Stochastic Differential Equations for Multi-dimensional Domain with reflecting boundary

by

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Introduction

In this paper we investigate multi-dimensional Skorohod stochastic differential equations (abbreviated: SDE's) on a domain D with reflecting boundary condition. If w is a given process in R^d, our problem is to find a solution ξ of the equation $\xi(t) = w(t) + \Phi(t)$ satisfying certain conditions (see (1.2) and (1.3) of §1) so that ξ is a reflecting process on $\overline{D}$ (the precise definition of the equation will be given in §1). This equation is called a multi-dimensional Skorohod equation on the analogy of the one-dimensional case first discussed by Skorohod [7] (see also [1][3]). First we consider a deterministic version of the Skorohod equation. This kind of problem has been discussed by Tanaka [8] when D is a convex domain and then by Lions and Sznitman [2] when D is a general domain satisfying Condition (A) and Condition (B) (see §1) together with the admissibility condition. Here the admissibility means roughly that D can be approximated in some sense by smooth domains.

One of the purposes of this paper is to prove the existence and uniqueness of solutions of the Skorohod equation $\xi = w + \phi$ only assuming Conditions (A) and (B). We do not need the admissibility condition. To prove this result we employ the method of Tanaka ([8]) instead of the penalty method ([2]); more precisely, we approximate w by step functions $\{w_m\}$, consider the problem $\xi_m = w_m + \phi_m$ which is easy to solve and then take a limit in m to obtain a solution of $\xi = w + \phi$. In the proof, the estimate of the total variation $|\phi_m|_t$ of ϕ_m plays the essential role.

Another problem of this paper is concerned with the Skorohod SDE. Let $\sigma: \overline{D} \to R^d \otimes R^d$ and $b: \overline{D} \to R^d$ be bounded and Lipschitz continuous. Assuming that D satisfies Conditions (A) and (B) we consider the Skorohod SDE

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt + d\Phi(t),$$

which gives rise to a reflecting Brownian motion, where $B(t)$ is a standard d-dimensional Brownian motion. Lions and Sznitman [2] proved the existence and uniqueness of solutions of the above SDE assuming the admissibility condition plus another additional condition (Condition (C) of §1). Our result is that we can remove both of these conditions to have a unique strong solution of the above SDE.

In §1 we introduce some conditions on D and give the definition of the Skorohod equation. Several remarks related to the conditions and the main results are also stated in this section. In §2 we give some lemmas which will be used in subsequent sections, and in §3 we estimate the total variation $|\phi_m|_t$. Solvability of

the Skorohod equation is treated in §4 (deterministic case) and in

§5 (stochastic case).

§1. Formulation of the problem and the results

Let D be a domain in R^d and define the set $\mathcal{N}_x$ of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \bigcup_{r>0} \mathcal{N}_{x,r},$$

$$\mathcal{N}_{x,r} = \{\underline{n} \in R^d : |\underline{n}| = 1, \ B(x - r\underline{n}, r) \cap D = \phi\},$$

where $B(z,r) = \{y \in R^d : |y - z| < r\}$, $z \in R^d$, $r > 0$. In general, it can happen that $\mathcal{N}_x = \phi$. We introduce two conditions on the domain D.

Condition (A) (uniform exterior sphere condition). There exists a constant $r_0 > 0$ such that

$$\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \phi \quad \text{for any} \quad x \in \partial D.$$

Condition (B). There exist constants $\delta > 0$ and $\beta \in [1, \infty)$ with the following property: for any $x \in \partial D$ there exists a unit vector $\underline{\ell}_x$ such that

$$\langle \underline{\ell}_x, \underline{n} \rangle \geq 1/\beta \quad \text{for any} \quad \underline{n} \in \bigcup_{y \in B(x,\delta) \cap \partial D} \mathcal{N}_y.$$

In what follows $\langle \cdot, \cdot \rangle$ denotes the usual inner product in R^d.

Remark 1.1. It is easy to see that the following two statements are equivalent (cf.[2: Remark 1.2]).

(i) $\underset{=}{n} \in \mathcal{K}_{x,r}$

(ii) $\langle y - x, \underset{=}{n} \rangle + \frac{1}{2r}|y - x|^2 \geq 0$ for any $y \in \overline{D}$.

Remark 1.2. D satisfies Condition (B) if it satisfies the following condition.

Condition (B'). There exist $\delta > 0$ and $\alpha \in [0,1)$ with the following property: for any $x \in \partial D$ there exists a unit vector $\underset{=x}{\ell}$ such that

$$C(y, \underset{=x}{\ell}, \alpha) \cap B(x, \delta) \subset \overline{D}, \quad ^\forall y \in B(x, \delta) \cap \partial D,$$

where $C(y, \underset{=x}{\ell}, \alpha)$ is the convex cone with vertex y, defined by

$$C(y, \underset{=x}{\ell}, \alpha) = \{z \in R^d : \langle z - y, \underset{=x}{\ell} \rangle \geq \alpha|z - y|\}.$$

Remark 1.3. Under Condition (A), there exists a unique $\overline{x} \in \overline{D}$ such that $|x - \overline{x}| = \text{dist}(x, \overline{D})$ for any $x \in R^d$ with $\text{dist}(x, \overline{D}) < r_0$, and $(\overline{x} - x)/|\overline{x} - x| \in \mathcal{K}_x$ if $x \notin \overline{D}$. The notation $\overline{x}$ is used in this sense throughout the paper (there will be no confusion with the closure $\overline{D}$).

The deterministic problem is stated as follows. Denote by

$W(R^d)$ (resp. $W(\overline{D})$) the space of continuous paths in R^d (resp. $\overline{D}$) and consider an equation

(1.1) $\xi(t) = w(t) + \phi(t)$,

where $w \in W(R^d)$ is given and satisfies $w(0) \in \overline{D}$; a solution of (1.1) is a pair of ξ and ϕ which should be found under the following two conditions (we often call ξ a solution of (1.1)).

(1.2) $\xi \in W(\overline{D})$.

(1.3) ϕ is an R^d-valued continuous function with bounded variation on each finite interval satisfying $\phi(0) = 0$ and

$$\phi(t) = \int_0^t \underset{\sim}{n}(s) d|\phi|_s,$$

$$|\phi|_t = \int_0^t \mathbf{1}_{\partial D}(\xi(s)) d|\phi|_s,$$

where

$$\underset{\sim}{n}(s) \in \mathcal{N}_{\xi(s)} \quad \text{if} \quad \xi(s) \in \partial D,$$

$|\phi|_t$ = the total variation of ϕ on $[0,t]$

$$= \sup \sum_{k=1}^n |\phi(t_k) - \phi(t_{k-1})|,$$

the supremum being taken over all partitions $0 = t_0 < t_1 < \cdots < t_n = t$.

Remark 1.4. For $w \in D(R_+ \to R^d)$ with $w(0) \in \overline{D}$, the problem (1.1) can be also posed if we replace $W(\cdot)$ in the additional conditions after (1.1) by $D(R_+ \to \cdot)$, where $D(R_+ \to R^d)$ (resp. $D(R_+ \to \overline{D})$) is the space of right continuous paths in R^d (resp. $\overline{D}$) with left-hand limits. For instance, when w is a right continuous step function with small jumps, a solution of (1.1) can be constructed as follows: let $w(t) = w(t_k)$, $t_k \leq t < t_{k+1}$, $k = 0, 1,$ $\cdots$, where $0 = t_0 < t_1 < \cdots$ and $\lim_{n \to \infty} t_n = \infty$ and assume that $\sup_{k \geq 1} |w(t_k) - w(t_{k-1})| < r_0$. Put

$$\xi(t) = \begin{cases} w(t) & , \ 0 \leq t < t_1, \\ \\ \overline{\xi(t_{k-1}) + w(t_k) - w(t_{k-1})}, & t_k \leq t < t_{k+1} \ (k \geq 1), \end{cases}$$

$$\phi(t) = \begin{cases} 0 & , \ 0 \leq t < t_1, \\ \\ \phi(t_{k-1}) + \overline{\xi(t_{k-1}) + w(t_k) - w(t_{k-1})} \\ \\ - \xi(t_{k-1}) - w(t_k) + w(t_{k-1}), & t_k \leq t < t_{k+1} \ (k \geq 1), \end{cases}$$

then if $t_n \leq t < t_{n+1}$, we have

$$|\phi|_t = \sum_{k=1}^{n} | \overline{\{\xi(t_{k-1}) + w(t_k) - w(t_{k-1})\}} -$$

$$- \xi(t_{k-1}) - w(t_k) + w(t_{k-1})| .$$

Since $\xi(t_k) \in D$ implies $\phi(t_k) - \phi(t_k-) = 0$, we have

$$|\phi|_t = \int_{(0,t]} \mathbb{1}_{\partial D}(\xi(s))d|\phi|_s,$$

where $\phi(t_k-) = \lim_{t \uparrow t_k} \phi(t)$. Therefore by Remark 1.3, we have

$$\phi(t) = \int_{(0,t]} \underline{n}(s)d|\phi|_s,$$

and (ξ,ϕ) is a solution of (1.1) for w.

One of the purpose of this paper is to prove the following theorem.

Theorem 4.1. If the domain D satisfies Conditions (A) and (B), then there exists a unique solution of (1.1) for any given $w \in W(R^d)$ with $w(0) \in \overline{D}$.

The above theorem was proved by Lions and Sznitman [2] under the additional condition that D is admissible. Employing Tanaka's method ([8]) we can remove the admissibility condition.

Next, given

$$\sigma: \overline{D} \to R^d \otimes R^d, \quad b: \overline{D} \to R^d,$$

we consider a Skorohod SDE on a probability space $(\Omega,\mathcal{F},P)$:

(1.4) $dX(t) = \sigma(X(t))dB(t) + b(X(t))dt + d\Phi(t),$

where the initial value $X(0) \in \overline{D}$ is assumed to be an $\mathcal{F}_0$-measurable

random variable and $B(t)$ is a d-dimensional $\mathcal{F}_t$-Brownian motion

with $B(0) = 0$. Here $\{\mathcal{F}_t\}$ is a filtration such that $\mathcal{F}_0$ contains

all P-negligible sets and $\mathcal{F}_t = \bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. A solution $(X(t),\Phi(t))$

should be found under the following conditions (1.5) - (1.7).

(1.5) $X(t)$ is a $\overline{D}$-valued $\mathcal{F}_t$-adapted continuous process.

(1.6) $\Phi(t)$ is an R^d-valued $\mathcal{F}_t$-adapted continuous process

with bounded variation on each finite interval such that

$\Phi(0) = 0$,

$$\Phi(t) = \int_0^t \underline{n}(X(s))d|\Phi|_s,$$

$$|\Phi|_t = \int_0^t \mathbb{1}_{\partial D}(X(s))d|\Phi|_s.$$

(1.7) $\underline{n}(s) \in \mathcal{N}_{X(s)}$ if $X(s) \in \partial D$.

Another main result of this paper is the following theorem.

Theorem 5.1.   Let D be a domain satisfying Conditions (A),
(B) and assume that σ and b are bounded and Lipschitz continuous.
Then there exists a unique strong solution of (1.4).

The meaning of the existence of a unique strong solution is
the same as in [1: p.149].

The above theorem was proved by Lions and Sznitman [2:
Theorem 3.1] under the admissibility condition and the following

Condition (C). See also Menaldi [4], Menaldi and Robin [5] for related problems.

Condition (C). There exists a function f in $C^2(R^d)$ which is bounded together with its first and second partial derivatives such that $\exists \gamma > 0$, $\forall x \in \partial D$, $\forall y \in \bar{D}$, $\forall \underline{n} \in \mathcal{N}_x$

$$(1.8) \qquad \langle y - x, \underline{n} \rangle + \frac{1}{\gamma} \langle \nabla f(x), \underline{n} \rangle |y - x|^2 \geq 0.$$

We can prove that if a domain D satisfies Conditions (A) and (B), then D satisfies the following condition.

Condition (C'). There exists a positive number δ' such that for each $x_0 \in \partial D$ we can find a function f in $C^2(R^d)$ satisfying (1.8) for any $x \in B(x_0, \delta') \cap \partial D$, $y \in B(x_0, \delta') \cap \bar{D}$ and $\underline{n} \in \mathcal{N}_x$.

In fact, it will be proved that D satisfies Condition (C') with $\delta' = \delta$, $\gamma = 2r_0 \beta^{-1}$, $\underline{\ell} = \underline{\ell}_{x_0}$ and $f(x) = \langle \underline{\ell}, x - x_0 \rangle$, where r_0, $\underline{\ell}$, β, δ are the same as in Conditions (A) and (B). Condition (C') means roughly that D satisfies (C) <u>locally</u>. Theorem 5.1 asserts that this local condition (C'), which is automatically satisfied under (A) and (B), is enough to have the existence and uniqueness of solutions of (1.4).

§2. Lemmas

In this section we prepare some lemmas which will be used in the following sections. The first lemma is well-known and the proof is omitted.

Lemma 2.1. (i) Let $f \in C^1(R)$ and $\theta = (\theta_t)_{t \geq 0}$ be a function in $D(R_+ \to R)$ with bounded variation. Then we have

(2.1) $df(\theta_t) = f'(\theta_{t-})d\theta_t + \Delta f(\theta_t) - f'(\theta_{t-})\Delta\theta_t,$

where $\Delta f(\theta_t) = f(\theta_t) - f(\theta_{t-})$, $\Delta\theta_t = \theta_t - \theta_{t-}$, $\theta_{t-} = \lim_{s \uparrow t} \theta_s$.
(ii) Let g, h be functions in $D(R_+ \to R^d)$ with bounded variation.
Then we have

(2.2) $\langle g(t),h(t)\rangle - \langle g(0),h(0)\rangle = \int_{(0,t]} \langle g(s),dh(s)\rangle$

$$+ \int_{(0,t]} \langle h(s),dg(s)\rangle - \sum_{s \leq t} \langle \Delta g(s),\Delta h(s)\rangle.$$

Lemma 2.2. Let $\psi \in D(R_+ \to R)$ be a non-decreasing function with $\psi(0) = 0$ and f be a non-negative Borel measurable function. If

(2.3) $f(t) \leq K_1 + K_2 \int_{(0,t)} f(s)d\psi(s) < \infty, \quad t \geq 0$

holds with some non-negative constants K_1 and K_2, then we have

$$(2.4) \qquad f(t) \leq K_1 \exp\{K_2 \psi(t)\}, \quad t \geq 0.$$

Proof. It is enough to show (2.4) in the case $K_1 > 0$. By (2.3) we have

$$\frac{K_2 f(t)}{K_1 + K_2 \int_{(0,t)} f(s)d\psi(s)} \leq K_2, \quad t \geq 0.$$

Integrating both sides with respect to $d\psi$ over $(0,t]$, we have

$$(2.5) \qquad \int_{(0,t]} \frac{K_2 f(u)}{K_1 + K_2 \int_{(0,u)} f(s)d\psi(s)} d\psi(u) \leq K_2 \psi(t), \quad t \geq 0.$$

If we set

$$F(x) = \log(x/K_1), \quad x > 0,$$

$$\theta_t = K_1 + K_2 \int_{(0,t]} f(s)d\psi(s),$$

then an application of (2.1) yields

$$dF(\theta_u) = F'(\theta_{u-})d\theta_u + \Delta F(\theta_u) - F'(\theta_{u-})\Delta\theta_u$$

$$= \frac{K_2 f(u)d\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)d\psi(s)} + \log\{1 + \frac{K_2 f(u)\Delta\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)d\psi(s)}\}$$

$$- \frac{K_2 f(u)\Delta\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)d\psi(s)} \leq \frac{K_2 f(u)d\psi(u)}{K_1 + K_2 \int_{(0,u)} f(s)d\psi(s)},$$

because $\log(1 + x) - x \leq 0$, $x \geq 0$. Therefore we have

$$\log(\theta_t/K_1) = F(\theta_t)$$

$$\leq \int_{(0,t]} \frac{K_2 f(u)}{K_1 + K_2 \int_{(0,u)} f(s)d\psi(s)} d\psi(u) \leq K_2\psi(t), \quad t > 0$$

by (2.5), that is, $\theta_t \leq K_1\exp\{K_2\psi(t)\}$. Thus by (2.3) we have

$$f(t) \leq \theta_t \leq K_1\exp\{K_2\psi(t)\}, \quad t \geq 0,$$

completing the proof.

Lemma 2.3. Suppose D satisfies Condition (A). (i) Let w, $w' \in D(R_+ \to R^d)$ and (ξ,ϕ), (ξ',ϕ') be solutions of the Skorohod equations $\xi = w + \phi$ and $\xi' = w' + \phi'$, respectively. Then we have

$$(2.6) \qquad |\xi(t) - \xi'(t)|^2 \leq |w(t) - w'(t)|^2$$

$$+ \frac{1}{r_0} \int_{(0,t]} |\xi(s) - \xi'(s)|^2 d(|\phi|_s + |\phi'|_s)$$

$$+ 2\int_{(0,t]} \langle w(t) - w(s) - w'(t) + w'(s), d\phi(s) - d\phi'(s)\rangle.$$

(ii) Let w, ξ, ϕ be as in (i). Then we have for $0 \leq s \leq t$,

$$(2.7) \qquad |\xi(t) - \xi(s)|^2 \leq |w(t) - w(s)|^2$$

$$+ \frac{1}{r_0} \int_{(s,t]} |\xi(u) - \xi(s)|^2 d|\phi|_u + 2 \int_{(s,t]} \langle w(t) - w(u), d\phi(u) \rangle.$$

Proof. (i) By (2.2) we have

$$|\xi(t) - \xi'(t)|^2 = |w(t) - w'(t)|^2 + |\phi(t) - \phi'(t)|^2$$

$$+ 2\langle w(t) - w'(t), \phi(t) - \phi'(t) \rangle$$

$$= |w(t) - w'(t)|^2 + 2 \int_{(0,t]} \langle \phi(s) - \phi'(s), d\phi(s) - d\phi'(s) \rangle$$

$$- \sum_{s \leq t} |\Delta\phi(s) - \Delta\phi'(s)|^2 + 2\langle w(t) - w'(t), \phi(t) - \phi'(t) \rangle,$$

$$\langle w(t) - w'(t), \phi(t) - \phi'(t) \rangle$$

$$= \int_{(0,t]} \langle w(t) - w'(t), d\phi(s) - d\phi'(s) \rangle$$

$$= \int_{(0,t]} \langle w(t) - w(s) - w'(t) + w'(s), d\phi(s) - d\phi'(s) \rangle$$

$$+ \int_{(0,t]} \langle w(s) - w'(s), d\phi(s) - d\phi'(s) \rangle.$$

Therefore

$$(2.8) \qquad |\xi(t) - \xi'(t)|^2 \leq |w(t) - w'(t)|^2$$

$$+ 2 \int_{(0,t]} \langle \xi(s) - \xi'(s), d\phi(s) - d\phi'(s) \rangle$$

$$+ 2 \int_{(0,t]} \langle w(t) - w(s) - w'(t) + w'(s), d\phi(s) - d\phi'(s) \rangle.$$

By Condition (A),(1.3) and Remark 1.1, we have

$$\int_{(0,t]} \langle \xi(s) - \xi'(s), d\phi(s) - d\phi'(s) \rangle$$

$$\leq \frac{1}{2r_0} \int_{(0,t]} |\xi(s) - \xi'(s)|^2 d(|\phi|_s + |\phi'|_s)$$

and hence we obtain (2.6).

(ii) By a method similar to (i), we have

$$|\xi(t) - \xi(s)|^2 = |w(t) - w(s)|^2 + |\phi(t) - \phi(s)|^2$$

$$+ 2\langle w(t) - w(s), \phi(t) - \phi(s) \rangle$$

$$= |w(t) - w(s)|^2 + |\phi(t) - \phi(s)|^2 + 2\int_{(s,t]} \langle w(t) - w(s), d\phi(u) \rangle.$$

Using (2.2),

$$|\phi(t) - \phi(s)|^2 = 2\int_{(s,t]} \langle \phi(u) - \phi(s), d\phi(u) \rangle - \sum_{s < u \leq t} |\Delta\phi(u)|^2$$

$$\leq 2\int_{(s,t]} \langle \phi(u) - \phi(s), d\phi(u) \rangle,$$

$$|\xi(t) - \xi(s)|^2 \leq |w(t) - w(s)|^2 + 2\int_{(s,t]} \langle \phi(u) - \phi(s), d\phi(u) \rangle$$

$$+ 2\int_{(s,t]} \langle w(t) - w(s), d\phi(u) \rangle$$

$$= |w(t) - w(s)|^2 + 2\int_{(s,t]} \langle \phi(u) - \phi(s), d\phi(u) \rangle$$

$$+ 2 \int_{(s,t]} \langle w(u) - w(s), d\phi(u) \rangle + 2 \int_{(s,t]} \langle w(t) - w(u), d\phi(u) \rangle$$

$$= |w(t) - w(s)|^2 + 2 \int_{(s,t]} \langle \xi(u) - \xi(s), d\phi(u) \rangle$$

$$+ 2 \int_{(s,t]} \langle w(t) - w(u), d\phi(u) \rangle .$$

By Condition (A),(1.3) and Remark 1.1 we have

$$\int_{(s,t]} \langle \xi(u) - \xi(s), d\phi(u) \rangle \leq \frac{1}{2r_0} \int_{(s,t]} |\xi(u) - \xi(s)|^2 d|\phi|_u$$

and hence we obtain (2.7). The proof of Lemma 2.3 is finished.

§3.   Estimate of $|\phi_m|_t$

Let D be a domain in R^d (not necessarily bounded) satisfying Conditions (A) and (B). For $w \in W(R^d)$ define $w_m \in D(R_+ \to R^d)$, $m \geq 1$, by

(3.1) $w_m(t) = w(k2^{-m})$, $k2^{-m} \leq t < (k + 1)2^{-m}$, $k \geq 0$

and consider the following Skorohod equation for w_m:

(3.2) $\xi_m(t) = w_m(t) + \phi_m(t)$, $t \geq 0$.

For simplicity we assume that w is uniformly continuous in $[0,\infty)$ for the time being (this assumption is not essential since we shall consider (3.2) for $0 \leq t \leq T$, T being arbitrary but fixed). Since w_m is a step function, a solution (ξ_m, ϕ_m) of (3.2) can be constructed explicitly as follows provided that m is sufficiently large (see Remark 1.4).

(3.3) $\xi_m(t) = \begin{cases} w_m(0) & , \ 0 \leq t < 2^{-m}, \\ \\ \overline{\xi_m((k-1)2^{-m}) + \Delta w_m(k2^{-m})}, & k2^{-m} \leq t < (k+1)2^{-m}, \end{cases}$

(3.4) $\phi_m(t) = \begin{cases} 0 & , \ 0 \leq t < 2^{-m}, \\ \\ \phi_m((k-1)2^{-m}) + \overline{\xi_m((k-1)2^{-m}) + \Delta w_m(k2^{-m})} \\ \\ \quad - \xi_m((k-1)2^{-m}) - \Delta w_m(k2^{-m}), & k2^{-m} \leq t < (k+1)2^{-m}. \end{cases}$

Here $\Delta w_m(t) = w_m(t) - w_m(t-)$, $w_m(t-) = \lim_{s \uparrow t} w_m(s)$. Notice that $|\Delta w_m(t)| < r_0$ for sufficiently large m because of uniform continuity of w and hence

$$\overline{\xi_m((k-1)2^{-m}) + \Delta w_m(k2^{-m})}$$

is uniquely defined (see Remark 1.3).

Remark 3.1. Under Condition (A), it is easy to see that for large m (3.4) implies

$$\Delta |\phi_m|_t = |\Delta\phi_m(t)| \leq |\Delta w_m(t)|.$$

The purpose of this section is to prove the following proposition which plays an essential role in the proof of Theorem 4.1 in §4.

Proposition 3.1. Let $T > 0$ be any fixed time. If the domain D satisfies Conditions (A) and (B), then for sufficiently large m we have

$$(3.5) \qquad |\phi_m|_t^s \leq K\Delta_{s-2^{-m},t}(w), \qquad 0 \leq s < t \leq T,$$

where $\Delta_{s,t}(w) = \sup_{s \leq t_1 < t_2 \leq t} |w(t_2) - w(t_1)|$ and K is a constant depending only on the constants r_0, β, δ in Conditions (A),(B),

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T, $\|w\|_T = \sup_{0 \leq t \leq T} |w(t)|$ and the modulus of uniform continuity of w on $[0,T]$.

Before proving the proposition, we prepare some lemmas under the same assumption as in Proposition 3.1. Define

$$T_{m,0} = \inf\{t > 0: \xi_m(t) \in \partial D\},$$

$$t_{m,n} = \inf\{t > T_{m,n-1}: |\xi_m(t) - \xi_m(T_{m,n-1})| \geq \delta/2\},$$

$$T_{m,n} = \inf\{t \geq t_{m,n}: \xi_m(t) \in \partial D\}.$$

Notice that $T_{m,n-1} < t_{m,n}$ but it can happen that $t_{m,n} = T_{m,n}$. Let

$$\Delta_{s,t}(w_m) = \sup_{s \leq t_1 < t_2 \leq t} |w_m(t_2) - w_m(t_1)|,$$

$$|\phi_m|_t^s = \sup \sum_{k=1}^n |\phi_m(t_k) - \phi_m(t_{k-1})|,$$

where the second supremum is taken over all partitions $s = t_0 < t_1 < \cdots < t_n = t$.

Lemma 3.1. For sufficiently large m, we have

$$(3.6) \qquad |\phi_m|_t^s \leq \beta\{\Delta_{s,t}(\xi_m) + \Delta_{s,t}(w_m)\}, \quad s, t \in [T_{m,n-1}, t_{m,n}]$$

$$\cup \ (t_{m,n}, T_{m,n}).$$

Proof. If $T_{m,n-1} \leqq s < t \leqq t_{m,n}$, then by Condition (B) we have

$$\langle \underline{\ell}, \xi_m(t) - \xi_m(s) \rangle = \langle \underline{\ell}, w_m(t) - w_m(s) \rangle + \langle \underline{\ell}, \phi_m(t) - \phi_m(s) \rangle$$

$$\geqq \langle \underline{\ell}, w_m(t) - w_m(s) \rangle + \beta^{-1} |\phi_m|_t^s,$$

where $\underline{\ell} = \underline{\ell}_{\xi_m(T_{m,n-1})}$ and m is taken to be sufficiently large so that $|\xi_m(t) - \xi_m(T_{m,n-1})| < \delta$ holds for $T_{m,n-1} \leqq t \leqq t_{m,n}$. Therefore

$$|\phi_m|_t^s \leqq \beta\{|\xi_m(t) - \xi_m(s)| + \Delta_{s,t}(w_m)\}, \quad T_{m,n-1} \leqq s < t \leqq t_{m,n}$$

Since ϕ_m is constant in the interval $(t_{m,n}, T_{m,n})$, we have

$$|\phi_m|_t^s \leqq \beta\{\Delta_{s,t}(\xi_m) + \Delta_{s,t}(w_m)\}, \quad s, t \in [T_{m,n-1}, t_{m,n}]$$
$$\cup (t_{m,n}, T_{m,n}).$$

Lemma 3.2. Let $T > 0$ be any fixed time. Then for sufficiently large m we have

$$(3.7) \qquad \Delta_{s,t}(\xi_m) \leqq \sqrt{2}\{(1 + \varepsilon^{-1})\Delta_{s,t}(w_m) + \varepsilon |\phi_m|_t^s\}$$

$$\cdot \exp(\gamma |\phi_m|_t^s), \quad 0 \leqq s < t \leqq T,$$

where $\gamma = (r_0)^{-1}$ and ε is an arbitrary positive number.

Proof. By (2.7) we have

$$|\xi_m(t) - \xi_m(s)|^2 \leq |w_m(t) - w_m(s)|^2 + \gamma \int_{(s,t]} |\xi_m(u) - \xi_m(s)|^2 d|\phi_m|_u$$

$$+ 2 \int_{(s,t]} \langle w_m(t) - w_m(u), d\phi_m(u) \rangle$$

$$\leq \Delta^2_{s,t}(w_m) + 2\Delta_{s,t}(w_m)|\phi_m|^s_t + \gamma \int_{(s,t]} |\xi_m(u) - \xi_m(s)|^2 d|\phi_m|_u$$

$$= \Delta^2_{s,t}(w_m) + 2\Delta_{s,t}(w_m)|\phi_m|^s_t + \gamma \int_{(s,t)} |\xi_m(u) - \xi_m(s)|^2 d|\phi_m|_u$$

$$+ \gamma |\xi_m(t) - \xi_m(s)|^2 \Delta|\phi_m|_t.$$

Since $\Delta|\phi_m|_t \leq |\Delta w_m(t)| < r_0/2 = (2\gamma)^{-1}$, $0 \leq t \leq T$, for sufficiently large m by Remark 3.1,

$$|\xi_m(t) - \xi_m(s)|^2 \leq 2\Delta^2_{s,t}(w_m) + 4\Delta_{s,t}(w_m)|\phi_m|^s_t$$

$$+ 2\gamma \int_{(s,t)} |\xi_m(u) - \xi_m(s)|^2 d|\phi_m|_u,$$

and hence, using Lemma 2.2, we have

$$(3.8) \qquad |\xi_m(t) - \xi_m(s)|^2 \leq 2\{\Delta^2_{s,t}(w_m) + 2\Delta_{s,t}(w_m)|\phi_m|^s_t\}$$

$$\cdot \exp(2\gamma|\phi_m|^s_t).$$

Therefore for any $\varepsilon > 0$,

$$|\xi_m(t) - \xi_m(s)|^2 \leq 2\{(1 + \varepsilon^{-2})\Delta_{s,t}^2(w_m) + \varepsilon^2(|\phi_m|_t^s)^2\}$$

$$\cdot \exp(2\gamma|\phi_m|_t^s),$$

that is,

$$|\xi_m(t) - \xi_m(s)| \leq \sqrt{2}\{(1 + \varepsilon^{-1})\Delta_{s,t}(w_m) + \varepsilon|\phi_m|_t^s\}$$

$$\cdot \exp(\gamma|\phi_m|_t^s), \quad 0 \leq s < t \leq T.$$

The proof is finished.

Lemma 3.3. Let $T > 0$ be any fixed time. Then, for sufficiently large m we have

$$(3.9) \qquad \Delta_{s,t}(\xi_m) \leq K\Delta_{s,t}(w_m), \quad s, t \in [T_{m,n-1}, t_{m,n}] \cup (t_{m,n}, T_{m,n}),$$

provided that $T_{m,n} \leq T$, where K is a constant depending only on the constants r_0, β, δ in Conditions (A),(B), T and $\|w\|_T$.

Proof. (3.7) combined with (3.6) implies

$$\Delta_{s,t}(\xi_m) \leq \sqrt{2}\{(1 + \varepsilon^{-1} + \beta\varepsilon)\Delta_{s,t}(w_m) + \beta\varepsilon\Delta_{s,t}(\xi_m)\}$$

$$\cdot \exp[\beta\gamma\{\Delta_{s,t}(\xi_m) + \Delta_{s,t}(w_m)\}], \quad s,t \in [T_{m,n-1}, t_{m,n}].$$

If we put $\eta = \Delta_{s,t}(\xi_m)$, then

(3.10) $\eta \leq \sqrt{2}\{(1 + \varepsilon^{-1} + \beta\varepsilon)\Delta_{s,t}(w_m) + \beta\varepsilon\eta\}\cdot\exp[\beta\gamma\{\eta + \Delta_{s,t}(w_m)\}]$,

$$s, \ t \in [T_{m,n-1}, t_{m,n}].$$

Since $0 \leq \eta < 2\delta$ for $s, \ t \in [T_{m,n-1}, t_{m,n}]$, we have

$$\eta \leq \sqrt{2}\{(1 + \varepsilon^{-1} + \beta\varepsilon)\Delta_{s,t}(w_m) + \beta\varepsilon\eta\}\cdot\exp\{2\beta\gamma(\delta + \|w\|_T)\}.$$

If we take ε as $0 < \varepsilon < (\sqrt{2}\beta)^{-1}\exp\{-2\beta\gamma(\delta + \|w\|_T)\}$,

$$\eta \leq \frac{\sqrt{2}(1 + \varepsilon^{-1} + \beta\varepsilon)\Delta_{s,t}(w_m)\cdot\exp\{2\beta\gamma(\delta + \|w\|_T)\}}{1 - \sqrt{2}\beta\varepsilon\cdot\exp\{2\beta\gamma(\delta + \|w\|_T)\}},$$

and hence

(3.11) $\Delta_{s,t}(\xi_m) \leq K_\varepsilon\Delta_{s,t}(w_m)$, $s, \ t \in [T_{m,n-1}, t_{m,n}]$,

where

$$K_\varepsilon = \frac{\sqrt{2}(1 + \varepsilon^{-1} + \beta\varepsilon)\cdot\exp\{2\beta\gamma(\delta + \|w\|_T)\}}{1 - \sqrt{2}\beta\varepsilon\cdot\exp\{2\beta\gamma(\delta + \|w\|_T)\}}.$$

If $t_{m,n} < T_{m,n}$, by the definitions of $t_{m,n}$, $T_{m,n}$, we have

$$\Delta_{s,t}(\xi_m) = \Delta_{s,t}(w_m), \quad s, \ t \in [t_{m,n}, T_{m,n}).$$

Therefore,

$$\Delta_{s,t}(\xi_m) \leq \Delta_{s,t_{m,n}}(\xi_m) + \Delta_{t_{m,n},t}(\xi_m)$$

$$\leq K_\varepsilon \Delta_{s,t_{m,n}}(w_m) + \Delta_{t_{m,n},t}(w_m)$$

$$\leq (K_\varepsilon + 1)\Delta_{s,t}(w_m), \quad T_{m,n-1} \leq s \leq t_{m,n} < t < T_{m,n},$$

and hence

$$\Delta_{s,t}(\xi_m) \leq K\Delta_{s,t}(w_m), \quad s, t \in [T_{m,n-1}, t_{m,n}] \cup (t_{m,n}, T_{m,n}),$$

where

$$K = \inf[K_\varepsilon + 1 : 0 < \varepsilon < (\sqrt{2}\beta)^{-1}\exp\{-2\beta\gamma(\delta + \|w\|_T)\}].$$

The proof of Lemma 3.3 is finished.

Now we are going to prove Proposition 3.1.

Proof of Proposition 3.1. (3.9) combined with (3.6) implies

$$(3.12) \qquad |\phi_m|_t^s \leq K'\Delta_{s,t}(w_m), \quad s, t \in [T_{m,n-1}, t_{m,n}] \cup (t_{m,n}, T_{m,n}),$$

where $K' = \beta(K + 1) + 1$. By (3.7) and (3.12) we have

$$(3.13) \qquad \delta/2 \leq |\xi_m(t_{m,n}) - \xi_m(T_{m,n-1})|$$

$$\leq \sqrt{2}\{(1 + \varepsilon^{-1})\Delta_{T_{m,n-1},t_{m,n}}(w_m) + \varepsilon|\phi_m|_{t_{m,n}}^{T_{m,n-1}}\}$$

$$\cdot \exp\{\gamma|\phi_m|_{t_{m,n}}^{T_{m,n-1}}\}$$

$$\leq \sqrt{2}(1 + \varepsilon^{-1} + K'\varepsilon)\Delta_{T_{m,n-1}t_{m,n}}(w_m)$$

$$\cdot \exp\{K'\gamma\Delta_{T_{m,n-1},t_{m,n}}(w_m)\}.$$

Now let η_ε be the positive root of the equation:

$$\sqrt{2}(1 + \varepsilon^{-1} + K'\varepsilon)\eta\cdot\exp(K'\gamma\eta) = \delta/2,$$

and put $\Delta = \sup\limits_{\varepsilon>0} \eta_\varepsilon$. It follows from (3.13) that

(3.14) $\Delta \leq \Delta_{T_{m,n-1},t_{m,n}}(w_m).$

Since w is continuous, there exist an integer $m_0 \geq 1$ and $h > 0$ such that for any $m \geq m_0$,

$$\Delta_{0,T,h}(w_m) < \Delta,$$

where $\Delta_{0,T,h}(w_m) = \sup\{|w_m(t) - w_m(s)|: 0 \leq s < t \leq T, |t - s| \leq h\}.$

Note that $T_{m,n} \leq T$ implies $T_{m,n} - T_{m,n-1} \geq h$ for $m \geq m_0$.
In fact, if $T_{m,n} - T_{m,n-1} < h$, we have

$$\Delta_{T_{m,n-1},t_{m,n}}(w_m) \leq \Delta_{T_{m,n-1},T_{m,n}}(w_m)$$

$$\leq \Delta_{0,T,h}(w_m) < \Delta,$$

which contradicts (3.14). Therefore if $m \geq m_0$, $T_{m,n} > T$ for

n > T/h. By (3.12),

$$|\phi_m|_t^s \leq (T/h + 1)K'\Delta_{s,t}(w_m) \leq K''\Delta_{s-2^{-m},t}(w), \quad 0 \leq s < t \leq T,$$

where $K'' = (T/h +1)K'$. The proof of Proposition 3.1 is finished.

§4.   Solvability of the deterministic problem

In this section we prove the following theorem.

Theorem 4.1. Suppose that a domain D in R^d satisfies
Conditions (A) and (B). Then for any $w \in W(R^d)$ with $w(0) \in \overline{D}$
there exists a unique solution $\xi(t,w)$ of the equation (1.1),
and $\xi(t,w)$ is continuous in (t,w).

Proof. The uniqueness was proved in [2: Theorem 1.1] under
Condition (A) only. In fact, the uniqueness follows immediately
from (2.6).

We prove the existence. Let $T > 0$ be any constant. In
what follows m is assumed to be so large that (3.5) holds.
First we notice that there exists a constant C, depending only
on the constants r_0, β, δ in Conditions (A),(B),   T, $\|w\|_T$ and
the modulus of uniform continuity of w on [0,T], such that

$$|\phi_m|_t \le C, \quad 0 \le t \le T;$$

in fact, it is enough to take $C = K\Delta_{0,T}(w)$ where K is the
constant in (3.5). For $m < n$ and $0 \le t \le T$ we see that

$$|w_m(t) - w_n(t)| \le \Delta_{0,t,2^{-m}}(w),$$

$$\int_{(0,t]} \langle w_m(t) - w_m(s) - w_n(t) + w_n(s), d\phi_m(s) - d\phi_n(s) \rangle$$

$$\le 4C\Delta_{0,t,2^{-m}}(w),$$

where

$$\Delta_{0,t,2^{-m}}(w) = \sup\{|w(u) - w(s)| : 0 \leq s < u \leq t, \ |u - s| \leq 2^{-m}\}.$$

Then by (2.6), setting $\gamma = (r_0)^{-1}$, we have

$$|\xi_m(t) - \xi_n(t)|^2 \leq |w_m(t) - w_n(t)|^2$$

$$+ \gamma \int_{(0,t]} |\xi_m(s) - \xi_n(s)|^2 d(|\phi_m|_s + |\phi_n|_s)$$

$$+ 2 \int_{(0,t]} \langle w_m(t) - w_m(s) - w_n(t) + w_n(s), d\phi_m(s) - d\phi_n(s)\rangle$$

$$\leq \Delta^2_{0,t,2^{-m}}(w) + \gamma \int_{(0,t)} |\xi_m(s) - \xi_n(s)|^2 d(|\phi_m|_s + |\phi_n|_s)$$

$$+ \gamma|\xi_m(t) - \xi_n(t)|^2 \Delta(|\phi_m|_t + |\phi_n|_t) + 8C\Delta_{0,t,2^{-m}}(w).$$

Since Remark 3.1 implies

$$\Delta(|\phi_m|_t + |\phi_n|_t) \leq r_0/2 = (2\gamma)^{-1}, \quad 0 \leq t \leq T$$

for sufficiently large m and n, we have

$$|\xi_m(t) - \xi_n(t)|^2 \leq 2\Delta^2_{0,t,2^{-m}}(w) + 16C\Delta_{0,t,2^{-m}}(w)$$

$$+ 2\gamma \int_{(0,t)} |\xi_m(s) - \xi_n(s)|^2 d(|\phi_m|_s + |\phi_n|_s).$$

Hence by Lemma 2.2, we have

$$|\xi_m(t) - \xi_n(t)|^2 \leq 2\{\Delta_{0,t,2^{-m}}(w) + 8C\}\Delta_{0,t,2^{-m}}(w)$$

$$\cdot \exp\{2\gamma(|\phi_m|_t + |\phi_n|_t)\}$$

$$\leq 2\{\Delta_{0,t,2^{-m}}(w) + 8C\}\Delta_{0,t,2^{-m}}(w) \cdot \exp(4C\gamma),$$

which tends to 0 as $m \to \infty$. Therefore ξ_m and ϕ_m converge uniformly on [0,T] as $m \to \infty$. We denote the limits by ξ and ϕ respectively. Then it follows immediately from Proposition 3.1 that ξ and ϕ are continuous in t. Therefore all we have to show is the following:

$$(4.1) \qquad |\phi|_t = \int_{(0,t]} \mathbb{1}_{\partial D}(\xi(s))d|\phi|_s,$$

$$(4.2) \qquad \phi(t) = \int_{(0,t]} \underline{n}(s)d|\phi|_s, \qquad \underline{n}(s) \in \mathcal{N}_{\xi(s)}, \qquad \xi(s) \in \partial D.$$

By Remark 1.1 and (1.3),

$$(4.3) \qquad \int_{(0,t]} \langle \eta(s) - \xi_n(s), d\phi_n(s) \rangle + \frac{1}{2r_0} \int_{(0,t]} |\eta(s) - \xi_n(s)|^2 d|\phi_n|_s$$

$$\geq 0,$$

for any $\eta \in W(\overline{D})$, $0 \leq t \leq T$. Put

$$I_1 = \int_{(0,t]} \langle \eta(s) - \xi_n(s), d\phi_n(s) \rangle,$$

$$I_2 = \int_{(0,t]} |\eta(s) - \xi_n(s)|^2 d|\phi_n|_s.$$

We now prove the following 1° and 2°.

1°. $\lim\limits_{n\to\infty} I_1 = \int_{(0,t]} \langle \eta(s) - \xi(s), d\phi(s) \rangle .$

Proof. We write

$$I_1 = \int_{(0,t]} \langle \zeta(s), d\phi_n(s) \rangle + \int_{(0,t]} \langle \xi(s) - \xi_n(s), d\phi_n(s) \rangle$$

$$= I_1' + I_1'',$$

where $\zeta(s) = \eta(s) - \xi(s)$. Then I_1'' is domonated in modulus by

$$C\|\xi - \xi_n\|_t = C \sup_{0 \le s \le t} |\xi(s) - \xi_n(s)|$$

which tends to 0 as $n \to \infty$. To handle I_1' let $0 = t_0 < t_1 < \cdots < t_m = t$ be an equi-partition ($t_k - t_{k-1} = t/m$) of $[0,t]$ and set $\zeta^m(s) = \zeta(t_k)$ for $t_k < s \le t_{k+1}$, $k = 0, 1, \cdots, m - 1$. For any fixed $\varepsilon > 0$ we take m so that $\|\zeta - \zeta^m\|_t < \varepsilon$ holds. Then

$$\left| \int_{(0,t]} \langle \zeta(s) - \zeta^m(s), d\phi_n(s) \rangle \right| \le C\|\zeta - \zeta^m\|_t \le C\varepsilon,$$

and similarly

$$\left| \int_{(0,t]} \langle \zeta(s), d\phi(s) \rangle \right| \le C\varepsilon,$$

because $|\phi|_t^s \le \varliminf\limits_{n\to\infty} |\phi_n|_t^s$. Therefore we have

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$$\left| I_1' - \int_{(0,t]} \langle \zeta(s), d\phi(s) \rangle \right|$$

$$\leq \left| \int_{(0,t]} \langle \zeta(s) - \zeta^m(s), d\phi_n(s) \rangle \right| + \left| \int_{(0,t]} \langle \zeta(s) - \zeta^m(s), d\phi(s) \rangle \right|$$

$$+ \left| \sum_{k=0}^{m-1} \zeta(t_k)\{\phi_n(t_{k+1}) - \phi_n(t_k)\} - \sum_{k=0}^{m-1} \zeta(t_k)\{\phi(t_{k+1}) - \phi(t_k)\} \right|$$

$$\leq 2C\varepsilon + o(1), \quad n \to \infty,$$

proving that $\lim\limits_{n\to\infty} I_1 = \lim\limits_{n\to\infty} I_1' = \int_{(0,t]} \langle \zeta(s), d\phi(s) \rangle$.

Let da_u be any weak limit on $[0,T]$ of $d|\phi_n|_u$ as $n \to \infty$ via some subsequence $n_1 < n_2 < \cdots$.

2°. $\lim I_2 = \int_{(0,t]} |\eta(s) - \xi(s)|^2 da_s$ if t is a point of continuity for a_u, where the limit is taken as $n \to \infty$ via $n_1 < n_2 < \cdots$.

Proof. We have

$$\left| \int_{(0,t]} |\eta(s) - \xi_n(s)|^2 d|\phi_n|_s - \int_{(0,t]} |\eta(s) - \xi(s)|^2 da_s \right|$$

$$\leq \left| \int_{(0,t]} |\eta(s) - \xi_n(s)|^2 d|\phi_n|_s - \int_{(0,t]} |\eta(s) - \xi(s)|^2 d|\phi_n|_s \right|$$

$$+ \left| \int_{(0,t]} |\eta(s) - \xi(s)|^2 d|\phi_n|_s - \int_{(0,t]} |\eta(s) - \xi(s)|^2 da_s \right|.$$

If t is a point of continuity for a_u, the second term tends to 0 as $n \to \infty$ via $n_1 < n_2 < \cdots$. If we put $\zeta_n(s) = \eta(s) - \xi_n(s)$,

then the first term is dominated by

$$\sup_{0 \leq s \leq t} |\{|\zeta_n(s)|^2 - |\zeta(s)|^2\}| \cdot C,$$

which tends to 0 as $n \to \infty$. Thus the proof of $2°$ is finished.

Now we are going to complete the proof of the theorem by making use of a method similar to that used in the proof of Theorem 1.1 of Lions and Sznitman [2]. From $|\phi|_t^s \leq \varliminf_{n \to \infty} |\phi_n|_t^s$ and the definition of the measure da_s, we have

$$(4.4) \qquad d|\phi|_s \leq da_s,$$

and hence there is an R^d-valued bounded measurable function h_s such that

$$(4.5) \qquad d\phi(s) = h_s da_s.$$

Making n tend to ∞ in (4.3) and then using $1°$, $2°$ and (4.5), we have

$$(4.6) \qquad \int_{(0,t]} \langle \eta(s) - \xi(s), h_s \rangle da_s + \frac{1}{2r_0} \int_{(0,t]} |\eta(s) - \xi(s)|^2 da_s$$

$$\geq 0, \quad 0 \leq t \leq T,$$

or equivalently

(4.7) $\qquad \langle \eta(s) - \xi(s), h_s \rangle + \frac{1}{2r_0} |\eta(s) - \xi(s)|^2 \geq 0, \quad da_s\text{-a.e.}$

Define a continuous function $\chi \ (0 \leq \chi \leq 1)$ on R^d by

$$\chi = \begin{cases} 1 & \text{on a compact set included in } D, \\ \\ 0 & \text{on } R^d \backslash D. \end{cases}$$

Since

$$\left| \int_{(0,t]} \chi(\xi(s)) d|\phi_n|_s - \int_{(0,t]} \chi(\xi_n(s)) d|\phi_n|_s \right|$$

$$\leq \sup_{0 \leq s \leq t} |\chi(\xi(s)) - \chi(\xi_n(s))| \cdot C \to 0, \quad n \to \infty,$$

we have

$$0 = \lim \int_{(0,t]} \chi(\xi_n(s)) d|\phi_n|_s = \lim \int_{(0,t]} \chi(\xi(s)) d|\phi_n|_s$$

$$= \int_{(0,t]} \chi(\xi(s)) da_s \geq 0.$$

Hence $\int_{(0,t]} \chi(\xi(s)) da_s = 0.$ Letting χ increase to the indicator function $\mathbb{1}_D$, we have

(4.8a) $\qquad \int_{(0,t]} \mathbb{1}_D(\xi(s)) da_s = 0,$

or equivalently

(4.8b) $\xi(s) \in \partial D$, da_s-a.e.

Thus $\xi(s) \in \partial D$ in (4.7). Therefore by Condition (A) and Remark
1.1 there exist $\theta(s) \geq 0$ and $\underline{n}(s) \in \mathcal{N}_{\xi(s)}$ (if $\xi(s) \in \partial D$) such
that $h_s = \theta(s)\underline{n}(s)$, da_s-a.e. Then (4.5) implies that $d|\phi|_s =$
$|h_s|da_s = \theta(s)da_s$ and hence

$$d\phi(s) = h_s da_s = \theta(s)\underline{n}(s)da_s = \underline{n}(s)d|\phi|_s,$$

which is nothing but (4.2). Now (4.1) is also clear. (4.1) and
(4.2) mean that ξ is a solution of (1.1). The continuity of
in (t,w) follows from Remark 4.1 below. The proof of Theorem
4.1 is finished.

Theorem 4.2. If (ξ,ϕ) is the solution of (1.1) for $w \in$
$W(R^d)$ with $w(0) \in \overline{D}$, for any finite $T > 0$, we have

$$|\phi|_t^s \leq K\Delta_{s,t}(w), \quad 0 \leq s < t \leq T,$$

where K is a constant depending only on the constants r_0, β,
δ in Conditions (A),(B), T, $\|w\|_T$ and the modulus of uniform
continuity of w on $[0,T]$.

The proof is immediate from Proposition 3.1.

Remark 4.1. Let (ξ,ϕ) and (ξ',ϕ') be the solutions of
the Skorohod equations $\xi = w + \phi$ and $\xi' = w' + \phi'$, respectively.

(i) By (2.6) and Gronwall's inequality, we immediately have

$$|\xi(t) - \xi'(t)|^2 \leq \{|w(t) - w'(t)|^2 + 4(C + C')\|w - w'\|_t\}$$

$$\cdot \exp\{(C + C')/r_0\}, \quad 0 \leq t \leq T,$$

where C is the constant appearing in the proof of Theorem 4.1 and C' is a similar constant for w'.

(ii) Similarly, (2.7) and an application of Gronwall's inequality yield

$$|\xi(t) - \xi(s)|^2 \leq \{|w(t) - w(s)|^2 + 2C\Delta_{s,t}(w)\} \cdot \exp(C/r_0),$$

$$0 \leq s < t \leq T.$$

Finally we assume that the domain D satisfies the following Condition (D) and study the Lipschitz continuity of the solution ξ of (1.1) in w with respect to the total variation in the w-space.

Condition (D). Condition (A) is satisfied and there exist constants $C_1 \geq 0$, $C_2 \in (0, r_0)$ such that

$$|\bar{x} - \bar{y}| \leq (1 + C_1\varepsilon)|x - y|$$

holds for any $x, y \in R^d$ with $|x - \bar{x}| \leq C_2$, $|y - \bar{y}| \leq C_2$, where $\varepsilon = \max\{|x - \bar{x}|, |y - \bar{y}|\}$.

Let w, $w' \in W(R^d)$ with $w(0)$, $w'(0) \in \overline{D}$ and (ξ,ϕ), (ξ',ϕ') be the unique solutions of the Skorohod equations $\xi = w + \phi$ and $\xi' = w' + \phi'$, respectively. Then we have the following proposition.

Proposition 4.1. Under Conditions (B) and (D), we have for each $T > 0$

$$(4.9) \qquad |\xi(t) - \xi'(t)| \leq K\{|w - w'|_t + |w(0) - w'(0)|\}, \quad 0 \leq t \leq T,$$

where $K > 0$ is a constant depending only on the constants C_1, C_2, r_0, β, δ in Conditions (B) and (D), T, $\{\Delta_{0,T,h}(w), 0 < h \leq T\}$ and $\{\Delta_{0,T,h}(w'), 0 < h \leq T\}$.

Remark 4.2. (i) We can take $K = \exp(2CC_1)$ where C and C_1 are the constants in Proposition 3.1 and Condition (D), respectively.
(ii) Any convex domain satisfies Condition (D) with $C_1 = 0$ and hence (4.9) holds with $K = 1$.
(iii) In most applications w is supposed to be a Brownian path in which case $|w|_t = \infty$, but (4.9) will be of some use if we consider the Skorohod equation $\xi(t) = w(t) + \int_0^t b(\xi(s))ds + \phi(t)$ with Lipschitz continuous b.

Proof of Proposition 4.1. Define w_m, ξ_m and ϕ_m by (3.1), (3.3) and (3.4), respectively and define also w'_m, ξ'_m and ϕ'_m similarly. Then by Condition (D),

$$|\xi_m(k2^{-m}) - \xi'_m(k2^{-m})|$$

$$= |\{\xi_m((k-1)2^{-m}) + w_m(k2^{-m}) - w_m((k-1)2^{-m})\}$$

$$- \{\xi_m'((k-1)2^{-m}) + w_m'(k2^{-m}) - w_m'((k-1)2^{-m})\}|$$

$$\leq [1 + C_1\{|\phi_m|_{k2^{-m}}^{(k-1)2^{-m}} + |\phi_m'|_{k2^{-m}}^{(k-1)2^{-m}}\}]$$

$$\cdot\{|\xi_m((k-1)2^{-m}) - \xi_m'((k-1)2^{-m})|$$

$$+ |w_m(k2^{-m}) - w_m((k-1)2^{-m}) - w_m'(k2^{-m}) + w_m'((k-1)2^{-m})|\}.$$

If we put

$$x_{m,k} = |\xi_m(k2^{-m}) - \xi_m'(k2^{-m})|,$$

$$a_{m,k} = |\phi_m|_{k2^{-m}}^{(k-1)2^{-m}} + |\phi_m'|_{k2^{-m}}^{(k-1)2^{-m}},$$

$$b_{m,k} = |w_m(k2^{-m}) - w_m((k-1)2^{-m}) - w_m'(k2^{-m}) + w_m'((k-1)2^{-m})|,$$

then

$$x_{m,k} \leq (1 + C_1 a_{m,k})(x_{m,k-1} + b_{m,k})$$

$$\leq \exp(C_1 a_{m,k})(x_{m,k-1} + b_{m,k})$$

$$\leq \exp(C_1 a_{m,k})\{\exp(C_1 a_{m,k-1})(x_{m,k-2} + b_{m,k-1}) + b_{m,k}\}$$

$$\leq \exp\{C_1(a_{m,k} + a_{m,k-1})\}(x_{m,k-2} + b_{m,k} + b_{m,k-1})$$

$$\leq \cdots$$

$$\leq \exp(C_1 \sum_{i=1}^{k} a_{m,i})(x_{m,0} + \sum_{i=1}^{k} b_{m,i}).$$

Since ξ_m, ϕ_m and w_m are constant in $[k2^{-m}, (k+1)2^{-m})$, we have

$$|\xi_m(t) - \xi_m'(t)| \leq \exp\{C_1(|\phi_m|_t + |\phi_m'|_t)\}$$

$$\cdot \{|w_m - w_m'|_t + |w(0) - w'(0)|\}.$$

By Proposition 3.1, we have for any $T > 0$,

$$|\phi_m|_t + |\phi_m'|_t \leq 2C, \quad 0 \leq t \leq T,$$

and hence, for $0 \leq t \leq T$,

$$|\xi_m(t) - \xi_m'(t)| \leq K\{|w_m - w_m'|_t + |w(0) - w'(0)|\},$$

where $K = \exp(2CC_1)$. Letting $m \to \infty$, we have (4.9). The proof is finished.

Before closing this section we state an interesting example, due to Saisho and Tanaka [6], of a domain satisfying Conditions (A) and (B).

Example. We write $x = (x_1, x_2, \cdots, x_n)$ for a point of R^{nd}, where each x_k denotes a point of R^d. Let D be a domain in

R^{nd} defined by

$$D = \{(x_1, x_2, \cdots, x_n): |x_i - x_j| > \rho \quad \text{for} \quad 1 \le \forall i < \forall j \le n\},$$

where ρ is a given positive constant. Saisho and Tanaka [6] proved that D satisfies Conditions (A) and (B). Suppose we are given $w_1, w_2, \cdots, w_n \in W(R^d)$ and write $\underline{w} = (w_1, w_2, \cdots, w_n)$. We assume that $\underline{w}(0) \in \overline{D}$, that is, $|w_i(0) - w_j(0)| \ge \rho$ for $1 \le \forall i < \forall j \le n$. Then by Theorem 4.1 there exists a unique solution of the Skorohod equation $\underline{\xi}(t) = \underline{w}(t) + \underline{\phi}(t)$ for the domain D. The component-wise consideration of this equation yields the following system of equations:

$$\xi_i(t) = w_i(t) + \sum_{j=1(\ne i)}^{n} \int_0^t (\xi_i(s) - \xi_j(s)) d\phi_{ij}(s), \quad 1 \le i \le n.$$

Here $\xi_i(t)$ and $\phi_{ij}(t)$ satisfy the following conditions.

(i) $\xi_i(t)$'s are continuous and $|\xi_i(t) - \xi_j(t)| \ge \rho$ for $1 \le i < j \le n$.

(ii) $\phi_{ij}(t)$'s are continuous non-decreasing functions satisfying $\phi_{ij}(t) = \phi_{ji}(t)$ and

$$\phi_{ij}(t) = \int_0^t \mathbf{1}(|\xi_i(s) - \xi_j(s)| = \rho) d\phi_{ij}(s).$$

When $w_1(t), w_2(t), \cdots, w_n(t)$ are independent d-dimensional Brownian

motions, $\underset{=}{\xi}(t)$ describes the motion of n mutually reflecting

Brownian balls of diameter ρ in the space R^d. For details see

Saisho and Tanaka [6].

§5. Solvability of the Skorohod SDE

Let D be a domain in R^d satisfying Conditions (A) and (B)
and suppose that we are given coefficients

$$\sigma: \overline{D} \to R^d \otimes R^d, \quad b: \overline{D} \to R^d$$

satisfying

$$|\sigma(x) - \sigma(y)| \leq L|x - y|, \quad |b(x) - b(y)| \leq L|x - y|,$$

$$|\sigma(x)| \leq L, \quad |b(x)| \leq L$$

for any $x, y \in \overline{D}$ with some constant $L \geq 0$. Let $(\Omega, \mathcal{F}, P)$ be a
complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ which is
assumed that each $\mathcal{F}_t$ contains all P-null sets and $\mathcal{F}_t = \bigcap_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$.
In this section we study a Skorohod SDE

(5.1) $dX(t) = \sigma(X(t))dB(t) + b(X(t))dt + d\Phi(t), \quad X(0) \in \overline{D},$

where B(t) is a d-dimensional $\mathcal{F}_t$-Brownian motion and X(0) is
$\mathcal{F}_0$-measurable. The main result is the following theorem.

Theorem 5.1. There exists a unique strong solution of (5.1).

The meaning of a strong solution is the same as in Definition
IV-1.6 of [1].

To prove the theorem we first consider a Skorohod SDE

(5.2) $dX_n(t) = \sigma(X_n(h_n(t)))dB(t) + b(X_n(h_n(t)))dt + d\Phi_n(t),$

with initial condition $X_n(0) = X(0),$ where

   $h_n(0) = 0,$

   $h_n(t) = (k - 1)2^{-n},$ $(k - 1)2^{-n} < t \leq k2^{-n},$ $k = 1,2,\cdots,$ $n \geq 1.$

By Theorem 4.1, we have a unique solution of (5.2); in fact, once $X_n(t)$ is obtained for $0 \leq t \leq k2^{-n},$ $X(t)$ for $k2^{-n} < t \leq (k+1)2^{-n}$ is uniquely determined as the solution of the Skorohod equation:

   $X_n(t) = X_n(k2^{-n}) + \sigma(X_n(k2^{-n}))\{B(t) - B(k2^{-n})\}$

   $+ b(X_n(k2^{-n}))(t - k2^{-n}) + \Phi_n(t).$

Put

(5.3) $Y_n(t) = X(0) + \int_0^t \sigma(X_n(h_n(s)))dB(s) + \int_0^t b(X_n(h_n(s)))ds,$

and denote by P_n the probability measure on $C([0,T] \to R^d \times \overline{D} \times R^d)$ introduced by the process $\{(B(t),X_n(t),Y_n(t)), 0 \leq t \leq T\},$ where T is an arbitrarily fixed time.

   Lemma 5.1. The family $\{P_n, n \geq 1\}$ is tight.

   Proof. It is easy to see that for $0 < \varepsilon < 1/2$

(5.4) $\displaystyle \lim_{\lambda \to \infty} \sup_{n \geq 1} P\{ \sup_{0 \leq s < t \leq T} \frac{|Y_n(t) - Y_n(s)|}{|t - s|^{\varepsilon}} > \lambda \} = 0;$

in fact, the martingale part of $Y_n(t)$ can be handled by transforming it to a Brownian motion by a time change and then by using a well-known result of Lévy on the modulus of uniform continuity of Brownian paths. The bounded variation part can be treated directly. Since $(X_n(t), \Phi_n(t))$ is a solution of the Skorohod equation $X_n(t) = Y_n(t) + \Phi_n(t)$, we have by Theorem 4.2

(5.5) $|\Phi_n|_t^s \leq K_n \Delta_{s,t}(Y_n), \quad 0 \leq s < t \leq T.$

K_n is random but can be controlled by the modulus of uniform continuity of $Y_n(t)$. Thus it follows from (5.4) that

(5.6) $\displaystyle \lim_{\lambda \to \infty} \sup_{n \geq 1} P\{ K_n > \lambda \} = 0.$

(5.4) implies that $P\{\Delta_{0,T,\varepsilon}(Y_n) > \lambda\} \to 0$ uniformly in n as $\varepsilon \downarrow 0$ for each $\lambda > 0$. Combining this with (5.5) and (5.6) we have $P\{\Delta_{0,T,\varepsilon}(\Phi_n) > \lambda\} \to 0$ and hence $P\{\Delta_{0,T,\varepsilon}(X_n) > \lambda\} \to 0$ uniformly in n as $\varepsilon \downarrow 0$. Therefore $\{P_n, n \geq 1\}$ is tight.

By Lemma 5.1 there exists a subsequence $n_1 < n_2 < \cdots$ such that P_{n_k} converges weakly as $k \to \infty$. To simplify the notation we assume that P_n itself converges weakly as $n \to \infty$. Then by Skorohod's realization theorem of almost sure convergence we can find, on a suitable probability space $(\widetilde{\Omega}, \widetilde{\mathfrak{F}}, \widetilde{P})$, a sequence of processes $(\widetilde{B}_n(t), \widetilde{X}_n(t), \widetilde{Y}_n(t))$, $n \geq 1$, satisfying the following two conditions.

(5.7)   For each   n   the process $(\widetilde{B}_n(t), \widetilde{X}_n(t), \widetilde{Y}_n(t))$, $0 \leq t \leq T$,

is equivalent in law to the process $(B(t), X_n(t), Y_n(t))$,

$0 \leq t \leq T$.

(5.8) $\widetilde{B}_n(t)$, $\widetilde{X}_n(t)$ and $\widetilde{Y}_n(t)$ converge uniformly in t (a.s.)

as $n \to \infty$ to some processes $\widetilde{B}(t)$, $\widetilde{X}(t)$ and $\widetilde{Y}(t)$

respectively.

We then define $\widetilde{\Phi}_n(t)$ and $\widetilde{\Phi}(t)$ by

(5.9) $\widetilde{X}_n(t) = \widetilde{Y}_n(t) + \widetilde{\Phi}_n(t)$, $\lim_{n \to \infty} \widetilde{\Phi}_n(t) = \widetilde{\Phi}(t)$.

From (5.3) and (5.7) it follows that

(5.10) $\widetilde{Y}_n(t) = \widetilde{X}(0) + \int_0^t \sigma(\widetilde{X}_n(h_n(s))) d\widetilde{B}_n(s) + \int_0^t b(\widetilde{X}_n(h_n(s))) ds$

and also that the first equation of (5.9) is an equation of Skorohod

type.

Lemma 5.2. $(\widetilde{X}(t), \widetilde{\Phi}(t))$ is a solution of the Skorohod SDE

(5.11) $\widetilde{X}(t) = \widetilde{X}(0) + \int_0^t \sigma(\widetilde{X}(s)) d\widetilde{B}(s) + \int_0^t b(\widetilde{X}(s)) ds + \widetilde{\Phi}(t)$.

Proof. Approximating stochastic integrals by Riemann-Stieltjes
sums, we can easily prove

$\int_0^t \sigma(\widetilde{X}_n(h_n(s))) d\widetilde{B}_n(s) \to \int_0^t \sigma(\widetilde{X}(s)) d\widetilde{B}(s)$ in probability, $n \to \infty$,

for each fixed t. Therefore, making n tend to ∞ in the first equation of (5.9), we obtain (5.11).

In the following lemma r_0, β and δ are the constants appearing in Conditions (A) and (B).

Lemma 5.3 (cf.[2: Remark 3.1]). D satisfies Condition (C') of §1 with $\gamma = 2r_0\beta^{-1}$.

Proof. Let $\underline{\ell} = \underline{\ell}_{x_0}$ be the unit vector appearing in Condition (B) and set $f(x) = \langle\underline{\ell}, x - x_0\rangle$. Then we have $\nabla f(x) = \underline{\ell}$. By Condition (B) and (A), we have

$$\langle y - x, \underline{n}\rangle + \frac{1}{\gamma}\langle\nabla f(x), \underline{n}\rangle |y - x|^2$$

$$= \langle y - x, \underline{n}\rangle + \frac{\beta}{2r_0}\langle\ell, \underline{n}\rangle |y - x|^2$$

$$\geq \langle y - x, \underline{n}\rangle + \frac{1}{2r_0}|y - x|^2 \geq 0, \quad \forall x \in B(x_0, \delta) \cap \partial D,$$
$$\forall y \in B(x_0, \delta) \cap \overline{D}, \quad \forall_{\underline{n}} \in \mathcal{N}_x,$$

completing the proof.

Let X_t and X_t' be solutions of (5.1) with the same initial value. Suppose that the supports of the coefficients σ and b are included in $B(x_0, \delta)$ for some $x_0 \in \partial D$. Then

$$\langle X_s - X_s', d\Phi_s - d\Phi_s'\rangle - \frac{1}{\gamma}|X_s - X_s'|^2\langle\underline{\ell}, d\Phi_s + d\Phi_s'\rangle$$

$$= - \{ \langle X'_s - X_s, d\Phi_s \rangle + \frac{1}{\gamma} |X_s - X'_s|^2 \langle \underline{\ell}, d\Phi_s \rangle \}$$

$$- \{ \langle X_s - X'_s, d\Phi'_s \rangle + \frac{1}{\gamma} |X_s - X'_s|^2 \langle \underline{\ell}, d\Phi'_s \rangle \}$$

$$\leq 0.$$

Therefore, making use of Condition (C') and employing a method similar to [2: pp.524-525], we can prove

$$E\{ \|X - X'\|_t^4 \} \leq Const. \int_0^t E\{ \|X - X'\|_s^4 \} ds,$$

which implies $X_t = X'_t$, $t \geq 0$. Thus we have the pathwise uniqueness of solutions of (5.1) under the restricted condition on σ and b. Combining this with Lemma 5.2, we have the following lemma by an arguement similar to [1: Theorem IV-1.1].

Lemma 5.4. Suppose that the supports of σ and b are included in $B(x_0, \delta)$ for some $x_0 \in \partial D$. Then there is a unique strong solution of (5.1).

Lemma 5.5. For any $x_0 \in R^d$, the pathwise uniqueness of solutions of (5.1) with initial value x_0 holds for $0 \leq t \leq \tau$ where τ is the exit time of $B(x_0, r)$. More precisely, if $X_i(t)$, $i = 1,2$ are $\mathcal{F}_t$-adapted solutions of (5.1) and if $\tau_i = \inf\{t > 0: X_i(t) \notin B(x_0, r)\}$, $i = 1,2$, then $\tau_1 = \tau_2$ and $X_1(t) = X_2(t)$ for $0 \leq t \leq \tau_1$, a.s.

Proof. For $0 < r < \delta$ we define Lipschitz continuous func-

tions $\hat{\sigma}: \overline{D} \to R^d \otimes R^d$ and $\hat{b}: \overline{D} \to R^d$ such that

$$\hat{\sigma}(x) = \begin{cases} \sigma(x) & \text{if } x \in B(x_0, r), \\ \\ 0 & \text{if } x \notin B(x_0, \delta), \end{cases}$$

$$\hat{b}(x) = \begin{cases} b(x) & \text{if } x \in B(x_0, r), \\ \\ 0 & \text{if } x \notin B(x_0, \delta), \end{cases}$$

and then consider the following Skorohod SDE:

$$(5.12) \qquad d\hat{Y}(t) = \hat{\sigma}(\hat{Y}(t))dB(t) + \hat{b}(\hat{Y}(t))dt + d\hat{\Psi}(t).$$

Lemma 5.4 implies that (5.12) has a unique strong solution. Let $X(t)$ be a solution of (5.1) starting at x_0 and put $\tau = \inf\{t > 0: X(t) \notin B(x_0, r)\}$. We may assume $\tau < \infty$ a.s.; otherwise, it is enough to consider $\tau \wedge n$. Next, define $\hat{B}(u) = B(\tau + u) - B(\tau)$ and $\hat{\mathcal{F}}_u = \mathcal{F}_{\tau+u} = \{\Lambda \in \mathcal{F}: \Lambda \cap \{\tau + u \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}$, $u \geq 0$. Then it is easy to see that $\hat{B}$ is an $\hat{\mathcal{F}}_u$-Brownian motion and that $\hat{\tau} = t - \tau \wedge t$ is an $\hat{\mathcal{F}}_u$-stopping time for each fixed $t \geq 0$. Since (5.12) has a unique strong solution,

$$(5.12') \qquad \hat{Y}(t) = X(\tau) + \int_0^t \hat{\sigma}(\hat{Y}(s))d\hat{B}(s) + \int_0^t \hat{b}(\hat{Y}(s))ds + \hat{\Psi}(t)$$

has a unique $\hat{\mathcal{F}}_u$-adapted solution. We put

$$Y(t) = \begin{cases} X(t) & , \ t \leq \tau, \\ \\ \widehat{Y}(t - \tau) & , \ t > \tau. \end{cases}$$

Then, the uniqueness assertion in the lemma follows from the pathwise uniqueness for (5.12) once we prove that Y is a solution of (5.12). Setting $\widehat{M}(t) = \int_0^t \widehat{\sigma}(\widehat{Y}(s))d\widehat{B}(s)$ and then noting that $\widehat{\tau}$ is an $\widehat{\mathscr{F}}_u$-stopping time, we have

$$\mathbf{1}_{\{\tau<t\}}\widehat{M}(\widehat{\tau}) = \mathbf{1}_{\{\tau<t\}}\int_0^\infty \mathbf{1}_{\{s\leq\widehat{\tau}\}}\widehat{\sigma}(\widehat{Y}(s))d\widehat{B}(s)$$

$$= \lim_{n\to\infty} \mathbf{1}_{\{\tau<t\}}\sum_{k=1}^\infty \mathbf{1}_{\{(k-1)2^{-n}\leq\widehat{\tau}\}}\widehat{\sigma}(\widehat{Y}((k-1)2^{-n}))$$

$$\cdot\{\widehat{B}(k2^{-n}) - \widehat{B}((k-1)2^{-n})\}$$

$$= \lim_{n\to\infty} \mathbf{1}_{\{\tau<t\}}\sum_{k=1}^\infty \mathbf{1}_{\{\tau+(k-1)2^{-n}\leq t\}}\widehat{\sigma}(Y(\tau + (k-1)2^{-n}))$$

$$\cdot\{B(\tau + k2^{-n}) - B(\tau + (k-1)2^{-n})\}$$

$$= \mathbf{1}_{\{\tau<t\}}\int_0^\infty \mathbf{1}_{\{\tau<s\leq t\}}\widehat{\sigma}(Y(s))dB(s),$$

and hence

$$Y(t) = \mathbf{1}_{\{t\leq\tau\}}Y(t) + \mathbf{1}_{\{t>\tau\}}Y(t)$$

$$= \mathbf{1}_{\{t\leq\tau\}}Y(t) + \mathbf{1}_{\{t>\tau\}}[X(\tau) + \int_0^\infty \mathbf{1}_{\{\tau<s\leq t\}}\widehat{\sigma}(Y(s))dB(s)$$

$$+ \int_0^\infty \mathbf{1}_{\{\tau<s\leq t\}}\widehat{b}(Y(s))ds + \widehat{\Psi}(t - \tau)]$$

$$= \mathbb{1}_{\{t \leq \tau\}} [x_0 + \int_0^t \hat{\sigma}(Y(s))dB(s) + \int_0^t \hat{b}(Y(s))ds + \Phi(t)]$$

$$+ \mathbb{1}_{\{t > \tau\}} [X(\tau) + \int_0^t \mathbb{1}_{\{\tau < s\}} \hat{\sigma}(Y(s))dB(s)$$

$$+ \int_0^t \mathbb{1}_{\{\tau < s\}} \hat{b}(Y(s))ds + \hat{\Psi}(t - \tau)]$$

$$= x_0 + \int_0^t \hat{\sigma}(Y(s))dB(s) + \int_0^t \hat{b}(Y(s))ds + \Psi(t),$$

where

$$\Psi(t) = \begin{cases} \Phi(t) & , \ t \leq \tau, \\ \\ \hat{\Psi}(t - \tau) + \Phi(\tau) & , \ t > \tau. \end{cases}$$

Thus Y solves (5.12). The proof of Lemma 5.5 is finished.

From Lemma 5.5 we can easily prove the following lemma.

Lemma 5.6. The pathwise uniqueness of solutions of (5.1) holds.

Using Lemma 5.2 and Lemma 5.6 we can finally prove Theorem 5.1 by a method similar to [1: Theorem IV-1.1].

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