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Hilbert transforms on one parameter groups of operators II

by

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ABSTRACT

Let X be a complete locally convex space and let $\{U_t : -\infty<t<\infty\}$ be a one parameter group of operators on X. Under some assumptions, there exists $\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_t x}{t} dt$ $(x \in X)$ in X which is denoted by Hx and called a Hilbert transform of x (*c.f.* [2]). In this paper, we study some properties of a Hilbert transform H on X, that is, the inversion formula and the closedness of H.

1. INTRODUCTION

In [2], we studied about the existence theorems of a Hilbert transform on a complete locally convex space. In this paper, we shall consider some properties of the Hilbert transform. For this , we define several terms some of which were already defined in [2].

Definition 1. *Let* $\mathbf{R}$ *be a real field. Let* X *be a complete locally convex space* *and let* $\{U_t; t \in \mathbf{R}\}$ *be a one parameter group of operators on* X, *that is,*

(i) $U_t : X \to X$ *is a continuous linear operator for all* $t \in \mathbf{R}$, *and* U_0 *is an identity operator on* X,

(ii) $U_t U_s = U_{t+s}$ *for all* $t, s \in \mathbf{R}$

(iii) *for any* $t \in \mathbf{R}$ *and any* $x \in X$, $(U_{t+h} - U_t)x$ *converges to* 0 *as* $h \to 0$ *in the topology of* X *(for short, in* X*)*

moreover, the following condition (iv) *is assumed in this paper:*

(iv) $U_t : X \to X$ *is continuous uniformly for* $t \in \mathbf{R}$, *that is, for any neighborhood* V *of* 0 *in* X, *there exists a neighborhood* W *of* 0 *in* X *such that*

$$U_t W \subset V \qquad \text{for } all\ t \in \mathbf{R}$$

Also, if there exists $\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} U_t x dt$ *in* X, *then we denote it by* $\bar{x}$

Definition 2. *A continuous linear operator* $H_{\varepsilon,N}$ $(0 < \varepsilon < N < \infty)$ *on* X *is defined as follows;*

$$H_{\varepsilon,N} x = \frac{1}{\pi} \int_{\varepsilon < |t| < N} \frac{U_t x}{t} dt \qquad (x \in X)$$

(this integral can be well defined since a mapping $t \in \mathbf{R} \to (U_t x)/t \in X$ *is continuous on a compact set* $\{t \in \mathbf{R}: \varepsilon \le |t| \le N\}$ *). Also, if there exists* $\lim_{\substack{\varepsilon \to 0+ \\ N \to \infty}} H_{\varepsilon,N} x$ *in* X, *we denote it by* Hx *and call it a Hilbert transform of* x. *And a domain of* H *(i.e.* $\{x \in X; Hx \text{ exists}\}$ *) is denoted by* $D(H)$

2. Special case (in Hilbert space)

In this section, we shall show several results in a Hilbert space, which are generalized in a complete locally convex space in the following section.

Theorem 1. *Let* $\{U_t : t \in \mathbf{R}\}$ *be a one parameter group of unitary operators on*

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a Hilbert space X (i.e. $U_t^* = U_{-t}$ for *all* $t \in \mathbf{R}$) . *Then , for any element x in X, there exists Hx in X. Moreover it is seen that*

$$\|Hx\|^2 = \|x - \bar{x}\|^2 \leq \|x\|^2$$

for all $x \in X$.

Proof. Let x be any element in X. Since $\{U_t : t \in \mathbf{R}\}$ be a one parameter group of unitary operators on a Hilbert space X, we, by Stone's theorem, see the following spectral representation of $U_t x$;

$$U_t x = \int_{-\infty}^{\infty} e^{i\lambda t}\, dE(\lambda)x$$

where $\{E(\lambda) : \lambda \in \mathbf{R}\}$ is a spectral family of a one parameter group of unitary operators $\{U_t : t \in \mathbf{R}\}$.

In order to show the first part of Theorem, it is sufficient to prove that $\|H_{\varepsilon,N}x - H_{\varepsilon',N'}x\|$ converges to 0 as $\varepsilon, \varepsilon' \to 0+$ and $N, N' \to \infty$. From the spectral representation of $U_t x$, we see that

$$\|H_{\varepsilon,N}x - H_{\varepsilon',N'}x\|^2$$

$$= \left\| \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_t x}{t} dt - \frac{1}{\pi}\int_{\varepsilon'<|t|<N'} \frac{U_t x}{t} dt \right\|^2$$

$$= \left\| \int_{-\infty}^{\infty} \left[\frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{e^{i\lambda t}}{t} dt - \frac{1}{\pi}\int_{\varepsilon'<|t|<N'} \frac{e^{i\lambda t}}{t} dt \right] dE(\lambda)x \right\|^2$$

$$= \left\| \int_{\infty}^{\infty} [g_{\varepsilon,N}(\lambda) - g_{\varepsilon',N'}(\lambda)] dE(\lambda)x \right\|^2$$

$$= \int_{\infty}^{\infty} |g_{\varepsilon,N}(\lambda) - g_{\varepsilon',N'}(\lambda)|^2 d\|E(\lambda)x\|^2 \qquad (1)$$

where $g_{\varepsilon,N}(\lambda) = \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{e^{i\lambda t}}{t} dt$.

It is clear that $g_{\varepsilon,N}(\lambda)$ has the following properties:

(i) $g_{\varepsilon,N}(\lambda)$ is a continuous function on R such that $|g_{\varepsilon,N}(\lambda)| \leq 1$ for all $\lambda \in \mathbf{R}$

(ii) if α and β are real numbers such that $0<\alpha<\beta<\infty$, then $g_{\varepsilon,N}$ uniformly converges to 1 (-1) for the closed interval $[\alpha,\beta]$ ($[-\beta,-\alpha]$) as $\varepsilon\to 0+, N\to\infty$,

and

(iii) $g_{\varepsilon,N}(0) = 0$.

From (1) and these properties of $g_{\varepsilon,N}$, we can easily see that $\|H_{\varepsilon,N}x - H_{\varepsilon',N'}x\|$ converges to 0 as $\varepsilon,\varepsilon'\to 0+$ and $N,N'\to\infty$. Hence $H_{\varepsilon,N}x$ converges to a certain element Hx in X as $\varepsilon\to 0+$ and $N\to\infty$.

Now we shall prove the second part of Theorem. We see that

$$\|Hx\|^2 = \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \|H_{\varepsilon,N}x\|^2$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \left\| \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{1}{t}\left(\int_{-\infty}^{\infty} e^{i\lambda t}\, dE(\lambda)x\right)dt \right\|^2$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \int_{\infty}^{\infty} |g_{\varepsilon,N}(\lambda)|^2 d\|E(\lambda)x\|^2$$

$$= \|x\|^2 - \|E(0+)x\|^2 + \|E(0-)x\|^2$$

Also we see that $\|E(0+)x\|^2 - \|E(0-)x\|^2 = \|\bar{x}\|^2$ and $\|\bar{x}\|^2 + \|x-\bar{x}\|^2 = \|x\|^2$. From this , the second part of Theorem immediately follows. This completes the proof.

Theorem 2. *Let $\{U_t : t\in\mathbf{R}\}$ be a one parameter group of unitary operators on a Hilbert space X . Then, for any x,y in X,*

(i) $(Hx,y) = -(x,Hy)$

(ii) $(Hx,Hy) = (x-\bar{x}, y-\bar{y})$

Proof. Let x and y be any elements in X. Then we see, from the unitarity of $\{U_t : t\in\mathbf{R}\}$, that, for any $x,y \in X$

$$(Hx,y) = \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} (H_{\varepsilon,N}x,y)$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \left(\frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_t x}{t}dt, y\right)$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{1}{t}(U_t x, y)dt$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{1}{t}(x, U_t^* y)dt$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{1}{t}(x, U_{-t} y)dt$$

$$= \lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} (x, \frac{1}{\pi}\int_{\varepsilon<|t|<N} \frac{U_{-t} y}{t}dt)$$

$$= -\lim_{\substack{\varepsilon\to 0+\\ N\to\infty}} (x, H_{\varepsilon,N} y)$$

$$= -(x, Hy)$$

Then (i) follows.

Also we see, from Theorem 1, that

$$4(Hx, Hy)$$

$$= \|Hx + Hy\|^2 - \|Hx - Hy\|^2 + i\|Hx + i\,Hy\|^2 - i\|Hx - iHy\|^2$$

$$= \|(x+y)-(x+y)^-\|^2 - \|(x-y)-(x-y)^-\|^2 + i\|(x+iy)-(x+iy)^-\|^2 - i\|(x-iy)-(x-iy)^-\|^2$$

$$= \|(x-\bar{x})+(y-\bar{y})\|^2 - \|(x-\bar{x})-(y-\bar{y})\|^2 + i\|(x-\bar{x})+i(y-\bar{y})\|^2 - i\|(x-\bar{x})-i(y-\bar{y})\|^2$$

$$= 4(x-\bar{x}, y-\bar{y})$$

Then (ii) follows. This completes the proof.

Theorem 3. *Let* $\{U_t : t\in\mathbf{R}\}$ *be a one parameter group of unitary operators on a Hilbert space* X *. Then it follows that*

$$H(Hx) = -(x-\bar{x}) \qquad \text{for all } x\in X$$

Proof. Let x be any elements in X. Then we see, from Theorem 1 and Theorem 2, that ,for any $y \in X$,

$$(H(Hx),y) = -(Hx,Hy) = -(x-\bar{x},y-\bar{y}) = -(x-\bar{x},y) + (x-\bar{x},\bar{y})$$

Since $(x-\bar{x},\bar{y}) = 0$, we obtain that

$$(H(Hx),y) = (-(x-\bar{x}),y) \qquad \text{for all } \ y \in X$$

Then we see that $H(Hx) = -(x-\bar{x})$. This completes the proof.

3. General case (in a complete locally convex space)

In this section, we shall try to generalize the theorems in the section 2. The following Lemma is fundamental for our theory.

Lemma 1. *Let $\{U_t : t \in \mathbf{R}\}$ be a one parameter group of operators on a complete locally convex space X. Let x be any element in X and let η,ε,N and M be a positive numbers such that* $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$ *(more precisely,* $0<\varepsilon-\eta<\varepsilon+\eta<N-\eta<N+\eta<M-N<M-\varepsilon<M+\varepsilon<M+N$). *Then it follows that*

$$H_{\eta,M}H_{\varepsilon,N}x = -\frac{1}{\pi^2}\left[\int_{-\frac{M-\varepsilon}{\varepsilon}}^{\frac{M-\varepsilon}{\varepsilon}} \frac{U_{\varepsilon t}x}{t} \log\left|\frac{t+1}{t-1}\right| dt - \int_{-\frac{M-N}{N}}^{\frac{M-N}{N}} \frac{U_{Nt}x}{t} \log\left|\frac{t+1}{t-1}\right| dt + R(\eta,\varepsilon,N,M;x)\right]$$

where

$$\begin{aligned} &R(\eta,\varepsilon,N,M;x) \\ &= \int_{\varepsilon-\eta}^{\varepsilon+\eta} \frac{U_a x + U_{-a}x}{a} \log\left|\frac{(a+\eta)(a+\varepsilon)}{\varepsilon\eta}\right| da + \int_{\varepsilon+\eta}^{N-\eta} \frac{U_a x - U_{-a}x}{a} \log\left|\frac{a+\eta}{a-\eta}\right| da \\ &\quad + \int_{N-\eta}^{N+\eta} \frac{U_a x + U_{-a}x}{a} \log\left|\frac{N\eta}{(a+N)(a-\eta)}\right| da + \int_{M-N}^{M-\varepsilon} \frac{U_a x + U_{-a}x}{a} \log\left|\frac{(a-M)(a-N)}{MN}\right| da \\ &\quad + \int_{M-\varepsilon}^{M+\varepsilon} \frac{U_a x + U_{-a}x}{a} \log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right| da + \int_{M+\varepsilon}^{M+N} \frac{U_a x + U_{-a}x}{a} \log\left|\frac{(a-M)(a-N)}{MN}\right| da \end{aligned}$$

Proof. Let x be any element in X, and let η,ε,N and M be a positive numbers such that $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$.

Then we see that,

$$H_{\eta,M}(H_{\varepsilon,N}x)$$

$$=\frac{1}{\pi}\int_{\eta<|s|<M}\frac{U_s}{s}\left(\frac{1}{\pi}\int_{\varepsilon<|t|<N}\frac{U_t x}{t}dt\right)ds$$

(*change variable* $t \to -t$).

$$=\frac{-1}{\pi^2}\iint_{\substack{\eta<|s|<M\\ \varepsilon<|t|<N}}\frac{U_{s-t}x}{st}\,dsdt$$

(*change variable* $s-t \to a, t \to v$ *respectively*)

$$=\frac{-1}{\pi^2}\iint_{\substack{\eta<|a+v|<M\\ \varepsilon<|v|<N}}\frac{U_a x}{v(a+v)}\,dadv$$

$$=-\frac{1}{\pi^2}\left[\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right]$$

$$-\frac{1}{\pi^2}\left[\int_{\varepsilon-\eta}^{\varepsilon+\eta}\left\{\left(\int_{-N}^{-(a+\eta)}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{-a+\eta}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right]$$

$$-\frac{1}{\pi^2}\left[\int_{\varepsilon+\eta}^{N-\eta}\left\{\left(\int_{-N}^{-(a+\eta)}+\int_{-(a-\eta)}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right.$$

$$\left.+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{-a-\eta}+\int_{-a+\eta}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right]$$

$$-\frac{1}{\pi^2}\left[\int_{N-\eta}^{N+\eta}\left\{\left(\int_{-(a-\eta)}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right.$$

$$\left.+\int_{-(N+\eta)}^{-(N-\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{-a-\eta}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right]$$

$$-\frac{1}{\pi^2}\left[\int_{N+\eta}^{M-N}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da+\int_{-(M-N)}^{-(N+\eta)}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right]$$

$$-\frac{1}{\pi^2}\left[\int_{M-N}^{M-\varepsilon}\left\{\left(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{M-a}\right)\frac{U_a x}{v(a+v)}dv\right\}da+\int_{-(M-\varepsilon)}^{-(M-N)}\left\{\left(\int_{-(M+a)}^{-\varepsilon}+\int_{\varepsilon}^{N}\right)\frac{U_a x}{v(a+v)}dv\right\}da\right]$$

$$-\frac{1}{\pi^2}\Big[\int_{M-\varepsilon}^{M+\varepsilon}\Big\{\int_{-N}^{-\varepsilon}\frac{U_a x}{v(a+v)}dv\Big\}da+\int_{-(M+\varepsilon)}^{-(M-\varepsilon)}\Big\{\int_{\varepsilon}^{N}\frac{U_a x}{v(a+v)}dv\Big\}da\Big]$$

$$-\frac{1}{\pi^2}\Big[\int_{M+\varepsilon}^{M+N}\Big\{\int_{-N}^{-(a-M)}\frac{U_a x}{v(a+v)}dv\Big\}da+\int_{-(M+N)}^{-(M+\varepsilon)}\Big\{\int_{-a-M}^{N}\frac{U_a x}{v(a+v)}dv\Big\}da\Big]$$

$$=-\frac{1}{\pi^2}I_1-\frac{1}{\pi^2}I_2-\frac{1}{\pi^2}I_3-\cdots-\frac{1}{\pi^2}I_8, \qquad say. \tag{1}$$

Now we shall successively calculate I_1, I_2, I_3 ... I_7 and I_8 .

First we see that

$$I_1=\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}\Big[\Big(\int_{-N}^{-\varepsilon}+\int_{\varepsilon}^{N}\Big)\Big(\frac{1}{v}-\frac{1}{a+v}\Big)dv\Big]da$$

$$=\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}\Big[\Big(log\Big|\frac{v}{a+v}\Big|\,\Big|_{-N}^{-\varepsilon}+\log\Big|\frac{v}{a+v}\Big|\,\Big|_{\varepsilon}^{N}\Big)da$$

$$=\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da-\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}\frac{U_a x}{a}log\Big|\frac{a+N}{a-N}\Big|da \tag{2}$$

Next we see that

$$I_2=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}\Big(log\Big|\frac{v}{a+v}\Big|\,\Big|_{-N}^{-(a+\eta)}+\log\Big|\frac{v}{a+v}\Big|\,\Big|_{\varepsilon}^{N}\Big)da$$

$$+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}\Big[\Big(log\Big|\frac{v}{a+v}\Big|\,\Big|_{-N}^{-\varepsilon}+\log\Big|\frac{v}{a+v}\Big|\,\Big|_{-a+\eta}^{N}\Big)da$$

$$=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}log\Big|\frac{(a+\varepsilon)(a+\eta)}{\varepsilon\eta}\Big|da+\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}log\Big|\frac{a-N}{a+N}\Big|da$$

$$+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}log\Big|\frac{\varepsilon(2a+\eta)}{(a-\varepsilon)(a+\eta)}\Big|da+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}log\Big|\frac{a-N}{a+N}\Big|da$$

$$=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}log\Big|\frac{(a+\varepsilon)}{a-\varepsilon}\Big|da-\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x}{a}log\Big|\frac{a+N}{a-N}\Big|da$$

$$+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da-\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}\frac{U_a x}{a}log\Big|\frac{a+N}{a-N}\Big|da$$

$$+\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x+U_{-a}x}{a}log\Big|\frac{(a+\varepsilon)(a+\eta)}{\varepsilon\eta}\Big|da \tag{3}$$

And we see that

$$I_3=\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\Big[\Big(\log\Big|\frac{v}{a+v}\Big|\Big|_{-N}^{-(a+\eta)}+\log\Big|\frac{v}{a+v}\Big|\Big|_{-(a-\eta)}^{-\varepsilon}+\log\Big|\frac{v}{a+v}\Big|\Big|_{\varepsilon}^{N}\Big)da$$

$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\Big[\Big(\log\Big|\frac{v}{a+v}\Big|\Big|_{-N}^{-\varepsilon}+\log\Big|\frac{v}{a+v}\Big|\Big|_{\varepsilon}^{-a-\eta}+\log\Big|\frac{v}{a+v}\Big|\Big|_{-a+\eta}^{N}\Big)da$$

$$=\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\Big|\frac{a+\eta}{a-\eta}\Big|da+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\Big|\frac{a-N}{a+N}\Big|da+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da$$

$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\Big|\frac{a-N}{a+N}\Big|da$$

$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\Big|\frac{(a-\eta)(2a+\eta)}{(2a-\eta)(a+\eta)}\Big|da$$

$$=\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da-\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x}{a}\log\Big|\frac{a+N}{a-N}\Big|da$$

$$+\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da-\int_{-(N-\eta)}^{-(\varepsilon+\eta)}\frac{U_a x}{a}\log\Big|\frac{a+N}{a-N}\Big|da$$

$$+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x-U_{-a}x}{a}\log\Big|\frac{a+\eta}{a-\eta}\Big|da \tag{4}$$

And we see that

$$I_4=\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\Big(\log\Big|\frac{v}{a+v}\Big|\Big|_{-(a-\eta)}^{-\varepsilon}+\log\Big|\frac{v}{a+v}\Big|\Big|_{\varepsilon}^{N}\Big)da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\Big(\log\Big|\frac{v}{a+v}\Big|\Big|_{-N}^{-\varepsilon}+\log\Big|\frac{v}{a+v}\Big|\Big|_{\varepsilon}^{-a-\eta}\Big)da$$

$$=\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da+\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\Big|\frac{N}{a+N}\Big|da+\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\Big|\frac{\eta}{a-\eta}\Big|da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\Big|\frac{a+\varepsilon}{a-\varepsilon}\Big|da-\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\Big|\frac{N}{a-N}\Big|da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\Big|\frac{a-\eta}{2a-\eta}\Big|da$$

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$$=\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{N-\eta}^{N+\eta}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$

$$+\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(N+\eta)}^{-(N-\eta)}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$

$$+\int_{N-\eta}^{N+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{N\eta}{(a-\eta)(a+N)}\right|da \qquad (5)$$

And we see that

$$I_5=\int_{N+\eta}^{M-N}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\,\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\,\Big|_{\varepsilon}^{N}\right)da$$

$$+\int_{-(M-N)}^{-(N+\eta)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\,\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\,\Big|_{\varepsilon}^{N}\right)da$$

$$=\int_{N+\eta}^{M-N}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{N+\eta}^{M-N}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da$$

$$+\int_{-(M-N)}^{-(N+\eta)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-(M-N)}^{-(N+\eta)}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da \qquad (6)$$

And we see that

$$I_6=\int_{M-N}^{M-\varepsilon}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\,\Big|_{-N}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\,\Big|_{\varepsilon}^{M-a}\right)da$$

$$+\int_{-(M-\varepsilon)}^{-(M-N)}\frac{U_a x}{a}\left(\log\left|\frac{v}{a+v}\right|\,\Big|_{-(M+a)}^{-\varepsilon}+\log\left|\frac{v}{a+v}\right|\,\Big|_{\varepsilon}^{N}\right)da$$

$$=\int_{M-N}^{M-\varepsilon}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da+\int_{-(M-\varepsilon)}^{-(M-N)}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da$$

$$+\int_{M-N}^{M-\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da \qquad (7)$$

And we see that

$$I_7=\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\,\Big|_{-N}^{-\varepsilon}da+\int_{-(M+\varepsilon)}^{-(M-\varepsilon)}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\,\Big|_{\varepsilon}^{N}da$$

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$$=\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right|da \tag{8}$$

Lastly we see that

$$I_8=\int_{M+\varepsilon}^{M+N}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\ \Big|_{-N}^{-(a-M)}da+\int_{-(M+N)}^{-(M+\varepsilon)}\frac{U_a x}{a}\log\left|\frac{v}{a+v}\right|\ \Big|_{-a-M}^{N}da$$

$$=\int_{M+\varepsilon}^{M+N}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da \tag{9}$$

Therefore we obtain, from (1),(2), ... , (8) and (9), that

$$-\pi^2 H_{\eta,M}H_{\varepsilon,N}x$$

$$=\Big(\int_{-(M-\varepsilon)}^{-(M-N)}+\int_{-M+N}^{-(N+\eta)}+\int_{-(N+\eta)}^{-(N-\eta)}+\int_{-(N-\eta)}^{-\varepsilon-\eta}+\int_{-\varepsilon-\eta}^{-\varepsilon+\eta}+\int_{-\varepsilon+\eta}^{\varepsilon-\eta}+\int_{\varepsilon-\eta}^{\varepsilon+\eta}+\int_{\varepsilon+\eta}^{N-\eta}+\int_{N-\eta}^{N+\eta}$$

$$+\int_{N+\eta}^{M-N}+\int_{M-N}^{M-\varepsilon}\Big)\Big(\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|\Big)da$$

$$-\Big(\int_{-M+N}^{-N-\eta}+\int_{-N-\eta}^{-N+\eta}+\int_{-N+\eta}^{-(\varepsilon+\eta)}+\int_{-(\varepsilon+\eta)}^{-(\varepsilon-\eta)}+\int_{-(\varepsilon-\eta)}^{\varepsilon-\eta}+\int_{\varepsilon-\eta}^{\varepsilon+\eta}+\int_{\varepsilon+\eta}^{N-\eta}+\int_{N-\eta}^{N+\eta}+\int_{N+\eta}^{M-N}\Big)$$

$$\Big(\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|\Big)da+R(\eta,\varepsilon,N,M;x)$$

$$=\int_{-(M-\varepsilon)}^{M-\varepsilon}\frac{U_a x}{a}\log\left|\frac{a+\varepsilon}{a-\varepsilon}\right|da-\int_{-M+N}^{M-N}\frac{U_a x}{a}\log\left|\frac{a+N}{a-N}\right|da+R(\eta,\varepsilon,N,M;x)$$

(*change variables* $a\to\varepsilon t$ and $a\to Nt$ *respectively*)

$$=\int_{-\frac{M-\varepsilon}{\varepsilon}}^{\frac{M-\varepsilon}{\varepsilon}}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt-\int_{-\frac{M-N}{N}}^{\frac{M-N}{N}}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+R(\eta,\varepsilon,N,M;x)$$

This completes the proof.

Lemma 2. *Let* $\{U_t : t\in\mathbf{R}\}$ *be a one parameter group of operators on a complete locally convex space* X. *Let* $R(\eta,\varepsilon,N,M;x)$ *be defined as in Lemma 1. Then it follows that*

(i) $\lim_{\substack{\eta\to 0+\\ M\to\infty}} R(\eta,\varepsilon,N,M;x)=0$ for *all* $x\in X$ and *all* $0<\varepsilon<\frac{1}{2}<2<N<\infty$, *and*

(ii) $\lim_{x\to 0} R(\eta,\varepsilon,N,M;x) = 0$ *uniformly* for $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$.

Proof. We can easily take a constant $K>0$ such that

$$\int_{\alpha-1}^{\alpha+1}\frac{1}{t}|\log|t\underline{+}1||\,dt<K \quad\text{and}\quad \int_{1-\beta}^{1+\beta}\frac{1}{t}|\log|t\underline{+}1||\,dt<K \tag{1}$$

for all $1<\alpha<\infty$ and all $0<\beta<1$. Also, we see that

$$\lim_{\alpha\to\infty}\int_{\alpha-1}^{\alpha+1}\frac{1}{t}|\log|t\underline{+}1||\,dt=0 \quad\text{and}\quad \lim_{\alpha\to 0}\int_{1-\alpha}^{1+\alpha}\frac{1}{t}|\log|t\underline{+}1||\,dt=0 \tag{2}$$

Let $R(\eta,\varepsilon,N,M;x)$ be defined as in Lemma 1, that is,

$$\begin{aligned}
&R(\eta,\varepsilon,N,M;x)\\
&=\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a+\eta)(a+\varepsilon)}{\varepsilon\eta}\right|da+\int_{\varepsilon+\eta}^{N-\eta}\frac{U_a x-U_{-a}x}{a}\log\left|\frac{a+\eta}{a-\eta}\right|da\\
&\quad+\int_{N-\eta}^{N+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{N\eta}{(a+N)(a-\eta)}\right|da+\int_{M-N}^{M-\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\\
&\quad+\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\right|da+\int_{M+\varepsilon}^{M+N}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a-M)(a-N)}{MN}\right|da\\
&=J_1+J_2+\cdots+J_6, \qquad \textit{say.}
\end{aligned}$$

Now we are going to estimate $R(\eta,\varepsilon,N,M;x)$. Let x be any element in X. Let q be any semi-norm from the system $\{q\}$ of semi-norms defining the topology of X.

Let ϑ be any positive number. Since $U_t:X\to X$ is continuous uniformly for $t\in\mathbf{R}$, we take a balanced convex neighborhood W of 0 in X such that

$$q(U_t x) < \vartheta \quad \text{for } \mathit{all}\ x\in W \text{ and } t\in\mathbf{R}$$

First we see

$$q(J_1)=q\left(\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{U_a x+U_{-a}x}{a}\log\left|\frac{(a+\eta)(a+\varepsilon)}{\varepsilon\eta}\right|da\right)$$

$$\le 2\vartheta\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{1}{a}\left|\log\left|\frac{a+\varepsilon}{\varepsilon}\right|\right|da+2\vartheta\int_{\varepsilon-\eta}^{\varepsilon+\eta}\frac{1}{a}\left|\log\left|\frac{a+\eta}{\eta}\right|\right|da$$

$$\leq 2\vartheta \int_{1-\frac{\eta}{\varepsilon}}^{1+\frac{\eta}{\varepsilon}} \frac{1}{t} |\log|t+1|| dt + 2\vartheta \int_{\frac{\varepsilon}{\eta}-1}^{\frac{\varepsilon}{\eta}+1} \frac{1}{t} |\log|t+1|| dt$$

which implies, by (1) and (2), that

$$\lim_{\substack{\eta \to 0+ \\ M \to \infty}} J_1 = 0 \qquad \text{for } all \ x \in X \text{ and } all \ 0 < \varepsilon < N < \infty \tag{3}$$

and

$$\lim_{x \to 0} J_1 = 0 \quad uniformly \text{ for } 0 < \eta < \varepsilon < 1/2 < 2 < N < 2N+1 < M < \infty \tag{3)'$$

Next we see that

$$q(J_2) = q\left(\int_{\varepsilon+\eta}^{N-\eta} \frac{U_a x - U_{-a} x}{a} log \left| \frac{a+\eta}{a-\eta} \right| da\right)$$

$$\leq 2\vartheta \int_{\frac{\varepsilon}{\eta}-1}^{\frac{\varepsilon}{\eta}+1} \frac{1}{t} \left|\log\left|\frac{t+1}{t-1}\right|\right| dt$$

This implies that

$$\lim_{\substack{\eta \to 0+ \\ M \to \infty}} J_2 = 0 \qquad \text{for } all \ x \in X \text{ and } all \ 0 < \varepsilon < N < \infty \tag{4}$$

and

$$\lim_{x \to 0} J_2 = 0 \quad uniformly \text{ for } 0 < \eta < \varepsilon < 1/2 < 2 < N < 2N+1 < M < \infty \tag{4)'$$

since $\int_{-\infty}^{\infty} \frac{1}{t} \left|\log\left|\frac{t+1}{t-1}\right|\right| dt = \pi^2$.

And we see that

$$q(J_3) = q\left(\int_{N-\eta}^{N+\eta} \frac{U_a x + U_{-a} x}{a} log \left| \frac{N\eta}{(a+N)(a-\eta)} \right| da\right)$$

$$\leq 2\vartheta \int_{1-\frac{\eta}{N}}^{1+\frac{\eta}{N}} \frac{1}{t} |\log|t+1|| dt + 2\vartheta \int_{\frac{N}{\eta}-1}^{\frac{N}{\eta}+1} \frac{1}{t} |\log|t-1|| dt$$

which implies, by (1) and (2), that

$$\lim_{\substack{\eta \to 0+ \\ M \to \infty}} J_3 = 0 \qquad \text{for } all \ x \in X \text{ and } all \ 0 < \varepsilon < N < \infty \tag{5}$$

and

$$\lim_{x\to 0} J_3=0 \quad \mathit{uniformly} \text{ for } 0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty \tag{5)'}$$

And we see that

$$q(\{J_4+J_6\})=q\Big(\int_{M-N}^{M-\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\Big|\frac{(a-M)(a-N)}{MN}\Big|da$$

$$+\int_{M+\varepsilon}^{M+N}\frac{U_a x+U_{-a}x}{a}\log\Big|\frac{(a-M)(a-N)}{MN}\Big|da\Big)$$

$$\leq 2\vartheta\int_{M-N}^{M+N}\frac{1}{a}\log\Big|\frac{(a-M)(a-N)}{MN}\Big|da$$

$$\leq 2\vartheta\int_{1-\frac{N}{M}}^{1+\frac{N}{M}}\frac{1}{t}\big|\log|t-1|\big|dt+2\vartheta\int_{\frac{M}{N}-1}^{\frac{M}{N}+1}\frac{1}{t}\big|\log|t-1|\big|dt$$

which implies, by (1) and (2), that

$$\lim_{\substack{\eta\to 0+\\ M\to\infty}}\{J_4+J_6\}=0 \qquad \text{for } \mathit{all}\ x\in X \text{ and } \mathit{all}\ 0<\varepsilon<N<\infty \tag{6}$$

and

$$\lim_{x\to 0}\{J_4+J_6\}=0 \quad \mathit{uniformly} \text{ for } 0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty \tag{6)'}$$

Lastly we see that

$$q(J_5)=q\Big(\int_{M-\varepsilon}^{M+\varepsilon}\frac{U_a x+U_{-a}x}{a}\log\Big|\frac{\varepsilon(a-N)}{N(a-\varepsilon)}\Big|da$$

$$\leq 2\vartheta\int_{M-\varepsilon}^{M+\varepsilon}\frac{1}{a}\log\Big|\frac{(a-N)}{N}\Big|da+2\vartheta\int_{M-\varepsilon}^{M+\varepsilon}\Big|\frac{1}{a}\log\Big|\frac{\varepsilon}{a-\varepsilon}\Big|\Big|da$$

$$\leq 2\vartheta\int_{\frac{M}{N}-1}^{\frac{M}{N}+1}\frac{1}{t}\big|\log|t-1|\big|dt+2\vartheta\int_{\frac{M}{\varepsilon}-1}^{\frac{M}{\varepsilon}+1}\frac{1}{t}\big|\log|t-1|\big|dt$$

which implies, by (1) and (2), that

$$\lim_{\substack{\eta\to 0+\\ M\to\infty}}J_5=0 \qquad \text{for } \mathit{all}\ x\in X \text{ and } \mathit{all}\ 0<\varepsilon<N<\infty \tag{7}$$

and

$$\lim_{x\to 0} J_5=0 \quad \textit{uniformly} \text{ for } 0<\eta<\varepsilon<1/2<2<N<2N+1<M<\infty \tag{7)'}$$

Hence we see, by (3),(3)',...,(7) and (7)', that (i) and (ii) are true, since $R(\eta,\varepsilon,N,M;x)=J_1+J_2+J_3+J_4+J_5+J_6$.

Hence this completes the proof.

Theorem 4. *Let* $\{U_t\ ;\ t\in\mathbf{R}\}$ *be a one parameter group of operators on a complete locally convex space* X. *Then, for any* $x\in X$, *there exists* $HH_{\varepsilon,N}x$ $(0<\varepsilon<N<\infty)$ *in* X, *and*

$$HH_{\varepsilon,N}x = -\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$

Proof. Let x be any element in X. By Lemma 1 and (i) in Lemma 2, we see that

$$HH_{\varepsilon,N}x=\lim_{\substack{\eta\to 0+\\ M\to\infty}} H_{\eta,M}H_{\varepsilon,N}x$$

$$=\lim_{\substack{\eta\to 0+\\ M\to\infty}}\Big[-\frac{1}{\pi^2}\Big[\int_{-\frac{M-\varepsilon}{\varepsilon}}^{\frac{M-\varepsilon}{\varepsilon}}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$

$$-\int_{-\frac{M-N}{N}}^{\frac{M-N}{N}}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+R(\eta,\varepsilon,N,M;x)\Big]$$

$$=-\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt+\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$

Hence this completes the proof.

The following Lemmas are useful to prove Theorem 5.

Lemma 3. *Let* $\{U_t : t\in\mathbf{R}\}$ *be a one parameter group of operators on a com-*

plete locally convex space X. Let x be any element in X such that $\bar{x}$ exists. Then, for any $\varphi \in L^1(\mathbf{R})$ such that $\int_{-\infty}^{\infty}\varphi(t)dt=1$, there exists $\lim_{N\to\infty}\int_{-\infty}^{\infty}U_{Nt}x\varphi(t)dt$, and which is equal to $\bar{x}$.

Proof. We define a characteristic function $\chi_{(a,b]}:\mathbf{R}\to\{0,1\}$ such that

$$\chi_{[a,b)}(t)=1 \ (for\ t\in[a,b)) \text{ and } 0\ (elsewhere)$$

First we assume that φ is represented by a linear combination of above characteristic functions *i.e.*

$$\varphi(t)=\sum_{i=1}^{m}c_i\chi_{(a_i,b_i]}(t)$$

where $\sum_{i=1}^{m}c_i(b_i-a_i)=\int_{-\infty}^{\infty}\varphi(t)dt=1$

Then we see that

$$\lim_{N\to\infty}\int_{-\infty}^{\infty}U_{Nt}x\varphi(t)dt=\lim_{N\to\infty}\sum_{i=1}^{m}c_i\int_{a_i}^{b_i}U_{Nt}xdt=\sum_{i=1}^{m}[c_i(b_i-a_i)\lim_{N\to\infty}\frac{1}{(b_i-a_i)N}\int_{a_iN}^{b_iN}U_txdt]$$
$$=\bar{x}\sum_{i=1}^{m}c_i(b_i-a_i)=\bar{x}$$

Next we shall consider about a general case. Let φ be any element in $L^1(\mathbf{R})$ such that $\int_{-\infty}^{\infty}\varphi(t)dt=1$. Let ε be any positive real and let q be any semi-norm from the system of semi-norms defining the topology of X. Then we can take $L>0$ and a linear combination $\varphi_0(t)=\sum_{i=1}^{m}c_i\chi_{(a_i,b_i]}(t)$ such that

$$q(U_tx)\le L \qquad \text{for all } t\in\mathbf{R}$$

and

$$\|\varphi-\varphi_0\|_1<\varepsilon$$

Therefore, we see that

$$q(\int_{-\infty}^{\infty}U_{Nt}x\varphi(t)dt-\int_{-\infty}^{\infty}U_{Mt}x\varphi(t)dt)$$

$$\le q(\int_{-\infty}^{\infty}U_{Nt}x(\varphi(t)-\varphi_0(t))dt)+q(\int_{-\infty}^{\infty}U_{Nt}x\varphi_0(t)dt-\int_{-\infty}^{\infty}U_{Mt}x\varphi_0(t)dt)$$

$$+q\left(\int_{-\infty}^{\infty} U_{Nt}x(\varphi(t)-\varphi_0(t))dt\right)$$

$$\leq 2L\varepsilon + q\left(\int_{-\infty}^{\infty} U_{Nt}x\varphi_0(t)dt - \int_{-\infty}^{\infty} U_{Mt}x\varphi_0(t)dt\right)$$

which implies that $\{\int_{-\infty}^{\infty} U_{Nt}x\varphi(t)dt\}_{N=1}^{\infty}$ is a Cauchy sequence in X, and has a certain limit y in X, since we have already proved that $\{\int_{-\infty}^{\infty} U_{Nt}x\varphi_0(t)dt\}_{N=1}^{\infty}$ was a Cauchy sequence in X. Moreover, it is clear that $y=\bar{x}$, so this completes the proof.

Lemma 4. *Let $\{U_t; t\in\mathbf{R}\}$ be a one parameter group of operators on a complete locally convex space X. Let x be an element in X such that there exists $\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi(t)dt$ for some $\varphi\in L^1(\mathbf{R})$ such that $\int_{-\infty}^{\infty}\varphi(t)dt=1$. Then there exists $\bar{x}$ in X and $\bar{x} = \lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi(t)dt$*

Proof. Let x^* be denoted by $\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi(x)dt$.

Firstly, we shall prove that $U_s x^* = x^*$ for all $s\in\mathbf{R}$. Let s be any fixed real number. Then we see that

$$\begin{aligned}
&U_s x^* - x^* \\
&= U_s\left[\lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi(t)dt\right] - x^* \\
&= U_s\left[\lim_{N\to\infty}\frac{1}{N}\int_{-\infty}^{\infty} U_t x\varphi\left(\frac{t}{N}\right)dt\right] - x^* \\
&= \lim_{N\to\infty}\frac{1}{N}\int_{-\infty}^{\infty} U_{t+s}x\varphi\left(\frac{t}{N}\right)dt - x^* \\
&= \lim_{N\to\infty}\frac{1}{N}\int_{-\infty}^{\infty} U_t x\varphi\left(\frac{t-s}{N}\right)dt - x^* \\
&= \lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi\left(t-\frac{s}{N}\right)dt - \lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi(t)dt \\
&= \lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\left(\varphi\left(t-\frac{s}{N}\right)-\varphi(t)\right)dt \\
&= 0, \qquad (1)
\end{aligned}$$

since $U_t: X\to X$ is continuous uniformly for $t\in\mathbf{R}$ and $\varphi(t-\frac{s}{N})-\varphi(t)\to 0$ in $L^1(\mathbf{R})$ as $N\to\infty$.

Hence we get that $U_s x^* = x^*$ for all $s \in \mathbf{R}$

Now let $D(t)$ be a function on $\mathbf{R}$ such that

$$D(t) = 1/2 \ (t \in [-1,1]) \quad \text{and } 0 \ (\textit{elsewhere})$$

Let V be any balanced convex neighborhood of 0 in X. By the continuity of $U_t : X \to X$ uniformly for $t \in \mathbf{R}$, there exists a balanced convex neighborhood W of 0 in X such that

$$U_t W \subset \frac{V}{3} \qquad \text{for } all \ t \in \mathbf{R} \tag{2}$$

Also, there exists $\eta > 0$ such that

$$\int_{-\infty}^{\infty} U_{Nt} x D(t) dt - \int_{-\infty}^{\infty} U_{Nt} (\int_{-\infty}^{\infty} D(t+s) \frac{1}{\eta} \varphi(\frac{s}{\eta}) ds) dt \in \frac{V}{3} \qquad \text{for } all \ N \geq 0 \tag{3}$$

since $\int_{-\infty}^{\infty} D(t+s) \frac{1}{\eta} \varphi(\frac{s}{\eta}) ds \to D(t) \ \ (\eta \to 0+)$ in $L^1(\mathbf{R})$.

And there exists $N_0 > 0$ such that

$$\int_{-\infty}^{\infty} U_{N\eta t} x \varphi(t) dt - x^* \in W \quad (N \geq N_0) \tag{4}$$

and

$$x^* - \int_{-\infty}^{\infty} U_{Nt} x \varphi(t) dt \ \in \frac{V}{3} \quad (N \geq N_0) \tag{5}$$

Then we see, by (1),(4) and (2), that

$$\begin{aligned} &\int_{-\infty}^{\infty} U_{Nt} (\int_{-\infty}^{\infty} D(t+s) \frac{1}{\eta} \varphi(\frac{s}{\eta}) ds) dt - x^* \\ &= \int_{-\infty}^{\infty} D(s) [\int_{-\infty}^{\infty} U_{Nt} x \frac{1}{\eta} \varphi(\frac{t+s}{\eta}) dt - x^*] ds \\ &= \frac{1}{2} \int_{-1}^{1} [U_{-Ns} (\int_{-\infty}^{\infty} U_{N\eta t} x \varphi(t) dt - x^*)] ds \\ &\in V/3 \qquad\qquad (N \geq N_0) \end{aligned} \tag{6}$$

Therefore we , by (3),(6) and (5), find that

$$\begin{aligned} &\frac{1}{2N} \int_{-N}^{N} U_t x dt - \int_{-\infty}^{\infty} U_{Nt} x \varphi(t) dt \\ &= [\int_{-\infty}^{\infty} U_{Nt} x D(t) dt - \int_{-\infty}^{\infty} U_{Nt} (\int_{-\infty}^{\infty} D(t+s) \frac{1}{\eta} \varphi(\frac{s}{\eta}) ds) dt] \\ &\qquad + [\int_{-\infty}^{\infty} U_{Nt} (\int_{-\infty}^{\infty} D(t+s) \frac{1}{\eta} \varphi(\frac{s}{\eta}) ds) dt - x^*] \\ &\qquad + [x^* - \int_{-\infty}^{\infty} U_{Nt} x \varphi(t) dt \end{aligned}$$

$$\in \frac{V}{3}+\frac{V}{3}+\frac{V}{3}=V \qquad (N \geq N_0)$$

Since V is arbitrary convex balanced neighborhood of 0 in X, this implies that $\bar{x}$ exists and $\bar{x} = \lim_{N\to\infty}\int_{-\infty}^{\infty} U_{Nt}x\varphi(t)dt$. This completes the proof.

Theorem 5. *Let $\{U_t; t\in\mathbf{R}\}$ be a one parameter group of operators on a complete locally convex space X. Let x be any element in $D(H)$. Then the following two statements are equivalent.*

(i) *there exists $\bar{x}$ in X,*

(ii) *Hx belongs to $D(H)$.*

Moreover, if there exists $\bar{x}$ in X, then $H^2x = -(x-\bar{x})$.

Proof. Let x be any element in $D(H)$. Since $H_{\varepsilon,N}$ is continuous, we see, by Theorem 4, that

$$\lim_{\substack{\varepsilon\to0+\\N\to\infty}} H_{\varepsilon,N}Hx = \lim_{\substack{\varepsilon\to0+\\N\to\infty}} H_{\varepsilon,N}(\lim_{\substack{\eta\to0+\\M\to\infty}} H_{\eta,M}x) = \lim_{\substack{\varepsilon\to0+\\N\to\infty}}\lim_{\substack{\eta\to0+\\M\to\infty}} H_{\eta,M}H_{\varepsilon,N}x$$

$$= -\frac{1}{\pi^2}\lim_{\varepsilon\to0+}\int_{-\infty}^{\infty}\frac{U_{\varepsilon t}x}{t}\log\left|\frac{t+1}{t-1}\right|dt + \frac{1}{\pi^2}\lim_{N\to\infty}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt$$

$$= -x + \lim_{\substack{\varepsilon\to0+\\N\to\infty}}\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{U_{Nt}x}{t}\log\left|\frac{t+1}{t-1}\right|dt \qquad (1)$$

Hence we get, by (1) and Lemma 3, that (i) implies (ii). Moreover, it immediately follows that (i) implies that $H^2x = -(\bar{x}-x)$.

Also, we get, by (1) and Lemma 4, that (ii) implies (i). This completes the proof.

Lemma 5. *Let $\{U_t ; t\in\mathbf{R}\}$ be a one parameter group of operators on a complete locally convex space X. Then, it follows that*

$$\lim_{x\to0} H_{\eta,M}H_{\varepsilon,N}x = 0 \qquad \textit{uniformly} \text{ for } 0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$$

Proof. Let ϑ be any positive number. Let q be any semi-norm from the system $\{q\}$ of semi-norms defining the topology of X. Since $U_t:X\to X$ is continuous uniformly for $t\in\mathbf{R}$, we can take a neighborhood W of 0 in X such that

$$q(U_t x)\leq\vartheta \qquad \text{for } all \;\; x\in W \;\text{ and } all \;\; t\in\mathbf{R}$$

Then, we see, by Lemma 2, that, for any $x\in W$ and any $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$

$$q(H_{\eta,M}H_{\varepsilon,N}x)$$

$$\leq q\Big(-\frac{1}{\pi^2}\Big[\int_{-\frac{M-\varepsilon}{\varepsilon}}^{\frac{M-\varepsilon}{\varepsilon}}\frac{U_{\varepsilon t}x}{t}log\left|\frac{t+1}{t-1}\right|dt-\int_{-\frac{M-N}{N}}^{\frac{M-N}{N}}\frac{U_{Nt}x}{t}log\left|\frac{t+1}{t-1}\right|dt+R(\eta,\varepsilon,N,M;x)\Big]\Big)$$

$$\leq\frac{\vartheta}{\pi^2}\Big[\int_{-\infty}^{\infty}\frac{1}{t}log\left|\frac{t+1}{t-1}\right|dt+\int_{-\infty}^{\infty}\frac{1}{t}log\left|\frac{t+1}{t-1}\right|dt+K\Big]$$

$$\leq\vartheta\Big(2+\frac{K}{\pi^2}\Big)$$

where K is a positive constant independent of η,ε,N and M (which can be taken by Lemma 2,(ii)). This completes the proof. This completes the proof.

Lemma 6. *Let $\{U_t\ ;\ t\in\mathbf{R}\}$ be a one parameter group of operators on a complete locally convex space X. Then, it follows that*

(i) for *any* $x\in X$ and *any* $0<\varepsilon<N<\infty$, $(H_{\varepsilon,N}x)^{-}=0$

(ii) for *any* $x\in D(H)$, $(Hx)^{-}=0$

Proof. Firstly we shall prove the first part of Lemma. Let x be any element in X. We see that, for large $T>0$,

$$I=\frac{1}{2T}\int_{-T}^{T}U_t H_{\varepsilon,N}x\,dt$$

$$=\frac{1}{2T}\int_{-T}^{T}U_t\Big[\frac{1}{\pi}\int_{\varepsilon<|s|<N}\frac{U_s x}{s}ds\Big]dt$$

$$= \frac{1}{2T}\int_{-T}^{T}\left[\frac{1}{\pi}\int_{\varepsilon<s<N}\frac{U_{t+s}x - U_{t-s}x}{s}ds\right]dt$$

$$= \frac{1}{2\pi}\int_{\varepsilon<s<N}\frac{1}{s}\left[\frac{1}{2T}\int_{-T}^{T}(U_{t+s}x - U_{t-s}x)dt\right]ds$$

$$= \frac{1}{2\pi}\int_{\varepsilon<s<N}\frac{1}{s}\left[\frac{1}{2T}\int_{-T-s}^{-T+s}U_t x dt + \frac{1}{2T}\int_{T-s}^{T+s}U_t x dt\right]ds \tag{1}$$

Let q be any semi-norm from the system $\{q\}$ of semi-norms defining the topology of X. By the uniformly continuity of $\{U_t : t \in \mathbf{R}\}$, we can take $C>0$ such that

$$q(U_t x) \leq C \quad \text{for } all \ t \in \mathbf{R}$$

Hence we get, by (1) and this, that

$$q(I) \leq \frac{1}{2\pi}\int_{\varepsilon<s<N}\frac{1}{s}\left[\frac{2Cs}{T}\right]ds = \frac{2C(N-\varepsilon)}{\pi T} \to 0 \qquad as \ T \to +\infty.$$

This implies that $(H_{\varepsilon,N}x)^- = 0$

Next we shall prove the second part of Lemma. Let x be any element in $D(H)$. Let V be any balanced convex neighborhood of 0 in X. By the uniformly continuity of U_t, there exists a balanced convex neighborhood W of 0 in$X$ such that

$$U_t W \subset \frac{V}{2} \quad \text{for } all \ t \in \mathbf{R} \tag{2}$$

Since $x \in D(H)$, there exist positive number ε_0 and N_0 such that

$$Hx - H_{\varepsilon_0,N_0}x \in W \tag{3}$$

And, by the first part of Theorem, we take $T_0>0$ such that

$$\frac{1}{2T}\int_{-T}^{T}U_t H_{\varepsilon_0,N_0}x dt \in \frac{V}{2} \qquad \text{for } all \ T \geq T_0 \tag{4}$$

Hence we see , by (2),(3) and (4), that , for any $T \geq T_0$,

$$\frac{1}{2T}\int_{-T}^{T}U_t Hx dt$$

$$= \frac{1}{2T}\int_{-T}^{T}U_t(H - H_{\varepsilon_0,N_0})x dt + \frac{1}{2T}\int_{-T}^{T}U_t H_{\varepsilon_0,N_0}x dt$$

$$\in \frac{V}{2} + \frac{V}{2} = V$$

which implies that $(Hx)^{-}$ exists in X and $(Hx)^{-}=0$. This completes the proof.

Theorem 6. *Let $\{U_t\ ;\ t \in \mathbf{R}\}$ be a one parameter group of operators on a complete locally convex space X. Then, the Hilbert transform H is a closed operator on X (though $D(H)$ is not always dense in X)*

Proof. Assume that $\{x_k\}_{k\in K}$ is any generalized sequence in $D(H)$ such that

$$x_k \to x \text{ and } Hx_k \to y \text{ in } X \tag{1}$$

It is sufficient to prove that $x \in D(H)$ and $Hx=y$. Let V be any balanced convex neighborhood of 0 in X. By Lemma 5, we take a balanced convex neighborhood W of 0 in X such that

$$H_{\eta,M} H_{\varepsilon,N} W \subset \frac{V}{4} \tag{2}$$

for all $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$.

And we take $k_0 \in K$ such that

$$Hx_k - y \in W \text{ for all } k \geq k_0 \tag{3}$$

By (2) and (3), we see that

$$H_{\eta,M} H_{\varepsilon,N}(Hx_k - y) \in \frac{V}{4} \tag{4}$$

for all $k \geq k_0$ and for all $0<\eta<\varepsilon<\frac{1}{2}<2<N<2N+1<M<\infty$.

Letting $\eta \to 0+, M \to \infty$ in(4), we see, by Theorem 4, that

$$-H_{\varepsilon,N} x_k - HH_{\varepsilon,N} y \in \frac{V}{3} \tag{5}$$

for all $k \geq k_0$ and for all $0<\varepsilon<\frac{1}{2}<2<N<\infty$.

And letting $k \to \infty$ in (5), we find that

$$-H_{\varepsilon,N}x - HH_{\varepsilon,N}y \in \frac{V}{2} \tag{6}$$

for all $0<\varepsilon<\frac{1}{2}<2<N<\infty$.

Next we shall prove that $\bar{y}=0$. Let G be any balanced convex neighborhood of 0 in X.

We can, by the uniformly continuity of U_t, take $k_0 \in K$ such that

$$\frac{1}{2T}\int_{-T}^{T} U_t(y - Hx_{k_0})dt \in \frac{G}{2} \qquad \text{for all } T>0, \tag{7}$$

and we can, from Lemma 6, take $T_0>0$ such that

$$\frac{1}{2T}\int_{-T}^{T} U_t(Hx_{k_0})dt \in \frac{G}{2} \qquad \text{for all } T \geq T_0. \tag{8}$$

By (7) and (8), we see that, for large T such that $T \geq T_0$,

$$\frac{1}{2T}\int_{-T}^{T} U_t y dt = [\frac{1}{2T}\int_{-T}^{T} U_t(y - Hx_k)dt] + [\frac{1}{2T}\int_{-T}^{T} U_t(Hx_{k_0})dt]$$

$$\in \frac{G}{2} + \frac{G}{2} = G$$

which implies that $\bar{y}=0$

From this and Theorem 4, we see that , for small ε and large N,

$$HH_{\varepsilon,N}y - (-y) \in \frac{V}{2}$$

Then it follows, from this and (6), that

$$H_{\varepsilon,N}x - y \in V$$

for small ε and large N, which implies that $x \in D(H)$ and $Hx=y$. This completes the proof.

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