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RATIONAL APPROXIMATIONS TO THE ROGERS-RAMANUJIAN CONTINUED FRACTION

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1. Introduction

Let $F(\alpha)$ be defined by

$$F(\alpha) = F(\alpha,x) = \sum_{n=0}^{\infty} \frac{\alpha^n x^{n^2}}{(1-x)(1-x^2)\dots(1-x^n)} \qquad (|x| < 1).$$

Then $F(\alpha)$ satisfies

$$F(\alpha) = F(\alpha x) + \alpha x\, F(\alpha x^2),$$

so that $F(\alpha)/F(\alpha x)$ can be developed in the Rogers-Ramanujian continued fraction

$$\frac{F(\alpha)}{F(\alpha x)} = 1 + \frac{\alpha x}{1} + \frac{\alpha x^2}{1} + \frac{\alpha x^3}{1} + \dots, \tag{1}$$

In particular, by virtue of the Rogers-Ramanujian identities, we have

$$1 + \frac{x}{1} + \frac{x^2}{1} + \dots = \frac{\sum_{n=0}^{\infty} \frac{x^{n^2}}{(1-x)(1-x^2)\dots(1-x^n)}}{\sum_{n=0}^{\infty} \frac{x^{n^2+n}}{(1-x)(1-x^2)\dots(1-x^n)}} = \prod_{n=0}^{\infty} \frac{(1-x^{5n+2})(1-x^{5n+3})}{(1-x^{5n+1})(1-x^{5n+4})}.$$

(For details see for example [1],[5].) We put for frevity

$$f(\alpha,x) = F(\alpha)/F(\alpha x).$$

In 1971 Osgood [8,9] proved that, if a, b, and d are non-zero integers with $|d| \geq 2$, then, for any $\varepsilon>0$, there is a positive constant q_0 =

$q_0(a,b,d,\varepsilon)$ such that

$$\left|f\left(\frac{a}{b},\frac{1}{d}\right)-\frac{p}{q}\right| > q^{-2-\varepsilon}$$

for all integers p, q $(\geq q_0)$.

For the values of the exponential function at rational points more precise results have been obtained (c.f. Bundschuh [2], Durand [3], Mahler [7], Shiokawa [10]): If a/b is a non-zero rational number, then there are explicit positive constants $B=B(a/b)$ and $C=C(a/b)$ such that

$$\left|e^{a/b}-\frac{p}{q}\right| > Cq^{-2-B/\log\log q}$$

for all integers p, q (≥ 3). Especially, Davis [3] proved that, if b is a non-zero integer and

$$C = \begin{cases} 1/|b| & \text{if } b \text{ is even,} \\ 1/|4b| & \text{otherwise,} \end{cases}$$

then, for any $\varepsilon>0$,

$$\left|e^{2/b}-\frac{p}{q}\right| < (C+\varepsilon)q^{-2}\frac{\log\log q}{\log q}$$

for infinitely many integers p, q, while there is a positive constant $q_0=q_0(b,\varepsilon)$ such that

$$\left|e^{2/b}-\frac{p}{q}\right| > (C-\varepsilon)q^{-2}\frac{\log\log q}{\log q}$$

for all integers p, q $(\geq q_0)$.

Comparing these results, we see that it would be interesting to replace, if possible, the ε in Osgood's theorem stated above by a function of q. In this connection, we prove in this paper the following theorems.

Theorem 1. Let a, b, c, and d be non-zero integers with

$$|d| > |c|^2. \tag{2}$$

Then $f(a/b,c/d)$ is an irrational number, and furthermore, there is a positive constant $C=C(a,b,c,d)$ such that

$$\left|f\left(\frac{a}{b},\frac{c}{d}\right)-\frac{p}{q}\right| > Cq^{-2-2A-B/\sqrt{\log q}}$$

for all integers p, q $(\geqq q_0)$, where

$$A=\frac{\log|c|}{\log|d/c^2|}$$

and

$$B=\frac{\log|a^2d| - A\log|b/a^2|}{\sqrt{\log|d/c^2|}}.$$

Corollary. Let a, b, and d be non-zero integers with $|d|\geqq 2$. Then there is a positive constant $C=C(a,b,d)$ such that

$$\left|f\left(\frac{a}{b},\frac{1}{d}\right)-\frac{p}{q}\right| > Cq^{-2-B/\sqrt{\log q}}$$

for all integers p, q $(\geqq 2)$, where

$$B=\frac{\log|a^2d|}{\sqrt{\log|d|}}.$$

Theorem 1 is in a sense best possible since we have the following theorem:

Theorem 2. Let a, b, and d be positive integers such that $(a,b)=1$, $d\geqq 2$, and a divides d, and let

$$C=\begin{cases}\sqrt{\dfrac{b}{a}} & \text{if } \left(\dfrac{a}{b}\right)^2 > d,\\[2ex] \sqrt{\dfrac{a}{bd}} & \text{otherwise.}\end{cases}$$

Then, for any $\varepsilon>0$,

$$\left|f\left(\frac{a}{b},\frac{1}{d}\right)-\frac{p}{q}\right| < (C+\varepsilon)q^{-2-\sqrt{\log d}\,/\sqrt{\log q}}$$

for infinitely many integers p, q $(\geqq 0)$, while there is a positive constant $q_0=q_0(a,b,d,\varepsilon)$ such that

$$\left|f\left(\frac{a}{b},\frac{1}{d}\right)-\frac{p}{q}\right| > (C-\varepsilon)q^{-2-\sqrt{\log d}\,/\sqrt{\log q}}$$

for all integers p, q $(\geq q_0)$.

2. A lemma.

We shall make use of the following lemma which itself is of some interest.

<u>Lemma</u>. Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers such that $|a_n a_{n+1}| > 4$ $(n \geq 1)$ and

$$\sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1} = \sigma < \infty.$$

Define as usual $p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $(n \geq 1)$ with $p_0 = q_{-1} = 0$, $p_{-1} = q_0 = 1$. Then $p_n/(a_2 a_3 \ldots a_n)$ and $q_n/(a_1 a_2 \ldots a_n)$ converge to finite non-zero limits, and they satisfy

$$e^{-4\sigma} < |p_n/(a_2 a_3 \ldots a_n)| < e^{2\sigma},$$

$$e^{-4\sigma} < |q_n/(a_1 a_2 \ldots a_n)| < e^{2\sigma},$$

so that the continued fraction

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots = \lim_{n\to\infty} \frac{p_n}{q_n}$$

is convergent.

<u>Proof</u>. By definition, $q_1 = a_1$, $q_2 = a_1 a_2 (1 + 1/(a_1 a_2))$, and we have

$$q_3 = a_1 a_2 a_3 \left(1 + \frac{1}{a_1 a_2}\right)\left(1 + \frac{\delta_2}{a_2 a_3}\right), \quad \frac{2}{3} < \delta_2 < 2,$$

in view of the inequalities $|a_n a_{n+1}| > 4$ $(n \geq 1)$ and $2/3 < 1/(1+xy) < 2$ if $|x| < 1/4$, $2/3 < y < 2$. Repeating this, we get

$$q_n = a_1 a_2 \ldots a_n \prod_{k=1}^{n} \left(1 + \frac{\delta_k}{a_k a_{k+1}}\right), \quad \frac{2}{3} < \delta_k < 2 \quad (n, k \geq 1).$$

If we regard q_n as a polynomial in n variables $a_1, a_2, \ldots, a_n$ and write it as $q_n = q_n(a_1 a_2 \dots a_n)$, we have $p_n = q_{n-1}(a_2, a_3, \dots, a_n)$; namely

$$p_n = a_2 a_3 \dots a_n \prod_{k=2}^{n} \left(1 + \frac{\gamma_k}{a_k a_{k+1}}\right), \quad \frac{2}{3} < \gamma_k < 2 \quad (n,k \geq 2).$$

Hence, the lemma follows from the absolute convergence of the series $\sum_{n=1}^{\infty} (a_n a_{n+1})^{-1}$.

To apply Lemma, we transform the continued fraction (1) by using the formula

$$\frac{b_1}{1} + \frac{b_2}{1} + \frac{b_3}{1} + \cdots\cdots = \frac{1}{\frac{1}{b_1}} + \frac{1}{\frac{b_1}{b_2}} + \frac{1}{\frac{b_2}{b_1 b_3}} + \frac{1}{\frac{b_1 b_3}{b_2 b_4}} + \cdots\cdots$$

(cf. [6;(2,3,24)]) and obtain the regular continued fraction

$$(1') \qquad f(\alpha,x) = 1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots\cdots,$$

where

$$(3) \qquad a_{2k-1} = \alpha^{-1} x^{-1}, \; a_{2k} = x^{-k} \quad (k \geq 1).$$

We note here that

$$(4) \qquad a_1 a_2 \dots a_{2k-1} = \alpha^{-k} x^{-k^2}, \; a_1 a_2 \dots a_{2k} = \alpha^{-k} x^{-k^2 - k} \quad (k \geq 1),$$

and hence

$$(5) \qquad \log|a_1 a_2 \dots a_n| = -\frac{n^2}{4} \log|x| - \frac{n}{2} \log|\alpha x| + O(1).$$

3. Proof of Theorem 1.

Let $\alpha = a/b$ and $x = c/d$ be as in Theorem 1. Then a_n, and hence, p_n, q_n are rational numbers for which $d_n p_n$, $d_n q_n$ are integers for all $n \geq 1$, where

$$d_{2k-1} = |a^k c^{k^2}|, \quad d_{2k} = |a^k c^{k^2+k}|,$$

so that

(6) $$\log d_n = \frac{n}{4}\log|c| + \frac{n}{2}\log|ac| + O(1).$$

Here and in what follows constants implied in O-symbols as well as positive constants m, n_0, c_0, $c_1,\ldots$ depend possibly on a, b, c, d (and ε in §4).

Since $a_n a_{n+1} = \alpha^{-1}x^{-n}$ $(n\geq 1)$ with $|x| < 1$, the series $\sum_{n=1}^{\infty}(a_n a_{n+1})^{-1}$ is absolutely convergent and there exists an integer $m\geq 1$ such that $|a_n a_{n+1}| > 4$ $(n\geq m)$. We may thus apply the lemma and find that the continued fraction

(7) $$\frac{1}{a_{n+1}} + \frac{1}{a_{n+2}} + \frac{1}{a_{n+3}} + \cdots = \theta_n, \text{ say,}$$

is convergent for each $n\geq m$ and

(8) $$\begin{aligned} e^{-6\sigma} &< |a_{n+k+1}\theta_{n+k}| < e^{6\sigma} \\ e^{-6\sigma} &< |a_{n+k+1}q_{n,k}/q_{n,k+1}| < e^{6\sigma} \end{aligned} \qquad (n\geq m,\ k\geq 1),$$

where $p_{n,k}/q_{n,k}$ is the k-th convergent of the continued fraction (7) and $\sigma = \sum_{n=1}^{\infty} |a_n a_{n+1}|^{-1}$. Hence

$$\left|\theta_n - \frac{p_{n,k}}{q_{n,k}}\right| = \frac{1}{|q_{n,k}(q_{n,k+1}+\theta_{n+k+1}q_{n,k})|} < \frac{2}{|q_{n,k}^2\ a_{n+k+1}|}.$$

for all sufficiently large k. But using again the lemma with (5) and (6), we get

$$\frac{\log|q_{n,k}^2\ a_{n+k+1}|}{\log|d_{n+k+1}q_{n,k}|} > 2 - \frac{2\log|c|}{\log|d|} - \frac{c_0}{k},$$

so that, for any $\varepsilon>0$,

$$\left|\theta_n - \frac{d_{n+k}p_{n,k}}{d_{n+k}q_{n,k}}\right| < |d_{n+k}q_{n,k}|^{-2+2(\log|c|)/\log|d|+\varepsilon}$$

for all sufficiently large k. This establishes the irrationality of θ_n $(n \geq m)$, since $d_{n+k}p_{n,k}$, $d_{n+k}q_{n,k}$ are integers and $2(\log|c|)/\log|d|<1$ by (2).

Now we may assume $p_m q_m \neq 0$, since at least one of $p_{n-1}q_{n-1}$, $p_n q_n$ is different from zero, because $a_n \neq 0$ $(n \geq 1)$. It follows from the formula $p_n = p_m q_{m,n-m} + p_{m-1}p_{m,n-m}$, $q_n = q_m q_{m,n-m} + q_{m-1}p_{m,n-m}$ that

$$\frac{p_n}{a_2 a_3 \cdots a_n} = \frac{p_m}{a_2 a_3 \cdots a_n} \frac{q_{m,n-m}}{a_{m+1} \cdots a_n}\left(1 + \frac{p_{m-1}}{p_m} \frac{p_{m,n-m}}{q_{m,n-m}}\right),$$

$$\frac{q_n}{a_1 a_2 \cdots a_n} = \frac{q_m}{a_1 a_2 \cdots a_n} \frac{q_{m,n-m}}{a_{m+1} \cdots a_n}\left(1 + \frac{q_{m-1}}{q_m} \frac{p_{m,n-m}}{q_{m,n-m}}\right).$$

By the lemma, quantities in the right-hand side above converge as $n \to \infty$ to finite limits which are different from zero, because of the fact that θ_m is irrational and $p_m q_m \neq 0$. Hence the continued fraction (1') converges to $f(a/b,c/d)$, which, as easily seen, is also irrational. Thus we have, using (5),

$$(9) \qquad \log|q_n| = \frac{n^2}{4} \log\left|\frac{d}{c}\right| + \frac{n}{2} \log\left|\frac{bd}{ac}\right| + O(1),$$

and so, using (6),

$$(10) \qquad \log\left|\frac{q_{n+1}}{d_{n+1}}\right| - \log\left|\frac{q_n}{d_n}\right| = \frac{n}{2} \log\left|\frac{d}{c^2}\right| + O(1).$$

Hence, noticing (2) and (8), we can choose $n_0 \geq m$ such that

$$(11) \qquad |\theta_n| < 1/2, \quad |q_{n-1}| < |q_n|, \quad |q_{n-1}/d_{n-1}| < |q_n/d_n| \quad (n \geq n_0).$$

Now let p,q be given non-zero integers. We may assume that $|q_{n_0}/d_{n_0}| < 4q$. Then by (10) and (11), there is an integer $n=n(q) \geq n_0$ such that

$$(12) \qquad |q_{n-1}/d_{n-1}| \leq 4q < |q_n/d_n|.$$

By virtue of the formula $p_n q_{n-1} - p_{n-1} q_n = \pm 1$, at least one of $p_{n-1}q - q_{n-1}p$, $p_n q - q_n p$ is different from zero. Assume first that $p_n q - q_n p \neq 0$. Then we have

$$d_n q_n \left(f\left(\frac{a}{b}, \frac{c}{d}\right) - \frac{p}{q}\right) = \frac{d_n(p_n q - q_n p)}{q} + d_n\left(q_n f\left(\frac{a}{b}, \frac{c}{d}\right) - p_n\right),$$

where $|d_n(p_n q - q_n p)| \geq 1$ and

$$\left|d_n\left(q_n f\left(\frac{a}{b}, \frac{c}{d}\right) - p_n\right)\right| = \frac{d_n}{|q_{n+1} + \theta_{n+1} q_n|} \leq \frac{2d_n}{|q_n|} < \frac{1}{2q},$$

so that

$$\text{(13)} \qquad \left|f\left(\frac{a}{b}, \frac{c}{d}\right) - \frac{p}{q}\right| > \frac{1}{2} q^{-1-(\log|d_n q_n|)/\log q}.$$

The same inequality will be obtained also in the case of $p_{n-1}q - q_{n-1}p \neq 0$.

It remains to estimate $|d_n q_n|$ from above in terms of q. Combining (3), (6), (9), and (12), we get

$$\log|d_n q_n| \leq \log q + \log(d_{n-1}d_n) + \log|a_n| + C_1$$

$$\leq \log q + \frac{n^2}{2}\log|c| + \frac{n}{2}\log|a^2 d| + C_2.$$

Here it follows from (12) with (6) and (9) that

$$\frac{n^2}{4}\log\left|\frac{d}{c^2}\right| + \frac{n}{2}\log\left|\frac{b}{a^2}\right| - C_3 < \log q$$

$$< \frac{n^2}{4}\log\left|\frac{d}{c^2}\right| + \frac{n}{2}\log\left|\frac{bd}{a^2 c^2}\right| + C_4,$$

so that

$$n = 2\sqrt{\log q} \Big/ \sqrt{\log|d/c^2|} + o(1),$$

and hence

$$n^2 \leq \frac{4\log q}{\log|c/d^2|} - \frac{4\sqrt{\log q}\,\log|b/a^2|}{\sqrt{\log|c/d^2|}\,\log|c/d^2|} + C_5.$$

Therefore, we obtain

$$\frac{\log|d_n q_n|}{\log q} < 1 + A + \frac{B}{\sqrt{\log q}},$$

which together with (13) leads to Theorem 1.

4. Proof of Theorem 2.

Let $\alpha=a/b$ and $x=1/d$ as in Theorem 2. Then $f(a/b,1/d)$ can be developed in the regular continued fraction

$$f(\frac{a}{b},\frac{1}{d}) = 1 + \frac{1}{\frac{b}{a}d +}\frac{1}{d +}\frac{1}{\frac{b}{a}d^2 +}\frac{1}{d^2 +}\cdots\frac{1}{\frac{b}{a}d^n +}\frac{1}{d^n +}\cdots,$$

whose partial denominators are positive integers, so that its convergents p_n/q_n $(n\geq 1)$ are just all the best approximations to $f(a/b,1/d)$. Thus we have only to estimate

$$(14) \qquad |f(\frac{a}{b},\frac{1}{d}) - \frac{p_n}{q_n}| = \frac{1}{|1 + \frac{\theta_{n+1}}{a_{n+1}} + \frac{q_{n-1}}{a_{n+1}q_n}|}\frac{1}{|q_n^2 a_{n+1}|}.$$

We note first that

$$\lim_{n\to\infty} \theta_{n+1}/a_{n+1} = \lim_{n\to\infty} q_{n-1}/(q_n a_{n+1}) = 0.$$

If $n=2k$, then by (3)

$$\log a_{2k+1} = k \log d + \log(db/a).$$

But by (4)

$$k = \frac{\sqrt{\log q_{2k}}}{\sqrt{\log d}} - \frac{\log(db/a)}{2\log d} + o(1),$$

and hence

$$\frac{\log a_{2k+1}}{\log q_{2k}} \geq \frac{\sqrt{\log d}}{\sqrt{\log q_{2k}}} + \frac{1}{2}\log(db/a) + o(1).$$

Similarly, we get

$$\frac{\log a_{2k}}{\log q_{2k-1}} \geq \frac{\sqrt{\log d}}{\sqrt{\log q_{2k-1}}} + \frac{1}{2}\log(a/b) + o(1).$$

(14) together with these estimates yields theorem 2.

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