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Functional Iimit Theorems for Weighted Sums of I.I.D Random Variables

by

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1 Introduction

Let $\{X_i\}_{i=0}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables with common distribution function $F(x)$ belonging to the domain of attraction of a stable law with index α, $0 < \alpha \leqq 2$. Let $\{c_i(\lambda)\}_{i=0}^{\infty}$, $\lambda > 0$, be a set of nonnegative numbers. In this paper, We shall study functional limit theorems for weighted sums $\Sigma_{i=0}^{\infty} c_i(\lambda)X_i$, especially, for $\{c_i(\lambda)\}$, the weights concernig the classical summability methods such as Cesàro, Abel, Borel and Euler (for which see, for example, Hardy[7]).

The Cesàro sum of $\{X_i\}$ is, of course, one of the main topics in probability theory and the Abel sum, which is often called the discounted sum, has also been well studied at least when $E[X_1^2] < \infty$. In comparison to them, the Borel and Euler sums are less studied, but there are some references in several situations, (for example, the almost sure convergence is investigated in [3], [4], [10], and the central limit problem in [6]).

Shukri[13] studied the general weights $\{c_i(\lambda)\}$ and gave conditions on $\{c_i(\lambda)\}$ for the validity of the stable limit theorem and its local version. Embrechts and Maejima[6] considered the rate of convergence in the central limit theorem under higher moment conditions. Furthermore Omey[11] proved the stable limit theorem for the Abel sum. As far as the authors know, functional limit theorems have been studied only for the Cesàro sum (the ordinary partial sum), and our main concern in this paper is for other weighted sums.

Recently, the idea of using the convergence of point processes to prove functional limit theorems has successfully been developed by Durrett and Resnick[5] and Kasahara and Watanabe[9]. The validity of our main theorems is based on some recent results by Kasahara and Watanabe[9].

In section 2, we give some preliminary results, and in sections 3 and 4, we study the general weighted sums. Functional limit theorems for the Abel sum are given in section 5, and the Borel and Euler sums are considered in section 6. Some additional comments are given in section 7.

We finally remark that although we treat only independent random variables in this paper, the same idea allows us to study some cases of dependent random variables.

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2. Preliminaries

In this section, we shall give some preliminary results on random variables belonging to the domain of attraction of a stable law. What we are going to mention in this section should be well known, but we believe it worthwhile to restate them for understanding the following sections.

In order that $F(x)$ belongs to the domain of attraction of a stable law with index α, $0 < \alpha \leqq 2$, it is necessary and sufficient that there exists a regularly varying function $\varphi(\lambda)$ of degree $1/\alpha$ such that as $\lambda \to \infty$, when $\alpha \neq 2$,

$$(2.1)\qquad \begin{cases} \lambda\{1 - F(\varphi(\lambda)x)\} \to C_+ x^{-\alpha}, & x > 0, \\ \lambda F(\varphi(\lambda)x) \to C_- |x|^{-\alpha}, & x < 0, \\ C_+ \geqq 0, \quad C_- \geqq 0, \quad C_+ + C_- > 0, \end{cases}$$

and when $\alpha = 2$,

$$(2.2)\qquad \begin{cases} \lambda\{1 - F(\varphi(\lambda)x)\} \to 0, & x > 0, \\ \lambda F(\varphi(\lambda)x) \to 0, & x < 0, \\ \lambda\{\int_{|x|\leqq 1} x^2 dF(\varphi(\lambda)x) - (\int_{|x|\leqq 1} x dF(\varphi(\lambda)x))^2\} \to \sigma^2 \ (> 0). \end{cases}$$

In what follows, we set

$$\nu(dx) = \begin{cases} \alpha C_+ I(x>0) x^{-\alpha-1} dx + \alpha C_- I(x<0) |x|^{-\alpha-1} dx, & \text{if } \alpha \neq 2, \\ 0, \quad \text{if } \alpha = 2, \end{cases}$$

where $I(\cdot)$ is an indicator function, and let $N(dudx)$ be a Poisson random measure on $(0, \infty)\times(\mathbb{R}\setminus\{0\})$ with intensity measure $du\nu(dx)$ when $\alpha \neq 2$. In case $\alpha = 2$, we consider a standard Brownian motion $B(t)$ with

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$B(0) = 0$ a.s. instead of $N(dudx)$.

We note that if we let

$$\nu_\lambda(dx) = \lambda dF(\varphi(\lambda)x),$$

then (2.1) is written as

$$\nu_\lambda(dx) \to \nu(dx)$$

in the vague topology on $[-\infty, \infty]\setminus\{0\}$, and

$$\int_{|x|\leq a} x^2\nu_\lambda(dx) - \frac{1}{\lambda}\left(\int_{|x|\leq a} x\nu_\lambda(dx)\right)^2 \to \int_{|x|\leq a} x^2\nu(dx)$$

for any $a > 0$. (2.2) is also written as

$$\nu_\lambda(dx) \to 0 \quad \text{vaguely on } [-\infty, \infty]\setminus\{0\}$$

$$\int_{|x|\leq 1} x^2\nu_\lambda(dx) - \frac{1}{\lambda}\left(\int_{|x|\leq 1} x\nu_\lambda(dx)\right)^2 \to \sigma^2.$$

It is easily seen that if $0 < \alpha < \beta$,

$$\int_{|x|\leq a} |x|^\beta\nu_\lambda(dx) \to \int_{|x|\leq a} |x|^\beta\nu(dx)$$

for any $a > 0$, and if $0 < \beta < \alpha$,

$$\int_{|x|\geq a} |x|^\beta\nu_\lambda(dx) \to \int_{|x|\geq a} |x|^\beta\nu(dx)$$

for any $a > 0$.

Let

$$S_\lambda(t) = \frac{1}{\varphi(\lambda)} \sum_{i\leq\lambda t} X_i,$$

$$S_\lambda^\delta(t) = S_\lambda(t) - \frac{[\lambda t]}{\lambda} a_\lambda(\delta),$$

where $[u]$ denotes the integral part of u and

(2.3) $$a_\lambda(\delta) = \lambda \int_{|x| \leqq \delta} x dF(\varphi(\lambda)x), \quad \delta > 0,$$

and let

$$Z^\delta(t) = \begin{cases} \int_0^{t+} \int_{|x|>\delta} xN(dudx) + \int_0^{t+} \int_{|x| \leqq \delta} x\tilde{N}(dudx), & \text{if } \alpha \neq 2, \\ \sigma B(t), & \text{if } \alpha = 2, \end{cases}$$

where

$$\tilde{N}(dudx) = N(dudx) - \hat{N}(dudx),$$

$\hat{N}(dudx)$ being $EN(dudx) = du\nu(dx)$.

The following is the well-known result by Skorohod[14].

<u>Proposition 2.1</u> ([14]). <u>As</u> $\lambda \to \infty$,

$$S_\lambda^\delta(t) \underset{J_1}{\overset{\mathcal{D}}{\to}} Z^\delta(t),$$

<u>where</u> $\underset{J_1}{\overset{\mathcal{D}}{\to}}$ <u>denotes the weak convergence in</u> J_1<u>-topology</u>.

The convergence of $S_\lambda(t)$ itself is reduced to that of $a_\lambda(\delta)$. When $0 < \alpha < 1$, we always have

(2.4) $$a_\lambda(\delta) \to \int_{|x| \leqq \delta} x\nu(dx).$$

When $1 < \alpha \leq 2$, if we assume $\int_{-\infty}^{\infty} x dF(x) = 0$, then

$$(2.5)\qquad a_\lambda(\delta) = -\lambda \int_{|x|>\delta} x dF(\varphi(\lambda)x) \to -\int_{|x|>\delta} x\nu(dx).$$

When $\alpha = 1$, we cannot get anything automatically. In what follows, we sometime assume the following.

<u>Assumption (A)</u>. <u>When</u> $1 < \alpha \leq 2$, $\int_{-\infty}^{\infty} x dF(x) = 0$. <u>When</u> $\alpha = 1$, $C_+ = C_-$ <u>and</u> $a_\lambda(1) \to 0$ <u>as</u> $\lambda \to \infty$.

Throughout this paper, we define $Z(t)$ by

$$\int_0^{t+}\int xN(dudx), \quad \text{if} \quad 0 < \alpha < 1,$$

$$(2.6)\qquad \int_0^{t+}\int_{|x|>\delta} xN(dudx) + \int_0^{t+}\int_{|x|\leq\delta} x\tilde{N}(dudx), \quad \text{if} \quad \alpha = 1,$$

$$(2.7)\qquad \int_0^{t+}\int_{|x|>\delta} xN(dudx) - \int_0^{t}\int_{|x|>\delta} x\hat{N}(dudx) + \int_0^{t+}\int_{|x|\leq\delta} x\tilde{N}(dudx),$$

$$\text{if} \quad 1 < \alpha < 2$$

and

$$\sigma B(t), \quad \text{if} \quad \alpha = 2.$$

<u>Remark</u>. In (2.7), the integral does not depend on choosing $\delta > 0$, and it can be expressed as

$$\lim_{a\to\infty} \int_0^{t+}\int_{|x|\leq a} x\tilde{N}(dudx).$$

If we assume assumption (A), the same is true for the case $\alpha = 1$.

We then have from Proposition 2.1, (2.4) and (2.5) the following.

Proposition 2.2. Under assumption (A), as $\lambda \to \infty$

$$S_\lambda(t) \underset{J_1}{\overset{\mathcal{D}}{\to}} Z(t).$$

3. Partial sums with general weights

In this section, we shall study the limit behaviour of the partial sums with general weights as an application of a recent result by Kasahara and Watanabe[9].

Let $\{f_\lambda(u)\}_{\lambda>0}$ and $f(u)$ be measurable functions defined on $(0, \infty)$, and assume that

(B) $f_\lambda(u)$ is uniformly bounded on any finite intervals

and

(C) $f_\lambda(u)$ converges continuously to $f(u)$ on $(0, \infty)$ almost surely, namely, it holds for almost all u that for any $\{u_\lambda\}$ tending to u, $f_\lambda(u_\lambda)$ converges to $f(u)$.

Denote

$$X_\lambda(t) = \frac{1}{\varphi(\lambda)} \sum_{i \leqq \lambda t} f_\lambda(\tfrac{i}{\lambda}) X_i,$$

$$A_\lambda^\delta(t) = \frac{1}{\lambda} \left(\sum_{i \leqq \lambda t} f_\lambda(\tfrac{i}{\lambda}) \right) a_\lambda(\delta),$$

where $a_\lambda(\delta)$ is the one defined in (2.3), and set

$$X_\lambda^\delta(t) = X_\lambda(t) - A_\lambda^\delta(t).$$

The following is a weighted version of Proposition 2.1.

Theorem 3.1. *Suppose that assumptions (B) and (C) are satisfied. For any $\delta > 0$, if $\alpha \neq 2$,*

$$X_\lambda^\delta(t) \overset{\mathscr{D}}{\underset{J_1}{\to}} \int_0^{t+} \int_{|x|>\delta} f(u) x N(dudx) + \int_0^{t+} \int_{|x|\leqq\delta} f(u) x \tilde{N}(dudx),$$

<u>and if</u> $\alpha = 2$,

$$X_\lambda^\delta(t) \underset{J_1}{\overset{\mathcal{D}}{\to}} \sigma\int_0^t f(u)dB(u).$$

This theorem can be proved by the same idea as in Durrett and Resnick[5], and also it is a special case of a result in a forthcoming paper of Kasahara and Watanabe[9], which, for the reference, we state below in the context of independent random variables.

Let $\{\xi_{\lambda i}\}_{i=1}^\infty$, $\lambda > 0$, be a collection of random variables which are independent for each $\lambda > 0$, and let $\nu(dx)$ be a Borel measure such that $\nu((-\infty, -\varepsilon]\cup[\varepsilon, \infty)) < \infty$ for any $\varepsilon > 0$. Also let $\{g_\lambda(u,x)\}_{\lambda>0}$, $g(u,x)$, $\{f_\lambda(u,x)\}_{\lambda>0}$ and $f(u,x)$ be measurable functions defined on $(0, \infty)\times\mathbb{R}$ such that $f_\lambda(u,0) = 0$. Suppose the following conditions (i)-(v) to be satisfied:

(i) $$\sum_{i\leqq\lambda t} P(\xi_{\lambda i} > x) \to t\nu((x, \infty))$$

at continuity points $x > 0$ of ν and

$$\sum_{i\leqq\lambda t} P(\xi_{\lambda i} < x) \to t\nu((-\infty, x))$$

at continuity points $x < 0$ of ν.

(ii) $g_\lambda(u,x)$ and $f_\lambda(u,x)$ converges continuously to $g(u,x)$ and $f(u,x)$ almost surely with respect to $du\nu(dx)$, respectively.

(iii) $$\lim_{\varepsilon\downarrow 0} \overline{\lim_{\lambda\to\infty}} \sum_{i\leqq\lambda t} E[|g_\lambda(\tfrac{i}{\lambda}, \xi_{\lambda i})|I(|\xi_{\lambda i}|<\varepsilon)] = 0$$

for any $t > 0$.

(iv) $$\lim_{\varepsilon\downarrow 0}\overline{\lim_{\lambda\to\infty}}\Big|\sum_{i\leq\lambda t}\{E[f_\lambda(\tfrac{i}{\lambda},\xi_{\lambda i})^2 I(|\xi_{\lambda i}|<\varepsilon)] - [E[f_\lambda(\tfrac{i}{\lambda},\ \xi_{\lambda i})I(|\xi_{\lambda i}|<\varepsilon)]]^2\} - h(t)\Big| = 0$$

for some continuous function $h(t)$.

(v) For any $T > 0$,

$$\int_0^T\int_{\mathbb{R}\setminus\{0\}}\{|g(u,x)| + |f(u,x)|^2\}du\nu(dx) < \infty,$$

$$\sup_{\substack{0\leq u\leq T\\ x\in\mathbb{R}}}|f_\lambda(u,x)| < \infty$$

and

$$\lim_{\varepsilon\downarrow 0}\overline{\lim_{\lambda\to\infty}}\sup_{\substack{0\leq u\leq T\\ 0\leq|x|\leq\varepsilon}}|f_\lambda(u,x)| = 0.$$

Proposition 3.2 ([9]). Under the assumptions stated above,

$$\Big(\sum_{i\leq\lambda t}g_\lambda(\tfrac{i}{\lambda},\ \xi_{\lambda i}),\quad \sum_{i\leq\lambda t}\{f_\lambda(\tfrac{i}{\lambda},\ \xi_{\lambda i}) - E[f_\lambda(\tfrac{i}{\lambda},\ \xi_{\lambda i})]\}\Big)$$

$$\xrightarrow{\mathscr{D}}\Big(\int_0^{t+}\int_{\mathbb{R}\setminus\{0\}}g(u,x)N(dudx),\quad \int_0^{t+}\int_{\mathbb{R}\setminus\{0\}}f(u,x)\tilde{N}(dudx) + M(t)\Big)$$

as $\lambda\to\infty$ in J_1-topology in $D([0,\infty)\to\mathbb{R}^2)$, where $N(dudx)$ is the Poisson random measure with intensity measure $du\nu(dx)$ and $M(t)$ is a continuous martingale independent of $N(dudx)$ such that $\langle M\rangle(t) = h(t)$.

Proof of Theorem 3.1. To apply Proposition 3.2, put

$$g_\lambda(u,x) = f_\lambda(u)xI(|x|>\delta),$$

$$g(u,x) = f(u)xI(|x|>\delta),$$

$$f_\lambda(u,x) = f_\lambda(u)xI(|x|\leq\delta),$$

$$f(u,x) = f(u)xI(|x|\leq\delta)$$

and

$$\xi_{\lambda i} = \frac{1}{\varphi(\lambda)} X_i.$$

Then we can rewrite

$$\begin{aligned}
X_\lambda^\delta(t) &= X_\lambda(t) - A_\lambda^\delta(t) \\
&= \sum_{i\leq\lambda t} f_\lambda(\tfrac{i}{\lambda})\{\frac{X_i}{\varphi(\lambda)} - \int_{|x|\leq\delta} x dF(\varphi(\lambda)x)\} \\
&= \sum_{i\leq\lambda t} f_\lambda(\tfrac{i}{\lambda})\xi_{\lambda i} I(|\xi_{\lambda i}|>\delta) \\
&\quad + \sum_{i\leq\lambda t} f_\lambda(\tfrac{i}{\lambda})\{\xi_{\lambda i} I(|\xi_{\lambda i}|\leq\delta) - E[\xi_{\lambda i} I(|\xi_{\lambda i}|\leq\delta)]\} \\
&= \sum_{i\leq\lambda t} g_\lambda(\tfrac{i}{\lambda}, \xi_{\lambda i}) + \sum_{i\leq\lambda t} \{f_\lambda(\tfrac{i}{\lambda}, \xi_{\lambda i}) - E[f_\lambda(\tfrac{i}{\lambda}, \xi_{\lambda i})]\},
\end{aligned}$$

to which Proposition 3.2 can be applied. □

Under assumption (A), we obtain the convergence of $X_\lambda(t)$ itself as follows. Define

$$\Delta(t) = \int_0^t f(u)dZ(u)$$

by

$$\int_0^{t+}\int_{\mathbb{R}} f(u)xN(dudx), \quad \text{if} \quad 0<\alpha<1,$$

$$\int_0^{t+}\int_{|x|>\delta} f(u)xN(dudx) + \int_0^{t+}\int_{|x|\leqq\delta} f(u)x\tilde{N}(dudx), \quad \text{if} \quad \alpha=1,$$

$$\int_0^{t+}\int_{|x|>\delta} f(u)xN(dudx) - \int_0^{t}\int_{|x|>\delta} f(u)x\hat{N}(dudx)$$

$$+\int_0^{t+}\int_{|x|\leqq\delta} f(u)x\tilde{N}(dudx), \quad \text{if} \quad 1<\alpha<2,$$

$$\sigma\int_0^t f(u)dB(u), \quad \text{if} \quad \alpha=2.$$

Theorem 3.3. Under assumptions (A), (B) and (C),

$$X_\lambda(t) \underset{J_1}{\overset{\mathscr{D}}{\to}} \Delta(t).$$

Proof. Because of Theorem 3.1, it is enough to calculate the limit of $A_\lambda^\delta(t)$. In the form of $A_\lambda^\delta(t)$ in (3.1), $a_\lambda(\delta)$ has the limit under assumptions (A), which is

$$\int_{|x|\leqq\delta} x\nu(dx), \quad \text{if} \quad 0<\alpha<1, \quad (\text{by } (2.4)),$$

$$-\int_{|x|>\delta} x\nu(dx), \quad \text{if} \quad 1<\alpha<2, \quad (\text{by}(2.5)),$$

$$0, \quad \text{if} \quad \alpha=1 \quad \text{or} \quad 2.$$

Hence it remains to calculate the limit of $\frac{1}{\lambda}\Sigma_{i\leqq\lambda t}f_\lambda(\frac{i}{\lambda})$. However, it follows from assumptions (B) and (C) that

$$\frac{1}{\lambda}\sum_{i\leqq\lambda t} f_\lambda(\frac{i}{\lambda}) = \int_0^{([\lambda t]+1)/\lambda} f_\lambda(\frac{[\lambda u]}{\lambda})du \to \int_0^t f(u)du$$

as $\lambda \to \infty$. Hence

$$\lim_{\lambda\to\infty} A_\lambda^\delta(t) = \begin{cases} \int_0^t \int_{|x|\leq\delta} f(u)x\hat{N}(dudx), & \text{if } 0 < \alpha < 1, \\ -\int_0^t \int_{|x|>\delta} f(u)x\hat{N}(dudx), & \text{if } 1 < \alpha < 2, \\ 0, \quad \text{if } \alpha = 1 \text{ or } 2, \end{cases}$$

and in fact, this convergence is uniform on any bounded interval of t. This together with Theorem 3.1 implies the theorem.

We conclude this section with mentioning convergence of the joint distribution of S_λ^δ (S_λ) and X_λ^δ (X_λ). Since the statements are similar, we state only the case of (S_λ, X_λ).

Theorem 3.4. *Under assumptions* (A), (B) *and* (C),

$$(S_\lambda(t), \ X_\lambda(t)) \underset{J_1}{\overset{\mathcal{D}}{\to}} (Z(t), \ \int_0^t f(u)dZ(u)).$$

Proof. The proof of the finite-dimensional convergence is reduced to that of Theorem 3.3, by the use of the Cramér-Wold device. The tightness is also reduced to that of the process considered in Theorem 3.3, because the tightness of $(S_\lambda(t), X_\lambda(t))$ is equivalent to that of any linear combination of components, (see Appendix (A.26) and (A.28) in Holley and Stroock[8]). □

The point we want to emphasize in this theorem is that the limit of $X_\lambda(t)$ is a functional of the limit of $S_\lambda(t)$.

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4. Infinite sums with general weights

In this section, we shall study the infinite sums with general weights. In addition to assumptions on $\{f_\lambda(u)\}_{\lambda>0}$ stated in the previous section, we further assume that

(D) $$\sup_{\substack{u \geqq 1 \\ \lambda \geqq 0}} |f_\lambda(u)| < \infty$$

and

(E) there exists $\gamma > \frac{1}{\alpha} - 1$ when $0 < \alpha \leqq 1$ or $\gamma \geqq 0$ when $1 < \alpha \leqq 2$ such that

$$\sup_\lambda \int_1^\infty |f_\lambda(\frac{[\lambda u]}{\lambda})| u^\gamma du < \infty. \tag{4.1}$$

It is easiliy verified that (4.1) is equivalent to

$$\sup_\lambda \frac{1}{\lambda^{\gamma+1}} \sum_{i \geqq \lambda} |f_\lambda(\frac{i}{\lambda})| i^\gamma < \infty. \tag{4.2}$$

Lemma 4.1. *Under assumption* (E), *there exists* β, $0 < \beta < \alpha$, *such that*

$$\sup_\lambda \int_1^\infty |f_\lambda(\frac{[\lambda u]}{\lambda})|^\beta du < \infty. \tag{4.3}$$

Proof. When $1 < \alpha \leqq 2$, (4.3) is true with $\beta = 1 < \alpha$. When $0 < \alpha \leqq 1$, take β in such a way that $(\gamma+1)^{-1} < \beta < \alpha$, which is possible because $(\gamma+1)^{-1} < \alpha$. Then, for any function $g(u) \geqq 0$, we have by the Hölder inequality that

$$\int_1^\infty g(u)^\beta du = \int_1^\infty (g(u)u^\gamma)^\beta u^{-\gamma\beta} du$$

$$\leq (\int_1^\infty g(u)u^\gamma du)^\beta (\int_1^\infty u^{-\gamma\beta/(1-\beta)} du)^{1-\beta}$$

$$\leq \text{const.}\times(\int_1^\infty g(u)u^\gamma du)^\beta,$$

which concludes the lemma. □

We now define

$$\Delta_\lambda^\delta \equiv \lim_{t\to\infty} X_\lambda^\delta(t) = \frac{1}{\varphi(\lambda)} \sum_{i=1}^\infty f_\lambda(\frac{i}{\lambda})(X_i - \frac{\varphi(\lambda)}{\lambda} a_\lambda(\delta))$$

and

$$\Delta_\lambda \equiv \lim_{t\to\infty} X_\lambda(t) = \frac{1}{\varphi(\lambda)} \sum_{i=1}^\infty f_\lambda(\frac{i}{\lambda})X_i.$$

The existence of almost sure limits of these two random varialbes is assured by the following lemma.

Lemma 4.2. Under assumptions (A), (D) and (E), Δ_λ^δ and Δ_λ exist almost surely.

Proof. We first show the existence of the limit Δ_λ^δ. Let

(4.4) $$Y_i^\lambda = X_i I(|X_i| > \delta\varphi(\lambda))$$

and

(4.5) $$Z_i^\lambda = X_i I(|X_i| \leq \delta\varphi(\lambda)) - E[X_i I(|X_i| \leq \delta\varphi(\lambda))].$$

Note that $\{Y_i^\lambda\}_{i=1}^\infty$ and $\{Z_i^\lambda\}_{i=1}^\infty$ are sequences of independent and identically distributed random variables and that

$$E[Z_i^\lambda] = 0,$$

$$\mathrm{Var}[Z_i^\lambda] = E[X_i^2 I(|X_i| \leqq \delta\varphi(\lambda))] - \{E[X_i I(|X_i| \leqq \delta\varphi(\lambda))]\}^2.$$

Then we have

$$X_\lambda^\delta(t) = \frac{1}{\varphi(\lambda)} \sum_{i \leqq \lambda t} f_\lambda(\tfrac{i}{\lambda}) Y_i^\lambda + \frac{1}{\varphi(\lambda)} \sum_{i \leqq \lambda t} f_\lambda(\tfrac{i}{\lambda}) Z_i^\lambda \equiv I_1(t) + I_2(t),$$

say.

As to $I_2(t)$,

$$E[(\sum_{i=1}^\infty f_\lambda(\tfrac{i}{\lambda}) Z_i^\lambda)^2] = E[|Z_1^\lambda|^2] \times \sum_{i=1}^\infty |f_\lambda(\tfrac{i}{\lambda})|^2,$$

where $E[|Z_1^\lambda|^2] < \infty$. By assumptions (D) and (E) ((4.2)), we see that $\Sigma_{i=1}^\infty |f_\lambda(\frac{i}{\lambda})|^2 < \infty$ and hence $I_2(t)$ converges as $t \to \infty$ in the sense of mean square. Since $I_2(t)$ is the sum of independent random variables, $I_2(t)$ also converges almost surely.

Next consider $I_1(t)$. Suppose $1 < \alpha \leqq 2$. We have

$$E[\sum_{i=1}^\infty |f_\lambda(\tfrac{i}{\lambda})| |Y_i^\lambda|] = E[|Y_1^\lambda|] \times \sum_{i=1}^\infty |f_\lambda(\tfrac{i}{\lambda})|,$$

where $E[|Y_1^\lambda|] < \infty$ since $\alpha > 1$ and $\Sigma_{i=1}^\infty |f_\lambda(\frac{i}{\lambda})| < \infty$ by (4.2) again. Therefore $I_1(t)$ converges almost surely, by the same reasoning as before.

Suppose $0 < \alpha \leqq 1$. Take β chosen in (4.3). Since $\beta < \alpha \leqq 1$, then we have

$$E[(\sum_{i=1}^{\infty}|f_\lambda(\tfrac{i}{\lambda})Y_i^\lambda|)^\beta] \leqq E[|Y_1^\lambda|^\beta]\times\sum_{i=1}^{\infty}|f_\lambda(\tfrac{i}{\lambda})|^\beta,$$

where $E[|Y_1^\lambda|^\beta] < \infty$ and $\Sigma_{i=1}^{\infty}|f_\lambda(\frac{i}{\lambda})|^\beta < \infty$ by Lemma 4.1. Thus $I_1(t)$ again converges almost surely.

The almost sure convergence of Δ_λ is given by that of Δ_λ^δ shown above and the absolute convergence of $\Sigma_{i=1}^{\infty}|f_\lambda(\frac{i}{\lambda})|$. □

We next define the limiting random variables

$$\Delta^\delta = \begin{cases} \int_0^\infty\int_{|x|>\delta} f(u)xN(dudx) + \int_0^\infty\int_{|x|\leqq\delta} f(u)x\tilde{N}(dudx), & \text{if } \alpha \neq 2, \\ \sigma\int_0^\infty f(u)dB(u), \quad \text{if } \alpha = 2. \end{cases}$$

The last integral is the usual Wiener integral. (Note that, under our assumptions, $f(u)$ is bounded on $(0, \infty)$ and $\int_{\mathbb{R}}|f(u)|du < \infty$, so that $\int_{\mathbb{R}}|f(u)|^2du < \infty$.) Δ^δ in case of $\alpha \neq 2$ is well-defined as follows.

Lemma 4.3. Under assumptions (C), (D) and (E), for $\alpha \neq 2$,

$$(4.6)\qquad \Delta^\delta = \lim_{t\to\infty}\{\int_0^{t+}\int_{|x|>\delta} f(u)xN(dudx) + \int_0^{t+}\int_{|x|\leqq\delta} f(u)x\tilde{N}(dudx)\}$$

exists in probability.

Proof. The proof is carried out in the same manner as in the proof of the previous lemma.

We first consider the second integral on the right hand side of (4.6). We have

$$E[(\int_1^\infty \int_{|x|>\delta} f(u)x\tilde{N}(dudx))^2] = \int_1^\infty f(u)^2 du \times \int_{|x|\leqq\delta} x^2\nu(dx) < \infty,$$

and thus

$$\lim_{t\to\infty} \int_0^{t+} \int_{|x|\leqq\delta} f(u)x\tilde{N}(dudx)$$

exists in probability.

The first integral on the right hand side of (4.6) is treated as follows. If $1 < \alpha < 2$,

$$E[\int_1^\infty \int_{|x|>\delta} |f(u)||x|N(dudx)] = \int_1^\infty |f(u)|du \times \int_{|x|>\delta} |x|\nu(dx) < \infty.$$

In case $0 < \alpha \leqq 1$, if we take β in (4.3), then we have

$$E[|\int_1^\infty \int_{|x|>\delta} f(u)xN(dudx)|^\beta] \leqq E[\int_1^\infty \int_{|x|>\delta} |f(u)|^\beta |x|^\beta N(dudx)]$$

$$= \int_1^\infty |f(u)|^\beta du \times \int_{|x|>\delta} |x|^\beta \nu(dx) < \infty,$$

where we have used $\int_1^\infty |f(u)|^\beta du < \infty$ by Lemma 4.1. In any case, $\int_0^{t+} \int_{|x|>\delta} f(u)xN(dudx)$ converges in probability as $t \to \infty$. □

The proofs of Lemma 4.2 and 4.3 also verify the following.

Lemma 4.4. *Under assumptions* (A)-(E),

$$\Delta \equiv \int_0^\infty f(u)dZ(u) = \lim_{t\to\infty} \int_0^t f(u)dZ(u)$$

exists in probability.

Under the above preliminaries, we easily see the following.

Theorem 4.1. Under assumptions (B)-(E), as $\lambda \to \infty$

$$(\Delta_\lambda^\delta, \quad S_\lambda^\delta(t)) \overset{\mathscr{D}}{\to} (\Delta^\delta, \quad Z^\delta(t))$$

in $\mathbb{R} \times D([0, \infty) \to \mathbb{R})$. If we further assume (A), then

$$(\Delta_\lambda, \quad S_\lambda(t)) \overset{\mathscr{D}}{\to} (\Delta, \quad Z(t))$$

in $\mathbb{R} \times D([0, \infty) \to \mathbb{R})$, namely,

$$\left(\frac{1}{\varphi(\lambda)} \sum_{i=1}^{\infty} f_\lambda(\tfrac{i}{\lambda}) X_i, \quad \frac{1}{\varphi(\lambda)} \sum_{i \leq \lambda t} X_i \right) \overset{\mathscr{D}}{\to} \left(\int_0^\infty f(u)\,dZ(u), \quad Z(t) \right).$$

Corollary 4.2. Suppose $\{f_\lambda^k(u)\}_{\lambda>0}$, $\{f^k(u)\}$, $k = 1, 2, \cdots, d$, satisfy assumptions (B)-(E). Then under assumption (A), as $\lambda \to \infty$

$$\left(\frac{1}{\varphi(\lambda)} \sum_{i=1}^{\infty} f_\lambda^1(\tfrac{i}{\lambda}) X_i, \quad \cdots, \quad \frac{1}{\varphi(\lambda)} \sum_{i=1}^{\infty} f_\lambda^d(\tfrac{i}{\lambda}) X_i \right)$$

$$\to \left(\int_0^\infty f^1(u)\,dZ(u), \quad \cdots, \quad \int_0^\infty f^d(u)\,dZ(u) \right).$$

5. The Abel sum

In this section, we apply the results given in the previous section to the Abel sum. Let

$$(5.1)\qquad f_\lambda(u) = f_\lambda(u;\ \gamma,\ t) = u^\gamma e^{-tu}, \quad \gamma > -\frac{1}{2}, \quad t > 0.$$

The case $\gamma = 0$ is the ordinary Abel weight. Set

$$(5.2)\qquad \Delta_\lambda \equiv \Delta_\lambda(\gamma,t) \equiv \frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} (\tfrac{i}{\lambda})^\gamma e^{-ti/\lambda} X_i$$

and

$$\Delta_\lambda^\delta \equiv \Delta_\lambda^\delta(\gamma,t) \equiv \frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} (\tfrac{i}{\lambda})^\gamma e^{-ti/\lambda} (X_i - \frac{\varphi(\lambda)}{\lambda} a_\lambda(\delta)),$$

where $a_\lambda(\delta)$ is the same one as in the previous sections.

In the rest part of this paper, we always assume assumption (A) for simplicity. However, of course, the problems without this assumption can be treated in the same manner as we did in the previous sections.

Theorem 5.1. *Under assumption (A), as* $\lambda \to \infty$,

$$\frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} (\tfrac{i}{\lambda})^\gamma e^{-ti/\lambda} X_i \to \int_0^\infty u^\gamma e^{-tu} dZ(u)$$

in the sense of convergence of joint distributions with respect to any finite number of $\gamma > -\frac{1}{2}$ *and* $t > 0$.

Proof. When $\gamma \geq 0$, since it can be easily checked that $f_\lambda(u)$ in (5.1) satisfies assumptions (B)-(E), Corollary 4.2 gives us the statement of the theorem. When $-\frac{1}{2} < \gamma < 0$, although $f_\lambda(u)$ is unbounded around 0, the assertion is also given by the standard argument (cf. Theorem 4.2 in [1]). □

<u>Remark</u>. If we replace $e^{-1/\lambda}$ by z in (5.2), we get the ordinary expression of the Abel sum and obtain that under assumption (A), as $z \to 1-0$,

$$\frac{(1-z)^\gamma}{\varphi(\frac{1}{1-z})} \sum_{i=0}^{\infty} i^\gamma z^{ti} X_i \to \int_0^\infty u^\gamma e^{-tu} dZ(u)$$

in the same sense as in Theorem 5.1.

We next fix $\gamma > -\frac{1}{2}$ and regard $\Delta_\lambda(\gamma,t)$ and $\Delta_\lambda^\delta(\gamma,t)$ as processes of $t > 0$. The following theorem on the weak convergence of $\{\Delta_\lambda(\gamma,t),\ t>0\}_{\lambda>0}$ is one of our main results in this paper.

<u>Theorem 5.2</u>. <u>Fix</u> $\gamma > -\frac{1}{2}$ <u>and</u> $\varepsilon > 0$ <u>arbitrarily</u>. <u>Then</u> $\Delta_\lambda(\gamma,t)$ <u>and</u> $\Delta_\lambda^\delta(\gamma,t)$ <u>are tight in</u> $C([\varepsilon,\infty))$ <u>as processes of</u> $t > 0$. <u>In particular, if we assume</u> (A), <u>then as</u> $\lambda \to \infty$

$$\{\Delta_\lambda(\gamma,t),\ t>0\} \to \{\int_0^\infty u^\gamma e^{-tu} dZ(u),\ t>0\}$$

<u>in</u> $C([\varepsilon,\infty))$.

This theorem extremely extends a result of Omey[11]. For proving the theorem, we need a lemma. Although we think that this lemma should be known, we cannot find the explicit proof in the literature. So, we give its proof below.

Lemma 5.1. Suppose that $\{X_n(t),\ t \in [0, 1]\}_{n>0}$ are monotone in t, $\{X(t),\ t \in [0, 1]\}$ is continuous a.s. and that $X_n(t)$ converges as $n \to \infty$ to $X(t)$ in the sense of all finite-dimensional distributions. Then $X_n(t)$ converges to $X(t)$ weakly in $D([0, 1] \to \mathbb{R})$.

Proof. By Theorem 15.5 in Billingsley[1], it suffices to check that

(i) for any $\eta > 0$, there exists $a > 0$ such that

$$P(|X_n(0)| > a) \leqq \eta, \quad n \geqq 1,$$

and

(ii) for any $\varepsilon > 0$ and $\eta > 0$, there exist δ with $0 < \delta < 1$ and n_0 such that

$$P(\sup_{\substack{0 \leqq s < t \leqq 1 \\ |s-t| \leqq \delta}} |X_n(t) - X_n(s)| \geqq \varepsilon) \leqq \eta, \quad n \geqq n_0.$$

Since (i) is obvious, we need only to show (ii). Without loss of generality, we assume that $X_n(t)$ is nondecreasing in t for each n. By the pathwise continuity of $X(t)$, for any $\varepsilon > 0$ and $\eta > 0$, there exists a division of $[0, 1]$: $0 = t_0 < t_1 < \cdots < t_d = 1$ such that

$$(5.3) \qquad P(\max_{1 \leqq i \leqq d} |X(t_i) - X(t_{i-1})| \geqq \frac{\varepsilon}{2}) \leqq \frac{\eta}{2}.$$

Hence it follows from the finite-dimensional convergence of $\{X_n(t)\}$ and (5.3) that there exists an n_0 such that for all $n \geqq n_0$,

$$(5.4) \qquad P(\max_{1 \leqq i \leqq d} |X_n(t_i) - X_n(t_{i-1})| \geqq \frac{\varepsilon}{2}) \leqq \eta.$$

Let $\delta = \min_{1 \leqq i \leqq d}(t_i - t_{i-1}) > 0$, and take s and t such that

$0 \leqq s < t \leqq 1$ and $t - s < \delta$. Then if $s \in [t_i, t_{i+1})$ for some i, then $t \in [t_i, t_{i+2})$. By the assumption that $X_n(t)$ is nondecreasing in t, we have

$$(5.5) \qquad 0 \leqq X_n(t) - X_n(s) \leqq X_n(t_{i+2}) - X_n(t_i)$$

$$\leqq 2 \max_{1 \leqq i \leqq d} |X_n(t_i) - X_n(t_{i-1})|.$$

Consequently by (5.4) and (5.5), we have

$$P(\sup_{\substack{0 \leqq s < t \leqq 1 \\ t-s<\delta}} |X_n(t) - X_n(s)| \geqq \varepsilon)$$

$$\leqq P(\max_{1 \leqq i \leqq d} |X_n(t_i) - X_n(t_{i-1})| \geqq \frac{\varepsilon}{2}) \leqq \eta$$

for any $n \geqq n_0$. The proof of the lemma is thus complete. □

<u>Proof of Theorem 5.2</u>. It is enough to prove the tightness of Δ_λ^δ. As in (4.4) and (4.5), we let

$$Y_i^\lambda = X_i I(|X_i| > \delta\varphi(\lambda))$$

and

$$Z_i^\lambda = X_i I(|X_i| \leqq \delta\varphi(\lambda)) - E[X_i I(|X_i| \leqq \delta\varphi(\lambda))].$$

We divide $\Delta_\lambda^\delta(\gamma, t)$ into three parts:

$$(5.6) \qquad \Delta_\lambda^\delta(\gamma, t) \equiv \Delta_\lambda^{(1)}(t) + \Delta_\lambda^{(2)}(t) + \Delta_\lambda^{(3)}(t),$$

where

$$\Delta_\lambda^{(1)}(t) = \frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} (\frac{i}{\lambda})^\gamma e^{-ti/\lambda} (Y_i^\lambda \vee 0),$$

$$\Delta_\lambda^{(2)}(t) = \frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} (\frac{i}{\lambda})^\gamma e^{-ti/\lambda} (Y_i^\lambda \wedge 0)$$

and

$$\Delta_\lambda^{(3)}(t) = \frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} (\frac{i}{\lambda})^\gamma e^{-ti/\lambda} Z_i^\lambda.$$

If we look at the proof of Theorem 5.1, we have

$$\Delta_\lambda^{(1)}(t) \to \int_0^\infty \int_{x>\delta} u^\gamma e^{-tu} xN(dudx)$$

and

$$\Delta_\lambda^{(2)}(t) \to \int_0^\infty \int_{x<-\delta} u^\gamma e^{-tu} xN(dudx)$$

in the sense of convergence of all finite-dimensional distributions. Since $\Delta_\lambda^{(i)}(t)$, $i = 1, 2$, are monotone in t and the corresponding limiting processes are continuous a.s., then it follows from Lemma 5.1 that $\Delta_\lambda^{(i)}(t)$, $i = 1, 2$, are tight in $D([\varepsilon, \infty))$. Since $\Delta_\lambda^{(i)}(t)$, $i = 1, 2$, are continuous processes, they are also tight in $C([\varepsilon, \infty))$.

To prove the tightness of $\Delta_\lambda^{(3)}(t)$, let $0 < \varepsilon \leqq s < t < \infty$. We have

$$E[(\Delta_\lambda^{(3)}(t) - \Delta_\lambda^{(3)}(s))^2]$$

$$= \frac{1}{\varphi(\lambda)^2} \sum_{i=0}^{\infty} (\frac{i}{\lambda})^{2\gamma} (e^{-ti/\lambda} - e^{-si/\lambda})^2 E[(Z_0^\lambda)^2].$$

Noting that

$$|e^{-ut} - e^{-us}| \leqq e^{-\varepsilon u} u|t-s|,$$

we have

$$E[(\Delta_\lambda^{(3)}(t) - \Delta_\lambda^{(3)}(s))^2]$$
$$\leqq \frac{1}{\varphi(\lambda)^2} (t-s)^2 \sum_{i=0}^{\infty} (\frac{i}{\lambda})^{2\gamma+2} e^{-2\varepsilon i/\lambda} E[(Z_0^\lambda)^2].$$

Hence as $\lambda \to \infty$

$$\frac{1}{\lambda} \sum_{i=0}^{\infty} (\frac{i}{\lambda})^{2\gamma+2} e^{-2\varepsilon i/\lambda} \to \int_0^\infty u^{2\gamma+2} e^{-2\varepsilon u} du < \infty$$

and

$$\frac{\lambda}{\varphi(\lambda)^2} E[(Z_0^\lambda)^2] = \int_{|x|\leqq\delta} x^2 dF(\varphi(\lambda)x) - \lambda(\int_{|x|\leqq\delta} x dF(\varphi(\lambda)x))^2$$
$$\to \begin{cases} \int_{|x|\leqq\delta} x^2 \nu(dx), & \text{if } \alpha \neq 2, \\ \sigma^2, & \text{if } \alpha = 2. \end{cases}$$

Therefore we have

$$E[(\Delta_\lambda^{(3)}(t) - \Delta_\lambda^{(3)}(s))^2] \leqq \text{const.}\times(t-s)^2,$$

so that $\Delta_\lambda^{(3)}(t)$ is tight in $C([\varepsilon, \infty))$ by Theorem 12.3 in [1]. Hence each summand of $\Delta_\lambda^\delta(\gamma,t)$ in (5.6) is tight in $C([\varepsilon, \infty))$ and so is $\Delta_\lambda^\delta(\gamma,t)$ by the definition of the tightness in the space C. □

We conclude this section with the following example of $f_\lambda(u)$ which is related to the Valiron sum (see, for example, [2]).

Let

$$f_\lambda(u) = f_\lambda(u;\ t) = e^{-tu^2}, \quad t > 0.$$

Then we have, under assumption (A), that

$$\Delta_\lambda(t) \equiv \frac{1}{\varphi(\lambda)} \sum_{i=0}^{\infty} e^{-t(i/\lambda)^2} X_i$$

converges weakly to as $\lambda \to \infty$ to

$$\Delta(t) \equiv \begin{cases} \int_0^\infty \int_{-\infty}^\infty e^{-tu^2} x N(dudx), & \text{if } 0 < \alpha < 1, \\ \int_0^\infty \int_{|x|>\delta} e^{-tu^2} x N(dudx) + \int_0^\infty \int_{|x|\leqq\delta} e^{-tu^2} x \tilde{N}(dudx), & \text{if } \alpha = 1, \\ \lim_{a\to\infty} \int_0^\infty \int_{|x|\leqq a} e^{-tu^2} x \tilde{N}(dudx), & \text{if } 1 < \alpha < 2, \\ \sigma \int_0^\infty e^{-tu^2} dB(u), & \text{if } \alpha = 2 \end{cases}$$

in $C([\varepsilon, \infty))$ for any $\varepsilon > 0$.

6. The Borel and Euler sums

In this section, we study the Borel and Euler sums. We start with the "random-walk" method, (see [3]). Let $\{\xi_i\}_{i=1}^{\infty}$ be a sequence of independent, aperiodic, nonnegative integer valued random variables having identical distribution. We assume throughout that ξ_1 has moments of any order and $\mu = E[\xi_1]$, $\tau^2 = \mathrm{Var}[\xi_1]$. Set $N_n = \xi_1 + \xi_2 + \cdots + \xi_n$. We call the weights

$$c_i(n) = P(N_n = i)$$

the random-walk method, which is related to a large class of summability methods including the Borel and Euler. (See also [2].) Our concern is the limiting distribution of

$$\Delta_n = \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} P(N_n = i)\ X_i .$$

To state the theorem, we denote

$$\Delta_\tau \equiv \begin{cases} \int_0^\infty \int_{-\infty}^\infty f_\tau(u) x N(dudx), & \text{if } 0 < \alpha < 1, \\ \int_0^\infty \int_{|x|>\delta} f_\tau(u) x N(dudx) + \int_0^\infty \int_{|x|\leqq\delta} f_\tau(u) x \tilde{N}(dudx), & \text{if } \alpha = 1, \\ \lim_{a\to\infty} \int_0^\infty \int_{|x|\leqq a} f_\tau(u) x \tilde{N}(dudx), & \text{if } 1 < \alpha < 2, \\ \sigma \int_0^\infty f_\tau(u) dB(u), & \text{if } \alpha = 2, \end{cases}$$

where

$$f_\tau(u) = \frac{1}{\sqrt{2\pi}\tau} e^{-u^2/(2\tau^2)}.$$

Our first theorem in this section is the following.

<u>Theorem 6.1</u>. <u>Suppose assumption</u> (A) <u>to be satisfied</u>. <u>Under the situation stated above, as</u> $n \to \infty$,

$$\Delta_n \overset{\mathcal{D}}{\to} \Delta_\tau + \Delta'_\tau,$$

<u>where</u> Δ'_τ <u>is an independent copy of</u> Δ_τ.

<u>Proof</u>. We divide Δ_n in the form of

$$\Delta_n = \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{[n\mu]-1} P(N_n = i)\ X_i + \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=[n\mu]}^{\infty} P(N_n = i)\ X_i$$

$$\equiv J_1(n) + J_2(n),$$

say. We first examine $J_2(n)$. Note that

$$\sum_{i=[n\mu]}^{\infty} P(N_n = i)\ X_i$$

is distributed as

$$\sum_{i=0}^{\infty} P(N_n = [n\mu] + i)\ X_i.$$

Then we have

$$J_2(n) = \frac{1}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} f_{\sqrt{n}}\left(\frac{i}{\sqrt{n}}\right) X_i,$$

where

$$f_{\sqrt{n}}(u) = \sqrt{n}\, P(N_n = [n\mu] + [\sqrt{n}u]).$$

To apply Theorem 4.1, we have to check that $f_{\sqrt{n}}(u)$ satisfies assumptions (B)-(E). The local limit theorem for $\{\xi_i\}$ gives us that as $n \to \infty$

$$f_{\sqrt{n}}\left(\frac{[\sqrt{n}u]}{\sqrt{n}}\right) = \sqrt{n}\, P(N_n = [n\mu] + [\sqrt{n}u])$$

$$\to \frac{1}{\sqrt{2\pi}\tau} e^{-u^2/(2\tau^2)} = f_\tau(u)$$

uniformly in u. Hence assumptions (B)-(D) are fulfilled. Assumptiom (E) ((4.2)) can be checked as follows. Let $\gamma > (\frac{1}{\alpha} - 1) \vee 0$. We have

$$(6.1) \qquad \frac{1}{(\sqrt{n})^{\gamma+1}} \sum_{i \geqq \sqrt{n}} f_{\sqrt{n}}\left(\frac{i}{\sqrt{n}}\right) i^\gamma \leqq \sum_{i=-\infty}^{\infty} \frac{1}{\sqrt{n}} f_{\sqrt{n}}\left(\frac{i}{\sqrt{n}}\right) \left|\frac{i}{\sqrt{n}}\right|^\gamma$$

$$= \sum_{i=-\infty}^{\infty} P(N_n = [n\mu] + i) \left|\frac{i}{\sqrt{n}\tau}\right|^\gamma \tau^\gamma$$

$$= \tau^\gamma E\left[\left|\frac{N_n - [n\mu]}{\sqrt{n}\tau}\right|^\gamma\right]$$

$$\to \tau^\gamma \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |x|^\gamma e^{-x^2/2}\, dx$$

as $n \to \infty$, provided that $E[|\xi_1|^\gamma] < \infty$. (See, for example, Petrov[10], p.103.) Therefore assumption (E) is also satisfied. It thus follows from Theorem 4.1 that $J_2(n)$ converges weakly to Δ_τ as $n \to \infty$.

$J_1(n)$ can be handled similarly. In fact, it is easily verified that

$$J_1(n) \overset{\mathcal{L}}{\sim} \frac{1}{\varphi(\sqrt{n})} \sum_{i=0}^{[n\mu]-1} \bar{f}_{\sqrt{n}}\left(\frac{i}{\sqrt{n}}\right) X_i \overset{\mathcal{L}}{\sim} \frac{1}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} \bar{f}_{\sqrt{n}}\left(\frac{i}{\sqrt{n}}\right) X_i,$$

where

$$\bar{f}_{\sqrt{n}}(u) = \sqrt{n}\, P(N_n = [n\mu] - [\sqrt{n}u]).$$

Since $\bar{f}_{\sqrt{n}}(u)$ satisfies assumptions (B)-(E) as $f_{\sqrt{n}}(u)$, then we see that $J_1(n)$ also converges weakly to Δ_τ. The independence of $J_1(n)$ and $J_2(n)$ concludes our statement. □

Stable limit theorems for the Borel and Euler sums $(e^{-\lambda}\sum_{i=0}^{\infty}(\lambda^i/i!)X_i$ and $\sum_{i=0}^{n}\binom{n}{i}p^i(1-p)^{n-i}X_i$, respectively) are given as corollaries of Theorem 6.1. (The central limit theorem and its rates of convergence are discussed in [6].)

Theorem 6.2. Let $0 < p < \infty$. Then we have

$$\Delta_p^1(\lambda) \equiv \frac{\sqrt{\lambda}}{\varphi(\sqrt{\lambda})} e^{-\lambda p} \sum_{i=0}^{\infty} \frac{(\lambda p)^i}{i!} X_i \to \Delta_\tau + \Delta'_\tau$$

as $\lambda \to \infty$, where $\tau = \sqrt{p}$.

The case $p = 1$ is the ordinary Borel sum.

Theorem 6.3. Let $0 < p < 1$. Then we have

$$\Delta_p^2(n) \equiv \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} X_i \to \Delta_\tau + \Delta'_\tau$$

as $n \to \infty$, where $\tau = \sqrt{p(1-p)}$.

Theorems 6.2 and 6.3 are given from Theorem 6.1 by taking ξ_1 as Poisson random variable with mean $0 < p < \infty$ and Bernoulli random variable with success probability $0 < p < 1$, respectively.

Convergence of the joint distribution can be treated similarly. Let $\{\xi_i\}$ and $\{\xi'_i\}$ be two sequences of random variables generating the random-walk methods and set $N_n = \Sigma_{i=1}^{n}\xi_i$, $N'_n = \Sigma_{i=1}^{n}\xi'_i$.

Theorem 6.4.

(i) If $E[\xi_1] = E[\xi'_1]$, then as $n \to \infty$

$$(6.2) \qquad \left(\frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} P(N_n = i)\, X_i,\ \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} P(N'_n = i)\, X_i \right) \to (\, \Delta_\tau + \Delta'_\tau,\ \Delta_{\tau'} + \Delta'_{\tau'}\,),$$

where $\tau^2 = \mathrm{Var}[\xi_1]$ and $(\tau')^2 = \mathrm{Var}[\xi'_1]$.

(ii) If $E[\xi_1] \neq E[\xi'_1]$, then the limits of components of (6.2) are independent.

Remark. (1) Let us consider the functional limit of the Borel sum in form of

$$\Delta_\lambda(t) \equiv \frac{\sqrt{\lambda}}{\varphi(\sqrt{\lambda})} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i}{i!} X_i, \qquad 0 < t < \infty.$$

Then, because of the statement (ii) in Theorem 6.4, $\Delta_\lambda(t)$ and $\Delta_\lambda(s)$ have independent limits if $t \neq s$.

(2) $\Delta_p^1(n)$ and $\Delta_q^2(n)$ in Theorems 6.2 and 6.3 have dependent limits only when $p = q$.

Proof of Theorem 6.4. (i) is shown by the same argument as in the proof of Theorem 6.1, and the staement (ii) is given by noticing that

$$\sum_{i=0}^{\infty} P(N_n = i)\, X_i \overset{\mathcal{L}}{\sim} \sum_{|i-n\mu| \leq a\sqrt{n}} P(N_n = i)\, X_i$$

as $a \to \infty$ uniformly with respect to n and hence that the main contribution from $\{X_i\}$ are independent if μ's are different. □

Theorem 6.5. If $t \neq s$, then

$$\frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} P(N_{[nt]} = i)\, X_i \quad \text{and} \quad \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{\infty} P(N_{[ns]} = i)\, X_i$$

have independent limits as $n \to \infty$.

The proof can be carried out by the same reasoning as in that of (ii) of Theorem 6.4.

Remark. Consider the process generated by the Euler sum:

$$\Delta_n(t) \equiv \frac{\sqrt{n}}{\varphi(\sqrt{n})} \sum_{i=0}^{[nt]} \binom{[nt]}{i} p^i (1-p)^{[nt]-i}\, X_i.$$

Then $\Delta_n(t)$ and $\Delta_n(s)$ have independent limits if $t \neq s$. Remark (1) after Theorem 6.4 is also verified by using Theorem 6.5.

7. Additional comments

In section 6, if we are interested only in the one-dimensional stable limit theorem for random-walk methods (Theorem 6.1), we may directly apply a theorem due to Shukri[13], which is the following with our notation. The underlying random variables $\{X_i\}$ are the same as before.

Proposition 7.1 ([13]). *Suppose assumption* (A) *and* $\varphi(\lambda)$ *in* (2.1) *has a form of* $\varphi(\lambda) = \lambda^{1/\alpha}L(\lambda)$, *where* $L(\lambda)$ *is slowly varying at infinity. Let*

$$d(n) = \left(\sum_{i=0}^{\infty} c_i(n)^{\alpha}\right)^{1/\alpha}.$$

If

$$(7.1) \qquad \sum_{i=0}^{\infty} \left|\frac{c_i(n)}{d(n)L(n)}\right|^{\alpha} L\left(\frac{c_i(n)}{d(n)L(n)}\right) \to 1 \quad \textit{as} \quad n \to \infty$$

and

$$(7.2) \qquad \frac{1}{d(n)} \sup_i c_i(n) \to 0 \quad \textit{as} \quad n \to \infty,$$

then

$$\frac{1}{d(n)L(n)} \sum_{i=0}^{\infty} c_i(n) X_i$$

converges weakly as $n \to \infty$ *to a stable random variable with index* α.

To get our Theorem 6.1 as an application of this proposition, we have to check (7.1) and (7.2) for our random-walk method generated by a sequence of random variables $\{\xi_i\}$ as in previous section. However (7.1) may not be checked to be verified except the case when $L(\cdot)$ is constant. In what follows, we assume that $L(\cdot) \equiv 1$. This means that we consider the problem for $\{X_i\}$ belonging to the domain of normal attraction of a stable law. In this case, (7.1) is automatically true, and thus it is enough to check (7.2). Note that

$$\sup_i c_i(n) \sim \text{const.}\times n^{-1/2} \quad \text{as} \quad n \to \infty \tag{7.3}$$

for any random-walk method, which is given by the local limit theorem for $\{\xi_i\}$:

$$c_i(n) = P(N_n = i) \sim \frac{1}{\sqrt{2\pi n}\tau} e^{-(i-n\mu)^2/(2n\tau^2)} \tag{7.4}$$

uniformly in i, where $\mu = E[\xi_1]$, $\tau^2 = \mathrm{Var}[\xi_1]$ and $N_n = \xi_1 + \cdots + \xi_n$. Hence the essential point to be checked is to estimate $\Sigma_{i=0}^{\infty} c_i(n)^{\alpha}$, and the proof for that is implicitly included in the arguments in the previous sections. However, as we think that this estimate may be of independent interest and also useful for other problems such as the convergence rates problem, we give its proof below. For getting (7.2), it is sufficient to show $d(n)^{-1} = o(n^{1/2})$ because of (7.3), but we can get the following exact estimate.

Theorem 7.2. *Let $r > 0$ and $\{c_i(n)\}$ be a random-walk method. We assume $E[|\xi_1|^2] < \infty$ and further $E[|\xi_1|^p] < \infty$ for some $p > \frac{1}{r} - 1$ when $r \geq \frac{1}{3}$. Then*

$$\sum_{i=0}^{\infty} c_i(n)^r \sim \text{const.}\times n^{(1-r)/2} \tag{7.5}$$

as $n \to \infty$.

Proof. We divide $\Sigma_{i=0}^{\infty} c_i(n)^r$ into four parts:

$$\sum_{i=0}^{\infty} c_i(n)^r = J_1(n) + J_2(n) + J_3(n) + J_4(n),$$

where for large $M > 0$,

$$J_1(n) = \sum_{i=0}^{[n\mu]-[M\sqrt{n}]-1} c_i(n)^r,$$

$$J_2(n) = \sum_{i=[n\mu]-[M\sqrt{n}]}^{[n\mu]-1} c_i(n)^r,$$

$$J_3(n) = \sum_{i=[n\mu]}^{[n\mu]+[M\sqrt{n}]} c_i(n)^r$$

and

$$J_4(n) = \sum_{i=[n\mu]+[M\sqrt{n}]+1}^{\infty} c_i(n)^r.$$

$J_2(n)$ and $J_3(n)$ give main contributions and thus we first estimate them. We have

$$J_3(n) = \sum_{i=0}^{[M\sqrt{n}]} P(N_n = [n\mu] + i)^r$$

$$= \int_0^{[M\sqrt{n}]+1} P(N_n = [n\mu] + [u])^r \, du$$

$$= \sqrt{n} \int_0^{M + 1/\sqrt{n}} P\left(\frac{N_n - [n\mu]}{\sqrt{n}\tau} = \frac{[\sqrt{n}v]}{\sqrt{n}\tau}\right)^r dv.$$

It follows from the local limit theorem (7.4) that

$$J_3(n) \sim n^{(1-r)/2} \int_0^M \frac{1}{\sqrt{2\pi}\tau} e^{-v^2/(2\tau^2)}\, dv = \text{const.}\times n^{(1-r)/2}.$$

Similarly,

$$J_2(n) = \sum_{i=1}^{[M\sqrt{n}]} P(N_n = [n\mu] - i)^r$$

$$= \sqrt{n} \int_0^M P\Big(\frac{N_n - [n\mu]}{\sqrt{n}\tau} = - \frac{[\sqrt{n}v]}{\sqrt{n}\tau}\Big)^r dv$$

$$\sim \text{const.}\times n^{(1-r)/2}.$$

As to $J_1(n)$ and $J_4(n)$, we consider two cases separately; $r < 1$ or $r \geqq 1$. If $r \geqq 1$, by (7.3),

$$J_1(n) + J_4(n) \leqq \{\sup_i c_i(n)\}^{r-1} \Big(\sum_{i=0}^{[n\mu]-[M\sqrt{n}]-1} + \sum_{i=[n\mu]+[M\sqrt{n}]+1}^{\infty} \Big) c_i(n)$$

$$\leqq \text{const.}\times n^{(1-r)/2}\, P(|N_n - [n\mu]| > [M\sqrt{n}])$$

$$\leqq \text{const.}\times n^{(1-r)/2} \Big(\frac{\tau}{M}\Big)^2 E\Big[\Big|\frac{N_n - [n\mu]}{\sqrt{n}\tau}\Big|^2\Big]$$

$$\sim \text{const.}\times n^{(1-r)/2} \frac{1}{M}$$

by the same reasoning as in (6.1). For large M, the contribution of $J_1(n) + J_4(n)$ is smaller than $J_2(n) + J_3(n)$, and so we get (7.5).

When $r < 1$, take δ such as $0 < \frac{1}{r} - 1 < \delta < p$, where $p = 2$ if $r > \frac{1}{3}$ and p is the one assumed in the theorem if $r \leqq \frac{1}{3}$. We have

$$J_1(n) = \sum_{i=[M\sqrt{n}]+1}^{[n\mu]} P(N_n = [n\mu] - i)^r$$

$$= \sum_{i=-[n\mu]}^{-[M\sqrt{n}]-1} P(N_n = [n\mu] + i)^r$$

$$\leqq \sum_{i=-\infty}^{-[M\sqrt{n}]-1} \{P(N_n = [n\mu] + i)\ |i|^\delta\}^r\ |i|^{-r\delta}$$

$$\leqq \{\sum_{i=-\infty}^{-[M\sqrt{n}]-1} P(N_n = [n\mu] + i)\ |i|^\delta\}^r\ \{\sum_{i=-\infty}^{-[M\sqrt{n}]-1} |i|^{-r\delta/(1-r)}\}^{1-r}$$

by the Hölder inequality. Therefore we have

$$J_1(n) \leqq \text{const.}\times\{\sum_{i=-\infty}^{-[M\sqrt{n}]-1} P(\frac{N_n - [n\mu]}{\sqrt{n}\tau} = \frac{i}{\sqrt{n}\tau})\ |\frac{i}{\sqrt{n}\tau}|^\delta\ (\tau\sqrt{n})^\delta\}^r$$

$$\times\ n^{(-r\delta+1-r)/2}$$

$$\leqq \text{const.}\times n^{(1-r)/2} \sum_{i=-\infty}^{-[M\sqrt{n}]-1} P(\frac{N_n - [n\mu]}{\sqrt{n}\tau} = \frac{i}{\sqrt{n}\tau})\,|\frac{i}{\sqrt{n}\tau}|^p\ (\frac{\tau}{M})^{p-\delta}$$

$$\leqq \text{const.}\times n^{(1-r)/2}\ (\frac{1}{M})^{p-\delta}\ E[|\frac{N_n - [n\mu]}{\sqrt{n}\tau}|^p]$$

$$\sim \text{const.}\times n^{(1-r)/2}\ (\frac{1}{M})^{p-\delta}.$$

$J_4(n)$ can be handled similarly, and hence, for large $M > 0$, the contributions of $J_1(n)$ and $J_4(n)$ are again smaller than $J_2(n) + J_3(n)$. The proof of the theorem is thus complete. □

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