Research Report

KSTS/RR-85/001
8 Jan., 1985

On the one-dimensional motion of the polytropic ideal gas Non-fixed on the boundary

by

Takeyuki Nagasawa

Takeyuki Nagasawa
Department of Mathematics, Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

Hiyoshi 3-14-1, Kohoku-ku, Yokohama, 223 Japan

KSTS/RR-85/001
January 8, 1985

On the one-dimensional motion of the polytropic ideal gas non-fixed on the boundary

Takeyuki NAGASAWA

Department of Mathematics, Faculty of Science and Technology,

Keio University, Yokohama 223, Japan

1. Introduction.

Recently, Greenberg and Li have shown the existence of the temporally global solution of the initial-boundary value problem with boundary damping for the quasi-linear wave equation in [1]. Stimulated by their work, the author attempts to discuss some types of the initial-boundary value problems, including boundary damping, for the equations describing the one-dimensional motion of the polytropic ideal gas. It is described by the following three equations in the Lagrangian coordinate corresponding to the conservation law of mass, momentum and energy:

$$u_t = v_x, \tag{1.1.1}$$

$$v_t = \sigma_x, \tag{1.1.2}$$

$$c\theta_t = \sigma v_x + \kappa\left(\frac{\theta_x}{u}\right)_x, \tag{1.1.3}$$

where u is the specific volume, v is the velocity, θ is the absolute temperature, σ is the stress which is the function of u, θ and v_x as follows:

$$\sigma = -R\frac{\theta}{u} + \mu\frac{v_x}{u}, \tag{1.2}$$

R is the gas constant, μ is the coefficient of viscosity, c is the heat capacity at constant volume and κ is the coefficient of heat conduction. The suffix denotes the partial differentiation with respect to the variable t or x. In the present paper, it is assumed that R, μ, c and κ are positive constants.

We shall consider the system (1.1) in the region $\{ (x,t) \in [0,1] \times [0,+\infty) \}$ under the initial conditions:

$$u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ \theta(x,0) = \theta_0(x), \tag{1.3}$$

and the some types of the boundary conditions. In [5] and [7], Kazhikhov and Shelukhin succeeded in solving the system (1.1) - (1.2) globally in time under the following boundary conditions:

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{1.4}$$

$$v(0,t) = v(1,t) = 0, \tag{1.5.0}$$

or (1.4) and

$$\sigma(0,t) = \sigma(1,t) = 0. \tag{1.5.1}$$

The condition (1.4) implies that the gas has the adiabatic ends. (1.5.0) implies that the ends are fixed, while (1.5.1) implies that the gas is put in a vacuum. In this paper, we shall discuss the system (1.1) - (1.3) under the boundary conditions (1.4) and (1.5.1) or

$$v(0,t) = \sigma(0,t),\ v(1,t) = -\sigma(1,t), \tag{1.5.2}$$

or

$$v(0,t) = \sigma(0,t),\ v(1,t) = 0, \tag{1.5.3}$$

or

$$v(0,t) = \sigma(0,t),\ \sigma(1,t) = 0, \tag{1.5.4}$$

or

$$\sigma(0,t) = 0,\ v(1,t) = 0. \tag{1.5.5}$$

The condition (1.5.2), boundary damping, implies that the ends are connected to some sort of dash pot, and (1.5.i) ($i = 3,4,5$) is the combination of the conditions (1.5.j) ($j = 0,1,2$).

From now on, the abbreviation "*the problem* (i)" stands for "*the initial-boundary value problem for* (1.1) - (1.4) *and* (1.5.i)" $(i = 0,1, \cdots ,5)$.

Firstly we prove the temporally global existence of a solution for the problems (2) - (5) (Theorem 1) under suitable assumptions by the improved argument of [7] established by Kazhikhov and Shelukhin.

Kazhikhov also showed the uniform boundedness and the stability of a solution for the problem (0), *i.e.*, any nonstationary solution converges to the stationary solution in some Sobolev norm as t increases with exponential rate, see [6]. For the problems (1) - (5), however, this result does not hold. We can easily construct the examples of the solution that are not uniformly bounded with respect to t. For instance,

$$u(x,t) = 1 + \frac{R}{\mu}\, t,\ v(x,t) = \frac{R}{\mu}\, x,\ \theta(x,t) = 1, \tag{1.6}$$

is the solution for the problem (1) with the initial data

$$u_0(x) = 1,\ v_0(x) = \frac{R}{\mu}\, x,\ \theta_0(x) = 1, \tag{1.7}$$

and the specific volume u grows infinity as t increases.

From a physical point of view, it is possible that the gas is rarefied under the conditions (1.5.i) $(i = 1,2, \cdots ,5)$, so u or $\int_0^1 u(x,t)dx$ may grow infinity (the amount $\int_0^1 u(x,t)dx$ means the volume of the region occupied by the gas). We will find this conjecture valid for any solution of the problems (1) - (5) under some assumptions (Theorem 2). In the isothermal case we will get the more exact growth rate of u or $\int_0^1 u(x,t)dx$.

2. Notation and results

We assume that all functions considered in this paper should be defined in $[0,1]$ or $[0,1] \times [0,T]$ $(0 < T < +\infty)$ and continuously differentiable as many as necessary. For non-negative integers r and s, we define

$$D_t^r D_x^s = \partial^{r+s}/\partial t^r \partial x^s,$$
$$\begin{cases} \Omega = (0,1),\ \overline{\Omega} = [0,1], \\ Q_T = \Omega \times (0,T),\ \overline{Q_T} = \overline{\Omega} \times [0,T]. \end{cases} \tag{2.1}$$

For a non-negative integer n and $\alpha \in (0,1)$,

$$\begin{cases} |u|^{(0)} = \sup\limits_{x\in\bar\Omega}|u(x)|, \\ |u|^{(\alpha)} = \sup\limits_{x,x'\in\bar\Omega,\, x\neq x'} \dfrac{|u(x) - u(x')|}{|x - x'|^{\alpha}} . \end{cases} \tag{2.2}$$

$$\begin{cases} ||u||^{(n)} = \sum\limits_{i=0}^{n} |D_x^i u|^{(0)}, \\ ||u||^{(n+\alpha)} = ||u||^{(n)} + |D_x^n u|^{(\alpha)}. \end{cases} \tag{2.3}$$

$$\begin{cases} |u|_T^{(0)} = \sup\limits_{(x,t)\in\bar Q_T}|u(x,t)|, \\ |u|_{x,T}^{(\alpha)} = \sup\limits_{(x,t),(x',t)\in\bar Q_T,\, x\neq x'} \dfrac{|u(x,t) - u(x',t)|}{|x - x'|^{\alpha}} , \\ |u|_{t,T}^{(\alpha/2)} = \sup\limits_{(x,t),(x,t')\in\bar Q_T,\, t\neq t'} \dfrac{|u(x,t) - u(x,t')|}{|t - t'|^{\alpha/2}} , \\ |u|_T^{(\alpha)} = |u|_{x,T}^{(\alpha)} + |u|_{t,T}^{(\alpha/2)}. \end{cases} \tag{2.4}$$

$$\begin{cases} ||u||_T^{(n)} = \sum\limits_{2r+s=0}^{n} |D_t^r D_x^s u|_T^{(0)}, \\ ||u||_T^{(n+\alpha)} = ||u||_T^{(n)} + \sum\limits_{2r+s=n} |D_t^r D_x^s u|_{x,T}^{(\alpha)} + \sum\limits_{2r+s=\max(n-1,0)}^{n} |D_t^r D_x^s u|_{t,T}^{(\alpha/2)}, \\ |||u|||_T^{(n+\alpha)} = \sum\limits_{r+s=0}^{n} |D_t^r D_x^s u|_T^{(0)} + \sum\limits_{r+s=n} |D_t^r D_x^s u|_T^{(\alpha)}. \end{cases} \tag{2.5}$$

$$\begin{cases} H^{n+\alpha} = \{\, u(x) \mid ||u||^{(n+\alpha)} < +\infty \,\}, \\ H_T^{n+\alpha} = \{\, u(x,t) \mid ||u||_T^{(n+\alpha)} < +\infty \,\}, \\ B_T^{n+\alpha} = \{\, u(x,t) \mid |||u|||_T^{(n+\alpha)} < +\infty \,\}. \end{cases} \tag{2.6}$$

Other notations, not described above, will be explained where they appear.

From now on, we always assume that for some $\alpha \in (0,1)$ the initial data satisfy

$$u_0 \in H^{1+\alpha},\ v_0 \in H^{2+\alpha},\ \theta_0 \in H^{2+\alpha}, \tag{2.7}$$

$$\begin{cases} 0 < \underline{u_0} \equiv \min\limits_{x\in\bar\Omega} u_0(x) \le u_0 \le \overline{u_0} \equiv |u_0|^{(0)}, \\ 0 < \underline{\theta_0} \equiv \min\limits_{x\in\bar\Omega} \theta_0(x) \le \theta_0 \le \overline{\theta_0} \equiv |\theta_0|^{(0)}, \end{cases} \tag{2.8}$$

and that the compatibility conditions hold for (1.3) - (1.5). We can establish the following theorem:

Theorem 1. *If the initial-boundary conditions satisfy the assumptions mentioned above,* *then there exists a temporally global unique solution* (u,v,θ) *for each problem* (2) - (5), *which belongs to* $B_T^{1+\alpha} \times H_T^{2+\alpha} \times H_T^{2+\alpha}$ *for any* $T \in (0,+\infty)$.

The same results for the problems (0) and (1) were established in [5] and [7].

Here we introduce the useful abbreviations as follows:

$$\begin{cases} m_u(t) = \min_{x\in\bar{\Omega}} u(x,t), \; M_u(t) = \max_{x\in\bar{\Omega}} u(x,t), \\ m_\theta(t) = \min_{x\in\bar{\Omega}} \theta(x,t), \; M_\theta(t) = \max_{x\in\bar{\Omega}} \theta(x,t). \end{cases} \tag{2.9}$$

Our result concerning the growth of u or $\int_0^1 u(x,t)dx$ is:

Theorem 2. *Under the same assumptions as in Theorem 1, there exists a positive constant* $K\ (> 1)$ *depending on* R, μ, c, κ *and initial data, such that* u *or* $\int_0^1 u(x,t)dx$ *of the solutions for the problems* (1) - (5) *grows to infinity as* t *increases with the following rate:*

(I) *Non-isothermal case.*

(i) *The problems* (2) *and* (3).

$$\left.\begin{matrix} K^{-1}t^{\frac{1}{4}(2-k_1)} \\ K^{-1}(\log(1+t))^{\frac{1}{4}} \\ K^{-1} \end{matrix}\right\} \le \int_0^1 u(x,t)dx \le K(1+t^{\frac{1}{4}}) \qquad \text{for} \quad \begin{cases} k_1 < 2, \\ k_1 = 2, \\ k_1 > 2. \end{cases} \tag{2.10}$$

$$\overline{\lim_{t\to+\infty}}\, t^{-s}\int_0^1 u(x,t)dx = +\infty \qquad \text{for} \quad s < \frac{1}{k_1+2} \,. \tag{2.11}$$

(ii) *The problem* (4).

$$\left.\begin{matrix} K^{-1}t^{1-k_1} \\ K^{-1}(\log(1+t))^{\max(1,k_2)} \end{matrix}\right\} \le \int_0^1 u(x,t)dx \le K(1+t) \qquad \text{for} \quad \begin{cases} k_1 < 1, \\ k_1 \ge 1. \end{cases} \tag{2.12}$$

$$\overline{\lim_{t\to+\infty}}\, t^{-s}\int_0^1 u(x,t)dx = +\infty \qquad \text{for} \quad s < \frac{1}{k_1+1} \,. \tag{2.13}$$

$$K^{-1}(\log(1+t))^{\max(1,k_2)} \le m_u(t) \le K(1+t). \tag{2.14}$$

(iii) *The problems* (1) *and* (5).

$$K^{-1}t \le \int_0^1 u(x,t)dx \le K(1+t). \tag{2.15}$$

$$K^{-1}(\log(1+t))^{\max(1,k_2)} \leq m_u(t) \leq K(1+t). \tag{2.16}$$

Here

$$k_1 = \frac{R}{c} > 0, \tag{2.17}$$

$$k_2 = \min\left\{\frac{R}{\mu\underline{\theta_0}}, \frac{4c}{R}\right\} \exp\left[-\frac{2}{\mu}\left\{\int_0^1 v_0 dx + \left(\int_0^1 (v_0^2 + 2c\theta_0)dx\right)^{1/2}\right\}\right]. \tag{2.18}$$

(II) *Isothermal case.*

(vi) *The problems* (2) *and* (3).

$$K^{-1}t^{1/2} \leq \int_0^1 u(x,t)dx \leq K(1+t^{1/2}). \tag{2.19}$$

(v) *The problems* (1), (4) *and* (5).

$$K^{-1}t \leq u(x,t) \leq K(1+t). \tag{2.20}$$

See also Remark at the hindmost part of Section 4.

In the following sections, C_i, $C_i(t)$ and $C_i(\epsilon_j,t)$ denote positive constants depending on their arguments and possibly R, μ, c, κ and initial data, and monotonically increasing with respect to t. For convenience we sometimes denote different constants by the same symbol C instead of C_i.

3. The proof of Theorem 1.

The proof of Theorem 1 is based on the local (in time) and unique existence theorem and on the a priori estimates. The local and unique existence theorem has been established by Nash [8], Itaya [2], [3] and Tani [9]. So it is enough to establish the a priori estimates in order to complete the proof of Theorem 1.

Proposition 3.1. *The following estimates on* (u,v,θ) *for the problems* (2) - (5) *hold.*

$$\min\{m_u(t),m_\theta(t)\} \geq C^{-1}(t), \tag{3.1}$$

$$\|u\|_T^{(1+\alpha)} + \|v\|_T^{(2+\alpha)} + \|\theta\|_T^{(2+\alpha)} \leq C(T). \tag{3.2}$$

We will proceed the argument under the assumption $u > 0$. It follows $\theta > 0$ from (1.1.3) by use of maximum principle.

Before proceeding the proof of Proposition 3.1, we study several properties of (u,v,θ) for the problems (1) - (5). They are also useful for the proof of Theorem 2. The proof of them is essentially the same as the case of the problem (0) established in [7], so we omit.

Lemma 3.1. (i) *We have the energy identities as follows:*

$$\int_0^1\left(\frac{1}{2}v_0^2 + c\theta_0\right)dx$$

$$= \begin{cases} \int_0^1\left(\frac{1}{2}v^2 + c\theta\right)dx + \int_0^t(v^2(0,\tau) + v^2(1,\tau))d\tau & \text{for the problem (2),} \\ \int_0^1\left(\frac{1}{2}v^2 + c\theta\right)dx + \int_0^t v^2(0,\tau)d\tau & \text{for the problems (3) and (4),} \\ \int_0^1\left(\frac{1}{2}v^2 + c\theta\right)dx & \text{for the problems (1) and (5).} \end{cases} \tag{3.3}$$

(ii) *We have the following estimates:*

$$U(t) + \int_0^t V(\tau)d\tau \le C(t), \tag{3.4}$$

$$M_\theta(t) \le C(1 + M_u(t)V(t)), \tag{3.5}$$

$$m_\theta(t) \ge C_1\left(1 + \int_0^t\frac{d\tau}{m_u(\tau)}\right)^{-1}, \tag{3.6}$$

where

$$U(t) \equiv \int_0^1\left\{\frac{1}{2}v^2 + R(u - \log u - 1) + c(\theta - \log\theta - 1)\right\}dx \ge 0, \tag{3.7}$$

$$V(t) \equiv \int_0^1\left(\mu\frac{v_x^2}{u\theta} + \kappa\frac{\theta_x^2}{u\theta^2}\right)dx \ge 0, \tag{3.8}$$

$$C_1 = \min\left\{\frac{1}{\theta_0}, \frac{4\mu c}{R^2}\right\}. \tag{3.9}$$

(iii) *We have useful expressions of* u *as follows:*

$$u(x,t) = \frac{1}{B(x,t)Y(t)}\left\{u_0(x) + \int_0^t\frac{R}{\mu}\theta(x,\tau)B(x,\tau)Y(\tau)d\tau\right\},$$

for the problems (2) and (3), (3.10)

$$u(x,t) = \frac{1}{D(x,t)}\left\{u_0(x) + \int_0^t \frac{R}{\mu}\,\theta(x,\tau)D(x,\tau)d\tau\right\},$$

for the problem (4), (3.11)

$$u(x,t) = \frac{1}{B(x,t)}\left\{u_0(x) + \int_0^t \frac{R}{\mu}\,\theta(x,\tau)B(x,\tau)d\tau\right\},$$

for the problems (1) and (5), (3.12)

where

$$B(x,t) = \exp\left\{\frac{1}{\mu}\int_0^x (v_0(\xi) - v(\xi,t))d\xi\right\}, \tag{3.13}$$

$$D(x,t) = \exp\left\{\frac{1}{\mu}\int_x^1 (v(\xi,t) - v_0(\xi))d\xi\right\}, \tag{3.14}$$

$$Y(t) = \exp\left\{-\frac{1}{\mu}\int_0^t v(0,\tau)d\tau\right\}. \tag{3.15}$$

Now we proceed the proof of (3.1).

Proof of (3.1). Properties mentioned above yield

$$C^{-1} \le B(x,t),\, D(x,t) \le C, \tag{3.16}$$

$$C^{-1} \le Y(t) \le C(t), \tag{3.17}$$

and therefore we have (3.1) by use of expressions of u and (3.6).

Q.E.D.

Moreover, Lemma 3.1, (3.16) and (3.17) yield

$$M_u(t) \le C(t)\Big(1 + \int_0^t M_u(\tau)V(\tau)d\tau\Big). \tag{3.18}$$

Applying Gronwall's inequality, by use of (3.4) we get the following estimate.

Lemma 3.2. *We have*

$$M_u(t) \le C(t). \tag{3.19}$$

Next we show (3.2) only for the problem (2). For the problems (3) - (5), we can lead it in a quite similar way. In proving it we need several lemmas concerning the estimates of derivatives of the solution.

Lemma 3.3. *We have*

$$\int_0^1 (v^4 + \theta^2 + u_x^2)dx + \int_0^t\int_0^1 (\theta_x^2 + v_x^2)dxd\tau$$
$$+ \int_0^t (M_\theta(\tau) + M_\theta^2(\tau) + v^4(0,\tau) + v^4(1,\tau)$$
$$+ v^2(0,\tau)\theta(0,\tau) + v^2(1,\tau)\theta(1,\tau))d\tau \le C(t). \tag{3.20}$$

Proof. We can obtain the lemma in the same manner as [7].

Q.E.D.

Lemma 3.4. *We have*

$$\int_0^1 v_x^2 dx + \int_0^t\int_0^1 v_{xx}^2 dxd\tau$$
$$\le C(t)\left[1 + \left\{\int_0^t v_t^2(0,\tau)d\tau\right\}^{1/2} + \left\{\int_0^t v_t^2(1,\tau)d\tau\right\}^{1/2}\right]. \tag{3.21}$$

Proof. We multiply (1.1.2) by v_{xx} and integrate in x over $\overline{\Omega}$, then using Schwarz' inequality with appropriate weights, we have

$$\left\{\int_0^1 \frac{1}{2}v_x^2 dx\right\}_t + \int_0^1 \frac{\mu}{2}\frac{v_{xx}^2}{u}dx$$
$$\le C(t)\left\{\int_0^1 \theta_x^2 dx + \left(m_\theta^2(t) + \max_{x\in\overline{\Omega}} v_x^2(x,t)\right)\int_0^1 u_x^2 dx\right\}$$
$$- v_t(0,t)v_x(0,t) + v_t(1,t)v_x(1,t).$$

Integrating both sides over $[0,t]$ and using (3.1), (3.20) and Schwarz' inequality, we get

$$\int_0^1 v_x^2 dx + 2\int_0^t\int_0^1 v_{xx}^2 dx \le C_2(t)\left[1 + \int_0^t \max_{x\in\overline{\Omega}} v_x^2(x,\tau)d\tau\right.$$
$$\left. + \left\{\int_0^t v_t^2(0,\tau)d\tau\cdot\int_0^t v_x^2(0,\tau)d\tau\right\}^{1/2} + \left\{\int_0^t v_t^2(1,\tau)d\tau\cdot\int_0^t v_x^2(1,\tau)d\tau\right\}^{1/2}\right]. \tag{3.22}$$

Let us evaluate the right-hand side. Firstly for any $\epsilon > 0$, we can obtain the following estimate by Schwarz' inequality with ϵ and (3.20):

$$\int_0^t \max_{x\in\overline{\Omega}} v_x^2(x,\tau)d\tau \le \int_0^t\left\{\int_0^1 v_x^2 dx + \int_0^1 2|v_x v_{xx}|dx\right\}d\tau$$

$$\leq \epsilon\int_0^t\int_0^1 v_{xx}^2 dxd\tau + C(\epsilon,t). \tag{3.23}$$

Secondly we evaluate the integration of the boundary value. From (1.2), we can rewrite the former relation of (1.5.2) as follows:

$$v(0,t) = -R\frac{\theta(0,t)}{u(0,t)} + \mu\frac{v_x(0,t)}{u(0,t)}\,.$$

Therefore (3.3), (3.19) and (3.20) yield

$$\int_0^t v_x^2(0,\tau)d\tau \leq C\left\{\sup_{0\leq\tau\leq t} M_u^2(\tau)\int_0^t v^2(0,\tau)d\tau + \int_0^t M_\theta^2(\tau)d\tau\right\} \leq C(t). \tag{3.24}$$

In the same manner we get

$$\int_0^t v_x^2(1,\tau)d\tau \leq C(t). \tag{3.25}$$

Substituting (3.23) with $\epsilon = C_2^{-1}(t)$, (3.24) and (3.25) into (3.22), we obtain the assertion.

Q.E.D.

Lemma 3.5. *For any* $\epsilon > 0$, *we have*

$$\int_0^1 \theta_x^2 dx + \int_0^t\int_0^1 \theta_{xx}^2 dxd\tau \leq \epsilon\int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau))d\tau + C(\epsilon,t). \tag{3.26}$$

Proof. Multiplying (1.1.3) by θ_{xx}, in a similar manner to the previous lemma we have

$$\left\{\int_0^1 \frac{c}{2}\theta_x^2 dx\right\}_t + \int_0^1 \frac{\kappa}{2}\,\frac{\theta_{xx}^2}{u}\,dx$$
$$\leq C(t)\left\{\max_{x\in\bar\Omega} v_x^2(x,t)\int_0^1(\theta^2 + v_x^2)dx + \max_{x\in\bar\Omega}\theta_x^2(x,t)\int_0^1 u_x^2 dx\right\}.$$

and then by (3.19) and (3.20),

$$\int_0^1 \theta_x^2 dx + 2\int_0^t\int_0^1 \theta_{xx}^2 dxd\tau$$
$$\leq C_3(t)\left[1 + \int_0^t\left\{\max_{x\in\bar\Omega} v_x^2(x,\tau)\left(1 + \int_0^1 v_x^2 dx\right) + \max_{x\in\bar\Omega}\theta_x^2(x,\tau)\right\}d\tau\right]. \tag{3.27}$$

Recalling (3.21) and (3.23), by Schwarz' inequality with an appropriate weight we have for any $\epsilon_1 > 0$,

$$\int_0^t \max_{x\in\bar\Omega} v_x^2(x,\tau)d\tau \leq \epsilon_1\int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau))d\tau + C(\epsilon_1,t), \tag{3.28}$$

and for any $\epsilon_2 > 0$,

$$\begin{aligned}
&\int_0^t \max_{x\in\bar\Omega} v_x^2(x,\tau)\cdot\int_0^1 v_x^2(x,\tau)dxd\tau \\
&\le \int_0^t \max_{x\in\bar\Omega} v_x^2(x,\tau)d\tau\cdot \max_{0\le\tau\le t}\int_0^1 v_x^2(x,\tau)dx \\
&\le \left\{\epsilon_2\int_0^t\int_0^1 v_{xx}^2 dxd\tau + C(\epsilon_2,t)\right\} \\
&\qquad\qquad \times C_4(t)\left[1 + \left\{\int_0^t v_t^2(0,\tau)d\tau\right\}^{¼} + \left\{\int_0^t v_t^2(1,\tau)d\tau\right\}^{¼}\right] \\
&\le \epsilon_3\int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau))d\tau + C(\epsilon_3,t), \qquad (3.29)
\end{aligned}$$

where

$$\epsilon_3 = \epsilon_2 C_4^2(t) + \text{any positive number.}$$

Moreover for any $\epsilon_4 > 0$ by (1.4), (3.20), we get in a similar way to the one deriving (3.23)

$$\begin{aligned}
\int_0^t \max_{x\in\bar\Omega} \theta_x^2(x,\tau)d\tau &\le \int_0^t\int_0^1 2\left|\theta_x\theta_{xx}\right|dxd\tau \\
&\le \epsilon_4\int_0^t\int_0^1 \theta_{xx}^2 dxd\tau + C(\epsilon_4,t). \qquad (3.30)
\end{aligned}$$

We set ϵ_j's such that

$$\epsilon = (\epsilon_1 + \epsilon_3)C_3(t),\ \epsilon_4 = C_3^{-1}(t),$$

then we obtain (3.26) by substituting (3.28), (3.29) and (3.30) into (3.27).

Q.E.D.

Lemma 3.6. *For any* $\epsilon > 0$, *we have*

$$\int_0^t\int_0^1 \theta_t^2 dxd\tau \le \epsilon\int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau))d\tau + C(\epsilon,t). \qquad (3.31)$$

Proof. From (1.1.3), (1.2) and (3.19), we get

$$\begin{aligned}
\int_0^t\int_0^1 \theta_t^2 dxd\tau \le C(t)\int_0^t\Bigg\{&\left(M_\theta^2(\tau) + \max_{x\in\bar\Omega} v_x^2(x,\tau)\right)\int_0^1 v_x^2 dx \\
&+ \max_{x\in\bar\Omega}\theta_x^2(x,\tau)\int_0^1 u_x^2 dx + \int_0^1 \theta_{xx}^2 dx\Bigg\}d\tau.
\end{aligned}$$

It yields the assertion by use of (3.20), (3.26), (3.29) and (3.30).

Q.E.D.

Lemma 3.7. *The function $v(x,t)$ possesses a generalized derivative $v_{xt} \in L^2(Q_t)$ for any $t \in [0,+\infty)$, and we have*

$$\int_0^1 v_t^2 dx + \int_0^t \int_0^1 v_{xt}^2 dx d\tau + \int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau)) d\tau \le C(t). \tag{3.32}$$

Proof. For a positive number Δt and a function $f(x,t)$, we denote a symbol Δf by

$$\Delta f \equiv f(x,t+\Delta t) - f(x,t).$$

After employing differences of (1.1.2) with respect to t, we multiply both sides by $\dfrac{\Delta v}{\Delta t}$. Integrating them over $\overline{Q_t}$, we get by the integration by parts with the boundary condition (1.5.2):

$$\left[\frac{1}{2}\int_0^1 \left(\frac{\Delta v}{\Delta t}\right)^2 dx\right]_{\tau=0}^{\tau=t} + \int_0^t \left\{\left(\frac{\Delta v}{\Delta t}\right)^2 (0,\tau) + \left(\frac{\Delta v}{\Delta t}\right)^2 (1,\tau)\right\} d\tau$$
$$= \int_0^t \int_0^1 \left\{ R \frac{1}{u(x,\tau+\Delta t)} \cdot \frac{\Delta\theta}{\Delta t} \cdot \frac{\Delta v_x}{\Delta t} + R\theta(x,\tau) \frac{\Delta(u^{-1})}{\Delta t} \cdot \frac{\Delta v_x}{\Delta t} \right.$$
$$\left. - \mu \frac{1}{u(x,\tau+\Delta t)} \left(\frac{\Delta v_x}{\Delta t}\right)^2 - \mu \frac{\Delta(u^{-1})}{\Delta t} v_x(x,\tau) \frac{\Delta v_x}{\Delta t} \right\} dx d\tau.$$

Letting Δt go to zero, we recognize that the generalized derivative v_{xt} exists in $L^2(Q_t)$, and by (1.1.1), (3.1) and Schwarz' inequality with appropriate weights

$$\frac{1}{2}\int_0^1 v_t^2 dx + \int_0^t \int_0^1 \frac{\mu}{2} \frac{v_{xt}^2}{u} dx d\tau + \int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau)) d\tau$$
$$\le \frac{1}{2}\int_0^1 v_t^2 dx \Big|_{t=0} + C(t) \int_0^t \left\{ \int_0^1 \theta_t^2 dx + \max_{x \in \overline{\Omega}} v_x^2(x,\tau) \int_0^1 (\theta^2 + v_x^2) dx \right\} d\tau.$$

We interpret the first term of the right-hand side as follows:

$$\int_0^1 v_t^2 dx \Big|_{t=0} = \int_0^1 \left\{ -R\left(\frac{\theta_0}{u_0}\right)' + \mu\left(\frac{v_0'}{u_0}\right)' \right\}^2 dx.$$

With the help of (3.19), (3.20), (3.28), (3.29) and (3.31), for any $\epsilon > 0$, we have

$$\int_0^1 v_t^2 dx + \int_0^t \int_0^1 v_{xt}^2 dx d\tau + 2\int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau)) d\tau$$
$$\le \epsilon \int_0^t (v_t^2(0,\tau) + v_t^2(1,\tau)) d\tau + C(\epsilon,t).$$

We complete the proof by setting $\epsilon = 1$.

Q.E.D.

Lemma 3.8. *We have*

$$\int_0^1 (v_x^2 + \theta_x^2)dx + \int_0^t\int_0^1 (v_{xx}^2 + \theta_{xx}^2 + \theta_t^2)dxd\tau \le C(t). \tag{3.33}$$

Proof. It is easily derived from Lemmas 3.4 - 3.7.

Q.E.D.

Proof of (3.2). Lemmas 3.3 and 3.8 yield (3.2) by the standard argument (see [4]).

Q.E.D.

The proof of Proposition 3.1 is now completed.

4. The proof of Theorem 2.

Firstly we show the following estimates which play an important role in the proof of Theorem 2.

Lemma 5.1. *We have*

$$\int_0^t\int_0^1 (v^2 + \theta)dxd\tau \le C\left\{1 + \left(\int_0^1 u\,dx\right)^n\right\}, \tag{4.1}$$

where

$$n = \begin{cases} 2 & \text{for the problems (2) and (3),} \\ 1 & \text{for the problems (1), (4) and (5).} \end{cases}$$

Proof. Firstly we establish the assertion for the problem (2). Integrating (1.1.2) over $[0,x]$ $(0 \le x \le 1)$ with the help of (1.1.1), (1.2) and (1.5.2), we have

$$\left\{\int_0^x v(\xi,t)d\xi\right\}_t + R\frac{\theta(x,t)}{u(x,t)} = \mu\frac{u_t(x,t)}{u(x,t)} - v(0,t).$$

Multiplying both sides by $u(x,t)$, and integrating over $\overline{Q_t}$, we get

$$\int_0^t\int_0^1 R\theta(x,\tau)dxd\tau = \mu\int_0^1(u(x,t) - u_0(x))dx - \int_0^t v(0,\tau)\int_0^1 u(x,\tau)dxd\tau$$
$$- \int_0^t\int_0^1 u(x,\tau)\left\{\int_0^x v(\xi,\tau)d\xi\right\}_t dxd\tau. \tag{4.2}$$

Integrating (1.1.1) and (1.1.2) over $\overline{\Omega}$, we obtain from (1.5.2),

$$\begin{cases} \left\{\int_0^1 u\ dx\right\}_t = v(1,t) - v(0,t), \\ \left\{\int_0^1 v\ dx\right\}_t = \sigma(1,t) - \sigma(0,t) = -v(1,t) - v(0.t). \end{cases} \tag{4.3}$$

Therefore we can turn the second term of the right-hand side of (4.2) as follows:

$$-\int_0^t v(0,\tau)\int_0^1 u(x,\tau)dxd\tau = \frac{1}{2}\int_0^t\left\{\int_0^1(u + v)dx\right\}_t\int_0^1 u\ dxd\tau$$
$$= \frac{1}{2}\left[\frac{1}{2}\left(\int_0^1 u\ dx\right)^2 + \int_0^1 v\ dx\int_0^1 u\ dx\right]_{\tau=0}^{\tau=t}$$
$$- \frac{1}{2}\int_0^t(v(1,\tau) - v(0,\tau))\int_0^1 v\ dxd\tau. \tag{4.4}$$

Here we use (1.1.1). In the same way, the third term becomes

$$-\int_0^t\int_0^1 u(x,\tau)\left\{\int_0^x v(\xi,\tau)d\xi\right\}_t dxd\tau$$
$$= -\left[\int_0^1 u(x,\tau)\int_0^x v(\xi,\tau)d\xi dx\right]_{\tau=0}^{\tau=t} + \int_0^t\int_0^1 v_x\int_0^x v(\xi,\tau)d\xi dxd\tau$$
$$= -\left[\int_0^1 u(x,\tau)\int_0^x v(\xi,\tau)d\xi dx\right]_{\tau=0}^{\tau=t}$$
$$+ \int_0^t v(1,\tau)\int_0^1 v\ dxd\tau - \int_0^t\int_0^1 v^2 dxd\tau. \tag{4.5}$$

It follows from (4.2), (4.4) and (4.5) that

$$\int_0^t\int_0^1(v^2 + R\theta)dxd\tau = \mu\int_0^1(u - u_0)dx$$
$$+ \left[\frac{1}{4}\left(\int_0^1 u\ dx\right)^2 + \frac{1}{2}\int_0^1 v\ dx\int_0^1 u\ dx - \int_0^1 u\int_0^x v\ d\xi dx\right]_{\tau=0}^{\tau=t}$$
$$+ \frac{1}{2}\int_0^t(v(1,\tau) + v(0,\tau))\int_0^1 v\ dxd\tau. \tag{4.6}$$

After some calculation by use of (4.3.2), we get

$$\frac{1}{2}\int_0^t(v(1,\tau) + v(0,\tau))\int_0^1 v\ dxd\tau = -\frac{1}{4}\left[\left(\int_0^1 v\ dx\right)^2\right]_{\tau=0}^{\tau=t}. \tag{4.7}$$

We substitute (4.7) into (4.6), and if we note that $u > 0$, we obtain the assertion from (4.6) by Schwarz' inequality and (3.3.1).

For the problem (3), the proof is derived quite similarly.

Next we concern the problem (1). We have by integration (1.1.2) over $\overline{Q_t}$ with (1.5.1),

$$\int_0^1 v\, dx \equiv \int_0^1 v_0 dx.$$

Therefore (if necessary we take $\tilde{v} \equiv v - \int_0^1 v_0 dx$ as an unknown function instead of v), we may assume

$$\int_0^1 v\, dx \equiv 0.$$

Hence in the same way as the problem (2), we can derive

$$\int_0^t\int_0^1 (v^2 + R\theta) dx d\tau = \mu\int_0^1 (u - u_0) dx - \left[\int_0^1 u \int_0^x v\, d\xi dx\right]_{\tau=0}^{\tau=t},$$

which yields the assertion.

For the problem (4), by means of the same procedure,

$$\begin{aligned}\int_0^t\int_0^1 &(v^2 + R\theta) dx d\tau \\ &= \mu\int_0^1 (u - u_0) dx + \left[\int_0^1 u \int_x^1 v\, d\xi dx\right]_{\tau=0}^{\tau=t} + \int_0^t v(0,\tau)\int_0^1 v\, dx d\tau.\end{aligned} \tag{4.8}$$

The last term is evaluated by Schwarz' inequality and (3.3.2) as follows:

$$\int_0^t v(0,\tau)\int_0^1 v\, dx d\tau \le C + \frac{1}{2}\int_0^t\int_0^1 v^2 dx d\tau. \tag{4.9}$$

Substituting (4.9) into (4.8), we establish the assertion by Schwarz' inequality and (3.3.2).

In the same way as the problem (1), we find the assertion is also valid for the problem (5).

Q.E.D.

Lemma 4.2. *We have*

$$\int_0^1 \theta\, dx \ge C \left(\int_0^1 u\, dx\right)^{-k_1}. \tag{4.10}$$

Proof. After dividing both sides of (1.1.3) by θ, integrating by parts them over $\overline{\Omega}$ by use of (1.2) and (1.4), we get

$$\left\{\int_0^1 (c\log\theta + R\log u)dx\right\}_t = V(t) \geq 0.$$

Integrating them over [0,t] and applying Jensen's inequality, we have

$$\int_0^1 (c\log\theta_0 + R\log u_0)dx \leq \int_0^1 (c\log\theta + R\log u)dx$$
$$\leq c\log\left\{\int_0^1 \theta\, dx\right\} + R\log\left\{\int_0^1 u\, dx\right\}.$$

We can derive (4.10) from the above estimate.

Q.E.D.

Now we proceed to prove Theorem 2.

(I) Non-isothermal case.

(i) The problems (2) and (3).

Here we prove (2.10) and (2.11) for the problem (2) only; the proof for the problem (3) is quite similar.

Integrating (1.1.1) over $\overline{Q_t}$ and applying Schwarz' inequality and (3.3.1), we have

$$\int_0^1 u\, dx \leq C(1 + t^{½}). \tag{4.11}$$

The above estimate and the estimate derived from (4.1) and (4.10) yield

$$\left\{1 + \left(\int_0^1 u\, dx\right)^2\right\} \geq C\int_0^t \left(\int_0^1 u\, dx\right)^{-k_1} d\tau$$
$$\geq C\int_0^t (1 + \tau^{½})^{-k_1} d\tau \geq \begin{cases} Ct^{1-\frac{k_1}{2}} & \text{for } k_1 < 2, \\ C\log(1 + t) & \text{for } k_1 = 2, \\ C \text{ if } t > 1 & \text{for } k_1 > 2. \end{cases} \tag{4.12}$$

(2.10) is obtained from (3.1), (4.11) and (4.12).

We prove (2.11) in the following way. If we assume that for some $s > 0$, there exists $C(s) \in [0,+\infty)$ such that

$$\overline{\lim_{t\to+\infty}}\, t^{-s}\int_0^1 u\, dx = C(s),$$

then we can establish in the same manner as the one mentioned above,

$$\left.\begin{matrix} C^{-1}t^{\frac{1}{2}(1-k_1 s)} \\ C^{-1}(\log(1+t))^{\frac{1}{2}} \\ C^{-1} \end{matrix}\right\} \le \int_0^1 u\,dx \le C(1+t^s) \qquad \text{for} \quad \begin{cases} s < k_1^{-1}, \\ s = k_1^{-1}, \\ s > k_1^{-1}. \end{cases} \tag{4.13}$$

But this can not be true when $s < \dfrac{1}{k_1 + 2}$.

Q.E.D.

(ii) The problem (4).

Integrating (3.11) over $\overline{\Omega}$, we have by the virtue of (3.3.2) and (3.16),

$$\int_0^1 u\,dx \le C(1+t), \tag{4.14}$$

and then we can obtain in a similar way to (i),

$$\left.\begin{matrix} C^{-1}t^{1-k_1} \\ C^{-1}\log(1+t) \\ C^{-1} \end{matrix}\right\} \le \int_0^1 u\,dx \le C(1+t) \qquad \text{for} \quad \begin{cases} k_1 < 1, \\ k_1 = 1, \\ k_1 > 1, \end{cases} \tag{4.15}$$

and (2.13).

From (3.6), (3.11) and (3.16), we get

$$m_\theta(t) \ge C_1\left(1 + \int_0^t \frac{C}{\underline{u_0}}\,d\tau\right)^{-1} \ge \frac{C}{1+t} .$$

Using this estimate, (3.11) and (3.16) again, we obtain

$$m_u(t) \ge C\left(1 + \int_0^t \frac{d\tau}{1+\tau}\right) \ge C\log(1+t). \tag{4.16}$$

On the other hand, (3.6), (3.11), (3.16) and the estimate

$$\frac{RD(x,\tau)}{\mu D(x,t)} \ge \frac{k_2}{C_1} ,$$

give us

$$m_u(t) \ge C + \frac{k_2}{C_1}\int_0^t m_\theta(\tau)\,d\tau$$

$$\geq C + \int_0^t \frac{k_2 m_u(\tau)d\tau}{m_u(\tau)\left(1 + \int_0^\tau \frac{ds}{m_u(s)}\right)}.$$

Applying Gronwall's inequality, and using (4.14) for the estimate of $m_u(\tau)$,

$$m_u(t) \geq C\exp\left\{\int_0^t \frac{k_2 d\tau}{m_u(\tau)\left(1 + \int_0^\tau \frac{ds}{m_u(s)}\right)}\right\}$$

$$= C\left(1 + \int_0^t \frac{d\tau}{m_u(\tau)}\right)^{k_2} \geq C\left(1 + \int_0^t \frac{d\tau}{C(1+\tau)}\right)^{k_2}$$

$$\geq C(\log(1 + t))^{k_2}. \tag{4.17}$$

We correct (4.15) for $k_1 \geq 1$ by (4.16) and (4.17) as follows:

$$C^{-1}(\log(1 + t))^{\max(1,k_2)} \leq \int_0^1 u\, dx \leq C(1 + t) \qquad \text{for } k_1 \geq 1. \tag{4.18}$$

The assertions (2.12) - (2.14) are derived from (4.14) - (4.18).

Q.E.D.

(iii) The problems (1) and (5).

In the same manner as (ii), we have from (3.3.3), (3.12) and (3.16),

$$\int_0^1 u\, dx \leq C(1 + t).$$

And we can establish the following estimate from (3.3.3) and (4.1),

$$\int_0^1 u\, dx \geq C\int_0^t\int_0^1 (v^2 + \theta)dxd\tau - 1 = Ct - 1.$$

These estimates and (3.1) yield (2.15). (2.16) is obtained in a similar way to (ii).

Q.E.D.

(II) Isothermal case.

(iv) The problem (2) and (3).

$$\int_0^1 u\, dx \leq C(1 + t^{½})$$

is established in the same manner as (i). The estimate from below is obtained from (4.1) with θ = const., *i.e.*:

$$1 + \left(\int_0^1 u\, dx\right)^2 \geq C\int_0^t\int_0^1 \theta\, dxd\tau = Ct.$$

Q.E.D.

(v) The problems (1), (4) and (5).

We get (2.20) by using (3.11) or (3.12) with θ = const. and (3.16).

Q.E.D.

Now we complete the proof of Theorem 2.

Remark. In cases (ii) and (iii), we can show slightly shaper (but more intricate) estimates than (2.14) and (2.16). Namely, these estimates and (3.6) yield

$$m_\theta(t) \geq C\left(1 + \int_0^t \frac{d\tau}{(\log(1+\tau))^{\max(1,\ell_2)}}\right)^{-1}.$$

Substituting into (3.11) or (3.12), we get desired estimates about $m_u(t)$. Yet the order function can not be described by an elementary function. Iterating the same procedure, we can also get more shaper (but much more intricate) estimates.

Acknowledgment.

The author would like to thank to Prof. A. Tani and Prof. N. Kikuchi for their interest in this work, the useful suggestions and the great encouragement.

References

1. Greenberg, J. M. and T. T. Li, The effect of boundary damping for the quasilinear wave equation, *J. Differential Equations*, **52** (1984), 66-75.
2. Itaya, N., On the Cauchy problem for the system of fundamental equations describing

the movement of compressible viscous fluid, *Kōdai Math. Sem. Rep.*, **23** (1971), 60-120.

3. Itaya, N., On the fundamental system of equations for compressible viscous fluid, *Sûgaku*, **28** (1976), 121-136 (Japanese).

4. Kawashima, S. and T. Nishida, Global solutions to the initial value problem for the equations of one-dimensional motion of viscous polytropic gases, *J. Math. Kyoto Univ.*, **21** (1981), 825-837.

5. Kazhykhov(Kazhikhov), A. V., Sur la solubilité globale des problèmes monodimensionnels aux valeurs initiales-limitées pour les èquations d'un gaz visquex et calorifère, *C. R. Acad. Sci. Paris Sér. A*, **284** (1977), 317-320.

6. Kazhikhov, A. V., To the theory of boundary value problems for equations of a one-dimensional non-stationary motion of a viscous heat-conductive gas, in "*Boundary value problems for equations of hydrodynamics*," Institute of Hydrodynamics, Novosibirsk, **50** (1981), 37-62 (Russian).

7. Kazhikhov, A. V. and V. V. Shelukhin, The unique solvability "in the large" with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas, *Prikl. Mat. Mekh.*, **41** (1977), 282-291 (Russian).

8. Nash, J., Le problème de Cauchy pour les équations differentielles d'un fluide général, *Bull. Soc. Math. France*, **90** (1962), 487-497.

9. Tani, A., On the first initial-boundary value problem of compressible viscous fluid motion, *Publ. Res. Inst. Math. Sci.*, **13** (1977), 193-253.