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Sojourns of multidimensional Gaussian processes with dependent components

by

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1. Introduction and a result

Let $\{X(t) = (X_1(t), X_2(t), \cdots, X_p(t))', \ t \geqq 0\}$ be a measurable separable p-dimensional stationary Gaussian process, where x' is the transposed vector of x. This paper deals with the limiting distribution of sojourn time by $X(t)$ in the fixed sphere, under some assumptions on the dependence among the components of p-dimensional process $X(t)$. In [4], the author studied the case of $p = 2$ and Berman[1] treated the p-dimensional stationary Gaussian process with independent components in the case of the expanding or shrinking spheres. The result in this paper is the direct extension of Theorem 1 (II) in [4] to the case of general dimension and the idea of the proof is the same.

Assumptions and notation are the following. Suppose that

$$EX(0) = 0,$$

$$R(t) \equiv EX(0)X(t)' = (R_{ij}(t))_{1\leqq i,j\leqq p},$$

$$R_{ij}(t) = \begin{cases} r(t), & \text{if } i = j, \\ \rho(t), & \text{if } i \neq j, \end{cases}$$

$r(t)$ and $\rho(t)$ are continuous,

$$r(0) = 1, \quad \rho(0) = \rho_0 \ (0 \leqq \rho_0 < 1),$$

$$(1.1) \qquad r(t) \sim t^{-\alpha}L(t) \quad \text{as} \quad t \to \infty,$$

$$(1.2) \qquad \rho(t) \sim \rho_\infty t^{-\alpha}L(t) \quad \text{as} \quad t \to \infty,$$

where $0 < \alpha < 1$, $0 \leqq \rho_\infty < 1$ and L is a slowly varying function at infinity. In [4], we assumed $\rho_\infty = 0$. However, it is not necessary to assume it by the same reasoning as in [7]. We shall also treat the functional limit theorem .

Define

$$M(t) = \int_0^t I[\, X(s) \in D \,]\, ds, \qquad t > 0,$$

where $I[\cdot]$ is the indicator function and

(1.3) $$D = \{\, (x_1, \cdots, x_p)' \mid \sum_{j=1}^{p} x_j^{\,2} \leqq 1 \,\}.$$

Our result is the following.

Theorem 1. *Let $Z(\tau)$, $\tau \in [0, \infty)$, be the Rosenblatt process with the representation*

$$Z(\tau) = \sqrt{(1-2\alpha)(2-2\alpha)}\,\Gamma(\tfrac{1+\alpha}{2})\{\Gamma(\tfrac{1-\alpha}{2})\Gamma(\alpha)\}^{-1}$$
$$\int_{-\infty}^{\infty} dB(u_1) \int_{-\infty}^{u_1} dB(u_2) \int_0^{\tau} \{(s-u_1)(s-u_2)\}^{-(\alpha+1)/2}$$
$$I[u_1 < s]\, I[u_2 < s]\, ds\,,$$

$B(\cdot)$ being a standard Brownian motion, and let $Z_j(\tau)$, $\tau \in [0, \infty)$, $j = 1, \cdots, p$ are independent copies of $Z(\tau)$, $\tau \in [0, \infty)$. Then, under the above conditions, as $t \to \infty$

$$\Delta_t(\tau) \equiv \{\mathrm{Var}\, M(t)\}^{-1/2} \{\, M(t\tau) - EM(t\tau) \,\}$$

converges weakly in the space $C[0, \infty)$ to

(1.4) $$\Delta(\tau) \equiv \{c_1^{\,2}(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0})^2 + \sum_{j=2}^{p} c_j^{\,2}(\frac{1-\rho_\infty}{1-\rho_0})^2\}^{-1/2}$$
$$\{c_1(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0})Z_1(\tau) + \sum_{j=2}^{p} c_j(\frac{1-\rho_\infty}{1-\rho_0})Z_j(\tau)\},$$

where $\{c_j,\ j = 1, \cdots, p\}$ *are non-zero constants determined later in the proof.*

2. Proof

Define by $\mathcal{F}$ the class of real-valued functions of p variables square-integrable with respect to $\Pi_{j=1}^{p}\phi(x_j)$, $\phi(x)$ being the standard normal density, and let $H_n(x)$, $n = 0, 1, 2, \cdots$ denote the n-th normalized Hermite polynomial defined by $H_n(x) = (-1)^n\phi(x)^{-1}(d^n/dx^n)\phi(x)$. Then $f \in \mathcal{F}$ has the expansion

$$(2.1)\qquad f(x_1, \cdots, x_p) = \sum_{n=0}^{\infty} \sum_{\Sigma_{j=1}^{p} n_j = n} c(n_1, \cdots, n_p) \prod_{j=1}^{p} H_{n_j}(x_j)$$

in the mean square sense, where

$$c(n_1, \cdots, n_p) = \{\prod_{j=1}^{p} n_j!\}^{-1} \int_{\mathbb{R}^p} f(x_1, \cdots, x_p) \prod_{j=1}^{p} \{H_{n_j}(x_j)\phi(x_j)dx_j\},$$

$$n_j \geqq 0, \quad \sum_{j=1}^{p} n_j = n.$$

The validity of (2.1) is well-known for $p = 1, 2$ (cf. [3]) and it can be shown for general p by the same reasoning as for $p = 1$, if we use the following lemma, (a special case of Lemma 3.2 in [5]).

Lemma 1. *Let* $(\xi_1, \cdots, \xi_{2p})$ *be* 2p *dimensional Gaussian random variables satisfying*

$$E\xi_j = 0,\ E\xi_j^{\ 2} = 1, \quad 1 \leqq j \leqq 2p,$$

$$E\xi_j\xi_{j+p} = r_j, \quad 1 \leqq j \leqq p,$$

$$E\xi_j\xi_k = 0, \quad \text{if} \quad (j,k) \notin \{(j,j),\ (j+p,j+p),\ (j,j+p);\ 1 \leqq j \leqq p\}.$$

Then

$$E[\prod_{j=1}^{p} H_{n_j}(\xi_j)\, H_{m_j}(\xi_{j+p})] = \prod_{j=1}^{p} \delta_{n_j m_j} \quad n! \quad r_j^{\,n_j},$$

where $\delta_{nm} = 1$ or 0 according as $n = m$ or $n \neq m$.

Let

(2.2) $$T = (t_{ij})_{1 \leqq i, j \leqq p},$$

where

$$t_{1j} = \frac{1}{\sqrt{p(1+(p-1)\rho_0)}}, \quad 1 \leqq j \leqq p,$$

$$t_{ij} = \frac{1}{\sqrt{i(i-1)(1-\rho_0)}}, \quad 2 \leqq i \leqq p, \quad 1 \leqq j < i,$$

$$t_{ii} = -\frac{i-1}{\sqrt{i(i-1)(1-\rho_0)}}, \quad 2 \leqq i \leqq p,$$

$$t_{ij} = 0, \quad 2 \leqq i \leqq p-1, \quad i < j \leqq p,$$

and define $Y(t) = (Y_1(t), \cdots, Y_p(t))'$ by $Y(t) = TX(t)$. Then

$$\tilde{R}(t) \equiv EY(0)Y(t)' = (\tilde{R}_{ij}(t))_{1 \leqq i, j \leqq p},$$

where

$$(2.3) \qquad \tilde{R}_{ij}(t) = \begin{cases} \dfrac{r(t) + (p-1)\rho(t)}{1 + (p-1)\rho_0}, & \text{if } i = j = 1, \\ \dfrac{r(t) - \rho(t)}{1 - \rho_0}, & \text{if } i = j = 2, 3, \cdots, p, \\ 0, & \text{if } i \neq j. \end{cases}$$

Therefore, $\{Y_j(t)\}$, $1 \leqq j \leqq p$, are independent Gaussian processes.

We then have

$$M(t) = \int_0^t I[\, X(s) \in D \,]\, ds = \int_0^t I[\, Y(s) \in \tilde{D} \,]\, ds,$$

where

$$\tilde{D} = \{\, (y_1, \cdots, y_p)' \mid T^{-1}(y_1, \cdots, y_p)' \in D \,\},$$

and by (2.1)

$$M(t) = \sum_{n=0}^{\infty} \sum_{\Sigma_{j=1}^{p} n_j = n} c(n_1, \cdots, n_p) \int_0^t \prod_{j=1}^{p} H_{n_j}(Y_j(s))\, ds,$$

where

$$(2.4) \qquad c(n_1, \cdots, n_p) = \{ \prod_{j=1}^{p} n_j! \}^{-1} \int_{\tilde{D}} \prod_{j=1}^{p} \{H_{n_j}(x_j)\phi(x_j)dx_j\}.$$

For proving the theorem, the following proposition is essential. For an $f \in \mathcal{F}$, define m by

$$m = \min\{\, n \geqq 1 \mid \text{there exists } c(n_1, \cdots, n_p) \neq 0 \text{ with } \Sigma_{j=1}^{p} n_j = n \text{ in the expansion (2.1)}\}.$$

We call this m the Hermite rank of $f \in \mathcal{F}$, follwing the definition due to Taqqu[6]

Proposition 1. Suppose that the Hermite rank of f is m and $r(t) \sim t^{-\alpha}L(t)$, $\rho(t) \sim \rho_\infty t^{-\alpha}L(t)$ for some α with $0 < \alpha < 1/m$. Write

$$K_t(\tau) = \int_0^{t\tau} f(Y_1(s),\cdots,Y_p(s))\, ds$$

and

$$I_t(\tau) = \sum_{\sum_{j=1}^p m_j = m} c(m_1, \cdots, m_p) \int_0^{t\tau} \prod_{j=1}^p H_{m_j}(Y_j(s))\, ds.$$

Then as $t \to \infty$

$$\{\mathrm{Var}\ K_t(1)\}^{-1/2} \{ K_t(\tau) - EK_t(\tau) \}$$

is asmptotically equal to

$$\{\mathrm{Var}\ I_t(1)\}^{-1/2}\, I_t(\tau)$$

in the sense of the finite dimensional distributions.

Proof. If we could show that for any $a_j \in \mathbb{R}$, $\tau_j \in [0, \infty)$, $j = 1,\cdots,h$,

$$\lim_{t\to\infty} E\Big| \sum_{j=1}^{h} a_j[\{\mathrm{Var}K_t(1)\}^{-1/2}\{K_t(\tau_j) - EK_t(\tau_j)\} - \{\mathrm{Var}I_t(1)\}^{-1/2} I_t(\tau_j)]\Big|^2 = 0,$$

then the proposition follows. For that, it is enough to show that

(2.5) $$\lim_{t\to\infty} E|\{\mathrm{Var}K_t(1)\}^{-1/2}\{K_t(\tau) - EK_t(\tau)\} - \{\mathrm{Var}I_t(1)\}^{-1/2}I_t(\tau)|^2 = 0.$$

Note that

$$EK_t(\tau) = c(0, \cdots, 0)t\tau,$$

and

$$K_t(\tau) - EK_t(\tau) = I_t(\tau) + R_t(\tau),$$

where

$$R_t(\tau) = \sum_{n=m+1}^{\infty} \sum_{\Sigma_{j=1}^p n_j = n} c(n_1, \cdots, n_p) \int_0^{t\tau} \prod_{j=1}^{p} H_{n_j}(Y_j(s))\, ds.$$

We have

$$E\{K_t(\tau) - EK_t(\tau)\}^2 = EI_t(\tau)^2 + ER_t(\tau)^2 + 2EI_t(\tau)R_t(\tau),$$

however by Lemma 1

$$EI_t(\tau)R_t(\tau) = \sum_{n=m+1}^{\infty} \sum_{\Sigma_{j=1}^p m_j = m} \sum_{\Sigma_{j=1}^p n_j = n} c(m_1, \cdots, m_p)c(n_1, \cdots, n_p) \int_0^{t\tau}\int_0^{t\tau} E[\prod_{j=1}^{p} H_{m_j}(Y_j(u))\, H_{n_j}(Y_j(v))]\, du dv$$

$$= 0.$$

Hence

(2.6) $$\mathrm{Var}\ K_t(\tau) = \mathrm{Var}\ I_t(\tau) + \mathrm{Var}\ R_t(\tau).$$

We have, by Lemma 1, (2.3), (1.1) and (1.2),

$$\mathrm{Var}\, I_t(\tau) = \sum_{\Sigma_{j=1}^p m_j = m} \sum_{\Sigma_{j=1}^p k_j = m} c(m_1, \cdots, m_p) c(k_1, \cdots, k_p)$$

$$\int_0^{t\tau}\int_0^{t\tau} E\{\prod_{j=1}^p H_{m_j}(Y_j(u))\, H_{k_j}(Y_j(v))\}\, dudv$$

$$= \sum_{\Sigma_{j=1}^p m_j = m} c(m_1, \cdots, m_p)^2 \int_0^{t\tau}\int_0^{t\tau} m_1!$$

$$\left(\frac{r(u-v) + (p-1)\rho(u-v)}{1 + (p-1)\rho_0}\right)^{m_1} \prod_{j=2}^p m_j!\, \left(\frac{r(u-v) - \rho(u-v)}{1 - \rho_0}\right)^{m_j}$$

$$dudv$$

$$= 2 \sum_{\Sigma_{j=1}^p m_j = m} c(m_1, \cdots, m_p)^2 \int_0^{t\tau} (t\tau - s) m_1!$$

$$\left(\frac{r(s) + (p-1)\rho(s)}{1 + (p-1)\rho_0}\right)^{m_1} \prod_{j=2}^p m_j!\, \left(\frac{r(s) - \rho(s)}{1 - \rho_0}\right)^{m_j} ds$$

$$\sim 2 \sum_{\Sigma_{j=1}^p m_j = m} c(m_1, \cdots, m_p)^2 \int_0^{t\tau} (t\tau - s) m_1!$$

$$\left(\frac{s^{-\alpha}L(s) + (p-1)\rho_\infty s^{-\alpha}L(s)}{1 + (p-1)\rho_0}\right)^{m_1}$$

$$\prod_{j=2}^p m_j!\, \left(\frac{s^{-\alpha}L(s) - \rho_\infty s^{-\alpha}L(s)}{1 - \rho_0}\right)^{m_j} ds$$

$$= 2 \sum_{\Sigma_{j=1}^p m_j = m} c(m_1, \cdots, m_p)^2\, m_1!\, \left(\frac{1 + (p-1)\rho_\infty}{1 + (p-1)\rho_0}\right)^{m_1}$$

$$\prod_{j=2}^p m_j!\, \left(\frac{1-\rho_\infty}{1-\rho_0}\right)^{m_j} \int_0^{t\tau} (t\tau - s) s^{-m\alpha} L(s)^m ds$$

$$\sim C(t\tau)^{2-m\alpha} L(t\tau)^m,$$

where

$$C = \frac{2}{(1-m\alpha)(2-m\alpha)} \sum_{\Sigma_{j=1}^{p} m_j = m} c(m_1, \cdots, m_p)^2 \, m_1! \left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)^{m_1} \prod_{j=2}^{p} m_j! \left(\frac{1-\rho_\infty}{1-\rho_0}\right)^{m_j}.$$

On the other hand,

$$\text{Var}\, R_t(\tau) = 2 \sum_{n=m+1}^{\infty} \sum_{\Sigma_{j=1}^{p} n_j = n} c(n_1, \cdots, n_p)^2 \int_0^{t\tau} (t\tau - s) \, n_1! \left(\frac{r(s)+(p-1)\rho(s)}{1+(p-1)\rho_0}\right)^{n_1} \prod_{j=2}^{p} n_j! \left(\frac{r(s)-\rho(s)}{1-\rho_0}\right)^{n_j} ds.$$

For any $n_1, \cdots, n_p$ with $\Sigma_{j=1}^{p} n_j \geqq m+1$, there exist $q_1, \cdots, q_p$ depending on $n_1, \cdots, n_p$ such that $q_j \leqq n_j$, $\Sigma_{j=1}^{p} q_j = m+1$. Then

$$\text{Var}\, R_t(\tau) \leqq 2 \sum_{n=m+1}^{\infty} \sum_{\Sigma_{j=1}^{p} n_j = n} c(n_1, \cdots, n_p)^2 \int_0^{t\tau} (t\tau - s) \, n_1! \left(\frac{r(s)+(p-1)\rho(s)}{1+(p-1)\rho_0}\right)^{q_1} \prod_{j=2}^{p} n_j! \left(\frac{r(s)-\rho(s)}{1-\rho_0}\right)^{q_j} ds$$

$$\sim 2 \sum_{n=m+1}^{\infty} \sum_{\Sigma_{j=1}^{p} n_j = n} c(n_1, \cdots, n_p)^2 \, n_1! \left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)^{q_1} \prod_{j=2}^{p} n_j! \left(\frac{1-\rho_\infty}{1-\rho_0}\right)^{q_j} \int_0^{t\tau} (t\tau - s) s^{-(m+1)\alpha} L(s)^{m+1} \, ds.$$

Note that the Parseval identity gives us

$$\sum_{n=0}^{\infty} \sum_{\Sigma_{j=1}^{p} n_j = n} c(n_1, \cdots, n_p)^2 \prod_{j=1}^{p} n_j!$$

$$= \int_{\mathbb{R}^p} f(x_1, \cdots, x_p)^2 \prod_{j=1}^{p} (\phi(x_j) dx_j) < \infty.$$

Hence for large t,

(2.7) $\quad \mathrm{Var}\ R_t(\tau) = o(\mathrm{Var}\ I_t(\tau)),$

which together with (2.6) gives us

$$\mathrm{Var}\ K_t(\tau) \sim \mathrm{Var}\ I_t(\tau).$$

Thus the left hand side of (2.5) turns out to be

$$\lim_{t\to\infty} \{\mathrm{Var}\ I_t(1)\}^{-1} E|K_t(\tau) - EK_t(\tau) - I_t(\tau)|^2$$

$$= \lim_{t\to\infty} \{\mathrm{Var}\ I_t(1)\}^{-1} ER_t(\tau)^2 = 0$$

by (2.7). The proof of the proposition is thus complete.

The limiting distribution in the theorem is determined by the following lemma.

<u>Lemma 2</u>. <u>Suppose that</u> (1.1) <u>and</u> (1.2) <u>are satisfied with</u> $0<\alpha<1/2$ <u>and let</u>

$$J_t(\tau) = \int_0^{t\tau} \sum_{j=1}^{p} c_j H_2(Y_j(s))\, ds, \qquad \tau \in [0, \infty).$$

<u>Then as</u> $t \to \infty$

$$\{\mathrm{Var}\ J_t(1)\}^{-1/2} J_t(\tau)$$

<u>converges weakly in the space</u> $C[0, \infty)$ <u>to</u> $\Delta(\tau)$ <u>defined in</u> (1.4).

<u>Proof</u>. Let

$$J_t^j(\tau) = \int_0^{t\tau} H_2(Y_j(s))\, ds, \quad j = 1, \cdots, p, \quad \tau \in [0, \infty).$$

Since $\{Y_j(t)\}$, $j = 1, \cdots, p$, are independent, then

$$\mathrm{Var}\, J_t(1) = \sum_{j=1}^{p} c_j^{\,2}\, \mathrm{Var}\, J_t^j(1).$$

We have

$$\begin{aligned}\mathrm{Var}\, J_t^1(1) &= \int_0^t \int_0^t EH_2(Y_1(u))H_2(Y_1(v))\, dudv \\ &= 2 \int_0^t \int_0^t \left(\frac{r(u-v) + (p-1)\rho(u-v)}{1 + (p-1)\rho_0}\right)^2 dudv \\ &= 4 \int_0^t (t-s)\left(\frac{r(s) + (p-1)\rho(s)}{1 + (p-1)\rho_0}\right)^2 ds \\ &\sim \frac{4}{(1-2\alpha)(2-2\alpha)} \left(\frac{1 + (p-1)\rho_\infty}{1 + (p-1)\rho_0}\right)^2 t^{2-2\alpha} L(t)^2\end{aligned}$$

and for $j = 2, \cdots, p$

$$\begin{aligned}\mathrm{Var}\, J_t^j(1) &= \int_0^t \int_0^t EH_2(Y_j(u))H_2(Y_j(v))\, dudv \\ &= 2 \int_0^t \int_0^t \left(\frac{r(u-v) - \rho(u-v)}{1 - \rho_0}\right)^2 dudv \\ &= 4 \int_0^t (t-s)\left(\frac{r(s) - \rho(s)}{1 - \rho_0}\right)^2 ds \\ &\sim \frac{4}{(1-2\alpha)(2-2\alpha)} \left(\frac{1-\rho_\infty}{1-\rho_0}\right)^2 t^{2-2\alpha} L(t)^2.\end{aligned}$$

Hence

$$\{\mathrm{Var}\ J_t(1)\}^{-1/2}\ J_t(\tau) \sim \{c_1{}^2\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)^2 + \sum_{j=2}^{p} c_j{}^2\left(\frac{1-\rho_\infty}{1-\rho_0}\right)^2\}^{-1/2}$$

$$\{c_1\left(\frac{1+(p-1)\rho_\infty}{1+(p-1)\rho_0}\right)\{\mathrm{Var}\ J_t^1(1)\}^{-1/2}\ J_t^1(\tau)$$

$$+ \sum_{j=2}^{p} c_j\left(\frac{1-\rho_\infty}{1-\rho_0}\right)\{\mathrm{Var}\ J_t^j(1)\}^{-1/2}\ J_t^j(\tau)\}.$$

By the non-central limit theorem ([2], [6]), under our conditions,

$$\{\mathrm{Var}\ J_t^j(1)\}^{-1/2}\ J_t^j(\tau) \overset{w}{\Longrightarrow} Z_j(\tau), \quad \tau \in [0, \infty)$$

in the sense of weak convergence in the space $C[0, \infty)$. Since $\{J_t^j(\tau),\ \tau \in [0, \infty)\}$ are independent, the proof of the lemma is complete.

Now, to prove the theorem, we have to calculate (2.4) for

$$\tilde{D} = \{(y_1, \cdots, y_p)' \mid T^{-1}(y_1, \cdots, y_p)' \in D\},$$

where T and D are defined in (2.2) and (1.3), respectively. Since

$$T^{-1} = (s_{ij})_{1 \leqq i,j \leqq p},$$

where

$$s_{i1} = \frac{\sqrt{1+(p-1)\rho_0}}{\sqrt{p}}, \quad 1 \leqq i \leqq p,$$

$$s_{ij} = \frac{\sqrt{1-\rho_0}}{\sqrt{i(i-1)}}, \quad 2 \leqq j \leqq p, \quad 1 \leqq i \leqq j,$$

$$s_{ii} = -\frac{(i-1)\sqrt{1-\rho_0}}{\sqrt{i(i-1)}}, \quad 2 \leqq i \leqq p,$$

$$s_{ij} = 0, \quad 2 \leqq j \leqq p-1, \quad j < i \leqq p,$$

then

$$\tilde{D} = \{(y_1, \cdots, y_p)' \mid (1 + (p-1)\rho_0)y_1^2 + (1 - \rho_0)\sum_{j=2}^{p} y_j^2 \leqq 1\}.$$

Obviously,

$$c(n_1, \cdots, n_p) = 0$$

for any $n_1, \cdots, n_p$ with $n_1 + \cdots + n_p = 1$, since $\tilde{D}$ is symmetric with respect to each axis. Also,

$$c(n_1, \cdots, n_p) = 0$$

for any $n_1, \cdots, n_p$ with $n_1 + \cdots + n_p = 2$ such that $n_j \leqq 1$ for all j.

Note that

$$\int_{-a}^{a} H_2(y)\phi(y)dy = -2a\phi(a),$$

and put

$$\alpha \equiv \alpha(y_2, \cdots, y_p) \equiv \{\frac{1 - (1-\rho_0)\Sigma_{j=2}^{p} y_j^2}{1 + (p-1)\rho_0}\}^{1/2}.$$

Then the standard calculation gives us

$$c(2,0,\cdots,0) = \frac{1}{2}\int\cdots\int_{D} H_2(y_1)\phi(y_1)\phi(y_2)\cdots\phi(y_p)dy_1dy_2\cdots dy_p$$

$$= - \int_A \cdots \int \alpha\phi(\alpha)\phi(y_2)\cdots\phi(y_p)dy_2\cdots dy_p < 0,$$

where

$$A = \{(y_2, \cdots, y_p)' \mid \sum_{j=2}^{p} y_j^2 \leqq (1-\rho_0)^{-1}\}.$$

Further put

$$\beta \equiv \beta(y_1, y_3, y_4, \cdots, y_p)$$

$$\equiv \{(1-\rho_0)^{-1}[1 - (1+(p-1)\rho_0)y_1^2] - \sum_{j=3}^{p} y_j^2\}^{1/2},$$

then we have

$$c(0,2,0,0,\cdots,0) = - \int_B \cdots \int \beta\phi(\beta)\phi(y_1)\phi(y_3)\phi(y_4)\cdots\phi(y_p)$$

$$dy_1 dy_3 dy_4 \cdots dy_p < 0$$

where

$$B = \{(y_1,y_3,y_4,\cdots,y_p)' \mid$$

$$(1-\rho_0)^{-1}[1 - (1+(p-1)\rho_0)y_1^2] \geqq \sum_{j=3}^{p} y_j^2\},$$

and

$$c(0,2,0,\cdots,0) = c(0,0,2,0,\cdots,0) = \cdots = c(0,0,\cdots,0,2).$$

Therefore the Hermite rank of $f(y_1, \cdots, y_p) = I[(y_1,\cdots,y_p)' \in \tilde{D}]$ is 2. If we determine $\{c_j\}$ in the theorem by $c_j = c(n_1,\cdots,n_p)$, where

$n_j = 2$, $n_i = 0$ for $i \neq j$, then Proposition 1 and Lemma 2 prove the finite dimensional convergence of $\Delta_t(\tau)$ to $\Delta(\tau)$.

It remains to prove the tightness of $\{\Delta_t(\tau), \tau \in [0, \infty)\}$. We have, for $0 < \tau_1 < \tau_2$

$$(2.8) \qquad E|\Delta_t(\tau_2) - \Delta_t(\tau_1)|^2$$

$$= \{\mathrm{Var}M(t)\}^{-1} E\left|\int_{t\tau_1}^{t\tau_2} (I[X(s) \in D] - EI[X(s) \in D])\, ds\right|^2.$$

Since the Hermite rank of $f(x_1,\cdots,x_p) \equiv I[(x_1,\cdots x_p)' \in D]$ is 2 in our case, as in the proof of Proposition 1, we have

$$(2.9) \qquad \mathrm{Var}M(t) \sim Ct^{2-2\alpha}L(t)^2$$

and

$$(2.10) \qquad E\left|\int_{t\tau_1}^{t\tau_2} (I[X(s) \in D] - EI[X(s) \in D])\, ds\right|^2$$

$$\leqq \text{const.} \times \int_0^{t(\tau_2-\tau_1)} (t(\tau_2-\tau_1) - s)s^{-2\alpha}L(s)^2\, ds$$

$$= \text{const.} \times \{t(\tau_2-\tau_1)\}^{2-2\alpha}L(t(\tau_2-\tau_1))^2.$$

It follows from (2.8)-(2.10) that

$$E|\Delta_t(\tau_2) - \Delta_t(\tau_1)|^2 \leqq \text{const.} \times (\tau_2-\tau_1)^{1+\gamma},$$

where $\gamma = 1 - 2\alpha > 0$, and hence the tightness of $\{\Delta_t(\tau)\}$ follows from the well-known criterion.

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