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Generalized Hilbert transforms in tempered distributions

by

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ABSTRACT

In this paper, we define a Hilbert transform Hu for any tempered distribution u. And we show that this Hilbert transform H has some suitable properties, *i.e.* the boundedness, the inversion formula and the signum rule.

I. Introduction

A Hilbert transform H of a function f on real field $\mathbf{R}$ is defined as :

$$Hf(x) = \lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} H_{\epsilon,N}f(x) = \lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} \frac{1}{\pi}\int_{\epsilon<|t|<N} \frac{f(x-t)}{t}dt \qquad (x\in\mathbf{R}).$$

The Hilbert transform H plays an important role in Fourier analysis. The properties of Hilbert transforms in the following Proposition are fundamental and well-known.

Let $L^p(\mathbf{R})$ be the class of all measurable functions f on $\mathbf{R}$ for which

$$\|f\|_{L^p} = \left(\int_{-\infty}^{\infty}|f(x)|^p dt\right)^{\frac{1}{p}} <\infty.$$

Proposition. *Let p be a real number such that $1<p<\infty$. Then*

(i) *[existence]* *for any $f\in L^p(\mathbf{R})$,*

$$Hf(x) = \lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} H_{\epsilon,N}f(x)$$

exists in the topology of $L^p(\mathbf{R})$,

(ii) *[boundedness]* *there exists a constant $C>0$ (independent of ϵ, N and f) such that*

$$\|Hf\|_{L^p}\le C\|f\|_{L^p} \quad (\|H_{\epsilon,N}f\|_{L^p}\le C\|f\|_{L^p}) \qquad \text{for } all\ f\in L^p(\mathbf{R}),$$

(iii) *[inversion formula]*

$$H(H(f)) = -f \qquad \text{for } all\ f\in L^p(\mathbf{R}),$$

(iv) *[signum rule]*

$$(Hf)^{\wedge} = -i\, sgn(x)\hat{f} \qquad \text{for all } f \in L^2(\mathbf{R}),$$

where $\hat{f}$ is a Fourier transform of f.

Many mathematicians have tried to define the Hilbert transforms naturally on more general space (see, for example, [2],[3],[4],[6],[7],[8],[10],[11] and [14]).

S.Koizumi ([10],[11]) introduced a generalized Hilbert transform H for $f \in W^2(\mathbf{R})$ through

$$Hf(x) = \lim_{\epsilon \to 0+} \frac{x+i}{\pi} \int_{\epsilon<|t|} \frac{f(x-t)}{t(t+i-x)} dt$$

where $W^2(\mathbf{R})$ (often called Wiener's class) is the class of all measurable functions f for which $\frac{f(x)}{1+|x|} \in L^2(\mathbf{R})$. And he obtained the similar results in the above Proposition for $W^2(\mathbf{R})$ instead of $L^p(\mathbf{R})$. Moreover, he studyed Hilbert transforms on the class of functions f for which $\frac{|f(x)|^p}{1+|x|^\alpha} \in L^1(\mathbf{R})$ for some $p \geq 1$, $\alpha > 0$.

Also, H.G.Tillmann ([14]) , E.J.Beltrami and M.R.Wohlers([2]) have studyed Hilbert transforms in connection with distribution theory. They showed that the Hilbert transform could be well defined on the subspace $\mathcal{D}_{L_p}$ of distributions which is firstly introduced by L. Schwartz (see [13]). The class $\mathcal{D}_{L^p}$ will be studied in the following section as $\mathcal{D}_{L^p_\delta}$. And they obtained the similar results in the above Proposition for $\mathcal{D}_{L^p}$ (or its dual space $\mathcal{D}^*_{L^p}$) instead of $L^p(\mathbf{R})$.

In this paper, we generally consider the Hilbert transform on tempered distributions $\mathcal{S}'$ (which includes $\mathcal{D}_{L^p}$ and $W^2(\mathbf{R})$) and show that it has the suitable properties as in the above Proposition.

II. A space $\mathcal{D}_{L^p_\delta}(\mathbf{R})$ and its dual space $\mathcal{D}_{L^p_\delta}(\mathbf{R})^*$

Let $\mathbf{R}$ be a real field. We denote by $\mathcal{D}'(\mathbf{R})$, or simply by $\mathcal{D}'$ (throughout this paper we consider only about one variable functions), the space of distributions. $\mathcal{D}'$ is the strong dual of $\mathcal{D}$, the space of infinitely differentiable functions with compact support in $\mathbf{R}$. And we denote a continuous bi-linear functional on $\mathcal{D}' \times \mathcal{D}$ by $<u,\phi>$ for all $u \in \mathcal{D}'$ and $\phi \in \mathcal{D}$.

$\mathcal{S}$ will denote the space of functions on $\mathbf{R}$ having derivatives of all order satisfying $\sup_{x\in\mathbf{R}} |x^\beta D^\alpha \phi(x)| < \infty$ for all indicies α and β of non-negative integers, where $D^\alpha = \frac{d^\alpha}{dx^\alpha}$. It is well-known that $\mathcal{S}$ is a

Fréchet space with the system of semi-norms $\{\sup_{x\in\mathbf{R}}|x^\beta D^\alpha\phi(x)|:\alpha\ \ \beta$ *are non-negative integers*$\}$. $\mathcal{S}'$ is the dual space of $\mathcal{S}$, called a space of tempered distributions.

The Fourier transformation $\hat{\phi}$ of a function $\phi\in\mathcal{S}$ is defined by

$$\hat{\phi}(t) = \int_{-\infty}^{\infty} e^{-itx}\phi(x)dx$$

Since the mapping $\phi\to\hat{\phi}$ of $\mathcal{S}$ onto $\mathcal{S}$ is linear continuous in the topology of $\mathcal{S}$, the Fourier transform $\hat{u}$ of a tempered distribution u can be defined as the tempered distribution $\hat{u}$ defined through

$$<\hat{u},\phi> = <u,\hat{\phi}> \qquad (\phi\in\mathcal{S}).$$

Definition 1. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. $L^p_{k,l}(\mathbf{R})$ denotes a subspace in $\mathcal{S}'$ of functions on* $\mathbf{R}$ *satisfying*

$$q^p_{k,l}(\phi)=\sup\{\ \|x^\alpha D^\beta\phi(x)\|_{L^p}\ :\ 0\le\alpha\le k,\ 0\le\beta\le l\}\ <\ \infty$$

where $D^\beta=\dfrac{d^\beta}{dx^\beta}$ in the sense of distributional derivative.

Moreover $C^{(l)}_k(\mathbf{R})$ denotes the space of functions on $\mathbf{R}$ *such that β-th derivative($0\le\beta\le l$) is continuous and*

$$\|\phi\|_{C^{(l)}_k}=\sup\{|x^\alpha D^\beta\phi(x)|: 0\le\alpha\le k,\ 0\le\beta\le l\}\ <\infty.$$

The following Lemmas 1 and 2 easily follow by the usual arguments of functional analysis.

Lemma 1. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. Then,*

(i) *$L^p_{k,l}(\mathbf{R})$ is a reflexive Banach with norm $q^p_{k,l}$,*

(ii) $\mathcal{S}\subset L^p_{k,l+1}(\mathbf{R})\subset L^p_{k,l}(\mathbf{R})\subset\mathcal{S}'$ and $\mathcal{S}\subset L^p_{k+1,l}(\mathbf{R})\subset L^p_{k,l}(\mathbf{R})\subset\mathcal{S}'$,

and

(iii) *each imbedding map in* (ii) *is continuous and $\mathcal{S}$ is a dense set in each space.*

Lemma 2. *If we define*

$$\hat{q}^p_{k,l}(\phi)=\sup\{\ \|D^\beta x^\alpha\phi(x)\|_{L^p}\ :\ 0\le\alpha\le k,\ 0\le\beta\le l\}$$

then $q^p_{k,l}$ and $\hat{q}^p_{k,l}$ are equivalent norms in $L^p_{k,l}(\mathbf{R})$.

Definition 2. *We can, by Lemma 1, define, for $1<p<\infty$ and non-negative integer k,*

$$\mathcal{D}_{L^p_k}(\mathbf{R}) = \lim_{l\to\infty} \text{proj } [L^p_{k,l}(\mathbf{R})].$$

Clearly, $\mathcal{D}_{L^p_k}(\mathbf{R})$ *is a Fréchet space with the system of countable semi-norms* $\{q^p_{k,l} : l=0,1,2,\ldots\}$ *. And* $\mathcal{D}_{L^p_k}(\mathbf{R})^*$ *is the dual space of* $\mathcal{D}_{L^p_k}(\mathbf{R})$.

If a generalized sequence $\{x_\lambda\}_{\lambda\in\Lambda}$ *in a Hausdorff topological vector space* $\mathbf{X}$ *converges to* x *as* $\lambda\to\lambda_0$ *in the topology of* $\mathbf{X}$, *we denote it by* $(\mathbf{X}) \lim_{\lambda\to\lambda_0} x_\lambda = x$.

By Lemma 1 and the properties of the projective limit, the following Lemma 3 immediately follows.

Lemma 3. *Let* p *be a real number such that* $1<p<\infty$ *. And let* l *and* k *be non-negative integers. Then,*

(i) $\mathcal{S} \subset \mathcal{D}_{L^p_{k+1}}(\mathbf{R}) \subset \mathcal{D}_{L^p_k}(\mathbf{R}) \subset \mathcal{S}'$

and

(ii) *each imbedding map in* (i) *is continuous and* $\mathcal{S}$ *is a dense set in each space.*

Theorem 1. *Let* p *be a real number such that* $1<p<\infty$ *. And let* l *and* k *be non-negative integers. Then,*

(i) $\mathcal{D}_{L^p_k}(\mathbf{R})^* = \lim_{l\to\infty} \text{ind } [L^p_{k,l}(\mathbf{R})^*]$

and

(ii) $\mathcal{D}_{L^p_k}(\mathbf{R})$ *is a reflexive Fréchet space.*

Proof. Since Lemma 1 shows that $\{L^p_{k,l}(\mathbf{R})^*\}_{l=0}^\infty$ is an increasing sequence of reflexive Banach spaces, $\lim_{l\to\infty} \text{ind } [L^p_{k,l}(\mathbf{R})^*]$ is a regular inductive limit ([9]).

By the properties of inductive limits and projective limits (see, for example, [5],[9] and [12]) and Lemma 1(i) , we get that

$$[\lim_{l\to\infty} \text{ind } [L^p_{k,l}(\mathbf{R})^*]]^* = \lim_{l\to\infty} \text{proj } [L^p_{k,l}(\mathbf{R})^{**}] = \lim_{l\to\infty} \text{proj } [L^p_{k,l}(\mathbf{R})] = \mathcal{D}_{L^p_k}(\mathbf{R}). \qquad (1)$$

Also, we see ([9]) that $\lim_{l\to\infty} \text{ind } [L^p_{k,l}(\mathbf{R})^*]$ is reflexive *i.e.*

$$[\lim_{l\to\infty} \text{ind } [L^p_{k,l}(\mathbf{R})^*]]^{**} = \lim_{l\to\infty} \text{ind } [L^p_{k,l}(\mathbf{R})^*]. \qquad (2)$$

Then, we, by (1) and (2), get that

$$[\mathcal{D}_{L_k^p}(\mathbf{R})]^* = [\lim_{l\to\infty} \text{ind } [L_{k,l}^p(\mathbf{R})^*]]^{**} = \lim_{l\to\infty} \text{ind } [L_{k,l}^p(\mathbf{R})^*]$$

and

$$[\mathcal{D}_{L_k^p}(\mathbf{R})]^{**} = [\lim_{l\to\infty} \text{ind } [L_{k,l}^p(\mathbf{R})^*]]^{***} = [\lim_{l\to\infty} \text{ind } [L_{k,l}^p(\mathbf{R})^*]]^* = \mathcal{D}_{L_k^p}(\mathbf{R}).$$

Therefore, we obtain (i) and (ii). This completes the proof.

Lemma 4. *Let q be a real number such that $1<q<\infty$. And let k and α be any non-negative integers. Let g be any function in $L^q(\mathbf{R})$ and P_k be any function having derivatives of all order such that*

$$\limsup_{x\to\pm\infty} \left|\frac{D^j P_k(x)}{(1+x^2)^{k/2}}\right| < \infty \quad \text{for } \textit{any non-negative integer } j\ .$$

Then there exist functions $g_j (j=0,1,2,\ldots,\alpha)$ such that

$$\| g_j(x)/(1+x^2)^{k/2} \|_{L^q} < \infty \qquad \text{and} \qquad P_k(x) D^\alpha g(x) = \sum_{j=0}^{\alpha} D^j g_j(x)\ . \tag{1}$$

Proof. We shall prove this Lemma by induction. Let $\alpha=0$. Since

$$P_k(x) g(x)/(1+x^2)^{k/2} \in L^q(\mathbf{R})$$

(1) immediately follows, if we put $g_0(x)=P_k(x)g(x)$.

Next we prove (1) for $\alpha+1$ under the assumption that (1) is true for α. Since DP_k is a function having derivatives of all order such that

$$\limsup_{x\to\pm\infty} \left|\frac{D^j DP_k(x)}{(1+x^2)^{k/2}}\right| < \infty \quad \text{for } \textit{all non-negative integer } j\ ,$$

we, by assumption, see that

$$\begin{aligned} P_k(x) D^{\alpha+1} g(x) &= D[P_k(x) D^\alpha g(x)] - [DP_k(x)][D^\alpha g(x)] \\ &= D(\sum_{j=0}^{\alpha} D^j g_j) - \sum_{j=0}^{\alpha} D^j g_j' \\ &= -g_0' + \sum_{j=1}^{\alpha} D^j (g_{j-1} - g_j') + D^{\alpha+1} g_\alpha \end{aligned}$$

where $\| g_j(x)/(1+x^2)^{k/2} \|_{L^q} < \infty$ and $\| g_j'(x)/(1+x^2)^{k/2} \|_{L^q} < \infty$ $(j=0,1,2,\ldots,\alpha)$.

This completes the proof.

Theorem 2. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. Then the following statements are equivalent:*

(i) $f \in \mathcal{D}_{L_k^p}(\mathbf{R})^*$

and

(ii) *there exist functions* $f_j (j=0,1,\ldots,l)$ *such that*

$$\|f_j/(1+x^2)^{k/2}\|_{L^q}<\infty \quad \text{and} \quad f=\sum_{j=0}^{l} D^j f_j$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Firstly, we shall prove that (ii) implies (i). Put $C=\max_{0\le j\le l}\|f_j/(1+x^2)^{k/2}\|_{L^q}$. We see, by Lemma 2, that ,for any $\phi\in\mathcal{D}$,

$$\begin{aligned}|<f,\phi>| &= |<\sum_{j=0}^{l} D^j f_j,\phi>| = |\sum_{j=0}^{l}(-1)^{-j}<f_j,D^j\phi>| \\ &= |\sum_{j=0}^{l}(-1)^{-j}<f_j/(1+x^2)^{k/2},(1+x^2)^{k/2}D^j\phi>| \\ &= \sum_{j=0}^{l}\|f_j/(1+x^2)^{k/2}\|_{L^q}\|(1+x^2)^{k/2}D^j\phi\|_{L^p} \\ &\le C\sum_{j=0}^{l}\|(1+x^2)^{k/2}D^j\phi\|_{L^p} \le C' q^p_{k,l}(\phi)\end{aligned}$$

which implies that (i) holds.

Next, we shall prove that (i) implies (ii). Assume that $f\in\mathcal{D}_{L^p_k}(\mathbf{R})^*$. Then, there exist $M>0$ and non-negative integer m such that, for any $\phi\in\mathcal{D}$,

$$|<f,\phi>|\le Mq^p_{k,m}(\phi)\le M\sup\{\ \|x^\alpha D^\beta\phi(x)\|_{L^p}\ :\ 0\le\alpha\le k,\ 0\le\beta\le m\} < \infty.$$

Since $\limsup_{|x|\to\infty}|[D^j(1+x^2)^{-k/2}](1+x^2)^{k/2}|<\infty$, $(j=0,1,2,\ldots)$, this implies that ,for any $\phi\in\mathcal{D}$,

$$\begin{aligned}|<f/(1+x^2)^{k/2},\phi>| &\le |<f,\phi/(1+x^2)^{k/2}>| \\ &\le M\sup_{\substack{0\le\alpha\le k\\ 0\le\beta\le l}}\|x^\alpha D^\beta(\phi/(1+x^2)^{k/2})\|_{L^p} \\ &\le M\sup_{\substack{0\le\alpha\le k\\ 0\le\beta\le l}}\|x^\alpha\sum_{j=0}^{\beta}\binom{\beta}{j}D^{\beta-j}(1/(1+x^2)^{k/2})D^j\phi\|_{L^p} \\ &\le M'\sup_{0\le\beta\le l}\|D^\beta\phi\|_{L^p} .\end{aligned}$$

Hence we see that

$$f(x)/(1+x^2)^{k/2}\in\mathcal{D}_{L^p_0}(\mathbf{R})^*$$

Then, from the theorem of L.Schwartz [13] , this implies that there exist functions $g_\alpha(\alpha=0,1,2,\ldots l)$ $(\in L^q(\mathbf{R})$) such that

$$f(x) = (1+x^2)^{k/2}\sum_{\alpha=0}^{l} D^\alpha g_\alpha .$$

Putting $P_k(x)=(1+x^2)^{k/2}$ in Lemma 4, we see that there exist functions $f_{\alpha,j}$

($\alpha=0,1,\ldots,l$ and $j=0,1,\ldots,\alpha$) such that

$$\|f_{\alpha,j}/(1+x^2)^{k/2}\|_{L^p}<\infty \quad \text{and} \quad f(x) = \sum_{\alpha=0}^{l}\sum_{j=0}^{\alpha} D^j f_{\alpha,j} = \sum_{j=0}^{l} D^j(\sum_{\alpha=j}^{l} f_{\alpha,j}) .$$

This completes the proof.

Lemma 5. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. Then,*

(i) $L^p_{k,l+1}(\mathbf{R})\subset C^{(l)}_k(\mathbf{R})$

and

(ii) $C^{(l)}_{k+1}(\mathbf{R})\subset L^p_{k,l}(\mathbf{R})$

Moreover each natural imbedding map in (i) *and* (ii) *is continuous.*

Proof. By Lemma 2 and the theorem of Barros-Neto J.([1]), we see that, for any $\phi\in L^p_{k,l+1}(\mathbf{R})$

$$\begin{aligned}
\|\phi\|_{C^{(l)}_k} &= \sup_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\sup_{x\in\mathbf{R}}|x^\alpha D^\beta\phi| \\
&= C\sup_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\sup_{x\in\mathbf{R}}|D^\beta x^\alpha\phi| \\
&\le C'\sup_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\{\|D^\beta x^\alpha\phi\|_{L^p}+\|D^{\beta+1}x^\alpha\phi\|_{L^p}\} \\
&\le C'\sup_{\substack{0\le\alpha\le k\\0\le\beta\le l+1}}\|D^\beta x^\alpha\phi\|_{L^p} \\
&\le C'' q^p_{k,l+1}(\phi)
\end{aligned}$$

which implies that (i) is true and the natural imbedding map is continuos.

Next we see that , for any $\phi\in C^{(l)}_{k+1}(\mathbf{R})$

$$\begin{aligned}
q^p_{k,l}(\phi) &= \sup_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\|x^\alpha D^\beta\phi\|_{L^p} \\
&= \sup_{\substack{0\le\alpha\le k\\0\le\beta\le l}}[\int_{-\infty}^{\infty}|\frac{(1+x^2)}{(1+x^2)}x^\alpha D^\beta\phi|^p dx]^{1/p} \\
&= [\int_{-\infty}^{\infty}\frac{1}{(1+x^2)^{p/2}}dx]^{1/p}[\sup_{\substack{0\le\alpha\le k\\0\le\beta\le l}}\sup_{x\in\mathbf{R}}|(1+x^2)^{1/2}x^\alpha D^\beta\phi|] \\
&\le C\|\phi\|_{C^{(l)}_{k+1}}
\end{aligned}$$

which implies that (ii) is true and the natural imbedding map is continuous. This completes the proof.

Theorem 3. *Let p be any $1<p<\infty$. Then,*

(i) $\lim_{k\to\infty}\text{proj}\,[\mathcal{D}_{L^p_k}(\mathbf{R})] = \mathcal{S}$

and

(ii) $\lim\limits_{k\to\infty} \text{ind}\, [\mathcal{D}_{L\xi}(\mathbf{R})^*] = \mathcal{S}'$.

Proof. Since

$$\lim_{k\to\infty} \text{proj}\, [\mathcal{D}_{L\xi}(\mathbf{R})] = \lim_{\substack{k\to\infty\\ l\to\infty}} \text{proj}\, [L_{L\xi,l}(\mathbf{R})] \quad \text{and} \quad \mathcal{S} = \lim_{\substack{k\to\infty\\ l\to\infty}} \text{proj}\, [C_k^l(\mathbf{R})]$$

we see, by Lemma 5, that (i) is true. Also, since

$$\lim_{k\to\infty} \text{proj}\, [\mathcal{D}_{L\xi}(\mathbf{R})^*] = \lim_{\substack{k\to\infty\\ l\to\infty}} \text{proj}\, [L_{L\xi,l}(\mathbf{R})^*] \quad \text{and} \quad \mathcal{S}' = \lim_{\substack{k\to\infty\\ l\to\infty}} \text{proj}\, [C_k^l(\mathbf{R})^*]$$

we see, by Lemma 5, that (ii) is true.

III. Generalized Hilbert transforms in $\mathcal{D}_{L\xi}(\mathbf{R})$

Definition 3. *Let $a=(a_1,\ldots,a_k)$ be a k-tuple of complex numbers such that $Im[a_j]\neq 0 (j=1,2,\ldots,k)$. where $Im[a_j]$ denotes the imaginary part of a complex number a_j.*

We define that, for any $\phi\in\mathcal{D}_{L\xi}(\mathbf{R})$

$$(H_a^{\epsilon,N}\phi)(x)=\frac{1}{\pi(x-a_1)\ldots(x-a_k)}\int_{\epsilon<|t|<N}(x-t-a_1)\ldots(x-t-a_k)\frac{\phi(x-t)}{t}dt,$$

specially, if $k=0$

$$(H^{\epsilon,N}\phi)(x)=\frac{1}{\pi}\int_{\epsilon<|t|<N}\frac{\phi(x-t)}{t}dt$$

The following Lemma easily follows.

Lemma 6. *Let $a=(a_1,\ldots,a_k)$ be a k-tuple of complex numbers such that $Im[a_j]\neq 0 (j=1,2,\ldots,k)$. A mapping $T_a: \mathcal{D}_{L\xi}(\mathbf{R})\to\mathcal{D}_{L\xi}(\mathbf{R})$ such that*

$$T_a\psi = \frac{\psi(x)}{(x-a_1)\ldots(x-a_k)} \quad \text{for all } \psi\in\mathcal{D}_{L\xi}(\mathbf{R})$$

is a bi-continuous surjection.

Theorem 4. *Let $a=(a_1,\ldots,a_k)$ be a k-tuple of complex numbers such that $Im[a_j]\neq 0 (j=1,2,\ldots,k)$. Then, for any $\phi\in\mathcal{D}_{L\xi}(\mathbf{R})$, $(\mathcal{D}_{L\xi})\lim\limits_{\substack{\epsilon\to 0+\\ N\to\infty}}(H_a^{\epsilon,N}\phi)$ exists in $\mathcal{D}_{L\xi}(\mathbf{R})$.*

Proof. Let $k=0$. By Proposition (i), we see that, for any $\phi\in\mathcal{D}_{L\xi}(\mathbf{R})$ and any $0<\epsilon'<\epsilon<N<N'<\infty$,

$$q_{\beta,l}^{\mathfrak{G}}(H^{\epsilon,N}\phi - H^{\epsilon',N'}\phi)$$
$$\leq \sup_{0\leq\beta\leq l} \| D^{\beta}\frac{1}{\pi}\int_{\substack{\epsilon'<|t|<\epsilon \\ N<|t|<N'}} \frac{\phi(x-t)}{t}dt\|_{L^p}$$
$$\leq \sup_{0\leq\beta\leq l} \| \frac{1}{\pi}\int_{\substack{\epsilon'<|t|<\epsilon \\ N<|t|<N'}} \frac{(D^{\beta}\phi)(x-t)}{t}dt\|_{L^p}$$
$$\to 0 \quad as \quad \epsilon,\epsilon' \to 0+ \quad \text{and} \quad N,N' \to \infty .$$

This implies that $\{H^{\epsilon,N}\phi\}$ is a Cauchy net as $\epsilon\to 0+, N\to\infty$ in $\mathcal{D}_{L^p}(\mathbf{R})$. Hence $\lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} (H_a^{\epsilon,N}\phi)$ exists in the topology of $\mathcal{D}_{L^p}(\mathbf{R})$.

In general case, by the above argument and Lemma 6, we see that, for any $\phi\in\mathcal{D}_{L^p}(\mathbf{R})$

$$(\mathcal{D}_{L^p})\lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} H_a^{\epsilon,N}\phi = (\mathcal{D}_{L^p})\lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} \frac{1}{\pi(x-a_1)\ldots(x-a_k)}\int_{\epsilon<|t|<N} \frac{(x-t-a_1)\ldots(x-t-a_k)}{t}\phi(x-t)dt$$
$$= (\mathcal{D}_{L^p})\lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} T_a H^{\epsilon,N}(T_a^{-1}\phi)$$
$$= T_a[(\mathcal{D}_{L^p})\lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} H^{\epsilon,N}(T_a^{-1}\phi)]$$

which exists since $T_a^{-1}\phi\in\mathcal{D}_{L^p}$.

Definition 4. *Let* $a=(a_1, \ldots ,a_k)$ *be a k-tuple of complex numbers such that* $Im[a_j]\neq 0 (j=1,2,\ldots,k)$.

We define a generalized Hilbert transform $H_a:\mathcal{D}_{L^p}(\mathbf{R})\to\mathcal{D}_{L^p}(\mathbf{R})$ *such that*

$$H_a\phi = (\mathcal{D}_{L^p})\lim_{\substack{\epsilon\to 0+ \\ N\to\infty}} H_a^{\epsilon,N}\phi \qquad (\phi\in\mathcal{D}_{L^p}(\mathbf{R})).$$

Note that a generalized Hilbert transform H_a is also represented by $T_a H T_a^{-1}$.

Theorem 5. *Let p be a real number such that* $1<p<\infty$. *And let l and k be non-negative integers. Let* $a=(a_1, \ldots ,a_k)$ *be a k-tuple of complex numbers such that* $Im[a_j]\neq 0 (j=1,2,\ldots,k)$.

Then,

(i) H_a *is a bounded linear operator on* $\mathcal{D}_{L^p}(\mathbf{R})$

and

(ii) $H_a(H_a\phi) = -\phi \qquad (\phi\in\mathcal{D}_{L^p}(\mathbf{R}))$.

Moreover, $H_a:\mathcal{D}_{L^p}(\mathbf{R})\to\mathcal{D}_{L^p}(\mathbf{R})$ *is a bi-continuous surjection such that* $H_a^{-1} = -H_a$.

Proof. It is sufficient to prove (i) and (ii) for $k=0$ since $H_a = T_a H T_a^{-1}$. Though this theorem for $k=0$ has been proved in [14] , we shall show the proof for the self-consistency.

By the similar way in the Theorem 4, we can easily obtain , from the Proposition (ii), that for any $\phi \in \mathcal{D}_{L^p_\delta}(\mathbf{R})$ and any $0<\epsilon<N<\infty$,

$$\begin{aligned} q^p_{\delta,l}(H^{\epsilon,N}\phi) &\le \sup_{0\le\beta\le l} \|\frac{1}{\pi}\int_{\epsilon<|t|<N} \frac{D^\beta\phi(x-t)}{t}dt\|_{L^p} \\ &\le C \sup_{0\le\beta\le l} \|D^\beta\phi\|_{L^p} \\ &\le C q^p_{\delta,l}(\phi) \end{aligned}$$

which implies (i) for $k=0$. Also, by Proposition (iii), (ii) immediately follows since $\mathcal{D}_{L^p_\delta}(\mathbf{R}) \subset L^p(\mathbf{R})$.

his completes the proof.

IV. Generalized Hilbert transforms in $\mathcal{D}'$

Definition 5. *Let p be any $1<p<\infty$ and k be any non-negative integer. And let $a=(a_1, \ldots, a_k)$ be a k-tuple of complex numbers such that $Im[a_j]\neq 0$ $(j=1,\ldots,k)$. Since the generalized Hilbert transform $H_a:\mathcal{D}_{L^p_\xi}(\mathbf{R}) \to \mathcal{D}_{L^p_\xi}(\mathbf{R})$ is linear continuous in the topology of $\mathcal{D}_{L^p_\xi}(\mathbf{R})$, we can define the generalized Hilbert transform $H_a^* u$ of $u \in \mathcal{D}_{L^p_\xi}(\mathbf{R})^*$ as the element of $\mathcal{D}_{L^p_\xi}(\mathbf{R})^*$ defined though*

$$<H_a^* u, \phi> \; = \; <u, H_a\phi> \qquad (\phi \in \mathcal{D}_{L^p_\xi}(\mathbf{R})).$$

Similarly, $H_a^{\epsilon,N\,*}$ is defined as the adjoint operator of $H_a^{\epsilon,N}$.

The following theorem immediately follows from the property of the adjoint operator and Theorem 5.

Theorem 6. *It follows that*

(i) H_a^* *is linear continuos in the topology of* $\mathcal{D}_{L^p_\xi}(\mathbf{R})^*$

(ii) $H_a^*(H_a^* u) = -u, \qquad (u \in \mathcal{D}_{L^p_\xi}(\mathbf{R})^*)$

Therefore, $H_a^{*-1} = -H_a^*$.

Theorem 7. *Let p be a real number such that $1<p<\infty$. And let l and k be non-negative integers. And let $a=(a_1, \ldots, a_k)$ be a k-tuple of complex numbers such that $Im[a_j]\neq 0$ $(j=1,\ldots,k)$. Then, for any $u \in \mathcal{D}_{L^p_\xi}(\mathbf{R})^*$*

$$H_a^* u = (\mathcal{D}_{L^p_\xi}^*) \lim_{\substack{\epsilon\to 0+\\ N\to\infty}} H_a^{\epsilon,N\,*} u \qquad (u \in \mathcal{D}_{L^p_\delta}(\mathbf{R})^*).$$

Proof. Firstly we shall prove this theorem, when $k=0$. Let u be any element in $\mathcal{D}_{L^q}(\mathbf{R})^*$

By theorem 2, we see that, for any $\phi \in \mathcal{D}_{L^p}(\mathbf{R})$

$$
\begin{aligned}
&|<(H^{\epsilon,N\,*}-H^*)u,\phi>| \\
&=|<\sum_{j=0}^{l} D^j u_j,(H^{\epsilon,N\,*}-H^*)\phi>| \\
&=\sum_{j=0}^{l}|<u_j,(H^{\epsilon,N\,*}-H^*)D^j\phi>| \\
&=\sum_{j=0}^{l}|<(H^{\epsilon,N\,*}-H^*)u_j,D^j\phi>| \\
&\leq \sum_{j=0}^{l}||(H^{\epsilon,N\,*}-H^*)u_j||_{L^q}||D^j\phi||_{L^p}
\end{aligned}
$$

where u_j $(j=1,2,\ldots,l)$ are defined as in the Theorem 2.

By Proposition(i), this implies that, for any bounded set $B \subset \mathcal{D}_{L^p}(\mathbf{R})$

$$
\begin{aligned}
&\sup_{\phi\in B}|<(H^{\epsilon,N\,*}-H^*)u,\phi>| \\
&=C\sum_{j=0}^{l}||(H^{\epsilon,N\,*}-H^*)u_j||_{L^q} \\
&\to 0 \qquad (as\ \epsilon\to 0+,\ N\to\infty).
\end{aligned}
$$

Hence we get that , for any $u \in \mathcal{D}_{L^q}(\mathbf{R})^*$

$$H_a u = (\mathcal{D}^*_{L^q}) \lim_{\substack{\epsilon\to 0+\\ N\to\infty}} H_a^{\epsilon,N} u.$$

In general case, we see that, for any $u \in \mathcal{D}_{L^q}(\mathbf{R})^*$

$$
\begin{aligned}
(\mathcal{D}^*_{L^q}) \lim_{\substack{\epsilon\to 0+\\ N\to\infty}} H_a^{\epsilon,N\,*}u &= (\mathcal{D}^*_{L^q}) \lim_{\substack{\epsilon\to 0+\\ N\to\infty}} ((T_a^{-1}H^{\epsilon,N\,*}T_a)u) \\
&= T_a^{-1}(\mathcal{D}^*_{L^q}) \lim_{\substack{\epsilon\to 0+\\ N\to\infty}} ((H^{\epsilon,N\,*}T_a)u) \\
&= T_a^{-1}((H^*T_a)u) \\
&= H_a^* u.
\end{aligned}
$$

This completes the proof.

Theorem 8. *Let p be a real number such that $1<p<\infty$. And let k be a non-negative integer. And let $a=(a_1,\ldots,a_k)$ be a k-tuple of complex numbers such that $Im[a_j]\neq 0$ $(j=1,\ldots,k)$. Then, for any u , $\in \mathcal{D}_{L^q}(\mathbf{R})^*$*

$$
(H_a^* u)\hat{}\,(\phi)=\begin{cases} -i<\hat{u},\phi> & \text{for } all\ \phi\in\mathcal{D}\ such\ that\ \mathrm{supp}[\phi]\subset(0,\infty) \\ i<\hat{u},\phi> & \text{for } all\ \phi\in\mathcal{D}\ such\ that\ \mathrm{supp}[\phi]\subset(-\infty,0) \end{cases}
$$

where $\hat{u}$ is the Fourier transform of u in $\mathcal{S}'$.

Proof. Let ϕ be any element in $\mathcal{D}$ such that supp$[\phi]\subset(0,\infty)$. From the properties of Fourier transforms and Proposition (iv), we see that

$$
\begin{aligned}
<(H_a^* u)^\wedge,\phi> &= <H_a^* u,\hat{\phi}> \\
&= <u, T_a^{-1} H T_a \hat{\phi}> \\
&= <u, T_a^{-1} H[[(i^{-1}D-a_1)(i^{-1}D-a_2)\ldots(i^{-1}D-a_k)\phi]^\wedge\ ]> \\
&= <u, T_a^{-1}[-i(i^{-1}D-a_1)(i^{-1}D-a_2)\ldots(i^{-1}D-a_k)\phi]^\wedge> \\
&= -i<u, T_a^{-1} T_a \hat{\phi}> \\
&= -i<u,\hat{\phi}> \\
&= -i<\hat{u},\phi>.
\end{aligned}
$$

In a similar way, we can prove this theorem when ϕ is any element in $\mathcal{D}$ such that supp$[\phi]\subset(-\infty,0)$. ence this completes the proof.

Corollary 1. *Let p be a real number such that $1<p<\infty$. Let k,m and n be nonnegative integers such that $k\leq m\leq n$. And let $a=(a_1, \cdots, a_m)$ and $b=(b_1, \ldots, b_n)$ are m-tuple and n-tuple of complex numbers such that $Im[a_i]\neq 0$ $(i=1,2,\ldots,m)$* and *$Im[b_i]\neq 0$ $(i=1,2,\ldots,n)$.*

Then, for any $u\in\mathcal{D}_{L^p_k}(\mathbf{R})^*$ $(\subset\mathcal{D}_{L^p_m}(\mathbf{R})^* \subset \mathcal{D}_{L^p_n}(\mathbf{R})^*$), $H_a u - H_b u$ is a polynomial.

Proof. By Theorem 8, we see that, for any $\phi\in\mathcal{D}$ with supp$[\phi]\subset(0,\infty)$

$$<(H_a^* u - H_b^* u)^\wedge,\phi> = <(H_a^* u)^\wedge,\phi> - <(H_b^* u)^\wedge,\phi> = -i<\hat{u},\phi> - (-i)<\hat{u},|(^*f> = 0. \tag{1}$$

Similarly we see that, for any $\phi\in\mathcal{D}$ with supp$[\phi]\subset(-\infty,0)$

$$<(H_a^* u - H_b^* u)^\wedge,\phi> = 0. \tag{2}$$

By (1) and (2), it follows that supp$[(H_a u - H_b u)^\wedge]=\{0\}$. This implies that $(H_a u - H_b u)^\wedge$ is a finite linear combination of a Delta function $\delta(x)$ and its derivatives. Therefore, $H_a u - H_b u$ is a certain polynomial. This completes the proof.

Remark. *Let u be any element in $\mathcal{S}'$. Since Theorem 3 shows that $u\in\mathcal{D}_{L^p_k}(\mathbf{R})$ for some k, the generalized Hilbert transform Hu can be defined by $H_a^* u$, where $a=(a_1, \cdots, a_k)$ is a k-tuple of complex numbers such that $Im[a_j]\neq 0$ $(j=1,2,\ldots,k)$. The above Corollary 1 shows that Hu is well defined under the identification of the difference of polynomials independently of choosing k and a.*

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