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The projection postulate in the linguistic interpretation of quantum mechanics

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Abstract

As the fundamental theory of quantum information science, recently I proposed measurement theory (i.e., quantum language, or the linguistic interpretation of quantum mechanics), which is characterized as the linguistic turn of the Copenhagen interpretation of quantum mechanics. This turn from physics to language does not only extend quantum theory to classical theory but also yield the quantum mechanical world view. Although the wave function collapse is prohibited in the linguistic interpretation, in this paper I show that the phenomenon like wave function collapse (or the projection postulate) can be realized without the phrase:"state after measurement".

Key phrases: Linguistic interpretation, Wave function collapse, von Neumann-Lüders projection postulate

1 Measurement theory (= quantum language)

1.1 Preparations

According to refs.[3]-[6], we briefly introduce measurement theory as follows.

Consider an operator algebra $B(H)$ (i.e., an operator algebra composed of all bounded linear operators on a Hilbert space H with the norm $\|F\|_{B(H)} = \sup_{\|u\|_H=1} \|Fu\|_H$), and consider the pair $[\mathcal{A}, \mathcal{N}]_{B(H)}$, called a *basic structure*. Here, $\mathcal{A}(\subseteq B(H))$ is a C^*-algebra, and $\mathcal{N}$ $(\mathcal{A} \subseteq \mathcal{N} \subseteq B(H))$ is a particular C^*-algebra (called a W^*-algebra) such that $\mathcal{N}$ is the weak closure of $\mathcal{A}$ in $B(H)$.
The measurement theory (=MT) is classified as follows.

$$
\text{(A)} \qquad \text{MT} = \begin{cases} (\text{A}_1)\text{: quantum system theory} \\ \qquad \scriptstyle(\text{when } \mathcal{A}=\mathcal{C}(H)) \\ (\text{A}_2)\text{: classical system theory} \\ \qquad \scriptstyle(\text{when } \mathcal{A}=C_0(\Omega)) \end{cases}
$$

That is, when $\mathcal{A} = \mathcal{C}(H)$, the C^*-algebra composed of all compact operators on a Hilbert space H, the (A_1) is called quantum measurement theory (or, quantum system theory), which can be regarded as the linguistic aspect of quantum mechanics. Also, when $\mathcal{A}$ is commutative (that is, when $\mathcal{A}$ is characterized by $C_0(\Omega)$, the C^*-algebra composed of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space Ω (*cf.* [9, 10])), the (A_2) is called classical measurement theory.
Also, note (*cf.* [9]) that, when $\mathcal{A} = \mathcal{C}(H)$,

(i) $\mathcal{A}^* = Tr(H)$ (=trace class), $\mathcal{N} = B(H)$, $\mathcal{N}_* = Tr(H)$ (i.e., pre-dual space)

Also, when $\mathcal{A} = C_0(\Omega)$,

(ii) $\mathcal{A}^*$ =the space of all signed measures on Ω", $\mathcal{N} = L^\infty(\Omega, \nu)(\subseteq B(L^2(\Omega, \nu)))$, $\mathcal{N}_* = L^1(\Omega, \nu)$, where ν is some measure on Ω (*cf.* [9]).

Let $\mathcal{A}(\subseteq B(H))$ be a C^*-algebra, and let $\mathcal{A}^*$ be the dual Banach space of $\mathcal{A}$. That is, $\mathcal{A}^* = \{\rho \mid \rho \text{ is a continuous linear functional on } \mathcal{A}\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid F \in \mathcal{A} \text{ such that } \|F\|_{\mathcal{A}}(= \|F\|_{B(H)}) \le 1\}$. Define the *mixed state* ρ $(\in \mathcal{A}^*)$ such that $\|\rho\|_{\mathcal{A}^*} = 1$ and $\rho(F) \ge 0$ for all $F \in \mathcal{A}$ such that $F \ge 0$. And define the mixed state space $\mathfrak{S}^m(\mathcal{A}^*)$ such that

$$\mathfrak{S}^m(\mathcal{A}^*)=\{\rho \in \mathcal{A}^* \mid \rho \text{ is a mixed state}\}.$$

A mixed state $\rho(\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that "$\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$" implies "$\rho = \rho_1 = \rho_2$". Put

$$\mathfrak{S}^p(\mathcal{A}^*)=\{\rho \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho \text{ is a pure state}\},$$

which is called a *state space.* It is well known (*cf.* [9]) that $\mathfrak{S}^p(\mathcal{C}(H)^*) = \{|u\rangle\langle u|$ (i.e., the Dirac notation) $\mid \|u\|_H = 1\}$, and $\mathfrak{S}^p(C_0(\Omega)^*) = \{\delta_{\omega_0} \mid \delta_{\omega_0}$ is a point measure at $\omega_0 \in \Omega\}$, where $\int_\Omega f(\omega)\delta_{\omega_0}(d\omega) = f(\omega_0)$ $(\forall f \in C_0(\Omega))$. The latter implies that $\mathfrak{S}^p(C_0(\Omega)^*)$ can be also identified with Ω (called a *spectrum space* or simply *spectrum*) such as

$$\underset{\text{(state space)}}{\mathfrak{S}^p(C_0(\Omega)^*)} \ni \delta_\omega \leftrightarrow \omega \in \underset{\text{(spectrum)}}{\Omega} \tag{1}$$

For instance, in the above (ii) we must clarify the meaning of the "value" of $F(\omega_0)$ for $F \in L^\infty(\Omega, \nu)$ and $\omega_0 \in \Omega$. An element $F(\in \mathcal{N})$ is said to be *essentially continuous at* $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$, if there uniquely exists a complex number α such that

(B) if ρ $(\in \mathcal{N}_*, \|\rho\|_{\mathcal{N}_*} = 1)$ converges to $\rho_0(\in \mathfrak{S}^p(\mathcal{A}^*))$ in the sense of weak* topology of $\mathcal{A}^*$, that is,

$$\rho(G) \longrightarrow \rho_0(G) \;\; (\forall G \in \mathcal{A}(\subseteq \mathcal{N})), \tag{2}$$

then $\rho(F)$ converges to α.

And the value of $\rho_0(F)$ is defined by the α.

According to the noted idea (cf. [1]), an *observable* $\mathsf{O} := (X, \mathcal{F}, F)$ in $\mathcal{N}$ is defined as follows:

(i) [σ-field] X is a set, $\mathcal{F}(\subseteq 2^X$, the power set of $X)$ is a σ-field of X, that is, "$\Xi_1, \Xi_2, ... \in \mathcal{F} \Rightarrow \cup_{n=1}^\infty \Xi_n \in \mathcal{F}$", "$\Xi \in \mathcal{F} \Rightarrow X \setminus \Xi \in \mathcal{F}$".

(ii) [Countable additivity] F is a mapping from $\mathcal{F}$ to $\mathcal{N}$ satisfying: (a): for every $\Xi \in \mathcal{F}$, $F(\Xi)$ is a non-negative element in $\mathcal{N}$ such that $0 \le F(\Xi) \le I$, (b): $F(\emptyset) = 0$ and $F(X) = I$, where 0 and I is the 0-element and the identity in $\mathcal{N}$ respectively. (c): for any countable decomposition $\{\Xi_1, \Xi_2, \ldots, \Xi_n, ...\}$ of Ξ (i.e., $\Xi, \Xi_n \in \mathcal{F}$ $(n = 1, 2, 3, ...)$, $\cup_{n=1}^\infty \Xi_n = \Xi$, $\Xi_i \cap \Xi_j = \emptyset$ $(i \ne j)$), it holds that $F(\Xi) = \sum_{n=1}^\infty F(\Xi_n)$ in the sense of weak* topology in $\mathcal{N}$.

1.2 Axiom 1 [Measurement] and Axiom 2 [Causality]

With any *system* S, a basic structure $[\mathcal{A}, \mathcal{N}]_{B(H)}$ can be associated in which the measurement theory (A) of that system can be formulated. A *state* of the system S is represented by an element $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$ and an *observable* is represented by an observable $\mathsf{O} := (X, \mathcal{F}, F)$ in $\mathcal{N}$. Also, the *measurement of the observable* O *for the system* S *with the state* ρ is denoted by $\mathsf{M}_\mathcal{N}(\mathsf{O}, S_{[\rho]})$ (or more precisely, $\mathsf{M}_\mathcal{N}(\mathsf{O} := (X, \mathcal{F}, F),\, S_{[\rho]})$). An observer can obtain a measured value x $(\in X)$ by the measurement $\mathsf{M}_\mathcal{N}(\mathsf{O}, S_{[\rho]})$.

The Axiom 1 presented below is a kind of mathematical generalization of Born's probabilistic interpretation of quantum mechanics. And thus, it is a statement without reality.

Now we can present Axiom 1 in the W^*-algebraic formulation as follows.

Axiom 1 [Measurement]. *The probability that a measured value* x $(\in X)$ *obtained by the measurement* $\mathsf{M}_\mathcal{N}(\mathsf{O} := (X, \mathcal{F}, F),\, S_{[\rho]})$ *belongs to a set* $\Xi(\in \mathcal{F})$ *is given by* $\rho(F(\Xi))$ *if* $F(\Xi)$ *is essentially continuous at* $\rho(\in \mathfrak{S}^p(\mathcal{A}^*))$.

Next, we explain Axiom 2. Let $[\mathcal{A}_1, \mathcal{N}_1]_{B(H_1)}$ and $[\mathcal{A}_2, \mathcal{N}_2]_{B(H_2)}$ be basic structures. A continuous linear operator $\Phi_{1,2} : \mathcal{N}_2$ (with weak* topology) $\to \mathcal{N}_1$(with weak* topology) is called a *Markov operator,* if it satisfies that (i): $\Phi_{1,2}(F_2) \ge 0$ for any non-negative element F_2 in $\mathcal{N}_2$, (ii): $\Phi_{1,2}(I_2) = I_1$, where I_k is the identity in $\mathcal{N}_k$, $(k = 1, 2)$. In addition to the above (i) and (ii), in this paper we assume that $\Phi_{1,2}(\mathcal{A}_2) \subseteq \mathcal{A}_1$ and $\sup\{\|\Phi_{1,2}(F_2)\|_{\mathcal{A}_1} \mid F_2 \in \mathcal{A}_2$ such that $\|F_2\|_{\mathcal{A}_2} \le 1\} = 1$.

It is clear that the dual operator $\Phi_{1,2}^* : \mathcal{A}_1^* \to \mathcal{A}_2^*$ satisfies that $\Phi_{1,2}^*(\mathfrak{S}^m(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^m(\mathcal{A}_2^*)$. If it holds that $\Phi_{1,2}^*(\mathfrak{S}^p(\mathcal{A}_1^*)) \subseteq \mathfrak{S}^p(\mathcal{A}_2^*)$, the $\Phi_{1,2}$ is said to be deterministic. If it is not deterministic, it is called non-deterministic or decoherence.

Here note that, for any observable $\mathsf{O}_2 := (X, \mathcal{F}, F_2)$ in $\mathcal{N}_2$, the $(X, \mathcal{F}, \Phi_{1,2}F_2)$ is an observable in $\mathcal{N}_1$.

Now Axiom 2 is presented as follows:

Axiom 2 [Causality]. *Let* $t_1 \le t_2$. *The causality is represented by a Markov operator* $\Phi_{t_1,t_2} : \mathcal{N}_{t_2} \to \mathcal{N}_{t_1}$.

1.3 The linguistic interpretation

In the above, Axioms 1 and 2 are kinds of spells, (i.e., incantation, magic words, metaphysical statements), and thus, it is nonsense to verify them experimentally. Therefore, what we should do is not "to understand" but "to use". After learning Axioms 1 and 2 by rote, we have to improve how to use them through trial and error.

We can do well even if we do not know the linguistic interpretation. However, it is better to know the linguistic interpretation (= the manual to use Axioms 1 and 2), if we would like to make progress quantum language early.

The essence of the manual is as follows:

(C) *Only one measurement is permitted.* And thus, the state after a measurement is meaningless since it can not be measured any longer. Thus, the collapse of the wavefunction is prohibited. We are not concerned with anything after measurement. Strictly speaking, the phrase "**after the measurement**" should not be used. Also, the causality should be assumed only in the side of system, however, a state never moves. Thus, the Heisenberg picture should be adopted, and thus, the Schrödinger picture should be prohibited.

and so on. For details, see [6].

2 The wave function collapse

2.1 Problem: the von Neumann-Lüders projection postulate

Let $[\mathcal{C}(H), B(H)]_{B(H)}$ be a quantum basic structure. Let $\mathbb{P} = [P_k]_{k=1}^{\infty}$ be a spectral decomposition in $B(H)$, that is, $P_k (\in B(H))$ is a projection ($\forall k = 1, 2, ...$) such that

$$\sum_{k=1}^{\infty} P_k = I$$

Put $\mathbb{N} = \{1, 2, ...\}$. Define the observable $\mathsf{O}_P = (\mathbb{N}, 2^{\mathbb{N}}, P)$ in $B(H)$ such that

$$P(\{k\}) = P_k \qquad (\forall k = 1, 2, ...) \tag{3}$$

Axiom 1 says:

(D_1) The probability that a measured value n $(\in \mathbb{N})$ is obtained by the measurement $\mathsf{M}_{B(H)}(\mathsf{O}_P :=(\mathbb{N}, 2^{\mathbb{N}}, P),\ S_{[\rho]})$ is given by

$$\mathrm{Tr}(\rho P_n)(= \langle u, P_n u\rangle), \quad (\text{ where } \rho = |u\rangle\langle u|)$$

Also, the von Neumann-Lüders projection postulate (*cf.* [7]) says:

(D_2) When a measured value n $(\in \mathbb{N})$ is obtained by the measurement $\mathsf{M}_{B(H)}(\mathsf{O}_P :=(\mathbb{N}, 2^{\mathbb{N}}, P),\ S_{[\rho]})$, the state ρ_a **after the measurement** is given by

$$\rho_a = \frac{P_n |u\rangle\langle u| P_n}{\|P_n u\|^2} \tag{4}$$

And furthermore, when a measurement $\mathsf{M}_{B(H)}(\mathsf{O}_F :=(X, \mathcal{F}, F),\ S_{[\rho_a]})$ is taken, the probability that a measured value belongs to $\Xi (\in \mathcal{F})$ is given by

$$\mathrm{Tr}(\rho_a F(\Xi)) \Big(= \langle \frac{P_n u}{\|P_n u\|}, F(\Xi) \frac{P_n u}{\|P_n u\|} \rangle \Big) \tag{5}$$

Problem 1. In the cases that O_P and O_F do not commute, it is obvious that the (5) does not hold. Thus, the (D_2) should be modified. Hence, we have the following problem:

(E) How should the projection postulate (=(D_1)+(D_2)) be modified? Or, how should it be understood?

In the following section, I, from the point-view of the linguistic interpretation, answer this problem.

2.2 The derivation of the von Neumann-Lüders projection postulate in the linguistic interpretation

Consider two basic structure $[\mathcal{C}(H), B(H)]_{B(H)}$ and $[\mathcal{C}(K \otimes H), B(K \otimes H)]_{B(K \otimes H)}$. Let $\mathbb{P} = [P_k]_{k=1}^{\infty}$ be a spectral decomposition in $B(H)$, and let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal system in a Hilbert space K. Define the predual Markov operator $\Psi_* : Tr(H) \to Tr(K \otimes H)$ by, for any $u \in H$,

$$\Psi_*(|u\rangle\langle u|) = |\sum_{k=1}^{\infty} (e_k \otimes P_k u)\rangle\langle \sum_{k=1}^{\infty} (e_k \otimes P_k u)| \tag{6}$$

or

$$\Psi_*(|u\rangle\langle u|) = \sum_{k=1}^{\infty} |e_k \otimes P_k u\rangle\langle e_k \otimes P_k u| \tag{7}$$

Thus the Markov operator $\Psi : B(K \otimes H) \to B(H)$ is defined by $\Psi = (\Psi_*)^*$.
Define the observable $\mathsf{O}_G = (\mathbb{N}, 2^{\mathbb{N}}, G)$ in $B(K)$ such that

$$G(\{k\}) = |e_k\rangle\langle e_k| \qquad (k \in \mathbb{N} = \{1, 2, ...\})$$

Let $\mathsf{O}_P = \{\mathbb{N}, 2^{\mathbb{N}}, P)$ be as in (3). Let $\mathsf{O}_F = (X, \mathcal{F}, F)$ be arbitrary observable in $B(H)$. Thus, we have the tensor observable $\mathsf{O}_G \otimes \mathsf{O}_F = (\mathbb{N} \times X, 2^{\mathbb{N}} \boxtimes \mathcal{F}, G \otimes F)$ in $B(K \otimes H)$.

Fix a pure state $\rho = |u\rangle\langle u|$ $(u \in H, \|u\|_H = 1)$. Consider the measurement $\mathsf{M}_{B(H)}(\Psi(\mathsf{O}_G \otimes \mathsf{O}_F), S_{[\rho]})$. Then, we see that

(F) the probability that a measured value (k, x) obtained by the measurement $\mathsf{M}_{B(H)}(\Psi(\mathsf{O}_G \otimes \mathsf{O}_F), S_{[\rho]})$belongs to $\{n\} \times \Xi$ is given by

$$
\begin{aligned}
&\mathrm{Tr}[(|u\rangle\langle u|)\Psi(G(\{n\}) \otimes F(\Xi))] \\
=&\mathrm{Tr}[(\Psi_*(|u\rangle\langle u|))(G(\{n\}) \otimes F(\Xi))] \\
=&\mathrm{Tr}[(|\sum_{k=1}^{\infty}(e_k \otimes P_k u)\rangle\langle\sum_{k=1}^{\infty}(e_k \otimes P_k u)|)(|e_n\rangle\langle e_n| \otimes F(\Xi))] \\
=&\langle P_n u, F(\Xi) P_n u\rangle \quad (\forall \Xi \in \mathcal{F}) \qquad (8)
\end{aligned}
$$

(In a similar way, the same result is easily obtained in the case of (7)).
The (8) implies the following (G_1) and (G_2):

(G_1) if $\Xi = X$, then

$$\mathrm{Tr}[\Psi_*(|u\rangle\langle u|)(G(\{n\}) \otimes F(X))] = \langle u, P_n u\rangle$$

(G_2) when a measured value (k, x) belongs to $\{n\} \times X$, the conditional probability such that $x \in \Xi$ is given by

$$\langle \frac{P_n u}{\|P_n u\|}, F(\Xi)\frac{P_n u}{\|P_n u\|}\rangle \quad (\forall \Xi \in \mathcal{F}) \qquad (9)$$

This is a direct consequence of Axioms 1 and 2.

Considering the correspondence: (D) $\Leftrightarrow$ (G), that is,

$$\mathsf{M}_{B(H)}(\mathsf{O}_P, S_{[\rho]}) \Leftrightarrow \mathsf{M}_{B(H)}(\Psi(\mathsf{O}_G \otimes \mathsf{O}_F), S_{[\rho]}),$$

namely,

$$(D_1) \Leftrightarrow (G_1), \quad (D_2) \Leftrightarrow (G_2)$$

there is a reason to assume that the true meaning of the (5) is just the (9) (since O_F is arbitrary).

Remark 1. Note the taboo phrase "**after the measurement**" is not used in (G_2) but in (D_2).

3 Conclusions

In this paper, I assert:

(H) Although the von Neumann-Lüders projection postulate (D_2) concerning the measurement $\mathsf{M}_{B(H)}(\mathsf{O}_P := (\mathbb{N}, 2^{\mathbb{N}}, P), S_{[\rho]})$ does not hold (i.e., (D_2) is wrong), the similar result (G_2) concerning $\mathsf{M}_{B(H)}(\Psi(\mathsf{O}_G \otimes \mathsf{O}_F), S_{[\rho]})$ holds

That is, the projection postulate (G_2) (without the phrase: "state after measurement") can be derived from Axioms 1 and 2.

I hope that my assertion will be examined from various points of view.

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