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On large scalar curvature

by

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Introduction.

For a compact Riemannian n-manifold (M,g), we set

$$\mu(M,g) := \inf \left\{ \int_M R_{\tilde{g}} dv_{\tilde{g}} / \left(\int_M dv_{\tilde{g}} \right)^{(n-2)/n} ; \ \tilde{g} \text{ is conformal to } g \right\},$$

where R_g is the scalar curvature of g. As Yamabe's problem is affirmatively and completely solved ([1], [6]), we have a metric g in any conformal class such that the scalar curvature R_g is constant, and $R_g \mathrm{Vol}(M,g)^{2/n} = \mu(M,g)$. This metric has relatively small scalar curvature, for $\mu(M,g) \leqq \mu(S^n,g_0) = n(n-1)\mathrm{Vol}(S^n,g_0)^{2/n}$ holds in general [1], where (S^n,g_0) is the Euclidean unit n-sphere. Moreover, recently, Schoen [6] has proved that $\mu(M,g) = \mu(S^n,g_0)$ if and only if (M,g) is conformally equivalent to (S^n,g_0), which was the unsolved part of Yamabe's problem.

We introduce a quantity $\mu(M)$ defined by

$$\mu(M) := \sup_g \mu(M,g).$$

M admits a metric of positive scalar curvature if and only if $\mu(M) > 0$. Also, $\mu(M) \leqq \mu(S^n) = \mu(S^n,g_0)$ from the above inequality by Aubin. Then compared with Schoen's result, it is interesting to ask whether there is an M other than S^n for which $\mu(M) = \mu(S^n)$. We give such an example.

Theorem A. *If* $n \geqq 3$, $\mu(S^1 \times S^{n-1}) = \mu(S^n)$.

Next, we consider the following problem: For a given constant c, does there exist a metric g of S^n $(n \geqq 3)$ such that $\mathrm{Vol}(S^n, g) = 1$, and that the scalar curvature $R_g = c$? If $c \leqq \mu(S^n)$, we can get such a metric by solving Yamabe's problem. Otherwise, Yamabe's method loses its power. Under a slightly relaxed condition on constancy, we prove the following.

Theorem B. For any $r > 0$ and any $\varepsilon > 0$, there exists a metric g of S^n $(n \geqq 3)$ such that $\mathrm{Vol}(S^n, g) = 1$, $R_g \geqq r$ and $\max_x R_g(x) / \min_x R_g(x) \leqq 1 + \varepsilon$.

§1. Preliminaries.

Let (N, g_N) be an $(n-1)$-dimensional compact Riemannian manifold whose scalar curvature is constant $(n-1)(n-2)$. We always assume $n \geqq 3$.

Suppose that f is a positive function on $\mathbb{R}$ which satisfies

$$(1.1) \quad 2(n-1)f\ddot{f} - n(n-1)\dot{f}^2 + (n-1)(n-2)f^2 = n(n-1).$$

Then, $(\mathbb{R} \times N, f(t)^{-2}(dt^2 + g_N))$ has constant scalar curvature $n(n-1)$. (1.1) is integrable and we have

$$(1.2) \quad \dot{f}^2 = -1 + f^2 - (2c/(n-2))f^n,$$

where c is an integral constant, which relates to the Ricci curvature as follows;

$$(1.3) \quad \mathrm{Ric}_{g(c)} - \frac{1}{n}R_{g(c)} = c\, f(t)^{n-2}(g_N - (n-1)dt^2),$$

where we put $g(c) = f(t)^{-2}(dt^2 + g_N)$ with f satisfying (1.2). From (1.2), we observe that, in order for f to be defined on whole $\mathbb{R}$, the constant c must be in the interval

$$(1.4) \quad 0 \leqq c \leqq ((n-2)/n)^{n/2} =: c_0.$$

Then, we have

$$(1.5)\quad \begin{cases} f(t) = \cosh(t+t_0) \text{ for some } t_0 \text{ if } c = 0; \\ f \text{ is a periodic function with prime period } T(c) \text{ if } 0<c<c_0; \\ f(t) = (n/(n-2))^{\frac{1}{2}} \text{ if } c = c_0. \end{cases}$$

Hence, if $0<c<c_0$, $g(c)$ is the lift of a metric (also denoted by $g(c)$) of $(\mathbb{R}/T(c)\mathbb{Z})\times N$. We denote by $V(c)$ the volume of $((\mathbb{R}/T(c)\mathbb{Z})\times N, g(c))$. Remark that $(\mathbb{R}\times N, g(0))$ is isometric to $S^n\setminus$(antipodal two points) if $(N,g_N) = (S^{n-1},g_0)$.

Lemma. $T(c)$ and $V(c)$ are continuous function in c, and

(i) $\lim_{c\to 0} T(c) = \infty$; $\lim_{c\to c_0} T(c) = 2\pi/\sqrt{n-2}$.

(ii) $\lim_{c\to 0} V(c) = \mathrm{Vol}(N,g_N)\mathrm{Vol}(S^n,g_0)/\mathrm{Vol}(S^{n-1},g_0)$;

$\lim_{c\to c_0} V(c) = 2\pi(n-2)^{(n-1)/2} n^{-n/2}\,\mathrm{Vol}(N,g_N)$.

Proof. We compute only $\lim_{c\to c_0} T(c)$ and $\lim_{c\to c_0} V(c)$. The other assertions are easy to prove.

For $0<c<c_0$, let $0<a(c)<b(c)$ be the two positive roots of the equation

$$(1.6)\qquad 1 - \lambda^2 + (2c/(n-2))\lambda^2 = 0.$$

We define a polynomial $F(\lambda,c)$ in λ with parameter c as

$$(1.7)\qquad \text{the left side of } (1.6) = (\lambda - a(c))(\lambda - b(c))\,F(\lambda,c).$$

It is easy to see that $\lim_{c\to c_0} a(c) = \lim_{c\to c_0} b(c) = \lambda_0 := (n/(n-2))^{\frac{1}{2}}$. Then by a direct calculation, we obtain

$$(1.8)\qquad \lim_{(\lambda,c)\to(\lambda_0,c_0)} F(\lambda,c) = n-2.$$

It follows from (1.2) that

$$(1.9)\quad T(c) = 2\int_{a(c)}^{b(c)} F(\lambda, c)^{-1/2} \frac{d\lambda}{\sqrt{-(\lambda - a(c))(\lambda - b(c))}}$$

and

$$(1.10)\quad V(c) = 2\,\mathrm{Vol}(N, g_N)\int_{a(c)}^{b(c)} \lambda^{-n} F(\lambda, c)^{-1/2} \frac{d\lambda}{\sqrt{-(\lambda - a(c))(\lambda - b(c))}}\,.$$

These combined with (1.8) and the elementary formula $\int_a^b d\lambda/\sqrt{-(\lambda-a)(\lambda-b)} = \pi$ for $a<b$, give the desired formulas. □

Remark. From the second variation formula for the functional $g \mapsto \int R_g dv_g / (\int dv_g)^{(n-2)/n}$, it follows that, if $T > 2\pi/\sqrt{n-2}$, the product metric $g_T = dt^2 + g_N$ of $S^1 \times N$ with length$(S^1, dt^2) = T$ is not a solution to Yamabe's problem (we call a metric g a solution to the Yamabe's problem, if the scalar curvature R_g is constant, and $R_g \mathrm{Vol}(g)^{2/n} = \mu(g)$.), although this metric has constant scalar curvature (cf. [/]). This fact corresponds to our computation $\lim_{c \to c_0} T(c) = 2\pi/\sqrt{n-2}$.

The following is a corollary of the above lemma.

Proposition. *Let $(S^1 \times N, g_T)$ be as above. If $T > 2\pi k/\sqrt{n-2}$ for some integer k, then there are at least $(k+1)$ metrics conformal to g_T with the same volume such that they have constant scalar curvatures but are not isometric to one another.*

Proof. From Lemma (i) and the assumption on T, we find $c_j \in (0, c_0)$, $j = 1, \ldots, k$ such that $T(c_j) = T/j$. By the natural j-fold covering $(\mathbb{R}/T\mathbb{Z}) \times N \to (\mathbb{R}/(T/j)\mathbb{Z}) \times N$, we lift the metric $g(c_j)$ to $(\mathbb{R}/T\mathbb{Z}) \times N$. Then the lifted metric, denoted by g_j , is conformal to g_T. Normalizing the volumes of g_T, $g_1, \ldots, g_k$ by homothetical changes, we get the desired metrics. $\square$

§2. Proof of Theorem A.

Let g_T be the product metric of $S^1 \times S^{n-1}$ with length$(S^1, dt^2) = T$. Let f be a positive function on $S^1 \times S^{n-1}$ such that $f^{-2}g_T$ is a solution to Yamabe's problem with constant scalar curvature $n(n-1)$. We lift the metric $f^{-2}g_T$ to the universal covering $\mathbb{R} \times S^{n-1}$. Since $f^{-2}(dt^2 + g_0) = (rf)^{-2}(dr^2 + r^2 g_0)$, $r = e^t$, and $dr^2 + r^2 g_0$ is the Euclidean flat metric of $\mathbb{R}^n \setminus 0 \cong \mathbb{R} \times S^{n-1}$, putting $u = (n(n-2)/4)^{(n-2)/4}(rf)^{(2-n)/2}$, we have

$$\Delta u + u^{\frac{n+2}{n-2}} = 0,$$

where Δ is the Laplacian of the flat metric $dr^2 + r^2 g_0$. Moreover, $u \to \infty$ as $r \to 0$, and $r^{n-2}u \to \infty$ as $r \to \infty$, because $\min f > 0$ and $\max f < \infty$. Then, we can apply a theorem of Gidas, Ni and Nirenberg (Theorem 4 of [4]), and conclude that f depends only on t; $f = f(t)$. Hence, $f^{-2}g_T$ is one of the metrics described in Proposition of §1. Now, it is easy to see that if $T \leqq 2\pi/\sqrt{n-2}$, then f is constant, and that if $T > 2\pi/\sqrt{n-2}$, take $c \in (0, c_0)$ such that $T(c) = T$, then $f^{-2}g_T = g(c)$ in §1 for $(N, g_N) = (S^{n-1}, g_0)$. Thus, for $T > 2\pi/\sqrt{n-2}$, $\mu(S^1 \times S^{n-1}, g_T) = n(n-1)V(c)^{n/2}$. Therefore, from Lemma in §1, we have $\lim_{T\to\infty} \mu(S^1 \times S^{n-1}, g_T) = n(n-1)\lim_{c\to 0} V(c)^{n/2} = n(n-1)\mathrm{Vol}(S^n, g_0)^{n/2} = \mu(S^n)$. That is, $\mu(S^1 \times S^{n-1}) \geqq \mu(S^n)$, so $\mu(S^1 \times S^{n-1}) = \mu(S^n)$. $\square$

§3. Proof of Theorem B.

Let $f_c : \mathbb{R} \longrightarrow \mathbb{R}_+$, $c \in (0, c_0)$ be the solution of

$$(3.1) \begin{cases} \dot{f}_c(t)^2 = -1 + f_c(t)^2 - (2c/(n-2)) f_c(t)^n \\ f_c(0) = a(c) \qquad \text{(see §1 for } a(c)). \end{cases}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\eta(t) \equiv 0$ for $t \leqq 0$, $\eta(t) > 0$ for $t > 0$, and $\eta(t) \equiv 1$ for $t \geqq 1$. We put

$$(3.2) \quad F_c(t) = \begin{cases} (\eta f_0 + (1-\eta) f_c)(t) & \text{for } -kT(c) \leqq t \text{ (see §1 for } T(c)); \\ ((1-\eta) f_0 + f_c)(t + kT(c) + 1) & \text{for } t \leqq -kT(c), \end{cases}$$

where k is an integer satisfying

$$(3.3) \quad k + 1 > (r(1+\varepsilon)/n(n-1))^{n/2} / \mathrm{Vol}(S^n, g_0).$$

Then, the Riemannian metric $g_c := F_c^{-2}(dt^2 + g_0)$ of $\mathbb{R} \times S^{n-1}$ is smoothly extendable to the two point compactification S^n. Its scalar curvature $R_c = R_c(t)$ is given by

$$(3.4) \quad R_c(t) = 2(n-1)F_c(t)\ddot{F}_c(t) - n(n-1)\dot{F}_c(t)^2 + (n-1)(n-2)F_c(t)^2.$$

From the definition of F_c, we have

$$(3.5) \quad R_c(t) = n(n-1) \quad \text{for } t \notin [0,1] \cup [-kT(c)-1, -kT(c)].$$

Note that $f_c | [0,1]$ converges uniformly to $f_0 | [0,1]$ up to second derivatives as $c \to 0$. Hence, R_c converges uniformly to the constant $n(n-1)$. Therefore, there is a small $c_1 > 0$ such that

$$(3.6) \quad \max R_{c_1} / \min R_{c_1} \leqq 1 + \varepsilon.$$

Also, from Lemma (ii) in §1 and (3.3), taking a smaller c_1 if necessary, we have

$$(3.7) \quad \mathrm{Vol}(g_{c_1}) = (r(1+\varepsilon)/n(n-1))^{n/2}.$$

Now, we define a metric g of S^n by $g := \mathrm{Vol}(g_{c_1})^{-2/n} g_{c_1}$. Then, clearly, $\mathrm{Vol}(S^n,g) = 1$, and $\max R_g/\min R_g \leqq 1 + \varepsilon$, from (3.6). Finally, $R_g = \mathrm{Vol}(g_{c_1})^{2/n} R_{c_1} \geqq (r(1+\varepsilon)/n(n-1))\min R_{c_1} \geqq r$. □

§4. Remarks on Theorem B.

We can also show that *if* $\mu(M) > 0$ *and* $n = \dim M \geqq 3$, *there exists, for any constant* r, *a metric* g *of* M *for which* $\mathrm{Vol}(M,g) = 1$ *and* $R_g > r$. But this is not interesting. A reason why we stick to (ε -) constancy of scalar curvature is in the following result.

Proposition. *Suppose that for any constant* r *and* $\varepsilon > 0$, *there is a metric* $g(r,\varepsilon)$ *such that* $\mathrm{Vol}(M,g(r,\varepsilon)) = 1$ *and* $|R_{g(r,\varepsilon)}(x) - r| < \varepsilon$ *for all* $x \in M$. *Then, for any nonconstant function* f, *there exists a metric* g *for which* $\mathrm{Vol}(M,g) = 1$ *and* $R_g = f$.

This proposition is a corollary of the following generalization of a theorem of Kazdan and Warner [5].

Theorem. *Let* g_0 *be a smooth Riemannian metric of a compact manifold* M *with* $\mathrm{Vol}(M,g_0) = 1$ *and with nonconstant scalar curvature* $R(g_0)$, *and* f *a smooth function satisfying* $\min f \leqq \min R(g_0)$ *and* $\max R(g_0) \leqq \max f$. *Then there is a smooth metric* g *such that* $\mathrm{Vol}(M,g) = 1$ *and* $R(g) = f$.

Before the proof, we give some other corollaries.

Corollary 1. *If* $n \geqq 3$, *then for any function* f *with* $\min f < \mu(M)$, *there is a metric* g *with* Vol(M,g) = 1 *and* R(g) =f.

Corollary 2. Let M_1 *and* M_2 *be compact manifold such that* $\mu(M_1) > 0$ *and* $\dim M_2 \geqq 1$. *Then, for any function* f *on* $M_1 \times M_2$, *there is a metric* g *with* $\mathrm{Vol}(M_1 \times M_2, g) = 1$ *and* $R(g) = f$.

Proof of Corollary 2. Under the assumption, it is not hard to see that for any constant c, there is a metric g_c of $M_1 \times M_2$ with $\mathrm{Vol}(M_1 \times M_2, g_c) = 1$ and $R(g_c) = c$. □

Proof of Theorem. The proof is similar to that given in [5]. So, we only sketch it. Let S^2T (resp. S^2_+T) denote the bundle of (resp. positive definite) symmetric covariant 2-tensors, and $\mathring{S}^2T = \{h \in S^2T;\ \mathrm{tr}_{g_0} h = 0\}$. Let $H_{2,p}(M;S^2T)$, $H_{2,p}(M;S^2_+T)$, ... be the Sobolev spaces of $H_{2,p}$-sections (i.e., up to 2nd derivatives are L_p) of S^2T, S^2_+T, ... We always assume that $p > n = \dim M$. $H_{2,p}(M;S^2_+T)$ is an open subset of $H_{2,p}(M;S^2T)$, and $\mathrm{Vol}: H_{2,p}(M;S^2_+T) \to \mathbb{R};\ g \mapsto \mathrm{Vol}(M,g)$ is a C^1-mapping whose differential is not zero. From this, there are a neighborhood U of 0 in $H_{2,p}(M;\mathring{S}^2T)$ and a C^1-function $\alpha: U \to \mathbb{R}$ such that, putting $\Phi(h) = g_0 + h + \alpha(h)g_0$ for $h \in U$, we have the following properties; (1) $\Phi(h) \in H_{2,p}(M;S^2_+T)$; (2) $\mathrm{Vol}(\Phi(h)) = 1$; (3) $\Phi(0) = g_0$; (4) $D\Phi$ at 0 is the inclusion map $H_{2,p}(M;\mathring{S}^2T) \hookrightarrow H_{2,p}(M;S^2T)$. We note that $\Phi(h)$ is a C^∞ metric if and only if h is a C^∞ section.

The scalar curvature $R: H_{2,p}(M;S^2_+T) \to L_p(M;\mathbb{R})$ is defined as a C^1-mapping for $p > n$. So we get a C^1-mapping $R \circ \Phi: U \to L_p(M;\mathbb{R})$, $U \subset H_{2,p}(M;\mathring{S}^2T)$. The differential $A: H_{2,p}(M;\mathring{S}^2T) \to L_p(M;\mathbb{R})$ of $R \circ \Phi$ at 0 is computed to be

$$A(h) = -\Delta h^i{}_i + h^{ij}{}_{;ij} - h^{ij}R_{ij}\ ,$$

where the connection, Ricci curvature, etc. are relative to g_0. The formal L_2-adjoint A^* is given by

$$A^*(u) = \bar{A}^*(u) + \beta(u)g_0.$$

where $\bar{A}^*(u) = -(\Delta u)g_0 + \nabla^2 u - u\,\mathrm{Ric}(g_0)$ and $\beta(u) = \int R(g_0)dv_{g_0}/n\,\mathrm{Vol}(g_0)$. $A^*: H_{4,p}(M;\mathbb{R}) \longrightarrow H_{2,p}(M;\overset{\circ}{S}{}^2T)$ is a continuous linear map. From the assumption that the scalar curvature $R(g_0)$ is not constant, we can show that $A\circ\bar{A}^*: H_{4,p}(M;\mathbb{R}) \longrightarrow L_p(M;\mathbb{R})$ is a linear homeomorphism, and $A\circ A^*: H_{4,p}(M;\mathbb{R}) \longrightarrow L_p(M;\mathbb{R})$ is injective (cf. [2], [3], [5]). Then, since $A\circ(A^* - \bar{A}^*)$ is a compact operator, we conclude that $A\circ A^*$ is invertible. (Remark. If g_0 is an Einstein metric with constant negative scalar curvature, then $A\circ A^*$ is not invertible. This is a difference from the case without volume constraint.)

Let $V = (A^*)^{-1}(U)$. Define a C^1-mapping $Q: V \rightarrow L_p(M;\mathbb{R})$ by $Q = R\circ\Phi\circ A^*$. Then the differential of Q at 0 is $A\circ A^*$. So, by the inverse function theorem for Banach spaces, Q is locally invertible. In particular, $Q(V)$ contains some ε-ball centered at $Q(0)$ in $L_p(M;\mathbb{R})$.

Now, for the function f given in Theorem, we have a diffeomorphism φ of M such that $|Q(0) - f\circ\varphi|_{L_p} < \varepsilon$ (see [5]). So, there is $u \in V \subset H_{4,p}(M;\mathbb{R})$ with $Q(u) = f\circ\varphi$. Although $Q(u)$ includes integrals of u, ∇u and $\nabla^2 u$, we can see that the elliptic regularty argument is applicable, by writing down $Q(u)$ explicitly. So, we see that u is smooth. Thus, $\bar{g} := \Phi\circ A^*(u)$ is a C^∞ Riemannian metric with $\mathrm{Vol}(M,\bar{g}) = 1$ and $R(\bar{g}) = f\circ\varphi$. Then the desired metric g is given by $g = (\varphi^{-1})^*\bar{g}$. □

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