Research Report

KSTS/RR-16/004
November 25, 2016
(Second edition, August 24, 2017)

On the theory of generalized Hilbert transforms
Chapter V
The spectre analysis and synthesis on the N.Wiener class S

by

Sumiyuki Koizumi

Sumiyuki Koizumi
Department of Mathematics
Faculty of Science and Technology
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

KSTS/RR-16/004
November 25, 2016
(Second edition, August 24, 2017)

ON THE THEORY OF GENERALIZED HILBERT TRANSRORMS
CHAPTER V
THE SPECTRE ANALYSIS AND SYNTHESIS ON THE N.WIENER CLASS S

Sumiyuki Koizumi

Note:
The Chapter V in this Research Report was published at November 25, 2016 firstly and the second edition with addition about properties $\sigma^*(u)$ is presented here at August 21, 2017 (c.f. pp.119~121).

KSTS/RR-16/004
November 25, 2016
(Second edition, August 24, 2017)

ON THE THEORY OF GENERALIZED HILBERT TRANSRORM V

THE SPECTRE ANALYSIS AND SYNTHESIS ON THE N.WIENER CLASS S

by
Sumiyuki Koizumi

Department of Mathematics,
Faculty of Science and Technology, Keio University,
3-14-1, Hiyoshi, Kohoku-Ku, Yokohama, 223, JAPAN

ABSTRACT
We shall continue the problem of spectrum of function of the N.Wiener class S after the preceding section 13 in this research report IV and we shall present the more fine and advanced results.

14. The Spectral Analysis on the N.Wiener class S.

We shall intend to construct the theory of spectral analysis on the N.Wiener class S nuder the hypothesis of which relaxed as in section 13 of IV in this series (c.f. pp.105~114).

We shall denote as before that the function $f(x)$ of the N.Wiener class S and $\varphi(x)$ of its correlation function and also that $s(u)$ and $\sigma(u)$ the G.F.T. of f and φ respectively.

We shall set the presupposed conditions as follows. There exists

$$(C_\lambda) \qquad c_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}\,dx \quad (\forall real\ \lambda)$$

and

$$(D_\lambda) \qquad d_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \varphi(x)e^{-i\lambda x}dx \quad (\forall real\ \lambda)$$

respectively.

Let us denote

$$\varphi_\varepsilon(x) = \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{i\lambda x}\,|\,s(u+\varepsilon; f) - s(u-\varepsilon; f)\,|^2\,du$$

and

$$\sigma_\varepsilon(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\varphi_\varepsilon(x)\frac{e^{-iux}-1}{-ix}dx+\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\varphi_\varepsilon(x)\frac{e^{-iux}}{-ix}dx.$$

Then we have

$$P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\varphi_\varepsilon(x)\frac{e^{-iux}-1}{-ix}dx=\frac{1}{2\varepsilon\sqrt{2\pi}}\int_0^u|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv$$

for any finite range of u and any positive number ε and we have

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_0^u|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv=\sigma_\varepsilon(u)+C_\varepsilon \qquad \text{a.e. u.}$$

where

$$C_\varepsilon=P.V.\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{\varphi_\varepsilon(x)}{ix}dx.$$

We shall refer the above formula to the Research Report III,pp.55~57.

Let us notice that $\sigma_\varepsilon(u)$ is defined on the space L^2 and so it is indeterminable on the set of measure 0. Therefore we shall intend to define $\sigma_\varepsilon(u)$ on the indeterminable set by the above formula for any point of u. Then $\sigma_\varepsilon(u)$ is defined everywhere and it is bounded continuous and monotone increasing function.

Now we shall consider the sequence $\{\sigma_\varepsilon(u)\}$ of which just defined

Since we have

$$\underset{\varepsilon\to 0}{l.i.m.}\sigma_\varepsilon(u)=\sigma(u) \qquad (L^2)$$

and the $\sigma_\varepsilon(u)$ is a bounded continuous and monotone increasing function of u for each fixed positive number ε. Then applying the Paley-Wiener Lemma [2](c.f. p.135),we could conclude that

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)=\sigma(u) \qquad a.e.\ u$$

and $\sigma(u)$ is also a bounded and monotone increasing function on the set of u where $\sigma(u)$ is defined.

14.1 The refinement of properties of $\sigma(u)$

First of all, we shall intend to clear the properties of $\sigma(u)$. The reader should refer

Lemma D_4 of III in this series of research report [3](c.f. pp.52~60).

Now we shall intend to refinement of the properties of the $\sigma(u)$ as follows.

Let us denote the set D of u where the sequence $\{\sigma_\varepsilon(u)\}$ is convergent and the set E of u where it is not convergent. Then we have $D \cup E = (-\infty, +\infty),\ \ m(E) = 0$.

We shall define

$$\underline{\sigma}(u) = \sup_{\substack{v<u \\ v\in D}} \sigma(v), \quad \overline{\sigma}(u) = \inf_{\substack{u<v \\ v\in D}} \sigma(v) \quad \text{and} \quad \sigma^*(u) = \frac{\underline{\sigma}(u) + \overline{\sigma}(u)}{2}$$

respectively.

We shall intend to investigate properties of the $\sigma^*(u)$

(i) The $\sigma^*(u)$ is defined at every point of u and bounded, monotone increasing function .

Proof. Since and $\overline{\sigma}(u)$ are $\underline{\sigma}(u)$ both to be bounded and so $\sigma^*(u)$ does too.

Since $\sigma(u)$ is bounded and monotone increasing on the set D, we have for any pair of (u', u'') for $u' < u''$

$$\sup_{\substack{v'<u' \\ v'\in D}} \sigma(v') \le \sup_{\substack{v''<u'' \\ v''\in D}} \sigma(v'')$$

and so we have

$$\underline{\sigma}(u') \le \underline{\sigma}(u'').$$

Then $\underline{\sigma}(u)$ is defined at every point of u and a bounded ,monotone increasing function.

Similarly we have the same property as $\overline{\sigma}(u)$ too.

Therefore we have

$$\sigma^*(u') = \frac{\underline{\sigma}(u') + \overline{\sigma}(u')}{2} \le \frac{\underline{\sigma}(u'') + \overline{\sigma}(u'')}{2} = \sigma^*(u'').$$

(ii) The $\sigma^*(u)$ satisfies at every point of u the following properties

$$\sigma^*(u-0) = \underline{\sigma}(u) \quad \text{and} \quad \sigma^*(u+0) = \underline{\sigma}(u).$$

Proof. Since $\sigma(u)$ is bounded, monotone increasing function on the set D, we have for any pair of (u', u'') such as $u' < u''$

$$\overline{\sigma}(u') = \inf_{\substack{u'<v' \\ v'\in D}} \sigma(v') = \inf_{\substack{u'<v'<u'' \\ v'\in D}} \sigma(v') \le \sup_{\substack{u'<v''<u'' \\ v''\in D}} \sigma(v'') = \sup_{\substack{v''<u'' \\ v''\in D}} \sigma(v'') = \underline{\sigma}(u'').$$

Therefore we have proved for any pair of (v,u) such as $v<u$,

$$\underline{\sigma}(v) \leq \overline{\sigma}(v) \leq \underline{\sigma}(u)$$

and so

$$\sigma^*(v) = \frac{\underline{\sigma}(v)+\overline{\sigma}(v)}{2} \leq \underline{\sigma}(u).$$

Then we shall take the limit such as $v \uparrow u$, we have by the property(i) of $\sigma^*(u)$

$$\sigma^*(u-0) \leq \underline{\sigma}(u).$$

Now we shall intend to prove the inverse inequality.

For any point v in the set D such as $v<u$, then we shall take a point v' such as $v<v'<u$. According to the case of the set to which v' belongs we shall treat as follows.

(a) In the case $v' \in D$, Since $\sigma(v')$ is monotone increasing on the set D, we have by the definition of $\underline{\sigma}(v')$ and $\underline{\sigma}(v')$ directly

$$\sigma(v) \leq \sigma(v') = \underline{\sigma}(v') = \overline{\sigma}(v').$$

and so we have

$$\sigma(v) \leq \sigma^*(v') = \frac{\underline{\sigma}(v')+\overline{\sigma}(v')}{2}.$$

For the sake of completeness we shall prove it as follows.

Lemma . We have

$$\sigma(u) = \underline{\sigma}(u) = \overline{\sigma}(u) \qquad (\forall u \in D)$$

Proof. We have for any point $v \in D$ such as $v<u, \sigma(v) \leq \sigma(u)$ and so

$$\underline{\sigma}(u) \leq \sigma(u).$$

On the other hand , since $u \in D$, for any $\eta_1 > 0$, there exists $\varepsilon' > 0$ such that

$$(\alpha) \qquad |\sigma_\varepsilon(u) - \sigma(u)| < \eta_1 \qquad (0 < \varepsilon < \varepsilon').$$

Since $\sigma_\varepsilon(u)$ is continuous, for any $\eta_2 > 0$, there exists a point $v \in D$, such that

$$(\beta) \qquad |\sigma_\varepsilon(u) - \sigma_\varepsilon(v)| < \eta_2 \qquad (v<u).$$

Since $v \in D$, for any $\eta_3 > 0$, there exists $\varepsilon'' > 0$ such that

$$(\gamma) \qquad |\sigma_\varepsilon(v) - \sigma(v)| < \eta_3 \qquad (0 < \varepsilon < \varepsilon'').$$

Combining these estimations (α), (β) and (γ) ,for any $\eta > 0$, we shall take $\eta_1 = \eta_2 = \eta_3 = \eta/3$, then we have

$$|\sigma(u)-\sigma(v)| \leq |\sigma(u)-\sigma_\varepsilon(u)| + |\sigma_\varepsilon(u)-\sigma_\varepsilon(v)| + |\sigma_\varepsilon(v)-\sigma(v)|$$
$$< \eta_1 + \eta_2 + \eta_3 = \eta, \qquad (0 < \varepsilon < min(\varepsilon', \varepsilon'')).$$

Therefore we have

$$\sigma(u) < \sigma(v) + \eta, \qquad (0 < \varepsilon < min(\varepsilon', \varepsilon'')$$

Thus we have

$$\sigma(u) \leq \underline{\sigma}(u).$$

Therefore we have proved $\sigma(u) = \underline{\sigma}(u)$. Similarly we shall prove $\sigma(u) = \overline{\sigma}(u)$.

(b) In the case $v' \notin D$, that is $v' \in E$, we have

$$\sigma^*(v') = \frac{\underline{\sigma}(v') + \overline{\sigma}(v')}{2}$$

and since $v < v'$, we have $\sigma(v) \leq \underline{\sigma}(v') \leq \overline{\sigma}(v')$ then we have

$$\sigma(v) \leq \sigma^*(v') = \frac{\underline{\sigma}(v') + \overline{\sigma}(v')}{2}.$$

Since in both cases we have the same inequality, therefore tending to $v' \uparrow u$, we shall conclude that

$$\sigma(v) \leq \sigma^*(u-0),$$

for any point v in the set D such as $v < u$.

Therefore we have

$$\underline{\sigma}(u) = \sup_{\substack{v<u \\ v\in D}} \sigma(v) \leq \sigma^*(u-0).$$

Thus we have proved the same inequality in both case, therefore we have

$$\sigma^*(u-0) = \underline{\sigma}(u).$$

Similarly we have

$$\sigma^*(u+0) = \overline{\sigma}(u).$$

(iii) The $\sigma^*(u) = \sigma(u)$ on the set D and continuous at any point u of the set D.

Proof. First of all we shall remark that

$$\underline{\sigma}(u) \leq \sigma(u) \leq \overline{\sigma}(u) \qquad (\forall u \in D).$$

Let us suppose that $\overline{\sigma}(u) - \underline{\sigma}(u) > 0$ at a point u of the set D. In particular let us suppose that $\sigma(u) - \underline{\sigma}(u) = \eta > 0$.

Since $\lim_{\varepsilon \to 0} \sigma_\varepsilon(u) = \sigma(u)$, for any $\eta_1 > 0$ there exists $\varepsilon' > 0$ such that

(a) $$|\sigma_\varepsilon(u) - \sigma(u)| < \eta_1 \qquad (0 < \varepsilon < \varepsilon').$$

Since $\sigma_\varepsilon(v) \to \sigma_\varepsilon(u)$ as $v \in D$ and $v \uparrow u$, we have for any $\varepsilon\ (0 < \varepsilon < \varepsilon')$ to be fixed and for any $\eta_2 > 0$ there exists $\delta > 0$ such that

(b) $$| \sigma_\varepsilon(v) - \sigma_\varepsilon(u) | < \eta_2 \qquad (u - \delta < v < u).$$

Since $\lim_{\varepsilon \to 0} \sigma_\varepsilon(v) = \sigma(v)$, for any $\eta_3 > 0$ there exists $\varepsilon'' > 0$ such that

(c) $$| \sigma_\varepsilon(v) - \sigma(v) | < \eta_3 \qquad (0 < \varepsilon < \varepsilon'')$$

Then we have for any $\varepsilon > 0$ to be fixed such as $0 < \varepsilon < min(\varepsilon', \varepsilon'')$

$$\sigma(v) > \sigma_\varepsilon(v) - \eta_3 > \sigma_\varepsilon(u) - \eta_2 - \eta_3 > \sigma(u) - \eta_1 - \eta_2 - \eta_3 = \sigma(u) - (\eta_1 + \eta_2 + \eta_3)$$

by combining the above estimations (a), (b) and (c). Then we could take $\eta_1 + \eta_2 + \eta_3 < \eta$ and we shall prove that there exists $v < u$ and $v \in D$ such that

$$\sigma(v) > \sigma(u) - \eta = \underline{\sigma}(u).$$

This lead to the contradiction that $\sigma(v) \leq \underline{\sigma}(u) \qquad (\forall v < u, v \in D)$.

Thus we shall conclude that

$$\underline{\sigma}(u) = \sigma(u)$$

at any point u of the set D.

Similarly we shall conclude that

$$\overline{\sigma}(u) = \sigma(u) \qquad (\forall u \in D)$$

and

$$\sigma^*(u) = \frac{\underline{\sigma}(u) + \overline{\sigma}(u)}{2} = \sigma(u)$$

By the results of (ii), we have

$$\sigma^*(u - 0) = \underline{\sigma}(u) \quad \text{and} \quad \sigma^*(u + 0) = \overline{\sigma}(u).$$

But since we have $\overline{\sigma}(u) = \underline{\sigma}(u) = \sigma(u)$ at any point of u of the set D and so

we have

$$\sigma^*(u - 0) = \sigma^*(u + 0) = \sigma^*(u)$$

at any point u of the set D.

Thus we shall prove that

$$\sigma^*(u) = \sigma(u)$$

on the set D and $\sigma^*(u)$ is continuous at any point u of the set D.

(iv) The $\sigma^*(u)$ is discontinuous of the first kind for every point u of the set E and the set E is at most countable.

Proof. In the first we shall intend to prove that

$$\underline{\sigma}(u) < \overline{\sigma}(u)$$

for any point u of the set E.

Let us suppose that $\underline{\sigma}(u)=\overline{\sigma}(u)$ at a point u of the set E. Then any pairs of $\varepsilon,\varepsilon'>0$, we have

$$\sigma_\varepsilon(u-\varepsilon')\le\sigma_\varepsilon(u)\le\sigma_\varepsilon(u+\varepsilon').$$

Since the measure of the set E is 0 and so , for any point of u of the set E, there exists a sequence of points $\{u\pm\varepsilon'\}$ such that $u\pm\varepsilon'\in D$ and $\{\varepsilon'\}\downarrow 0$. In the first we shall intend $\{\varepsilon\}\downarrow 0$, then we have

$$\sigma(u-\varepsilon')\le\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\sigma(u+\varepsilon')$$

Next we shall intend the $\{\varepsilon'\}\downarrow 0$. Since the $\sigma(u)$ is bounded monotone increasing function on the set D, we have

$$\lim_{\varepsilon'\to 0}\sigma(u-\varepsilon')=\sup_{\substack{v<u\\ v\in D}}\sigma(v)=\underline{\sigma}(u)$$

and

$$\lim_{\varepsilon'\to 0}\sigma(u+\varepsilon')=\inf_{\substack{u<v\\ v\in D}}\sigma(v)=\overline{\sigma}(u)$$

Therefore we have

$$\underline{\sigma}(u)\le\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)\le\overline{\sigma}(u).$$

Since we have assumed that $\underline{\sigma}(u)=\overline{\sigma}(u)$ at the point u, we have

$$\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)=\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(u)$$

and so $\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)$ exists. Thus it lead to the contradiction of the point u of set E where $\lim_{\varepsilon\to 0}\sigma_\varepsilon(u)$ does not exists.

Next we shall intend to prove that the set E is at most countable.
By the results of (ii), we have

$$\sigma^*(u+0)-\sigma^*(u-0)=\overline{\sigma}(u)-\underline{\sigma}(u)>0$$

for any point u of the set E.

Since the $\sigma^*(u)$ is bounded, monotone increasing function, we shall conclude that the $\sigma^*(u)$ is discontinuous of the first kind at each point u of the set E and the set E is at most countable.

Thus we have proved that the function $\sigma^*(u)$ could be defined everywhere and satisfies the following properties.

The function $\sigma^*(u)$ is bounded, monotone increasing function of u and furthermore

(a) On any point u of the set D, we have $\sigma^*(u)=\sigma(u)$ and moreover

$$\sigma^*(u-0)=\sigma^*(u+0)=\sigma^*(u)$$

and so it is continuous there.

(b) On any point u of the set E, we have

$$\sigma^*(u-0)=\underline{\sigma}(u) \quad \text{and} \quad \sigma^*(u+0)=\overline{\sigma}(u)$$

and it is discontinuous of the first kind and has the magnitude of jump

$$\sigma^*(u+0)-\sigma^*(u-0)=\overline{\sigma}(u)-\underline{\sigma}(u)>0.$$

It should be remarked that the set E is at most countable.

. Hereafter we shall denote $\sigma(u)$ instead of $\sigma^*(u)$. Thus we have defined $\sigma(u)$ on the set $D\cup E=(-\infty,\ \infty)$ and satisfy the above properties (a) and (b).

14.2 The spectral synthesis of function of the N.Wiener class S.

Let us begin to prove the general properties of function $f(x)$ in the N.Wiener class S under the hypothesis stated in the beginning of section 14.1 as follows.

There exists

$$(C_\lambda) \qquad c_\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}\,dx \quad (\forall real\ \lambda)$$

and

$$(D_\lambda) \qquad d_\lambda=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \varphi(x)e^{-i\lambda x}dx \quad (\forall real\ \lambda)$$

respectively.

Then we shall prove

Theorem E. Let us suppose that the function $f(x)$ of which belongs to the N.Wiener class S and satisfies the conditions (C_λ) and (D_λ) for all real λ.

Then we have

(I) $$|c_\lambda|^2\le d_\lambda \qquad (\forall real\ \lambda)$$

(II) There exist at most countable set of real number $\lambda = \{\lambda_n\}$ $(n = 0,1,2,\cdots\cdots)$ and it satisfies followings

(i) $$c_{\lambda_n} \neq 0 \quad (n = 0,1,2,\cdots\cdots),$$

where we shall denote the c_n instead of c_{λ_n} $(n = 0,1,2,\cdots\cdots)$ and $\lambda_0 = 0$ with $c_0 = 0$ may be permitted.

(ii) $$c_\lambda = d_\lambda = 0 \quad (\forall \lambda \notin \Lambda)$$

There exist B_2 almost periodic function $g(x)$ of which can be expressed as its Fourier series

(iii) $$g(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

If we decompose $f(x)$ as follows

(iv) $$f(x) = g(x) + h(x)$$

then we have

(v) $$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |h(x)|^2 dx = \frac{1}{2\pi} \sum_n (d_n - |c_n|^2)$$

After Prof.N.Wiener we shall denote $f(x)$ and $s(u)$ as its G.F.T. and also $\varphi(x)$ and $\sigma(u)$ as its correlation function and G.F.T. respectively. N.Wiener[1](c.f. p.159) also introduced the following functions

(21.175) $$\varphi_\varepsilon(x) = \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{i\lambda x} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 \, du$$

and

(21.22) $$\sigma_\varepsilon(u) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx + \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_{1}^{A} \right] \varphi_\varepsilon(x) \frac{e^{-iux}}{-ix} dx$$

Then R.E.A.C.Paley-N.Wiener[2](c.f. p.135) proved that

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon(u) = \sigma(u) \quad (a.e.\ \ u).$$

We have proved in III in this research report [3] (c.f. pp.52~60)

Lemma D_4. Let us suppose that $f(x)$ belongs to the class S, then the following limit

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{u} |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv = \lim_{\varepsilon\to 0} (\sigma_\varepsilon(u) - \sigma_\varepsilon(-\varepsilon))$$

exists and equals to

$$\sigma(u)-\sigma(-0) \qquad (a.e. \ \ u)$$

over any finite range of u.

Now we shall begin to refine Lemma D_4 for the sake of completeness and prove Lemma E. Let us suppose that $f(x)$ belongs to the class S, then the following

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\pm\varepsilon}^{u\pm\varepsilon}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv=\lim_{\varepsilon\to 0}(\sigma_\varepsilon(u\pm\varepsilon)-\sigma_\varepsilon(\pm\varepsilon))$$

exists and equals to

$$\sigma(u\pm 0)-\sigma(\pm 0)$$

for any point u, respectively.

Proof of Lemma E.

(i) Let us suppose that $0\in D$, then we have $\lim_{\varepsilon\to 0}\sigma_\varepsilon(0)=\sigma(0)$,where $\{\varepsilon\}$ is any sequence of positive number ε and tend to 0.

Since $\{\sigma_\varepsilon(u)\}$ is bonded, monotone increasing as function of u for each positive number ε, we have

$$\sigma_\varepsilon(-u)\le\sigma_\varepsilon(-\varepsilon)\le\sigma_\varepsilon(0) \qquad (0<\varepsilon<u).$$

Then tending $\varepsilon\to 0$ for any point u such as $-u\in D$ where $\sigma_\varepsilon(-u)\to\sigma(-u)$, we have

$$\sigma(-u)\le\underline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(-\varepsilon)\le\overline{\lim}_{\varepsilon\to 0}\sigma_\varepsilon(-\varepsilon)\le\sigma(0).$$

In the last tending $u\to 0$ such as $-u\in D$ where $\sigma(-u)\to\sigma(-0)=\sigma(0)$,thus we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(-\varepsilon)=\sigma(-0).$$

Similarly we have

$$\lim_{\varepsilon\to 0}\sigma_\varepsilon(+\varepsilon)=\sigma(+0).$$

(ii) Let us suppose that $0\notin D$, that is $0\in E$. Since the following limit $\lim_{\varepsilon\to 0}\sigma_\varepsilon(0)$ does not exist, we have defined $\sigma(0)$ by using $\sigma^*(0)$ as follows

$$\sigma(0)\equiv\sigma^*(0)=\frac{\underline{\sigma}(0)+\overline{\sigma}(0)}{2}$$

Let us put $\sigma(+0)-\sigma(-0)=d>0$ and define

$$\sigma^{\sim}(u)=\sigma(u)-dh(u)$$

where $h(u)$ is the Heaviside operator, that is as follows

$$h(u)=\begin{cases}1 & (u>0)\\ \frac{1}{2} & (u=0)\\ 0 & (u<0).\end{cases}$$

Then $\sigma^{\sim}(u)$ is continuous at $u=0$. Because we have

$$\sigma^{\sim}(-0)=\sigma(-0)$$

$$\sigma^{\sim}(0)=\sigma(0)-\frac{\sigma(+0)-\sigma(-0)}{2}=\frac{\sigma(-0)+\sigma(+0)}{2}-\frac{\sigma(+0)-\sigma(-0)}{2}=\sigma(-0)$$

$$\sigma^{\sim}(+0)=\sigma(+0)-(\sigma(+0)-\sigma(-0))=\sigma(-0)$$

respectively. Now let us put

$$\sigma_{\varepsilon}^{\sim}(u)=\sigma_{\varepsilon}(u)-dh(u)$$

and consider the sequence $\{\sigma_{\varepsilon}^{\sim}(u)\}$ instead of $\{\sigma_{\varepsilon}(u)\}$, then it is continuous at any point u of the set D. Because we have

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}^{\sim}(u)=\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(u)-dh(u)=\sigma(u)-dh(u)=\sigma^{\sim}(u)$$

and $\sigma^{\sim}(u)$ is continuous at $u=0$. Then we have by using the results of (i)

$$\lim_{\varepsilon\to 0}\sigma^{\sim}(-\varepsilon)=\sigma^{\sim}(-0)$$

where

$$\sigma_{\varepsilon}^{\sim}(-\varepsilon)=\sigma_{\varepsilon}(-\varepsilon)-dh(-\varepsilon)=\sigma_{\varepsilon}(-\varepsilon)\quad\text{and}\quad\sigma^{\sim}(-0)=\sigma(-0)$$

Therefore we have

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(-\varepsilon)=\sigma(-0)$$

Similarly we have

$$\lim_{\varepsilon\to 0}\sigma^{\sim}(+\varepsilon)=\sigma^{\sim}(+0)$$

where

$$\sigma_{\varepsilon}^{\sim}(+\varepsilon)=\sigma_{\varepsilon}(+\varepsilon)-dh(+\varepsilon)=\sigma_{\varepsilon}(+\varepsilon)-(\sigma(+0)-\sigma(-0))$$

and

$$\sigma^{\sim}(+0)=\sigma(+0)-d=\sigma(+0)-(\sigma(+0)-\sigma(-0))=\sigma(-0)$$

Thus we have proved

$$\lim_{\varepsilon\to 0}\{\sigma_{\varepsilon}(+\varepsilon)-(\sigma(+0)-\sigma(-0))\}=\sigma(-0)$$

and thus we have proved

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(+\varepsilon)=\sigma(+0)$$

In general we shall prove by the same argument as above at the point $u=0$,

$$\lim_{\varepsilon\to 0}\sigma_{\varepsilon}(u\pm\varepsilon)=\sigma(u\pm 0)$$

for any point $u\notin D$, that is $u\in E$. (Notice: In these cases, we should remark that we would use $\sigma(\pm 0)$ etc. instead as the ordinary notations $\sigma(0\pm)$ etc. respectively).

Proof of the Theorem E. First of all let us1 notice that we have

$$c_{\lambda}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx \qquad (\forall real\ \ \lambda).$$

Then we have by the hypotheses (C_{λ}) and the one sided Wiener formula,

$$|c_{\lambda}|^2=|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx|^2=|\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(v+\varepsilon;f)-s(v-\varepsilon;f)\}dv|^2$$

$$\le\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(v+\varepsilon;f)-s(v-\varepsilon;f)|^2dv=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}$$

and we have by the hypotheses (D_{λ}) and one-sided Wiener formula too

$$d_{\lambda}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x;f)e^{-i\lambda x}dx$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{\sigma(u+\varepsilon;\varphi)-\sigma(u-\varepsilon;\varphi)\}du=\frac{\sigma(\lambda+0;\varphi)-\sigma(\lambda-0;\varphi)}{\sqrt{2\pi}}$$

for all real λ where we should apply the Lemma E in each of the last formula.

Then we have

$$|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-i\lambda x}dx|^2\le\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi(x;f)e^{-i\lambda x}dx$$

for all real λ. Therefore we have

$$|c_{\lambda}|^2\le d_{\lambda} \qquad (\forall real\ \lambda)$$

According to Lemma E, we shall expect as follows. Let us put the set $E=\{\lambda_n\}$ $(n=0,1,2,\cdots\cdots)$ as the set Λ and the set D as its complement. Let us put

$$d_\lambda = \begin{cases} 0, & \lambda \in D \\ \dfrac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}}, & (n=0,1,2,\cdots\cdots) \end{cases}$$

and

$$c_\lambda = \begin{cases} 0, & \lambda \in D \\ c_n & (n=0,1,2,\cdots\cdots) \end{cases}$$

Then we shall acknowledge that these settings are true and coincide with the assertions of Theorem E.

We have

$$\sum_n |c_n|^2 \le \sum_n d_n = \sum_n \frac{\sigma(\lambda_n+0;\varphi)-\sigma(\lambda_n-0;\varphi)}{\sqrt{2\pi}} \le \frac{\sigma(+\infty;\varphi)-\sigma(-\infty;\varphi)}{\sqrt{2\pi}} < \infty.$$

Therefore we shall conclude that there exists a B_2-almost periodic function $g(x)$ of which Fourier series is as follows

$$g(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

Then if we put

$$f(x)-g(x)=h(x)$$

say, and we shall consider correlation functions $\varphi(x;f)$, $\psi(x;g)$ and $\chi(x;h)$ of f,g,h ; their G.F.T. $\sigma(u;\varphi),\sigma(u;\psi)$ and $\sigma(u;\chi)$ of φ,ψ,χ respectively. Let us remark that

$$c_\lambda = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x)e^{-i\lambda x}dx \qquad (\forall real\ \lambda).$$

Then by repeating the same arguments as IV in this research report [4] (c.f. pp.105~108), we shall prove

$$\varphi(x;f)=\psi(x;g)+\chi(x;h)$$

and

$$\sigma(u;\varphi)=\sigma(u;\psi)+\sigma(u;\chi)$$

respectively.

Then by the Lemma E, $\sigma(u;\varphi)$ is continuous on the set D and discontinuous of the first kind with jump on the set E such as

$$d_n = \frac{\sigma(\lambda_n + 0;\varphi) - \sigma(\lambda_n - 0;\varphi)}{\sqrt{2\pi}}$$

$$= \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda_n - \varepsilon}^{\lambda_n + \varepsilon} \{\sigma(u+\varepsilon;\varphi) - \sigma(u-\varepsilon;\varphi)\} du \qquad (n = 0,1,2,\cdots\cdots)$$

On the other hand , since $\sigma(u;\psi)$ is G.F.T. of $\psi(x;g)$ and $\psi(x;g)$ is the correlation function of B_2-almost periodic function $g(x)$, we have

$$\sigma(u;\psi) = \begin{cases} \sqrt{2\pi} \sum_{\lambda_n < u} |c_n|^2 & (u \neq \lambda_m) \\ \sqrt{2\pi} \left(\sum_{\lambda_n < u} |c_n|^2 + \frac{1}{2} |c_m|^2 \right) & (u = \lambda_m) \end{cases}$$

Then $\sigma(u;\psi)$ is continuous on the set D and discontinuous of the first kind with jump on the set E such as

$$|c_n|^2 \le \frac{\sigma(\lambda_n + 0;\psi) - \sigma(\lambda_n - 0;\psi)}{\sqrt{2\pi}} \qquad (n = 0,1,2,\cdots\cdots)$$

Thus we have

$$|c_n|^2 \le d_n \qquad (n = 0,1,2,\cdots\cdots)$$

on the set E.

Furthermore we have

$$c_\lambda = d_\lambda = 0,$$

on the set D.

Therefore we shall prove that $\sigma(u;\chi)$ is bounded, monotone increasing function. Since $\sigma(u;\chi)$ is G.F.T. of $\chi(x;h)$ and $\chi(x;h)$ is the correlation function of $h(x)$, we have by the N.Wiener Theorem[1](Theorem 24,pp. 146~149)

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |h(x)|^2 \, dx = \frac{1}{2\pi} \sum_n (d_n - |c_n|^2).$$

In particular, if it is satisfied the condition

$$d_n - |c_n|^2 = 0 \qquad (n = 0,1,2,\cdots\cdots)$$

then we have

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |h(x)|^2 dx = 0.$$

Thus we have proved the Theorem E completely.

Reference

[1] N.Wiener, The Fourier Integral and Certain of its Applications, Cambridge Univ. (1932).

[2] R.E.A.C.Paley and N. Wiener, Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Pub. Vol. XIX (1934).

[3] S.Koizumi, ON THE THEORY OF GENERALIZED HILBERT TRANSRORMS III THE GENERALIZED HARMONIC ANALYSIS IN THE COMPLEX DOMAIN (1) Research Report, Department of Mathematics, Faculty of Science and Technology, Keio Univ. (2013)

[4] S.Koizumi, ON THE THEORY OF GENERALIZED HILBERT TRANSRORMS IV THE GENERALIZED HARMONIC ANALYSIS IN THE COMPLEX DOMAIN (2) Ibid (2013).

Department of Mathematics
Faculty of Science and Technology
Keio University

Research Report

2016

[16/001] Shiro Ishikawa,
Linguistic interpretation of quantum mechanics: Quantum Language [*Ver. 2*],
KSTS/RR-16/001, January 8, 2016

[16/002] Yuka Hashimoto, Takashi Nodera,
Inexact shift-invert Arnoldi method for evolution equations,
KSTS/RR-16/002, May 6, 2016

[16/003] Yuka Hashimoto, Takashi Nodera,
A Note on Inexact Rational Krylov Method for Evolution Equations,
KSTS/RR-16/003, November 9, 2016

[16/004] Sumiyuki Koizumi,
On the theory of generalized Hilbert transforms (*Chapter V: The spectre analysis and synthesis on the N.Wiener class S*),
KSTS/RR-16/004, November 25, 2016 (Second edition, August 24, 2017)

[16/005] Shiro Ishikawa,
History of Western Philosophy from the quantum theoretical point of view,
KSTS/RR-16/005, December 6, 2016

2017

[17/001] Yuka Hashimoto, Takashi Nodera,
Inexact Shift-invert Rational Krylov Method for Evolution Equations,
KSTS/RR-17/001, January 27, 2017 (Revised July 24, 2017)

[17/002] Dai Togashi, Takashi Nodera,
Convergence analysis of the GKB-GCV algorithm,
KSTS/RR-17/002, March 27, 2017

[17/003] Shiro Ishikawa,
Linguistic solution to the mind-body problem,
KSTS/RR-17/003, April 3, 2017

[17/004] Shiro Ishikawa,
History of Western Philosophy from the quantum theoretical point of view; Version 2,
KSTS/RR-17/004, May 12, 2017

[17/005] Sumiyuki Koizumi,
On the theory of generalized Hilbert transforms (*Chapter VI: The spectre analysis and synthesis on the N.Wiener class S* (2)),
KSTS/RR-17/005, June 8, 2017 (Second edition, September 1, 2017)