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Fourier Coefficients of L^p Functions on SU(1,1)

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Takeshi Kawazoe

Takeshi Kawazoe
Department of Mathematics, Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

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§1. Introduction.

Let G be SU(1,1) and $A^{p,r}(D)$ ($0<p<\infty$ and $r \in \mathbf{R}$) the weighted Bergman space on the unit disc $D=\{z \in \mathbf{C};\ |z|<1\}$ (cf. §2.1). For $n \in \frac{1}{2}\mathbf{Z}$ and $n \geqq 1$ we define the operators $T_n(g)$ ($g \in G$) on the Hilbert spaces $A^{2,n-1}(D)$ by

$$T_n(g)F(z)=(\overline{\alpha}z+\overline{\beta})^{-2n}F\left(\frac{\alpha z+\beta}{\overline{\beta}z+\overline{\alpha}}\right),\ \text{where } g^{-1}=\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix} \in G \text{ and } F \in A^{2,n-1}(D).$$

Then $(T_n, A^{2,n-1}(D))$ ($n \in \frac{1}{2}\mathbf{Z}$ and $n \geqq 1$) are irreducible unitary representations of G called the holomorphic discrete series of G (cf. [Su],P.235). Now we define the matrix coefficients of $T_n(g)$ by

$$\tilde{f}^n_{pq}(g)=[T_n(g)e^n_q, e^n_p]_{n-1} \qquad (g \in G),$$

where $\{e^n_m;\ m \in \mathbf{N}\}$ is the complete orthonormal system of $A^{2,n-1}(D)$ satisfying $T_n(k_\theta)e^n_m=e^{-(n+m)\theta}e^n_m$ for all $k_\theta=\mathrm{diag}(e^{i\theta/2},e^{-i\theta/2})$ ($0\leqq\theta<4\pi$) and $[\ ,\ ]_{n-1}$ is the obvious inner product of $A^{2,n-1}(D)$. For $n \in \frac{1}{2}\mathbf{Z}$ and $n \leqq -1$ we put $\tilde{f}^n_{pq}=\overline{\tilde{f}^{-n}_{pq}}$. These are the matrix coefficients of the anti-holomorphic discrete series of G (cf. [Su],P.318). Then by the Plancherel formula on G we see that $L^2(G)$, the space of all square integrable functions with respect to a G-invariant Haar measure dg on G, can be written as the direct sum of the continuous (principal) part $L^2_p(G)$, which consists of wave packets, and the discrete part $°L^2(G)$, which is the L^2-span of $\{\tilde{f}^n_{pq};\ n \in \frac{1}{2}\mathbf{Z},\ |n| \geqq 1$ and $p,\ q \in \mathbf{N}\}$, that is,

$$L^2(G)=L^2_p(G) \oplus °L^2(G) \quad \ni \quad f=f_p + °f.$$

Here we put $f^n_{pq}=\tilde{f}^n_{pq}/\|\tilde{f}^n_{pq}\|_2$ and for $f \in L^2(G)$,

$$\hat{f}(n;p,q)=\int_G f(g)\overline{f^n_{pq}(g)}dg \qquad (n \in \tfrac{1}{2}\mathbf{Z},\ |n| \geqq 1 \text{ and } p,\ q \in \mathbf{N}).$$

Obviously these are the Fourier coefficients of the Fourier series of $°L^2(G)$ and $°f=\sum_{n,p,q}\hat{f}(n;p,q)f^n_{pq}$ (cf. [EM]). Moreover in the case of $n \geqq 1$, by the Parseval's formula on $°L^2(G)$ or equivalently, by the reproducing formula on

*) Not for review. A generalization of this report will be submitted for publication elsewhere.

$A^{2,n-1}(D)$, we see that the mapping Φ_m^n ($m \in \mathbf{N}$) defined by

$$\begin{array}{ccc} L^2\text{-span of } \{f_{\ell m}^n ;\ \ell \in \mathbf{N}\} & \xrightarrow{\Phi_m^n} & A^{2,n-1}(D) \\ \omega & & \omega \\ f(g) = \sum_{\ell=0}^{\infty} \hat{f}(n;\ell,m) f_{\ell m}^n(g) & \longmapsto & F(z) = \int_G \overline{f}(g) T_n(g) e_m^n(z) dg \\ & & = C \sum_{\ell=0}^{\infty} \overline{\hat{f}}(n;\ell,m) e_m^n(z) \end{array}$$

is bijective and norm-preserving in the sense that the L^2-norm of f on G equals to the $A^{2,n-1}(D)$-norm of $F(z)=\Phi_m^n(f)$ up to a constant. Thus the Fourier coefficients $\hat{f}(n;\ell,m)$ ($\ell \in \mathbf{N}$) of the L^2-span of $\{f_{\ell m}^n ;\ \ell \in \mathbf{N}\}$ are determined by the right hand side, that is, they correspond to the Taylor coefficients of $A^{2,n-1}(D)$ (see Theorem 5.1).

The purpose of this paper is to characterize the Fourier coefficients $\hat{f}(n;\ell,0)$ ($\ell \in \mathbf{N}$) for f in $L^p(G)$ ($1 \leqq p \leqq 2$). As in the case of p=2, if $(n,p) \neq (1,1)$, we see easily that these coefficients correspond to the Taylor coefficients of $A^{p,np/2-1}(D)$ instead of $A^{2,n-1}(D)$ (see Theorem 8.1). However, when $(n,p)=(1,1)$, this characterization is nonsense, because the spaec $A^{1,-1/2}(D)$ vanishes. Therefore we need another argument.

Now we note that for a fixed $g \in G$ each $T_n(g)e_m^1(z)$ belongs to the Hardy space $H^1(D)$ (cf. §2.1) and the H^1-norm is bounded uniformly on $g \in G$. Thus it is easy to see that

$$\Phi_0^1(L^1(G)) \subset H^1(D).$$

Then the Fourier coefficients $\hat{f}(1;\ell,0)$ ($\ell \in \mathbf{N}$) must satisfy the same property of the Taylor coefficients of $H^1(D)$, for example, they satisfy the Hardy's inequality (cf. [D], p.48). In particular, we can obtain an order of $\hat{f}(1;\ell,0)$ as $\ell \to \infty$. In §7 we shall give an exact characterization of $\Phi_0^1(L^1(G))$ and show that the order is best possible. To do this we shall use the fractional derivatives and the fractional integrals of holomorphic functions on D. Actually $\Phi_0^1(L^1(G))$ coinsides with the following space on D.

$$H_0^1(D) = \{ F \ ;\ \text{holomorphic on D and } \| z^{-1}(zF(z))^{[2r+1]} \|_{1,r} < \infty \} \ (r > -\tfrac{1}{2}),$$

where $\|\cdot\|_{1,r}$ is the norm of the Bergman space $A^{1,r}(D)$ and for a holomorphic function F on D $F^{[\alpha]}$ is the fractional derivative of F of order α (see §9.2).

§2. Preliminaries.

2.1. Functions on the unit disc. Let D denote the unit disc $\{z \in \mathbf{C}; |z| < 1\}$. For a function F on D we put

$$\|F\|_{p,r}=\left(\frac{1}{\pi}\int_D |F(z)|^p(1-|z|^2)^{2r}dz\right)^{1/p},$$

$$\|F\|_{H^p} = \lim_{r\to -\frac{1}{2}} (2r+1)\|F\|_{p,r},$$

$$\|F\|_{\infty} = \sup_{z\in D} |F(z)|,$$

where $0<p<\infty$, $r \in \mathbf{R}$ and dz is the Euclidean measure on D. Let $L^{p,r}(D)$, $L_p(D)$ and $L^{\infty}(D)$ denote the spaces of all functions F on D with finite $\|F\|_{p,r}$, $\|F\|_{H^p}$ and $\|F\|_{\infty}$ respectively. Let A(D) denote the space of all analytic functions on D. Then the weighted Bergman spaces $A^{p,r}(D)$ and the Hardy spaces $H^p(D)$ are defined as the intersection of A(D) and $L^{p,r}(D)$ and one of A(D) and $L_p(D)$ respectively. Obviously $A^{p,r}(D)=\{0\}$ for $r \leqq -\frac{1}{2}$ and $\|F\|_{H^p}=\sup_{0\leqq r<1} \left(\frac{1}{\pi}\int_0^{2\pi}|F(re^{i\theta})|^p d\theta\right)^{1/p}$. For $1\leqq p<\infty$, these spaces are Banach spaces in the above norms, especially, $A^{2,r}(D)$ and $H^2(D)$ are Hilbert spaces. We denote the inner product of $A^{2,r}(D)$ $(r > -\frac{1}{2})$ by

$$[F,G]_r= \frac{1}{\pi}\int_D F(z)\overline{G(z)}(1-|z|^2)^{2r}dz.$$

Then

$$e_\ell^{r+1}(z)=B(\ell+1,2r+1)^{-\frac{1}{2}}z^\ell=\left(\frac{\Gamma(\ell+2+2r)}{\Gamma(\ell+1)\Gamma(2r+1)}\right)^{\frac{1}{2}}z^\ell \qquad (\ell \in \mathbf{N})$$

is a complete orthonormal system of $A^{2,r}(D)$. Moreover for all F in $A^{2,r}(D)$ the following reproducing formula holds (cf. [FR]):

$$F(z)=\frac{(2r+1)}{\pi}\int_D F(\zeta)(1-\bar{\zeta}z)^{-2(1+r)}(1-|\zeta|^2)^{2r}d\zeta. \tag{2.1}$$

Now let F be in $L^{p,r}(D)$ ($0<p<\infty$ and $r \in \mathbf{R}$). Then for each fixed $0<\eta<1$ $F(\eta e^{i\theta})$ belongs to $L^p([0,2\pi))$ and thus, has the unique Fourier series $F(\eta e^{i\theta}) \sim \sum_{\ell=-\infty}^{\infty} b_\ell(\eta)e^{i\ell\theta}$. Then putting $a_\ell(|z|)=b_\ell(|z|)|z|^{-\ell}$ ($\ell \in \mathbf{N}$), we see that F(z) has the unique expansion $F(z) \sim \sum_{\ell=-\infty}^{\infty} a_\ell(|z|)z^\ell$. Here we put $L_+^{p,r}(D)=\{F \in L^{p,r}(D);\ F(z) \sim \sum_{\ell=0}^{\infty} a_\ell(|z|)z^\ell\}$. The we have

$$A^{p,r}(D) \subset L_+^{p,r}(D) \subset L^{p,r}(D).$$

2.2. Functions on SU(1,1). Let G be $SU(1,1)=\{g \in M_2(\mathbf{C});\ {}^t\bar{g}\begin{pmatrix}1&0\\0&-1\end{pmatrix}g=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and $\det g=1\}$ and G=KAN the Iwasawa decomposition of G, where the subgroups K, A and N of G are given by

$$K=\left\{ k_\theta=\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2}\end{pmatrix};\ 0\leqq\theta<4\pi \right\},$$

$$A=\left\{ a_t=\begin{pmatrix} \mathrm{ch}\, t/2 & \mathrm{sh}\, t/2 \\ \mathrm{sh}\, t/2 & \mathrm{ch}\, t/2\end{pmatrix};\ t \in \mathbf{R} \right\},$$

$$N=\left\{ n_x=\begin{pmatrix} 1+ix/2 & -ix/2 \\ ix/2 & 1-ix/2\end{pmatrix};\ x \in \mathbf{R} \right\}.$$

Then the Haar measure dg of G is given by the following integral formulas according to the decompositions G=KAN and $G=KCL(A^+)K$, where $A^+=\{a_t;\ t > 0\}$. For any continuous functions f on G with compact support

$$\begin{aligned}\int_G f(g)dg &= \frac{1}{4\pi}\int_0^{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(k_\theta a_t n_x)e^t d\theta dt dx \\ &= \frac{1}{8\pi}\int_0^{4\pi}\int_0^{+\infty}\int_0^{4\pi} f(k_\theta a_t k_{\theta'})\,\mathrm{sh}\, t\, d\theta dt d\theta',\end{aligned} \tag{2.2}$$

where $d\theta$, $d\theta'$ and dt, dx denote the Euclidean measures on $[0,4\pi)$ and $\mathbf{R}$ respectively (cf. [Su], pp. 251-252). For a function f on G we put

$$\| f\|_p=\left(\int_G |f(g)|^p dg\right)^{1/p} \quad (0<p<\infty),$$

$$\| f\|_\infty= \sup_{g \in G} |f(g)|.$$

Let $L^p(G)$ $(0<p\leqq\infty)$ denote the space of all functions f on G with finite $\|f\|_p$. Then for $1\leqq p\leqq\infty$ these spaces are Banach spaces in the obvious norms, especially, $L^2(G)$ is a Hilbert space. We denote the inner product of $L^2(G)$ by

$$(f,h)=\int_G f(g)\overline{h(g)}dg.$$

By the Plancherel formula for $L^2(G)$ (cf. [Su], p.344) each f in $L^2(G)$ can be written as the sum of the integral part f_P of f and the discrete part $°f$ of f, that is,

$$L^2(G)=L^2_P(G)\oplus °L^2(G)\ni f=f_P+°f.$$

Let $n\in\frac{1}{2}\mathbf{Z}$. Then $\chi_n(k_\theta)=e^{in\theta}$ $(0\leqq\theta<4\pi)$ is an irreducible unitary representation of K. For p, $q\in\frac{1}{2}\mathbf{Z}$ we say that a function f on G is of (p,q)-type if it satisfies

$$f(k_\theta g k_{\theta'})=\chi_p(k_\theta)f(g)\chi_q(k_{\theta'}) \qquad (k_\theta,\ k_{\theta'}\in K \text{ and } g\in G).$$

Here we put $E_pf(g)=E^L_pf(g)=\frac{1}{4\pi}\int_0^{4\pi}\chi_p(k_\theta)f(k_\theta g)d\theta$ and $fE_q(g)=E^R_qf(g)=\frac{1}{4\pi}\int_0^{4\pi}\chi_q(k_\theta)f(gk_\theta)d\theta$ $(g\in G)$. Then obviously E^L_p and E^R_q are projections to the left (-p)-type and the right (-q)-type respectively. E_pfE_q is of (-p,-q)-type. We put for $n\in\frac{1}{2}\mathbf{Z}$ and $0<p\leqq\infty$,

$$L^p_n(G)=\{f\in L^p(G);\ f=\mathbf{E}_nf\},\ \text{where } \mathbf{E}_n=\sum_{k=n}^{\infty}E^L_n\ (n\geqq0) \text{ and } =\sum_{k=n}^{-\infty}E^L_n\ (n<0).$$

Then it is easy to see that

$$L^p_n(G)\subset \mathbf{E}_nL^p(G).$$

When p=2, noting that $\|f\|_2^2=\sum_{p,q\in\frac{1}{2}\mathbf{Z}}\|E_pfE_q\|_2^2$ for $f\in L^2(G)$, we see that

$$L^2_n(G)=\mathbf{E}_nL^2(G).$$

2.3. Discrete series of SU(1,1). The group G=SU(1,1) acts on the unit disc D by the linear transformations

$$z \longmapsto gz=\frac{\alpha z+\beta}{\bar{\beta}z+\bar{\alpha}}, \text{ where } g=\begin{pmatrix}\alpha & \beta\\ \bar{\beta} & \bar{\alpha}\end{pmatrix} \in G.$$

This action is transitive and the isotropy subgroup of G at 0 equals to K, that is, G/K is homeomorphic to D. We put $J(g,z)=\bar{\beta}z+\bar{\alpha}$ and define the operators $T_n(g)$ ($g \in G$) on $A^{2,n-1}(D)$ ($n \in \frac{1}{2}\mathbf{Z}$ and $n \geq 1$) as follows.

$$T_n(g)F(z)=J(g^{-1},z)^{-2n}F(g^{-1}z) \quad (z \in D) \quad \text{for } F \in A^{2,n-1}(D).$$

Then $(T_n, A^{2,n-1}(D))$ ($n \in \frac{1}{2}\mathbf{Z}$ and $n \geq 1$) are irreducible unitary representations of G called the holomorphic discrete series of G. For p, q $\in \mathbf{N}$ we put

$$\tilde{f}^n_{pq}(g)=[T_n(g)e^n_q, e^n_p]_{n-1} \qquad (g \in G) \text{ and}$$

$$f^n_{pq}(g)=\tilde{f}^n_{pq}(g)/\| \tilde{f}^n_{pq}\|_2.$$

Explicitly we see that $\| \tilde{f}^n_{pq}\|_2^2=4\pi(2n-1)^{-1}$ and for $p \geq q$

$$f^n_{pq}(g)=2^{-1}\pi^{-\frac{1}{2}}(2n-1)^{\frac{1}{2}}\left(\frac{\Gamma(p+1)\Gamma(2n+p)}{\Gamma(q+1)\Gamma(2n+q)}\right)^{\frac{1}{2}} \frac{1}{(p-q)!} (\text{th } t/2)^{p-q}(\text{ch } t/2)^{-2n}$$

$$\times F(-q,2n+p,p-q+1;(\text{th } t/2)^2)e^{-in(\theta+\theta')}e^{-ip\theta}e^{-iq\theta'},$$

where $g=k_\theta a_t k_{\theta'} \in KCL(A^+)K$ (cf. [Sa], p.91). Each f^n_{pq} is a $(-(n+p),-(n+q))$-type normalized L^2-function on G. In particular we see that

$$T_n(g)e^n_m(z)=2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}\sum_{k=0}^{\infty} f^n_{km}(g)e^n_k(z),$$
$$f^n_{\ell 0}(g)=2^{-1}\pi^{-\frac{1}{2}}\overline{e^n_\ell(g0)}(\text{ch } t/2)^{-2n}e^{-in(\theta+\theta')}, \tag{2.3}$$

where $z \in D$ and $g=k_\theta a_t k_{\theta'} \in KCL(A^+)K$. For $n \in \frac{1}{2}\mathbf{Z}$ and $n \leq -1$ we put $f^n_{pq}=\overline{f^{-n}_{pq}}$ (p, q $\in \mathbf{N}$). These are the matrix coefficients of the anti-holomorphic discrete series of G (cf. [Su], p.318). By the Plancherel formula for

$L^2(G)$ we see that for $f \in L^2(G)$

$$°L^2_\omega(G) = \text{the } L^2\text{-span of } \{f^n_{pq};\ n \in \tfrac{1}{2}\mathbf{Z},\ |n| \geq 1 \text{ and } p,\ q \in \mathbf{N}\}$$

$$°f = \sum_{\substack{n \in \frac{1}{2}\mathbf{Z},\ |n|\geq 1\\ p,q \in \mathbf{N}}} \hat{f}(n;p,q) f^n_{pq}, \tag{2.4}$$

where $\hat{f}(n;p,q)=(f,f^n_{pq})=(°f,f^n_{pq})$. We call $f(n;p,q)$'s by the *Fourier coefficients* of the Fourier series (2.4) of f on G. We define a projection $°P_n$ as follows. For $f \in L^2(G)$

$$°P_n(f)=°P_n(°f)= \sum_{p,q \in \mathbf{N}} \hat{f}(n;p,q) f^n_{pq}.$$

Obviously $°P_n(f)=°P_n(\mathbf{E}_n f \mathbf{E}_n)$ and

$$°P_n L^2(G) = \text{the } L^2\text{-span of } \{f^n_{pq};\ p,\ q \in \mathbf{N}\}.$$

We denote the extension of $°P_n \mid L^2(G) \cap L^p(G)$ to $L^p(G)$ $(0<p\leqq\infty)$ by the same letter. Then it is easy to see that

$$°P_n L^p(G) \supset \text{ the } L^p\text{-span of } \{f^n_{pq};\ p,\ q \in \mathbf{N}\}.$$

Let $n \in \frac{1}{2}\mathbf{Z}$ and $n \geqq \frac{1}{2}$. Then we see that $(T_n, H^{1/n}(D))$ is an irreducible †) representation of G and for each $g \in G$ $T_n(g)$ preserves the "norm" of $H^{1/n}(D)$. In particular, when $n=\frac{1}{2}$, $(T_{\frac{1}{2}}, H^2(D))$ is an irreducible unitary representation of G called the limit of discrete series of G (cf. [Sa]). Because $\| e^n_m \|_{H^{1/n}} = B(m+1,2n-1)^{-\frac{1}{2}}$ $(n \geqq 1$ and $m \in \mathbf{N})$ and $T_n(g)$ $(g \in G)$ preserves the norm, we see that

$$\| T_n(g) e^n_m \|_{H^{1/n}} = B(m+1,2n-1)^{-\frac{1}{2}} \quad (n \geqq 1,\ m \in \mathbf{N} \text{ and } g \in G). \tag{2.5}$$

†) When $0<p<1$, we can define the metric in $H^p(D)$ by $d(F,G)=\| F-G \|^p_{H^p}$ for $F,G \in H^p(D)$. Then these metrics turn $H^p(D)$ into F-spaces (cf. [Sh], p.187).

§3. Φ_m^n-transform.

Let $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$ and $m \in \mathbf{N}$. We note that for each $z \in D$ $|T_n(g)e_m^n(z)| = O((\mathrm{ch}\ t/2)^{-2n})$ $(g=k_\theta a_t k_{\theta'} \in KCL(A^+)K)$. Then by Hölder's inequality we can well define $\Phi_m^n(f)$ for $f \in L^p(G)$ $(1\leq p<\infty)$ as follows.

$$\Phi_m^n(f)(z)=(T_n(\cdot)e_m^n(z),f)$$
$$=2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}\sum_{\ell=0}^{\infty}\overline{\hat{f}}(n;\ell,m)e_\ell^n(z).$$

Clearly, $\Phi_m^n(f)$ belongs to $A(D)$ and

$$\Phi_m^n(f_{pq}^{n'})=\delta_{nn'}\delta_{mq}2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}e_p^n \tag{3.1}$$

for $n,n' \in \frac{1}{2}\mathbf{Z}$, $n,n' \geq 1$ and $m,p,q \in \mathbf{N}$. Moreover for $f \in L^1(G)$ we have $\|\Phi_m^n(f)\|_{2,n-1} \leq \|f\|_1$, because $T_n(g)$ preserves the norm $\|\cdot\|_{2,n-1}$.

Conversely, for F in $A^{2,n-1}(D)$ we define $\Psi_m^n(F)$ as follows.

$$\Psi_m^n(F)(g)=4^{-1}\pi^{-1}(2n-1)[T_n(g)e_m^n,F]_{n-1}$$
$$=2^{-1}\pi^{-\frac{1}{2}}(2n-1)^{\frac{1}{2}}\sum_{\ell=0}^{\infty}f_{\ell m}^n(g)[e_\ell^n,F]_{n-1}.$$

This operator is well defined, because for each $g \in G$ $T_n(g)e_m^n$ belongs to $A^{2,n-1}(D)$. We denote the extension of Ψ_m^n to the image of Φ_m^n by the same letter. Then using an approximating argument, we see that

$$\Psi_m^n \circ \Phi_m^n(f)=\sum_{\ell=0}^{\infty}\hat{f}(n;\ell,m)f_{\ell m}^n={}^{\circ}P_n(f)E_{n+m}.$$

Therefore we can obtain the following

Lemma 3.1. *Let* $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$, $m \in \mathbf{N}$ *and* $1\leq p<\infty$. *Then*

(i) $\Psi_m^n \circ \Phi_m^n={}^{\circ}P_n \circ E_{n+m}^R$ *on* $L^p(G)$,

(ii) $\Phi^n_m(L^p(G))=\Phi^n_m({}^\circ P_n L^p(G)E_{n+m})$,

(iii) Φ^n_m *is injective on* ${}^\circ P_n L^p(G)E_{n+m}$ *and the inverse is given by* Ψ^n_m.

§4. $\tilde{\Phi}^n_0$ and $[\ ]_{n-1}$-transforms.

Let $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$ and $1 \leq p < \infty$. We shall consider a decomposition of f in $L^p(G)E_n$ according to the K-types. By choosing suitable functions a_ℓ on $\mathbf{R}$, we can decompose f as follows.

$$\begin{aligned} f(g)&=f(k_\theta a_t k_{\theta'}) \\ &= \sum_{\ell=-\infty}^{\infty} 2^{-1}\pi^{-\frac{1}{2}}B(\ell+1,2n-1)^{-\frac{1}{2}}a_\ell(th\tfrac{t}{2})(ch\tfrac{t}{2})^{-2n}(th\tfrac{t}{2})^{\ell}e^{-in(\theta+\theta')}e^{-i\ell\theta}. \end{aligned}$$

We note that when $\ell \geq 0$, each term equals to $a_\ell(th\frac{t}{2})f^n_{\ell 0}(g)$ (see (2.3)). Here we put

$$\tilde{\Phi}^n_0(f)(z)=2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}}\sum_{\ell=-\infty}^{\infty}\overline{a}_\ell(|z|)B(\ell+1,2n-1)^{-\frac{1}{2}}z^\ell. \tag{4.1}$$

Then we have

$$f(g)=4^{-1}\pi^{-1}(2n-1)^{\frac{1}{2}}(ch\tfrac{t}{2})^{-2n}\overline{\tilde{\Phi}^n_0(f)(g0)}e^{-in(\theta+\theta')} \qquad (g=k_\theta a_t k_{\theta'})$$

and by (2.2)

$$\begin{aligned} \| f\|^p_p&=2(4^{-1}\pi^{-1}(2n-1)^{\frac{1}{2}})^p\int_0^{4\pi}\int_0^1|\tilde{\Phi}^n_0(re^{i\theta})|^p(1-r^2)^{np-2}rdrd\theta \\ &=2^{2-2p}\pi^{1-p}(2n-1)^{p/2}\int_D|\tilde{\Phi}^n_0(z)|^p(1-|z|^2)^{np-2}dz \\ &=c_{n,p}\|\tilde{\Phi}^n_0(f)\|^p_{p,np/2-1}. \end{aligned} \tag{4.2}$$

On the other hand, for each F in $L^{p,np/2-1}(D)$ we can easily define an f in $L^p(G)E_n$ such that $\tilde{\Phi}^n_0(f)=F$ and (4.2) holds. Therefore we see that the maping $\tilde{\Phi}_0^n$ is *norm-preserving* in the sense of (4.2) of $L^p(G)E_n$ onto $L^{p,np/2-1}(D)$. Applying the same argument to $L_n^p(G)$ and the L^p-span of $\{f_{\ell 0}^n; \ell \in \mathbf{N}\}$, we can obtain the following

<u>Proposition 4.1.</u> *Let* $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$ *and* $1 \leq p < \infty$. *Then*

$$\begin{array}{ccc} L^p(G)E_n & \xrightarrow{\tilde{\Phi}_0^n} & L^{p,np/2-1}(D) \\ \cup & & \cup \\ L_n^p(G)E_n & \xrightarrow{\tilde{\Phi}_0^n} & L_+^{p,np/2-1}(D) \\ \cup & & \cup \\ L^p\text{-span of } \{f_{\ell 0}^n; \ell \in \mathbf{N}\} & \xrightarrow{\tilde{\Phi}_0^n = \Phi_0^n} & A^{p,np/2-1}(D), \end{array}$$

where each mapping is bijective and norm-preserving.

Let $1 \leq p \leq 2$. Then for F in $L^{p,\frac{1}{2}np-1}(D)$ we define $[F]_{n-1}$ as follows.

$$[F]_{n-1}(z) = \frac{(2n-1)}{\pi} \int_D F(\zeta)(1-\bar{\zeta}z)^{-2n}(1-|\zeta|^2)^{2n-2} d\zeta. \tag{4.3}$$

Because of $\frac{1}{2}np-1 \leq n-1$ for $1 \leq p \leq 2$ this is well defined. If F is of the form in the right hand side of (4.1), we see that

$$[F]_{n-1}(z) = 4\pi^{\frac{1}{2}}(2n-1)^{\frac{1}{2}} \sum_{\ell=0}^{\infty} B(\ell+1,2n-1)^{-\frac{1}{2}} \frac{\int_0^1 \bar{a}_\ell(r) r^{2\ell+1}(1-r^2)^{2n-2} dr}{(2n-1)B(\ell+1,2n-1)} z^\ell.$$

In particular, we have

<u>Lemma 4.2.</u> $\Phi_0^n = [\cdot]_{n-1} \circ \tilde{\Phi}_0^n$.

Now we denote the images of $L^{p,\frac{1}{2}np-1}(D)$ and $L_+^{p,\frac{1}{2}np-1}(D)$ under $[\ ]_{n-1}$ by $[L^{p,\frac{1}{2}np-1}(D)]_{n-1}$ and $[L_+^{p,\frac{1}{2}np-1}(D)]_{n-1}$ respectively. Then by Proposition 4.1, Lemma 4.2 and the reproducing formula (2.1) we obtain the following

<u>Proposition 4.3</u>. *Let* $n \in \frac{1}{2}\mathbf{Z}$, $n \geqq 1$ *and* $1 \leqq p \leqq 2$. *Then*

$$\begin{array}{l} \Phi_0^n(L^p(G)) = [L^{p,\frac{1}{2}np-1}(D)]_{n-1} \\ \quad \cup \\ \Phi_0^n(L_n^p(G)) = [L_+^{p,\frac{1}{2}np-1}(D)]_{n-1} \\ \quad \cup \\ \Phi_0^n(L^p\text{-span of } \{f_{\ell 0}^n; \ell \in \mathbf{N}\}) = A^{p,\frac{1}{2}np-1}(D). \end{array}$$

§5. <u>The case of</u> $L^2(G)$.

Let notations be as above. Then by the facts we stated in §2 and §3 we see that

$$\begin{aligned} \Phi_m^n(L^2(G)) &= \Phi_m^n({}^\circ P_n L^2(G) E_{n+m}) \\ &= \Phi_m^n(L^2\text{-span of } \{f_{\ell m}^n; \ell \in \mathbf{N}\}) \\ &= \Phi_m^n(\{\sum_{\ell=0}^\infty a_\ell f_{\ell m}^n;\ \sum_{\ell=0}^\infty |a_\ell|^2 < \infty\}) \\ &= \{2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}} \sum_{\ell=0}^\infty a_\ell e_\ell^n(z);\ \sum_{\ell=0}^\infty |a_\ell|^2 < \infty\} \\ &= \{\sum_{\ell=0}^\infty b_\ell z^\ell;\ 4^{-1}\pi^{-1}(2n-1) \sum_{\ell=0}^\infty B(\ell+1, 2n-1)|b_\ell|^2 < \infty\} \\ &= A^{2,n-1}(D). \end{aligned}$$

Therefore we have by Proposition 4.3 that

<u>Theorem 5.1</u>. *Let* $n \in \frac{1}{2}\mathbf{Z}$, $n \geqq 1$ *and* $m \in \mathbf{N}$.

(i) ${}^\circ P_n L^2(G) E_{n+m} = {}^\circ P_n L_n^2(G) E_{n+m} = L^2$-span of $\{f_{\ell m}^n;\ \ell \in \mathbf{N}\}$,

(ii) Φ_m^n: ${}^\circ P_n L^2(G) E_{n+m} \longrightarrow A^{2,n-1}(D)$ *is bijective and norm-preserving,*

(iii) $\Phi_0^n(L^2(G)) = [L^{2,n-1}(D)]_{n-1} = [L_+^{2,n-1}(D)]_{n-1} = A^{2,n-1}(D)$.

Remark 5.2. We can replace Φ_0^n in (iii) of Theorem 5.1 by Φ_m^n (see Remark 8.3).

One of our purposes is to study whether Theorem 5.1 is valid or not in the case of $L^p(G)$ $(1 \leqq p \leqq 2)$. To do this we shall consider a generalization of the $[\ ]_{n-1}$-transform in the next section.

§6. $[\ ]_\rho$-transform.

Let $r \leqq \rho$ and $0<p<\infty$. For $F \in L^{p,r}(D)$ we put

$$[F]_\rho(z)=\frac{(2\rho+1)}{\pi}\int_D F(\zeta)(1-\bar{\zeta}z)^{-2(\rho+1)}(1-|\zeta|^2)^{2\rho}d\zeta.$$

Because of $r \leqq \rho$ this is well defined and $[F]_\rho \in A(D)$. By (2.1) we have

Lemma 6.1. *Let* $r \leqq \rho$, $\rho > -\frac{1}{2}$ *and* $0<p<\infty$. *Then*

$$A^{p,r}(D)=[A^{p,r}(D)]_\rho \subset [L_+^{p,r}(D)]_\rho \subset [L^{p,r}(D)]_\rho \subset A(D).$$

Proposition 6.2.

(i) *If* $0 \leqq r<\rho$ *or* $-\frac{1}{2}<r<0$, $r+\frac{1}{2} \leqq \rho$, *then*

$$A^{1,r}(D)=[L_+^{1,r}(D)]_\rho=[L^{1,r}(D)]_\rho.$$

(ii) *If* $\rho \geqq 0$, *then*

$$[L^{1,-\frac{1}{2}}(D)]_\rho \subset H^1(D).$$

(iii) *If* $0 \leqq r<\rho$ *or* $-\frac{1}{2}<r<0$, $r+\frac{1}{2} \leqq \rho$, *then for any* $0<\varepsilon \leqq p-1$ *and* $1<p<\infty$,

$$[L^{p,r}(D)]_\rho \subset A^{p-\varepsilon,r}(D).$$

(iv) *If* $-\frac{1}{2}<r\leqq\rho$ *and* $\rho \geqq 0$, *then for any* $1\leqq p<\infty$,

$$[L^{\infty}(D)]_{\rho} \subset A^{p,r}(D).$$

(pr) (i) From Lemma 6.1 it is enough to show that $[L^{1,r}(D)]_{\rho} \subset L^{1,r}(D)$. Let F be in $L^{1,r}(D)$. Then by the definition

$$[F]_{\rho}(\xi e^{i\psi}) = \frac{(2\rho+1)}{\pi} \int_0^1 \int_0^{2\pi} F(\eta e^{i\theta})(1-\eta\xi e^{i(\psi-\theta)})^{-2(\rho+1)}(1-\eta^2)^{2\rho}\eta d\eta d\theta.$$

Therefore,

$$\| [F]_{\rho} \|_{1,r} \leqq \frac{(2\rho+1)}{\pi} \int_0^1 \int_0^{2\pi} |F(\eta e^{i\theta})| I_{\rho,r}(\eta)(1-\eta^2)^{2\rho}\eta d\eta d\theta,$$

where

$$I_{\rho,r}(\eta) = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |(1-\eta\xi e^{i\psi})|^{-2(\rho+1)}(1-\xi^2)^{2r}\xi d\xi d\psi.$$

Lemma 6.3. *Let* $0\leqq r\leqq\rho$ *or* $-\frac{1}{2}<r<0$, $r+\frac{1}{2} \leqq \rho$. *Then for* $0\leqq\eta<1$,

$$|I_{\rho,r}(\eta)| \leqq C \begin{cases} (1+(2r+1)^{-1})(1+(\rho-r)^{-1})(1-\eta^2)^{2(r-\rho)} & (r<\rho) \\ (1+(2r+1)^{-1})(1+\log(1-\eta^2)) & (r=\rho), \end{cases}$$

where C *does not depend on* r, ρ and η.

(pr) We note that $|I_{\rho,r}(\eta)| \leqq C \int_0^1 (1-\eta^2\xi^2)^{-2\rho-1}(1-\xi^2)^{2r}\xi d\xi$ (cf. [DRS], Theorem A). Therefore,

Case 1: $0\leqq\eta\leqq\frac{1}{2}$. Since $(1-\eta^2\xi^2) \geqq 3/4$, $|I_{\rho,r}(\eta)| \leqq C \int_0^1 (1-\xi^2)^{2r}\xi d\xi \leqq C(2r+1)^{-1}$.

Case 2_+: $\frac{1}{2}<\eta<1$ and $r \geqq 0$. Since $(1-\xi^2) \leqq (1-\eta^2\xi^2)$, $|I_{\rho,r}(\eta)| \leqq C \int_0^1 (1-\eta^2\xi^2)^{2(r-\rho)-1}\xi d\xi$. Thus, it is bounded by $C(\rho-r)^{-1}(1-\eta^2)^{2(r-\rho)}$, when $\rho > r$ and by $C\log(1-\eta^2)$, when $\rho=r$.

Case 2_-: $\frac{1}{2}<\eta<1$ and $r < 0$. By changing variables, we have that $|I_{\rho,r}(\eta)| \leqq C\eta^{-4r-2} \int_0^{\eta^2} x^{2r}((1-\eta^2)+x)^{-2\rho-1}dx \leqq C\eta^{-4r-2} \int_0^1 x^{2r}((1-\eta^2)+x)^{-2\rho-1}dx$. Since $\rho \geqq r+\frac{1}{2}$, $((1-\eta^2)+x)^{2(r-\rho)+1} \leqq (1-\eta^2)^{2(r-\rho)+1}$. Then the integral is bounded by $(1-\eta^2)^{2(r-\rho)+1} \int_0^1 x^{2r}((1-\eta^2)+x)^{-2(r+1)}dx = (2r+1)^{-1}(2-\eta^2)^{-(2r+1)}(1-\eta^2)^{2(r-\rho)}$.

Therefore the desired result is obtained. Q.E.D.

Then we have under the assumption of (i) that

$$\| [F]_\rho \|_{1,r} \leqq C_{\rho,r} \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |F(\eta e^{i\theta})| (1-\eta^2)^{2r} \eta d\eta d\theta$$

$$= C_{\rho,r} \| F \|_{1,r} < \infty,$$

where $C_{\rho,r} = C(2\rho+1)(1+(2r+1)^{-1})(1+(\rho-r)^{-1})$. Thus $[F]_\rho$ belongs to $L^{1,r}(D)$.

(ii) Let $\rho > 0$. If we take an $r > -\frac{1}{2}$ sufficiently small, we may assume that $\rho \geqq r+\frac{1}{2}$. Then applying the previous result to F in $L^{1,-\frac{1}{2}}(D) \subset L^{1,r}(D)$, we see that

$$\| [F]_\rho \|_{1,r} \leqq C(2\rho+1)(2r+1)^{-1} \| F \|_{1,r}.$$

In particular,

$$\| [F]_\rho \|_{H^1} = \lim_{r \to -\frac{1}{2}} (2r+1) \| [F]_\rho \|_{1,r} \leqq C(2\rho+1) \| F \|_{1,-\frac{1}{2}} < \infty.$$

Thus $[F]_\rho$ belongs to $H^1(D)$. When $\rho=0$, the assertion is clear from (2.5) and the fact that $\Phi_0^1(L^1(G)) = [L^{1,-\frac{1}{2}}(D)]_0$ (see Proposition 4.3).

(iv) Let F be in $L^\infty(D)$. Then by Lemma 6.3, we have

$$|[F]_\rho(\eta e^{i\theta})| \leqq (2\rho+1) \| F \|_\infty |I_{\rho,\rho}(\eta)|$$

$$\leqq C_\rho \| F \|_\infty (1+\log(1-\eta^2)), \text{ where } C_\rho = 2C(\rho+1).$$

Therefore if we take an $\varepsilon > 0$ such that $0<\varepsilon<2r+1$, we see that for each $1 \leqq p < \infty$,

$$\| [F]_\rho \|_{p,r}^p \leqq 2C_\rho^p \| F \|_\infty^p \int_0^1 (1+\log(1-\eta^2))^p (1-\eta^2)^\varepsilon (1-\eta^2)^{2r-\varepsilon} \eta d\eta$$

$$\leqq CC_\rho^p \| F \|_\infty^p \int_0^1 (1-\eta^2)^{2r-\varepsilon} \eta d\eta < \infty.$$

Thus $[F]_\rho$ belongs to $A^{p,r}(D)$.

(iii) We note that r and ρ under the assumption of (iii) also satisfy the assumptions of (i) and (iv). Therefore, by (i) and (iii) we see that $[\ ]_\rho \in (L^{1,r}(D), L^{1,r}(D))$ †) and $(L^\infty(D), L^{p',r}(D))$ for any $1 \leq p' < \infty$. By an interpolation argument (cf. [BL], p.17) we obtain that $[\ ]_\rho \in (L^{p,r}(D), L^{q,r}(D))$, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{p'}(1-\frac{1}{p})$. Since p' is arbitrary for $1 \leq p' < \infty$, we can obtain the desired result.

Q.E.D.

Applying Lemma 6.1 and Proposition 6.2 to the right hand side of Proposition 4.3, we can obtain from Lemma 3.1 that

<u>Corollary 6.4</u>.

(1) *If* n=1 *and* p=1, *then*

$$\Phi_0^1(L^1(G)) = [L^{1,-\frac{1}{2}}(D)]_0 \subset H^1(D).$$

(2) *If* $n \in \frac{1}{2}\mathbf{Z}$, $n > 1$ *and* $1 \leq p \leq 2$, *or* n=1 *and* p=2, *then*

(i) $°P_n L^p(G) E_n = °P_n L^p_n(G) E_n = L^p$-span of $\{f^n_{\ell 0};\ \ell \in \mathbf{N}\}$,

(ii) Φ_0^n: $°P_n L^p(G) E_n \longrightarrow A^{p,\frac{1}{2}np-1}(D)$ *is bijective and norm-preserving,*

(iii) $\Phi_0^n(L^p(G)) = [L^{p,\frac{1}{2}np-1}(D)]_{n-1} = [L_+^{p,\frac{1}{2}np-1}(D)]_{n-1} = A^{p,\frac{1}{2}np-1}(D)$.

(pr) (1) is obvious from Proposition 4.3 and Proposition 6.2 (ii). If we could prove (2) (iii), the rest of the assertions in (2) follows from Lemma 3.1 and Propositions 4.1 and 4.3. We shall prove (2) (iii) by an interpolation argument. When p=2, $[\ ]_{n-1} \in (L^{2,n-1}(D), L^{2,n-1}(D))$ by Theorem 5.1 and when p=1, $[\ ]_{n-1} \in (L^{1,\frac{1}{2}n-1}(D), L^{1,\frac{1}{2}n-1}(D))$ $(n > 1)$ by Proposition 6.2 (i). Therefore by the interpolation between these two cases (cf. [BL], p.17), we

†) "$T \in (A,B)$" means that T is a bounded linear operator of A to B.

see that $[\ ]_{n-1} \in (L^{p,\frac{1}{2}np-1}(D), L^{p,\frac{1}{2}np-1}(D))$ for $1 \leqq p \leqq 2$ and $n > 1$. In particular, $[L^{p,\frac{1}{2}np-1}(D)]_{n-1} \subset A^{p,\frac{1}{2}np-1}(D)$ by Lemma 6.1. Thus when $1 \leqq p \leqq 2$, $n > 1$, (2) (iii) follows from Proposition 4.3. The rest is obtained in Theorem 5.1.

Q.E.D.

<u>Corollary 6.5</u>. *Let* $n \in \frac{1}{2}\mathbf{Z}$ *and* $n \geqq 1$. *Then*

$$A^{1,\frac{1}{2}n-1}(D) \subset H^{1/n}(D).$$

(Pr) Let f be in $L^1(G)$. Then it is easy to see from (2.5) that $\| \Phi^n_m(f) \|_{H^{1/n}} \leqq B(m+1,2n-1)^{-\frac{1}{2}} \| f \|_1$. Therefore, $\Phi^n_0(L^1(G)) \subset H^{1/n}(D)$. The desired result is obvious from Corollary 6.4 (2).

Q.E.D.

<u>Remark 6.6</u>. If we define the metric in $H^p(D)$ $(0<p<1)$ as in the footnote in §2.3, the inclusion in Corollary 6.5 is dense. In fact $(T_n, H^{1/n}(D))$ $(n \in \frac{1}{2}\mathbf{Z}$ and $n \geqq \frac{1}{2})$ are irreducible representations of G and $\Phi^n_0(L^1(G))$ are G-invariant subspaces of $H^{1/n}(D)$: for $f \in L^1(G)$, $T_n(g)\Phi^n_0(f)=\Phi^n_0(L_g f)$, where $L_g f(x)=f(g^{-1}x)$ for $g, x \in G$. Then $L_g f \in L^1(G)$ and thus, $T_n(g)\Phi^n_0(f) \in \Phi^n_0(L^1(G))$.

§7. <u>The case of T_1</u>.

For $0<p<\infty$, $r \in R$ and $\alpha \geqq 0$, we put

$$A^{p,r}_\alpha(D) = \{F \in A(D);\ z^{-1}(zF(z))^{[\alpha]} \in A^{p,r}(D)\ \},$$

where $F^{[\alpha]}$ is the fractional derivative of $F \in A(D)$ of order α, that is, if $F(z)=\sum_{\ell=0}^{\infty} a_\ell z^\ell \in A(D)$, $F^{[\alpha]}(z)$ is defined by

$$F^{[\alpha]}(z) = \sum_{\ell=0}^{\infty} \frac{\Gamma(1+\ell+\alpha)}{\Gamma(1+\ell)} a_\ell z^\ell$$

(cf. §9). Obviously $A^{p,r}_0(D)=A^{p,r}(D)$.

Proposition 7.1. *Let* $1 \leqq p \leqq 2$, $\alpha > \frac{1}{p} - 1$ *and* $\alpha \geqq 0$. *Then*

$$\Phi_0^1(L^p(G)) = \Phi_0^1(L_1^p(G)) = A_\alpha^{p,\frac{1}{2}p(1+\alpha)-1}(D).$$

In particular, if $p > 1$, *we can take* $\alpha = 0$ *and thus the right hand side equals to* $A^{p,\frac{1}{2}p-1}(D)$.

(Pr) Case 1: p=1. Let $\alpha > 0$. We shall prove that $\Phi_0^1(L^1(G)) \subset A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)$. First we note that

$$z^{-1}(zT_1(g)e_0^1(z))^{[\alpha]}(z) = \frac{\Gamma(\alpha+2)(1-\eta^2)e^{-i(\theta+\theta')}}{(1-\eta e^{i\theta}z)^{\alpha+2}},$$

where $g = k_\theta a_t k_{\theta'} \in KCL(A^+)K$ and $\eta = \mathrm{th}\frac{t}{2}$. Then we see from Lemma 6.3 that

$$\| z^{-1}(zT_1(g)e_0^1(z))^{[\alpha]} \|_{1,\frac{1}{2}\alpha-\frac{1}{2}} \leqq (1-\eta^2)|I_{\frac{1}{2}\alpha,\frac{1}{2}(\alpha-1)}(\eta)|\Gamma(\alpha+2)$$

$$\leqq C_\alpha < \infty, \quad \text{where } C_\alpha = C(1+\alpha^{-1})\ \Gamma(\alpha+2).$$

Therefore for $f \in L^1(G)$ we have

$$\| z^{-1}(z\Phi_0^1(f)(z))^{[\alpha]} \|_{1,\frac{1}{2}\alpha-\frac{1}{2}} \leqq (|f|, \| z^{-1}(zT_1(\cdot)e_0^1(z))^{[\alpha]} \|_{1,\frac{1}{2}\alpha-\frac{1}{2}})$$

$$\leqq C_\alpha \| f \|_1 < \infty.$$

This means that $\Phi_0^1(f)$ belongs to $A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)$, because $\Phi_0^1(f)$ belongs to $A(D)$.

Next we shall prove that $A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D) \subset \Phi_0^1(L_1^1(G))$. Let $F(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell$ be in $A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)$ and we put

$$°f(g) = 4^{-1}\pi^{-1}(\mathrm{ch}\tfrac{t}{2})^{-2}\overline{F}(g0)e^{-i(\theta+\theta')}$$

$$= 4^{-1}\pi^{-1}(1-\eta^2)\Big(\sum_{\ell=0}^{\infty} \overline{a}_\ell \eta^\ell e^{-i\ell\theta}\Big)e^{-i(\theta+\theta')},$$

where $g = k_\theta a_t k_{\theta'}$ and $\eta = \mathrm{th}\frac{t}{2}$. Then we see that $\Phi_0^1(°f) = F$ (see §3 and §4). Here we define two functions f and f_P on G as follows.

$$f(g)=4^{-1}\pi^{-1}(1-\eta^2)^{1+\alpha}\Big(\sum_{\ell=0}^{\infty}\frac{\Gamma(\ell+\alpha+2)}{\Gamma(1+\alpha)\Gamma(\ell+2)}\,\bar{a}_\ell\eta^\ell e^{-i\ell\theta}\Big)e^{-i(\theta+\theta')}$$

and

$$f_p(g)=f(g)-{}^\circ f(g)$$

$$=4^{-1}\pi^{-1}(1-\eta^2)\Big(\sum_{\ell=0}^{\infty}\bar{a}_\ell\Big(\frac{\Gamma(\ell+2+\alpha)}{\Gamma(1+\alpha)\Gamma(\ell+2)}(1-\eta^2)^{\alpha}-1\Big)\eta^\ell e^{-i\ell\theta}\Big)e^{-i(\theta+\theta')}.$$

Then because of $\int_0^1\Big(\frac{\Gamma(\ell+2+\alpha)}{\Gamma(1+\alpha)\Gamma(\ell+2)}(1-\eta^2)^{\alpha}-1\Big)\eta^{2\ell+1}d\eta=0$ for all $\ell\in\mathbf{N}$ we see that $\Phi_0^1(f_p)=0$, that is,

$$\Phi_0^1(f)=\Phi_0^1({}^\circ f)=F.$$

On the other hand,

$$\|f\|_1=2\int_0^1\int_0^{4\pi}4^{-1}\pi^{-1}(1-\eta^2)^{1+\alpha}\Gamma(1+\alpha)^{-1}|z^{-1}(zF(z))^{[\alpha]}(\eta e^{i\theta})|(1-\eta^2)^{-2}\eta d\eta d\theta$$

$$=\Gamma(1+\alpha)^{-1}\pi^{-1}\int_D|z^{-1}(zF(z))^{[\alpha]}(z)|(1-|z|^2)^{\alpha-1}dz$$

$$=\Gamma(1+\alpha)^{-1}\|z^{-1}(zF(z))^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}}<\infty.$$

Therefore, we see that f belongs to $L_1^1(G)$ and $\Phi_0^1(f)=F$.
Then we obtain that $\Phi_0^1(L_1^1(G))=A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)$. The desired result follows from Lemma 6.1.

Case 2: $1<p\leq 2$. By the same argument in Case 1 we see that $A_\alpha^{p,\frac{1}{2}p(1+\alpha)-1}(D)\subset\Phi_0^1(L_1^p(G))$. Therefore it is enough to show that $\Phi_0^1(L^p(G))\subset A_\alpha^{p,\frac{1}{2}p(1+\alpha)-1}(D)$ for $\alpha>\frac{1}{p}-1$. First we shall prove the following lemma.

<u>Lemma 7.2</u>. *Let* $\beta\geq 0$ *and* F *be in* $A^{2,0}(D)$. *Then* $z^{-1}(zF(z))^{[\beta]}$ *belongs to* $A^{2,\beta}(D)$.

(Pr) Let $F(z)=\sum_{\ell=0}^{\infty}a_\ell z^\ell\in A^{2,0}(D)$. This means that $\sum_{\ell=0}^{\infty}B(\ell+1,1)|a_\ell|^2<\infty$. If we put $z^{-1}(zF(z))^{[\beta]}(z)=\sum_{\ell=0}^{\infty}b_\ell z^\ell$, then $b_\ell=\frac{\Gamma(\ell+2+\beta)}{\Gamma(\ell+2)}a_\ell$ $(\ell\in\mathbf{N})$. Thus we have $\sum_{\ell=0}^{\infty}B(\ell+1,1)\big(\frac{\Gamma(\ell+2)}{\Gamma(\ell+2+\beta)}\big)^2|b_\ell|^2<\infty$. Here we note that $\frac{\Gamma(\ell+2\beta+2)\Gamma(\ell+2)}{\Gamma(\ell+2+\beta)\Gamma(\ell+2+\beta)}$ $(\ell\in\mathbf{N})$ are bounded below uniformly on ℓ. Therefore $\sum_{\ell=0}^{\infty}B(\ell+1,2\beta+1)|b_\ell|^2<\infty$. This means that $z^{-1}(zF(z))^{[\beta]}$ belongs to $A^{2,\beta}(D)$. Q.E.D.

Now we define an operator Φ_α by $\Phi_\alpha(f)(z)=z^{-1}(z\Phi_0^1(f)(z))^{[\alpha]}(z)$ for a function f on G. Then $\Phi_\alpha \in (L^1(G), L^{1,\frac{1}{2}\alpha-\frac{1}{2}}_\alpha(D))$ by Case 1 and $\Phi_\alpha \in (L^2(G), L^{2,\alpha}(D))$ by Theorem 5.1 and Lemma 7.2. Therefore by the interpolation between these two cases we see that $\Phi_\alpha \in (L^p(G), L^{p,\frac{1}{2}p(1+\alpha)-1}(D))$ for $1\leqq p\leqq 2$. In particular, we obtain that $\Phi_0^1(L^p(G)) \subset A_\alpha^{p,\frac{1}{2}p(1+\alpha)-1}(D)$. Q.E.D.

In what follows we put

$$H_0^1(D)=A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)=\{\, F \in A(D);\ \| F^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}} < \infty \,\}^{\dagger)} \quad (\alpha > 0).$$

Corollary 7.3.

(i) *If* $\alpha > 0$, *then*

$$\Phi_0^1(L^1(G))=\Phi_0^1(L_1^1(G))=[L^{1,-\frac{1}{2}}(D)]_0=[L_+^{1,-\frac{1}{2}}(D)]_0=A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)=H_0^1(D) \subset H^1(D).$$

(ii) *If* $1<p\leqq 2$ *and* $\alpha > \frac{1}{p}-1$, *then*

$$\Phi_0^1(L^p(G))=\Phi_0^1(L_1^p(G))=[L^{p,\frac{1}{2}p-1}(D)]_0=[L_+^{p,\frac{1}{2}p-1}(D)]_0$$

$$=A_\alpha^{p,\frac{1}{2}p(1+\alpha)-1}(D)=A^{p,\frac{1}{2}p-1}(D).$$

(iii) *If* $1\leqq p\leqq 2$, $\beta > \frac{1}{p}-1$ *and* $\alpha > 0$, *then*

$$A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D) \subset A_\beta^{p,\frac{1}{2}p(1+\beta)-1}(D) \quad \text{and} \quad H_0^1(D) \subset A^{p,\frac{1}{2}p-1}(D) \quad (1<p\leqq 2).$$

(iv) *If* $1\leqq p\leqq 2$,

$$\Phi_0^1(L^1(G)) \subset \Phi_0^1(L^p(G)).$$

(Pr) (i) and (ii) are obvious from Proposition 7.1 and Corollary 6.4. When $1\leqq p\leqq 2$ and $\beta > \frac{1}{p}-1$, we see from Lemma 6.3 that

†) See the proof of Theorem 9.5.

$$\| z^{-1}(zT_1(g)e_0^1(z))^{[\beta]}\|_{p,\frac{1}{2}p(1+\beta)-1} \leq (1-\eta^2)\,|I_{p(1+\frac{1}{2}\beta)-1,p(\frac{1}{2}+\frac{1}{2}\beta)-1}(\eta)|^{1/p}\Gamma(\beta+2)$$

$$\leq C_\beta < \infty.$$

where $g=k_\theta a_t k_\theta$, and $\eta=\mathrm{th}\frac{t}{2}$. Therefore as in the proof of Case 1 in Proposition 7.1, we can obtain that $\Phi_0^1(L^1(G)) \subset A_\beta^{p,\frac{1}{2}p(1+\beta)-1}(D)$. Then (iii) follows from (i). (iv) is obvious from (i), (ii) and (iii). Q.E.D.

Remark 7.4. The inclusion $H_0^1(D) \subset H^1(D)$ is dense. As stated in §2.3, $(T_1,H^1(D))$ is an irreducible representation of G and $\Phi_0^1(L^1(G))=H_0^1(D)$ is a G-invariant subspace of $H^1(D)$ (cf. Remark 6.6).

§8. Main theorems.

Summarizing Corollary 6.4 and Corollary 7.3, we conclude the following

Theorem 8.1.

(1) *Let* $n \in \frac{1}{2}\mathbf{Z}$, $n \geq 1$, $1\leq p\leq 2$ *and* $(n,p)\neq(1,1)$

(i) $°P_nL^p(G)E_n = °P_nL_n^p(G)E_n = L^p$-span of $\{f_{\ell 0}^n;\ \ell \in \mathbf{N}\}$,

(ii) Φ_0^n: $°P_nL^p(G)E_n \longrightarrow A^{p,\frac{1}{2}np-1}(D)$ *is bijective and norm-preserving,*

(iii) $\Phi_0^n(L^p(G))=[L^{p,\frac{1}{2}np-1}(D)]_{n-1}=[L_+^{p,\frac{1}{2}np-1}(D)]_{n-1}=A^{p,\frac{1}{2}np-1}(D)$.

(2) *Let* $n=1$ *and* $p=1$.

(i) $°P_1L^1(G)E_1 = °P_1L_1^1(G)E_1 \neq L^1$-span of $\{f_{\ell 0}^1;\ \ell \in \mathbf{N}\}$,

(ii) Φ_0^1: $°P_1L^1(G)E_1 \longrightarrow H_0^1(D)$ *is bijective,*

(iii) $\Phi_0^1(L^1(G))=[L^{1,-\frac{1}{2}}(D)]_0=[L_+^{1,-\frac{1}{2}}(D)]_0=H_0^1(D)$.

As we have stated in §1, the Fourier coefficients $\hat{f}(n;\ell,0)$ ($\ell \in \mathbf{N}$) for $f \in L^p(G)$ ($1 \leq p \leq 2$) are characterized by the right hand sides of (iii)'s in Theorem 8.1. In particular, when p=1, we have the following

<u>Theorem 8.2</u>. *Let* $n \in \frac{1}{2}\mathbf{Z}$ *and* $n \geq 1$.

(1) *If* f *belongs to* $L^1(G)$, *then* $\hat{f}(n;\ell,0)=o(\ell^{-\frac{1}{2}})$.

(2) *If* $a_{n\ell}=O(\ell^{-\frac{1}{2}-\gamma})$ $(\gamma > \frac{1}{2})$, *then there exists a function* f *in* $L^1(G)$ *such that* $a_{n\ell}=\hat{f}(n;\ell,0)$ ($\ell \in \mathbf{N}$).

Moreover, these orders: $-\frac{1}{2}$ *and* $-\frac{1}{2}-\gamma$ $(\gamma > \frac{1}{2})$ *are best possible.*

(pr) We note that for $f \in L^1(G)$

$$\Phi_0^n(f)(z)=\sum_{\ell=0}^{\infty} a_\ell z^\ell = 2\pi^{\frac{1}{2}}(2n-1)^{-\frac{1}{2}} \sum_{\ell=0}^{\infty} \overline{\hat{f}(n;\ell,0)}B(\ell+1,2n-1)^{-\frac{1}{2}} z^\ell$$

(cf. §3), $z^{-1}(z\Phi_0^1(f)(z))^{[\alpha]} \in A_\alpha^{1,\frac{1}{2}\alpha-\frac{1}{2}}(D)$ and $\Phi_0^n(f) \in A^{1,\frac{1}{2}n-1}(D)$ $(n > 1)$ by Theroem 8.1. Therefore, (1) is obvious from [DRS], Theorem 4. On the other hand, under the assumption of (2), we see that $F(z)=\sum_{\ell=0}^{\infty} a_{n\ell}z^\ell$ belongs to $\Phi_0^n(L^1(G))$ by the same theorem. Thus the desired result is clear. Q.E.D.

<u>Remark 8.3</u>. (1) By considering anti-holomorphic functions on D, we can obtain quite similar conclusions for the cases of $n \in \frac{1}{2}\mathbf{Z}$ and $n \leq -1$.
(2) When p=2, we can replace Φ_0^n by Φ_m^n ($m \in \mathbf{N}$). We regard any elements Z in the Lie algebra of G=SU(1,1) as the left invariant differential operators D_Z on G by the formula:

$$\mathbf{D}_Z f(g)=\frac{d}{dt}f(g\exp(tZ))\Big|_{t=0}.$$

Here we put $\mathbf{D}_-=\mathbf{D}_X+i\mathbf{D}_Y$, where $X=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $Y=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$. Then we see that

$$\mathbf{D}_- f_{pq}^n=-2(q(2n+q-1))^{\frac{1}{2}}f_{pq-1}^n \qquad (n \in \tfrac{1}{2}\mathbf{Z},\ n \geq 1 \text{ and } p,\ q \in \mathbf{N})$$

(cf. [Su], p.216). Then we can obtain that

$$\phi_m^n(L^p(G))=\phi_m^n({}^\circ P_n L^p(G)E_{n+m})=\phi_0^n(\mathbf{D}_-{}^m {}^\circ P_n L^p(G)E_{n+m}).$$

Therefore, to replace ϕ_0^n in Theorem 8.1 by ϕ_m^n ($m \varepsilon \mathbf{N}$), we have to characterize the image of $\mathbf{D}_-{}^m$ on ${}^\circ P_n L^p(G)E_{n+m}$. When p=2, it is easy to see that $D_-{}^m$: ${}^\circ P_n L^2(G)E_{n+m} \longrightarrow {}^\circ P_n L^2(G)E_n$ is bijective and norm-preserving. Thus we have

$$\phi_m^n(L^2(G))=\phi_0^n(D_-{}^m {}^\circ P_n L^2(G)E_{n+m})=\phi_0^n({}^\circ P_n L^2(G)E_n)=\phi_0^n(L^2(G)),$$

and ϕ_0^n can be replaced by ϕ_m^n.

§9. Appendix.

Let notations be as in the preceding sections.

9.1. Bounded operators. For $0<p<\infty$ and $r \varepsilon \mathbf{R}$ we put

$$T_{p,r}(g)F(z)=J(g^{-1},z)^{-\frac{4(1+r)}{p}} F(g^{-1}z), \quad \text{where } g \varepsilon G \text{ and } F \varepsilon L^{p,r}(D).$$

Obviously, if $\frac{4(1+r)}{p}=2n \varepsilon \mathbf{Z}$ and $n \geqq \frac{1}{2}$, $T_{p,r}$ coinsides with T_n defined in §2.3.

Lemma 9.1. *Let* $0<p<\infty$ *and* $r \varepsilon \mathbf{R}$. *Then each* $T_{p,r}(g)$ ($g \varepsilon G$) *is* $\|\cdot\|_{p,r}$-*norm-preserving. In particular,* $T_{p,r}(g)$ *belongs to* $(L^{p,r}(D),L^{p,r}(D))$.

(Pr) We note that $(1-|z|^2)dz$ is a G-invariant measure on D and $J(g^{-1},gz)^{-2}\times(1-|gz|^2)=(1-|z|^2)$ ($g \varepsilon G$, $z \varepsilon D$). Thus the desired result follows from the definition of $\|\cdot\|_{p,r}$. Q.E.D.

Let $n \varepsilon \frac{1}{2}\mathbf{Z}$ and $n \geqq \frac{1}{2}$. For a function f on G we define the operator $T_n(f)$ by

$$T_n(f)=\int_G \overline{f}(g)T_n(g)dg.$$

Now we assume that $f \varepsilon L^1(G)$. Then the following proposition easily follows from the definitions of the Bergman and Hardy spaces and Lemma 9.1. Actually for each norm $\|\cdot\|$, we see that $\|T_n(f)F\| \leqq \int_G |f(g)|\,\|T_n(g)F\|\,dg \leqq \|f\|_1 \times \|F\|$.

<u>Proposition 9.2</u>. *Let* $n \in \frac{1}{2}\mathbf{Z}$, $n \geqq \frac{1}{2}$ *and* f *be in* $L^1(G)$. *Then*

(1) *If* $1 \leqq p < \infty$ *and* $(n,p) \neq (1,1)$, $(\frac{1}{2},2)$, *then* $T_n(f)$ *belongs to* $(A^{p,\frac{1}{2}np-1}(D), A^{p,\frac{1}{2}np-1}(D))$.

(2) $T_1(f)$ *belongs to* $(H^1(D), H^1(D))$.

(3) $T_{\frac{1}{2}}(f)$ *belongs to* $(H^2(D), H^2(D))$.

Moreover, in each case the operator norm of $T_n(f)$ *is bounded by* $\|f\|_1$.

Now we note that for $f \in L^1(G)$ and $h \in L^p(G)$ $(1 \leqq p \leqq \infty)$ their convolution $f*h(x) = \int_G f(xy^{-1})h(y)dy$ belongs to $L^p(G)$ and then

$$\Phi_m^n(f*h)(z) = T_n(f)\Phi_m^n(h) \qquad (z \in D).$$

Then using Theorem 8.1, we can obtain the following

<u>Proposition 9.3</u>. *Let* $1 \leqq p \leqq 2$, $n \in \frac{1}{2}\mathbf{Z}$ *and* $n \geqq 1$. *Then*

(1) *For any* f *in* $L^p(G)$, $T_n(f)$ $(n > 1)$ *belongs to* $(A^{1,\frac{1}{2}n-1}(D), A^{p,\frac{1}{2}np-1}(D))$ *and the operator norm is bounded by* $C\|f\|_p$.

(2) *For any* f *in* ${}^\circ L^p(G)$ $(p > 1)$, $T_1(f)$ *belongs to* $(H_0^1(D), A^{p,\frac{1}{2}p-1}(D))$, *where we define the norm of* $H_0^1(D)$ *by* $\|z^{-1}(zF(z))^{[\alpha]}\|_{1,\frac{1}{2}\alpha-\frac{1}{2}}$ $(F \in H_0^1(D))$ *for a fixed* $\alpha > 0$. *Moreover, the operator norm is bounded by* $C\|f\|_p$.

(Pr) (2) Let H be in $H_0^1(D)$. Then by (2)(iii) of Theorem 8.1 there exists an ${}^\circ h \in {}^\circ P_1 L^1(G) E_1$ $(\not\subset L^1(G))$ such that $\Phi_0^1({}^\circ h) = H$. More precisely, by the proof of Proposition 7.1, there exists an $h \in L^1(G)$ such that $h = h_p + {}^\circ h$ and $\|h\|_1 = C\|H\|_{H_0^1}$, where $\|\cdot\|_{H_0^1}$ is the norm of $H_0^1(D)$ defiend above. Then

$$\|T_1(f)H\|_{p,\frac{1}{2}p-1} = \|T_1(f)\Phi_0^1({}^\circ h)\|_{p,\frac{1}{2}p-1}$$

$$= \|\Phi_0^1(f*{}^\circ h)\|_{p,\frac{1}{2}p-1}$$

$$=\| {}^{\circ}P_1(f*{}^{\circ}h)E_1\|_p = \| f*({}^{\circ}P_1({}^{\circ}h)E_1)\|_p$$

$$=\| f*{}^{\circ}h\|_p.$$

Here we note that $f \in {}^{\circ}L^p(G)$, and thus $f*{}^{\circ}h=f*h$. Then we have

$$\| T_1(f)H\|_{p,\frac{1}{2}p-1} = \| f*h\|_p$$

$$\leqq \| f\|_p \| h\|_1 = C\| f\|_p \| H\|_{H_0^1}.$$

Thus the desired result is obtained.

(1) This case is easier than the first one. We can take an $h \in {}^{\circ}P_n L^1(G)E_n \subset L^1(G)$ such that $\| h\|_1 = C\| H\|_{1,\frac{1}{2}n-1}$ for each $H \in A^{1,\frac{1}{2}n-1}(D)$. The rest of the proof is quite similar to the first one. Q.E.D.

The next proposition follows from the Kunze-Stein phenomenon (cf. [C]) and the same argument as above.

Proposition 9.4. *Let* $1 \leqq p < 2$, $n \in \frac{1}{2}\mathbf{Z}$ *and* $n \geqq 1$. *Then*

(1) *For any* f *in* $L^p(G)$, $T_n(f)$ *belongs to* $(A^{2,n-1}(D), A^{2,n-1}(D))$ *and the operator norm is bounded by* $C\| f\|_p$.

(2) *For any* f *in* $L^2(G)$, $T_n(f)$ *belongs to* $(A^{p,\frac{1}{2}np-1}(D), A^{2,n-1}(D))$ *and the operator norm is bounded by* $C\| f\|_2$.

9.2. Fractional derivatives and integrals. Let $\beta \geqq 0$. For $F(z) = \sum_{\ell=0}^{\infty} a_\ell z^\ell \in A(D)$, we define the fractional derivative (resp. integral) of F of order β as follows.

$$F^{[\beta]}(z) = \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1+\beta)}{\Gamma(\ell+1)} a_\ell z^\ell \quad (\text{resp. } F_{[\beta]}(z) = \sum_{\ell=0}^{\infty} \frac{\Gamma(\ell+1)}{\Gamma(\ell+1+\beta)} a_\ell z^\ell).$$

Then by using the $[\ ]_\rho$-transform in §6, we see that for $\rho > -\frac{1}{2}$

$$F^{[\beta]}(z)=\frac{\Gamma(2\rho+1)}{\Gamma(2\rho+1-\beta)}\,[|z|^{2\beta}(1-|z|^2)^{-\beta}F(z)]_\rho$$
$$F_{[\beta]}(z)=\frac{\Gamma(2\rho+1)}{\Gamma(2\rho+1+\beta)}\,z^{2\rho+1}[(1-|z|^2)^{\beta}z^{-(2\rho+1)}F(z)]_\rho, \tag{9.1}$$

where in the second formula $2\rho+1 \in \mathbf{N}$ and F has a $(2\rho+1)$th zero at $z=0$.

Theorem 9.5. *Let* $r > -\frac{1}{2}$ *and* $\beta \geqq 0$. *Then*
(1) *If* F *belongs to* $A^{1,r}(D)$, *then* $F^{[\beta]}$ *belongs to* $A^{1,r+\frac{1}{2}\beta}(D)$.
(2) *If* F *belongs to* $A^{1,r+\frac{1}{2}\beta}(D)$, *then* $F_{[\beta]}$ *belongs to* $A^{1,r}(D)$.
Moreover, if we replace $A^{1,-\frac{1}{2}}(D)$ *by* $H^1_0(D)$, *the assertions are also valid for the case of* $r=-\frac{1}{2}$.

(Pr) When $\beta=0$, the assertions are trivial. Let $\beta > 0$.
(1) Let F be in $A^{1,r}(D)$ $(r > -\frac{1}{2})$. Then since $|z|^{2\beta}(1-|z|^2)^{-\beta}F(z)$ belongs to $L^{1,r+\frac{1}{2}\beta}_+(D)$, we see from (9.1) and Proposition 6.2 (i) that $F^{[\beta]}$ belongs to $[L^{1,r+\frac{1}{2}\beta}_+(D)]_\rho=A^{1,r+\frac{1}{2}\beta}(D)$, where we can take a sufficiently large $\rho \geqq 0$ such that $2\rho+1-\beta > 0$, $r < \rho$ and $r+\frac{1}{2} \leqq \rho$. Next let F be in $H^1_0(D)$ and put $F(z)=F(0)+(F(z)-F(0))=F_0+F_1(z)$. Then it is easy to see that $z^{-1}F_1(z) \in H^1_0(D)$. Thus, by the definition of $H^1_0(D)$, $z^{-1}(zz^{-1}F_1(z))^{[\beta]}$ must be in $A^{1,-\frac{1}{2}+\frac{1}{2}\beta}(D)$. In particular, $(F_1)^{[\beta]}$ belongs to $A^{1,-\frac{1}{2}+\frac{1}{2}\beta}(D)$. Therefore, we see that $F^{[\beta]}=(F_0)^{[\beta]}+(F_1)^{[\beta]}$ belongs to $A^{1,-\frac{1}{2}+\frac{1}{2}\beta}(D)$.
(2) Let F be in $A^{1,r+\frac{1}{2}\beta}(D)$ $(r \geqq -\frac{1}{2})$. If $F(z)$ has a simple zero at $z=0$, we see that $(1-|z|^2)^{\beta}z^{-1}F(z)$ belongs to $L^{1,r}_+(D)$. Thus by (9.1) and Theorem 8.1 we see that $F_{[\beta]}$ belongs to $[L^{1,r}_+(D)]_0=A^{1,r}(D)$ $(r > -\frac{1}{2})$ and $H^1_0(D)$ $(r=-\frac{1}{2})$. For a general F we shall devide it as $F=F_0+F_1$ as in the first case. Then it is easy to see that $F_1(z)$ has a simple zero at $z=0$ and belongs to $A^{1,r+\frac{1}{2}\beta}(D)$. Thus $(F_1)_{[\beta]}$ must be in $A^{1,r}(D)$ $(r > -\frac{1}{2})$ and $H^1_0(D)$ $(r=-\frac{1}{2})$. Therefore, $F_{[\beta]}=(F_0)_{[\beta]}+(F_1)_{[\beta]}$ belongs to the same space. Q.E.D.

Remark 9.6. This theorem is obtained by [DRS], Theorem 5.

Corollary 9.7. *Let* $1\leqq p\leqq 2$, $r > -\frac{1}{2}$ *and* $\beta \geqq 0$. *Then*
(1) *If* F *belongs to* $A^{p,r}(D)$, *then* $F^{[\beta]}$ *belongs to* $A^{p,r+\frac{1}{2}\beta p}(D)$.
(2) *If* F *belongs to* $A^{p,r+\frac{1}{2}\beta p}(D)$, *then* $F_{[\beta]}$ *belongs to* $A^{p,r}(D)$.

(Pr) When p=2, the assertions follow from the same argument in Lemma 7.2, and when p=1, they are just the above theorem. Therefore the desired result follows by the interpolation between these cases. Q.E.D.

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