " *kinetic energy* " + " *potential* ".

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§1. Introduction.

On any compact energy surface of a Hamiltonian system, we can always find a periodic solution if the Hamiltonian H is a physical one, that is $H = T + U$ = "kinetic energy" + "potential" [H2] [GZ].

Solutions of the Hamiltonian system of given energy e correspond to geodesics with respect to the Jacobi metric

$$ds^2 = (e - U) T .$$

And orthogonal geodesic chords of a compact manifold with boundary, which is diffeomorphic to $\{ U \leq e \}$, yields the periodic solution of the Hamiltonian system. Here an *orthogonal geodesic chord* means a geodesic starting from and ending at a point of the boundary orthogonally.

A partial result has been obtained by Seifert [S] for the case $\{ U \leq e \} \approx D^n$, the n-disk. In a footnote of the paper, Seifert conjectured that one may have n periodic solutions by the method of Lusternik-Schnirelmann [LS]. This problem is still open, although a partial result for the system near the rotationally symmetric one is obtained [H3]. (Remark that this is not the so called "Seifert Conjecture".)

[LS] gives n orthogonal geodesic chords of any convex body in $\mathbf{R}^n$ and Bos [B] extends it to convex n disks.

In this note, we define a number $\nu(M)$, where M is a compact C^∞ manifold with boundary, and show that there exist at least $\nu(M)$ orthogonal geodesic chords of M if the boundary is convex with respect to the Riemannian metric given on M.

Furthermore we have $\nu(D^n) = n$ and

$$\nu(M) \geq n$$

if M is contractible ($n = dim\ M$).

For $n =$ 1 and 2, compact contractible manifold with boundary is always a disk. For $n =$ 3, this question is equivalent to the Poincaré Conjecture. For $n \geq$ 4, there are examples of compact contractible manifolds with boundary, which are not homeomorphic to a disk [Ma]. In this case, the boundary is a homology sphere with nontrivial fundamental group.

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§2. Numbers determined by compact manifold with boundary.

Let M be a compact C^∞ manifold with boundary

$$B \equiv \partial M \neq \emptyset.$$

We define four numbers $\nu(M)$, $\nu_\pi(M)$, $\nu_H(M)$, and $\nu_\Pi(M)$ as follows.

We put $Y_M \equiv \Omega(M;B,B) \equiv \{\omega:[0,1] \to M;\text{ continuous and } \omega(0),\omega(1) \in B\}$ endowed with compact open topology and regard B as a subset of Y_M by the usual way. For simplicity the coefficient field $\mathbf{Z}_2$ is understood.

(*i*)
$$\nu_\pi(M) = \begin{cases} 1 & \text{if } \pi_k(Y_M, B) \neq 0 \quad \text{for some } k \geq 1, \\ 0 & \text{otherwise} \end{cases}$$

(*ii*)
$$\nu_H(M) = \mathrm{Max}\{ k \geq 0 ;\ ^\exists \alpha_1,\alpha_2,\ldots,\alpha_{k-1} \in H^*(Y_M) \text{ with } \deg \alpha_j > 0, j = 1,\cdots,k-1;\ ^\exists a \in H_*(Y_M, B) \text{ such that } (\alpha_1 \cup \cdots \cup \alpha_{k-1}) \cap a \neq 0 \}$$

We remark that

(2.1)
$$\nu_H(M) \geq 1 \qquad \text{iff} \quad H_*(Y_M, B) \neq 0$$

(*iii*) Let $\xi : Y_M \to Y_M$ be the involution defined by

$$\xi\omega = \omega^{-1} \equiv \omega(1- \cdot).$$

$$\nu_\Pi(M) = \mathrm{Max}\{ k \geq 0 ;\ ^\exists \alpha_1,\alpha_2,\ldots,\alpha_{k-1} \in H_\Pi{}^*(Y_M), \text{ with } \deg \alpha_j > 0, \text{and } ^\exists a \in H_*{}^\Pi(Y_M, B) \text{ such that } (\alpha_1 \cup \cdots \cup \alpha_{k-1}) \cap a \neq 0 \}$$

H_*^Π ($H_\Pi{}^*$) means $\mathbf{Z}_2$-equivariant (co)homology (see [H1] [N]).

Finally we define a nonnegative integer

(*iv*)
$$\nu(M) = \mathrm{Max}\{ \nu_\pi(M), \nu_H(M), \nu_\Pi(M) \}$$

Lemma 2 of [H2] gives

(2.2)
$$\nu_\pi(M) \geq 1$$

or

(2.3) $$\nu_H(M) \geq 1 \qquad (\Leftrightarrow H_*(Y_M, B) \neq 0),$$

hence we have

(2.4) $$\nu(M) \geq 1.$$

We also have

Lemma 1. *Let L be a closed manifold and N a compact manifold with boundary. Then, if* $\nu_H(N) \geq 1$, *we have*

$$\nu_H(N\times L) \geq (\text{cup length}\, L) + \nu_H(N).$$

(*proof*) Let $\alpha_1, \cdots, \alpha_{k-1} \in H^*(N)$, $\deg \alpha_j > 0$ and $a \in H_*(N, \partial N)$ with

$$a' = (\alpha_1 \cup \cdots \cup \alpha_{k-1}) \cap a \neq 0.$$

And let $\beta_1, \ldots, \beta_l \in H^*(L)$, $\deg \beta_j > 0$, with $\beta' = \beta_1 \cup \cdots \cup \beta_l \neq 0$. We choose $b \in H_*(L)$ so that $<(\beta_1 \cup \cdots \cup \beta_l), b> = 1$, then $\beta' \cap b \neq 0$. Now $a\times b \in H_*((N, \partial N)\times L) = H_*(N\times L, \partial(N\times L))$. And we have

$$\begin{aligned}
&((1\times\beta_1)\cup\cdots\cup(1\times\beta_l)\cup(\alpha_1\times 1)\cup\cdots\cup(\alpha_{k-1}\times 1)) \cap (a\times b) \\
&= (1\times\beta')\cap(((\alpha_1\times 1)\cup\cdots\cup(\alpha_{k-1}\times 1)) \cap (a\times b) \\
&= (1\times\beta')\cap(a'\times b) \\
&= a'\times(\beta'\cap b) \\
&\neq 0 \qquad\qquad \text{in} \quad H_*((N\times L), \partial(N\times L)).
\end{aligned}$$

This proves the lemma. Q.E.D.

§3. Orthogonal Geodesic Chords

First we have

Therem 1. *Let M be a compact Riemannian manifold with convex boundary. Then there exist at least $\nu(M)$ orthogonal geodesic chords of M.*

Before giving the proof, we set a path space and distances on it. Let Λ be the set of all p.w. (piecewise) C^∞ curve $\lambda : [0,1] \to M$ with $\lambda(0)$ and $\lambda(1) \in \partial M$.

For $\lambda_1, \lambda_2 \in \Lambda$, we define

$$d_\infty(\lambda_1, \lambda_2) = \operatorname*{Max}_{0 \le t \le 1} d(\lambda_1(t), \lambda_2(t)) ,$$

where d is the distance derived from the Riemannian metric on M , and

$$d_1(\lambda_1, \lambda_2) = d_\infty(\lambda_1, \lambda_2) + \left(\int_0^1 (|\dot\lambda_1(t)| - |\dot\lambda_2(t)|)^2 \, dt \right)^{1/2} .$$

Let E be the energy functional $E : \Lambda \to [0, \infty)$ defined by

$$E(\lambda) = \frac{1}{2} \int_0^1 |\dot\lambda(t)|^2 \, dt .$$

E is continuous w.r.t. d_1 (§16 in [Mi]).

Now we have a sequence of continuous mappings

$$(\Lambda, d_1) \overset{i}{\subset} (\Lambda, d_\infty) \overset{j}{\subset} (Y_M, d_\infty) .$$

We remark that these i and j both give homotopy equivalences by Theorem 17.1 in [Mi].

For $K \ge 0$, we put

$$\Lambda^K = \{ \lambda \in \Lambda \,;\, E(\lambda) \le K \} .$$

Then Lemma A.1.4 in [K], slightly changed so as to suit for our situation, that is, with convex boundary, gives

Lemma 2. *We take the distance d_∞ on Λ . For given $K \ge 0$, there is a deformation (continuous map)*

$$\mathcal{D} : \Lambda^K \times [0, 1] \to \Lambda^K$$

satisfying

(i) $\mathcal{D}(\cdot, 0) = i_d : \Lambda^K \to \Lambda^K$

(ii) $\mathcal{D}$ *is* E *- decreasing, that is*

$$E(\mathcal{D}(\lambda, t)) \leq E(\mathcal{D}(\lambda, s)) \quad \text{if} \quad t \geq s$$

(iii) $\mathcal{D}(\lambda, 1) = \lambda$ *if and only if* λ *is an orthogonal geodesic chord (or constant curve on* $B = \partial M$ *).*

(iv) *Let* $0 < \kappa < K$ *and* C_κ *be the set of all orthogonal geodesic chords of Energy* κ *. Then for any open neighborhood* W *of* C_κ *(one may choose* $W = \varnothing$ *in the case* $C_\kappa = \varnothing$ *), there exists* $\rho >$) *such that*

$$\mathcal{D}\, \Lambda^{\kappa+\rho} \subset W \cup \Lambda^{\kappa-\rho}$$

where $\mathcal{D} \equiv \mathcal{D}(\cdot, 1) : \Lambda^K \to \Lambda^K$.

(v) $\mathcal{D}(\cdot, t) : \Lambda^K \to \Lambda^K$ *is equivariant for any* $t \in [0, 1]$.

This lemma means that the convexity of the boundary plays the role of the condition (C) of Palais-Smale.

(Proof of Theorem 1)

First we prove $\nu_\pi(M) = 1$ gives at least one non-constant orthogonal geodesic chord.

As usual there is an $\epsilon > 0$ with

$$\pi_k(\Lambda, \Lambda^\epsilon) \cong \pi_k(\Lambda, \Lambda^0) \cong \pi_k(Y_M, B)$$

(see the remark above). Let a be the non-trivial element of $\pi_k(\Lambda, \Lambda^\epsilon)$. We put

$$\kappa_a = \inf_{f \in a} \sup E(f(D^k)) \ .$$

Although E is not continuous w.r.t. d_∞ , this is finite because

$$i : (\Lambda, d_1) \subset (\Lambda, d_\infty)$$

is a homotopy equivalence. As usual Lemma 2, in particular (iv), gives that κ_a is a critical value of E , that is, $C_{\kappa_a} \neq \varnothing$, and $\kappa_a \geq \epsilon$ by the fact that a is non-trivial in the relative homotopy $\pi_k(\Lambda, \Lambda^\epsilon)$.

The case $\nu_H(M)$ is also the usual one.

Now consider the equivariant case.

We put $\Lambda_\Pi \equiv S^\infty \times_\Pi \Lambda$, the orbit space of $S^\infty \times \Lambda$ under the involution $(\zeta, \lambda) \mapsto (-\zeta, \xi\lambda)$.

Since E is invariant under ξ , we can define

$$\tilde{E} : \Lambda_\Pi \to [0, \infty)$$

by $\tilde{E}[\zeta, \lambda] = E(\lambda)$, where $[\zeta, \lambda]$ is the equivalence class represented by (ζ, λ) .

Let a be the nontrivial element of $H_*^\Pi(\Lambda, \Lambda^\epsilon) = H_*(\Lambda_\Pi, \Lambda_\Pi^\epsilon)$. We put

$$\tilde{\kappa_a} = \inf_{z \in a} \tilde{E}(|z|) ,$$

where $|z| = \bigcup_i \mathrm{Im}\, \sigma_i$, $z = \sum_i \sigma_i$, $\sigma_i : \Delta_q \to \Lambda_\Pi$, singular simplex.

Then, by Lemma 2 in [H3], $C_{\tilde{\kappa_a}} \neq \emptyset$ and $\tilde{\kappa_a} \geq \epsilon$.

Furthermore, assume that there exists $\alpha \in H^*(\Lambda_\Pi)$ with $\deg\alpha > 0$, such that $b \equiv \alpha \cap a \neq 0$ in $H_*(\Lambda_\Pi, \Lambda_\Pi^\epsilon)$. Then $\tilde{\kappa_b}$ is also a critical value with $\tilde{\kappa_b} \geq \epsilon$.

In this case, in general, $\tilde{\kappa_b} \leq \tilde{\kappa_a}$ and if $\tilde{\kappa_b} = \tilde{\kappa_a}$, then there are infinitely many critical points on the level (a version of Lemma 2 in [H3]).

Remarking that

$$\alpha_1 \cap (\alpha_2 \cap a) = (\alpha_1 \cup \alpha_2) \cap a ,$$

we have at least $\nu_\Pi(M)$ critical points with $\geq \epsilon$ (non-constant orthogonal geodesic chords). Q.E.D.

So (2.4) gives

Corollary. *Any compact Riemannian manifold with convex boundary has an orthogonal geodesic chord.*

This is Theorem C of [GZ].

[GZ] also gives another proof for Bos' theorem. There is an example of compact Riemannian manifold with boundary with no orthogonal geodesic chords[B], hence the assumption of convex boundary is necessary.

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Since we have

$$(3.1) \qquad \nu(D^n) = n$$

(the proof is given later), Theorem 1 also gives Bos' theorem.

Now we have

Theorem 2. *Let M be a compact contractible manifold with convex boundary. Then there exist* dim M *orthogonal geodesic chords of M.*

In this case we shall show

$$(3.2) \qquad \nu_{\Pi}(M) \geq \dim M,$$

giving Theorem 2 using Theorem 1.

To prove Theorem 2, we prepare the following

Lemma 3. *Let B be a closed manifold of dimension* n . *Put $B^2 = B \times B$ and define the involution $\xi : B^2 \to B^2$ by*

$$\xi(x, y) = (y, x).$$

Then there exist $a \in H_{2n}{}^{\Pi}(B^2, \Delta)$ and $\theta \in H^1{}_{\Pi}(B)$ with $\theta^n \cap a \neq 0$ in $H_n{}^{\Pi}(B^2, \Delta)$, where Δ denotes the diagonal set.

(*Proof*) This is Theorem 2 of [H1], which needs the work by Steenrod [N]. Q.E.D.

(Proof of Theorem 2)

We put $B = \partial M$. Let

$$\pi : Y_M \to B^2$$

be the projection defined by

$$\omega \to (\omega(0), \omega(1)).$$

Since the mapping

$$\pi' : S^\infty \times Y_M \to S^\infty \times B^2$$

defined by

$$(\zeta, \omega) \to (\zeta, \pi(\omega))$$

is equivariant, we have the mapping

$$\tilde{\pi} : S^\infty \underset{\Pi}{\times} Y_M \to S^\infty \underset{\Pi}{\times} B^2.$$

It is easy to see that $\tilde{\pi}$ gives a fibre space.

We fix base points $b \in B$ and $* \in S^\infty$ and take

$$\tilde{*} = [*, (b, b)]$$

as the base point of $S^\infty \underset{\Pi}{\times} B^2$. Then we can identify the fibre $\tilde{\pi}^{-1}(\tilde{*})$ and ΩM . Thus we have a fibration

(3.3) $$\Omega M \to S^\infty \underset{\Pi}{\times} Y_M \overset{\tilde{\pi}}{\to} S^\infty \underset{\Pi}{\times} B^2.$$

Until we obtain the fibration (3.3), we don't use the assumption that M is contractible.

Now, since $M \simeq b$, we have $\Omega M \simeq b$. Hence the spectral sequence of the (co)homology of the fibration is a very simple one and we can easily have the conclusion

$$\nu_\Pi(M) \geq \dim B + 1 = \dim M$$

using Lemma 3.

This is (3.2), giving the theorem. Q.E.D.

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§4. Open Problems.

In this section, we propose some open problems.

Let M be a compact manifold with boundary.

(i) $\nu_H(M) \geq 1$?

Although this gives only one orthogonal geodesic chord by Lemma 2 and it is already obtained from (2.2), there is a little meaning because it appears in the assumption of Lemma 1.

(ii) $\nu_\Pi(M) \geq \nu_H(M)$?

For the case $M = D^n$, we have

(4.1) $$\nu_\Pi(D^n) = n.$$

In fact (3.2) gives $\nu_\Pi(D^n) \geq n$. And if $\nu_\Pi(D^n) > n$, we have always $n+1$ orthogonal geodesic chords. But for the solid ellipsoid whose lengths of axes are all different, we have exactly n orthogonal geodesic chords if the Riemannian metric considered is the standard one.

One can easily obtain $\nu_H(D^n) = 2$, hence in this case we have (ii).

This corresponds to the fact that

$$\text{cup length } S^{n-1} = 1$$

but

$$\text{cup length } \mathbf{R}P^{n-1} = n-1,$$

showing the importance of $\mathbf{Z}_2$-action, in particular, $\mathbf{Z}_2$-equivariant cohomology under an involution.

(iii) *Study the structure of* $H_\Pi{}^*(Y_M)$.

For contractible M, this is isomorphic to $H_\Pi{}^*(\partial M \times \partial M)$, which is completely determined by the Isomorphism Theorem by Steenrod [N]. For general M, it seems to be difficult to determine the equivariant cohomology, but this is indispensable in order to count $\nu_\Pi(M)$.

References

[B] W. Bos, Kritische Sehnen anf Riemannschen Elementarraumstucken, Math. Ann. 151(1963), 431-451.

[GZ] H. Gluck and W. Ziller, Existence of periodic motions of conservative system, Seminar on Minimal Submanidolds, Princeton University Press (1983), 65-98.

[H1] K. Hayashi, Double normals of a compact submanifold, Tokyo J. Math., 5(1982), 419-425.

[H2] ________ , Periodic solution of classical Hamiltonian systems, Tokyo J. Math., 6(1983), 473-486.

[H3] ________ , The existence of periodic orbits on the sphere, Tokyo J. Math., 7(1984), 359-369.

[K] W. Klingenberg, Lectures on Closed Geodesics, Springer, Berlin-Heidelberg-New York, 1978.

[LS] L. Lyusternik and L. Schnirelmann, Méthodes Topologiques dans les Problèmes Variationnels, Moscow (1930), (Russian); Hermann, Paris (1934).

[Ma] B. Mazur, A note on some contractible 4-manifolds, Ann. Math., 73(1961), 221-228.

[Mi] J. Milnor, Morse Theory, Ann. of Math. Studies, no. 51, Princeton Univ. Press, 1974.

[N] M. Nakaoka, Fudoten Teiri to Sono Shuhen, Iwanami, Tokyo, 1977 (in Japanese).