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Abstract

We aim to obtain explicit representations of locally risk-minimizing by using Malliavin calculus for Lévy processes. For incomplete market models whose asset price is described by a solution to a stochastic differential equation driven by a Lévy process, we derive general formulas of locally risk-minimizing including Malliavin derivatives; and calculate its concrete expressions for call options, Asian options and lookback options.

Keywords: Incomplete markets, local risk-minimization, call options, Asian options, lookback options, Lévy processes, Malliavin calculus, Clark-Ocone formula.

1 Introduction

Locally risk-minimizing (LRM, for short) is a very well-known hedging method for contingent claims in a quadratic way. Theoretical aspects of LRM has been developed to a high degree. On the other hand, the necessity of researches on its explicit representations has been increasing. From this insight, we aim to obtain explicit representations of LRM for incomplete market models whose asset price process is described by a solution to a stochastic differential equation (SDE, for short) driven by a Lévy process, as a typical framework of incomplete market models. In particular, we use Malliavin calculus for Lévy processes to achieve our purpose.

LRM has more than two decades history. There is so much literature on this topic. Among other things, Schweizer [12] and [13] are useful to understand an outline. LRM has an intimate relationship with Föllmer-Schweizer decomposition (FS decomposition, for short), which is a kind of orthogonal decomposition of a random variable into a stochastic integration and an orthogonal martingale. As the first step, we focus on deriving a representation of FS decomposition under some mild conditions by using the martingale representation

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theorem. In order to compute its explicit expressions, we use Malliavin calculus. Note that we adopt the approach, undertaken by Solé, Utzet and Vives [15], of Malliavin calculus for Lévy processes on canonical Lévy space. As a result, using the Clark-Ocone type formula under change of measure shown by Suzuki [16], [17], we will formulate general representations of LRM including Malliavin derivatives of the claim to be hedged.

In the second half of this paper, we derive formulas on representations of LRM for three typical options. Firstly, we shall study call options, whose payoff is not smooth as a function of the asset price at the maturity. Thus, the chain rule is not available to calculate Malliavin derivatives for call options. Instead, we use the mollifier approximation. Moreover, we illustrate a concrete expression of LRM for the models whose asset price process is a solution to an SDE with deterministic coefficients. Next, Asian options will be discussed. Thirdly, we shall deal with lookback options, whose payoff is depending on the running maximum of the asset price process. Actually, we need complicated calculations to get Malliavin derivatives of the running maximum. For lookback options, we shall focus only on the exponential Lévy case; and derive Malliavin derivatives by using an approximation method.

Summarizing the above, our main contribution is threefold as follows:

1. formulating representations of LRM with Malliavin derivatives for Lévy markets,

2. illustrating how to calculate Malliavin derivatives for non-smooth functions of a random variable, and the running maximum of processes by using approximation methods.

3. introducing concrete representations of LRM of call options, Asian options and lookback options for Lévy markets.

This paper is structured as follows: In section 2, we prepare some terminologies; and give model descriptions, mathematical preliminaries and standing assumptions. We also introduce in section 2 examples satisfying our standing assumptions. General representations of LRM are introduced in section 3. Call options, Asian options and lookback options are studied in Sections 4, 5 and 6, respectively. Section 7 is devoted to concluding remarks.

2 Preliminaries

2.1 Model description

We begin with preparation of the probabilistic framework and the underlying Lévy process X under which we discuss Malliavin calculus in the sequel. Let $T > 0$ be a finite time horizon, $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ a one-dimensional Wiener space on $[0, T]$; and W its coordinate mapping process, that is, a one-dimensional standard Brownian motion with $W_0 = 0$. Let $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical Lévy space (see [15] and Delong and Imkeller [7]) for a pure jump Lévy process J on

$[0, T]$ with Lévy measure ν, that is, $\Omega_J = \cup_{n=0}^{\infty}([0, T] \times \mathbb{R}_0)^n$, where $\mathbb{R}_0 := \mathbb{R} \setminus \{0\}$; and $J_t(\omega_J) = \sum_{i=1}^{n} z_i \mathbf{1}_{\{t_i \le t\}}$ for $t \in [0, T]$ and $\omega_J = ((t_1, z_1), \ldots, (t_n, z_n)) \in ([0, T] \times \mathbb{R}_0)^n$. Note that $([0, T] \times \mathbb{R}_0)^0$ represents an empty sequence. Now, we assume that $\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty$; and denote $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_W \times \Omega_J, \mathcal{F}_W \times \mathcal{F}_J, \mathbb{P}_W \times \mathbb{P}_J)$. Let $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]}$ be the canonical filtration completed for $\mathbb{P}$. Let X be a square integrable centered Lévy process on $(\Omega, \mathcal{F}, \mathbb{P})$ represented as

$$X_t = \sigma W_t + J_t - t \int_{\mathbb{R}_0} z \nu(dz), \tag{2.1}$$

where $\sigma > 0$. Denoting by N the Poisson random measure defined as $N(t, A) := \sum_{s \le t} \mathbf{1}_A(\Delta X_s)$, $A \in \mathcal{B}(\mathbb{R}_0)$ and $t \in [0, T]$, where $\Delta X_s := X_s - X_{s-}$, we have $J_t = \int_0^t \int_{\mathbb{R}_0} z N(ds, dz)$. In addition, we define its compensated measure as $\widetilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt$. Thus, we can rewrite (2.1) as

$$X_t = \sigma W_t + \int_0^t \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz). \tag{2.2}$$

We consider, throughout this paper, a financial market being composed of one risk-free asset and one risky asset with finite time horizon T. For simplicity, we assume that the interest rate of the market is given by 0, that is, the price of the risk-free asset is 1 at all times. The fluctuation of the risky asset is assumed to be given by a solution to the following stochastic differential equation (SDE, for short):

$$dS_t = S_{t-} \left[\alpha_t dt + \beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z} \widetilde{N}(dt, dz) \right], \quad S_0 > 0, \tag{2.3}$$

where α, β and γ are predictable processes. Recall that γ is a stochastic process measurable with respect to the σ-algebra generated by $A \times (s, u] \times B$, $A \in \mathcal{F}_s$, $0 \le s < u \le T$, $B \in \mathcal{B}(\mathbb{R}_0)$. Now, we assume the following:

Assumption 2.1

1. *(2.3) has a solution S satisfying the so-called structure condition (SC, for short). That is, S is a special semimartingale with the canonical decomposition $S = S_0 + M + A$ such that*

$$\left\| [M]_T^{1/2} + \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})} < \infty, \tag{2.4}$$

where $dM_t = S_{t-}(\beta_t dW_t + \int_{\mathbb{R}_0} \gamma_{t,z} \widetilde{N}(dt, dz))$ and $dA_t = S_{t-}\alpha_t dt$. Moreover, defining a process $\lambda_t := \dfrac{\alpha_t}{S_{t-}(\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz))}$, we have $A = \int \lambda d\langle M \rangle$. Thirdly, the mean-variance trade-off process $K_t := \int_0^t \lambda_s^2 d\langle M \rangle_s$ is finite, that is, K_T is finite $\mathbb{P}$-a.s.

2. *$\gamma_{t,z} > -1$, (t, z, ω)-a.e., that is, $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \mathbf{1}_{\{\gamma_{t,z} \le -1\}} \nu(dz) dt\right] = 0$.*

Remark 2.2 1. *The SC is closely related to the no-arbitrage condition. For more details on the SC, see [12] and [13].*

2. *The process K as well as A is continuous.*

3. *(2.4) implies that* $\sup_{t\in[0,T]}|S_t| \in L^2(\mathbb{P})$ *by Theorem V.2 of Protter [9].*

4. *Condition 2 ensures that* $S_t > 0$ *for any* $t \in [0,T]$.

2.2 Locally risk-minimizing

We define locally risk-minimizing (LRM, for short) for a contingent claim $F \in L^2(\mathbb{P})$. The following definition is based on Theorem 1.6 of [13].

Definition 2.3 1. Θ_S *denotes the space of all* $\mathbb{R}$*-valued predictable processes* ξ *satisfying* $\mathbb{E}\left[\int_0^T \xi_t^2 d\langle M\rangle_t + (\int_0^T |\xi_t dA_t|)^2\right] < \infty$.

2. *An* L^2*-strategy is given by a pair* $\varphi = (\xi, \eta)$, *where* $\xi \in \Theta_S$ *and* η *is an adapted process such that* $V(\varphi) := \xi S + \eta$ *is a right continuous process with* $\mathbb{E}[V_t^2(\varphi)] < \infty$ *for every* $t \in [0,T]$. *Note that* ξ_t *(resp.* η_t*) represents the amount of units of the risky asset (resp. the risk-free asset) an investor holds at time* t.

3. *For* $F \in L^2(\mathbb{P})$, *the process* $C^F(\varphi)$ *defined by* $C_t^F(\varphi) := F1_{\{t=T\}} + V_t(\varphi) - \int_0^t \xi_s dS_s$ *is called the cost process of* $\varphi = (\xi, \eta)$ *for* F.

4. *An* L^2*-strategy* φ *is said locally risk-minimizing for* F *if* $V_T(\varphi) = 0$ *and* $C^F(\varphi)$ *is a martingale orthogonal to* M, *that is,* $[C^F(\varphi), M]$ *is a uniformly integrable martingale.*

The above definition of LRM is a simplified version, since the original one, introduced in [12] and [13], is rather complicated

Now, we focus on a representation of LRM. To this end, we define Föllmer-Schweizer decomposition (FS decomposition, for short).

Definition 2.4 *An* $F \in L^2(\mathbb{P})$ *admits a Föllmer-Schweizer decomposition if it can be described by*

$$F = F_0 + \int_0^T \xi_t^F dS_t + L_T^F, \tag{2.5}$$

where $F_0 \in \mathbb{R}$, $\xi^F \in \Theta_S$ *and* L^F *is a square-integrable martingale orthogonal to* M *with* $L_0^F = 0$.

Proposition 5.2 of [13] shows the following:

Proposition 2.5 (Proposition 5.2 of [13]) *Under Assumption 2.1, an LRM $\varphi = (\xi, \eta)$ for F exists if and only if F admits an FS decomposition; and its relationship is given by*

$$\xi_t = \xi_t^F, \quad \eta_t = F_0 + \int_0^t \xi_s^F dS_s + L_t^F - F1_{\{t=T\}} - \xi_t^F S_t.$$

As a result, it suffices to obtain a representation of ξ^F in (2.5) in order to obtain LRM. Henceforth, we identify ξ^F with LRM. To this end, we consider the process $Z := \mathcal{E}(-\int \lambda dM)$, where $\mathcal{E}(Y)$ represents the stochastic exponential of Y, that is, Z is a solution to the SDE $dZ_t = -\lambda_t Z_{t-} dM_t$. In addition to Assumption 2.1, we suppose the following:

Assumption 2.6 *Z is a positive square integrable martingale; and $Z_T F \in L^2(\mathbb{P})$.*

A martingale measure $\mathbb{P}^* \sim \mathbb{P}$ is called minimal if any square-integrable $\mathbb{P}$-martingale orthogonal to M remains a martingale under $\mathbb{P}^*$. We can see the following:

Lemma 2.7 *Under Assumption 2.1, if Z is a positive square integrable martingale, then a minimal martingale measure $\mathbb{P}^*$ exists with $d\mathbb{P}^* = Z_T d\mathbb{P}$.*

Proof. Since $d(ZS) = S_- dZ + Z_- dM + Z_- \lambda d\langle M \rangle - Z_- \lambda d[M]$, the product process ZS is a $\mathbb{P}$-local martingale. So that, defining a probability measure $\mathbb{P}^*$ as $d\mathbb{P}^* = Z_T d\mathbb{P}$, we have that S is a $\mathbb{P}^*$-martingale, since $\sup_{t \in [0,T]} |S_t|$ and Z_T are in $L^2(\mathbb{P})$. Next, for any L a square-integrable $\mathbb{P}$-martingale with null at 0 orthogonal to M, LZ is a $\mathbb{P}$-local martingale. By the square integrability of L, L remains a martingale under $\mathbb{P}^*$. Thus, $\mathbb{P}^*$ is a minimal martingale measure. $\square$

Example 2.8 *We introduce a model framework under which Assumption 2.1 is satisfied, and Z is a positive square integrable martingale. We consider the following three conditions:*

1. *$\gamma_{t,z} > -1$, (t, z, ω)-a.e.*
2. *$\sup_{t \in [0,T]} (|\alpha_t| + \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)) < C$ for some $C > 0$.*
3. *There exists an $\varepsilon > 0$ such that*

$$\frac{\alpha_t \gamma_{t,z}}{\beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz)} < 1 - \varepsilon \quad \text{and} \quad \beta_t^2 + \int_{\mathbb{R}_0} \gamma_{t,z}^2 \nu(dz) > \varepsilon, (t, z, \omega)\text{-a.e.}$$

The above condition 2 ensures the existence of a unique solution S to (2.3) satisfying $\sup_{t \in [0,T]} |S_t| \in L^2(\mathbb{P})$ by Theorem 117 of Situ [14]. The first condition of Assumption 2.1 is seen as follows: Firstly, we have $\left\| \int_0^T |dA_s| \right\|_{L^2(\mathbb{P})}^2 \leq C^2 T^2 \mathbb{E}[\sup_{t \in [0,T]} |S_t|^2] <$

∞. *Next, by the Burkholder-Davis-Gundy inequality, there exists a* $C > 0$ *such that*

$$\begin{aligned}\mathbb{E}[[M]_T] &\leq C\mathbb{E}\left[\sup_{t\in[0,T]}|M_t|^2\right] \\ &\leq C\left\{\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right]+|S_0|^2+\mathbb{E}\left[\sup_{t\in[0,T]}|A_t|^2\right]\right\}<\infty\end{aligned}$$

Thus, all conditions of Assumption 2.1 are satisfied.

On the other hand, the above condition 3 guarantees the positivity of Z. *Noting that* Z *is a solution to* $dZ_t = -\lambda_t Z_{t-} dM_t$, *we have* $\sup_{t\in[0,T]}|Z_t| \in L^2(\mathbb{P})$ *by using Theorem 117 of [14] again. In addition, since* $\mathbb{E}[\int_0^T \lambda_t^2 d[M]_t] < \infty$ *by conditions 2 and 3, the process* $-\int_0^\cdot \lambda_s dM_s$ *is a square integrable martingale by Lemma on p.171 of [9]. Thus, the process* $-\int_0^\cdot \lambda_s Z_{s-} dM_s$ *is a local martingale, that is, so is* Z. *Theorem I.51 of [9] implies that* Z *is a square integrable martingale. Hence, a minimal martingale measure exists by Lemma 2.7.*

2.3 Barndorff-Nielsen and Shephard model

We introduce what we call Barndorff-Nielsen and Shephard model as one more example which satisfies Assumption 2.1 and the square integrable martingale property of Z. This is an Ornstein-Uhlenbeck type stochastic volatility model, undertaken by Barndorff-Nielsen and Shephard [1], [2]. Let H be a subordinator without drift, that is, a non-decreasing, pure jump and no diffusion component Lévy process with $H_0 = 0$. Note that its Lévy measure ν satisfies $\nu((-\infty,0)) = 0$ and $\int_0^\infty (z\wedge 1)\nu(dz) < \infty$ by Proposition 3.10 of Cont and Tankov [6]. In addition, we assume that $\int_0^\infty z^2\nu(dz) < \infty$, that is, the square integrability of H. Suppose that the underlying Lévy process X is given as $X = W + \widetilde{H}$, where $\widetilde{H}$ is the compensated process of H. Now, we define a process Σ^2 as a solution to the following SDE:

$$\Sigma_t^2 = \Sigma_0^2 - R\int_0^t \Sigma_s^2 ds + H_t,$$

where $\Sigma_0^2 > 0$ and $R > 0$. By simple calculations, we have $\Sigma_t^2 = e^{-Rt}\Sigma_0^2 + \int_0^t e^{-R(t-s)}dH_s$. In addition, we define

$$L_t := \mu t - \frac{1}{2}\int_0^t \Sigma_s^2 ds + \int_0^t \Sigma_s dW_s + \rho H_t,$$

where $\mu \in \mathbb{R}$ and $\rho \leq 0$. Note that we restrict the coefficient of the second term to $-\frac{1}{2}$ for the sake of simplicity. Now, the asset price process S is assumed to be given by $S_t = S_0\exp(L_t)$ with $S_0 > 0$, that is, a solution to the following SDE:

$$dS_t = S_{t-}\left\{\alpha dt + \Sigma_t dW_t + \int_{\mathbb{R}_0}(e^{\rho z}-1)\widetilde{N}(dt,dz)\right\}, \qquad (2.6)$$

where $\alpha := \mu + \int_{\mathbb{R}_0}(e^{\rho z}-1)\nu(dz)$. Note that the SDE (2.6) does not satisfy condition 2 of Example 2.8. The goal of this subsection is to confirm that the above model satisfies Assumption 2.1 and that Z is a positive square integrable martingale under the following additional assumptions:

Assumption 2.9 *1.* $\int_1^\infty \exp\left\{2\frac{1-e^{-RT}}{R}z\right\}\nu(dz) < \infty$.

2. $\alpha > 0$ *or* $e^{-RT}\sigma_0^2 + \int_{\mathbb{R}_0}(e^{\rho z}-1)^2\nu(dz) > |\alpha|$.

Remark 2.10 *There are two typical examples of the Barndorff-Nielsen and Shephard models. One is the case where Σ_t^2 follows an inverse Gaussian distribution, that is, the process Σ^2 is given as an IG-OU process. The corresponding Lévy measure is given as*

$$\nu(dz) = \frac{a}{2\sqrt{2\pi}}z^{-\frac{3}{2}}(1+b^2z)\exp\left\{-\frac{1}{2}b^2z\right\}\mathbf{1}_{\{z>0\}}dz,$$

where a and b are positive constants. Whenever $\frac{1}{2}b^2 > 2\frac{1-e^{-RT}}{R}$, Condition 1 of Assumption 2.9 is satisfied as well as $\int_0^\infty z^2\nu(dz) < \infty$.

The other is the Gamma-OU case. In this case, Σ_t^2 follows a Gamma distribution; and $\nu(dz)$ is given as $\nu(dz) = abe^{-bz}\mathbf{1}_{\{z>0\}}dz$ for $a > 0$ and $b > 0$. If $b > 2\frac{1-e^{-RT}}{R}$, then condition 1 of Assumption 2.9 is satisfied. For more details, see Schoutens [11].

As for Assumption 2.1, it suffices to see $\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] < \infty$ by the same manner as Example 2.8. On the other hand, the second condition of Assumption 2.9 ensures the positivity of Z. Since $\mathbb{E}[\int_0^T \lambda_t^2 d[M]_t] < \infty$, the square integrable martingale property of Z is shown by the same way as Example 2.8.

Lemma 2.11 $\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] < \infty$.

Proof. *Step 1.* Denoting, for $t \in [0,T]$

$$\begin{aligned}\widehat{M}_t &:= \int_0^t \Sigma_s dW_s - \frac{1}{2}\int_0^t \Sigma_s^2 ds + \rho H_t + t\int_{\mathbb{R}_0}[-e^{\rho z}+1]\nu(dz) \\ &= \int_0^t \Sigma_s dW_s - \frac{1}{2}\int_0^t \Sigma_s^2 ds + \rho\int_0^t\int_{\mathbb{R}_0} z\widetilde{N}(ds,dz) + t\int_{\mathbb{R}_0}[\rho z - e^{\rho z}+1]\nu(dz),\end{aligned}$$

we see that $e^{\widehat{M}}$ is a martingale. From the view of Theorem 1.4 of Ishikawa [8], we have only to make sure the following three conditions:
(1) $\int_0^\infty(1-e^{\rho z})^2\nu(dz) < \infty$,
(2) $\int_0^\infty(\rho z e^{\rho z} + 1 - e^{\rho z})\nu(dz) < \infty$,
(3) $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Sigma_s^2 ds\right)\right] < \infty$.
Since $1-e^{\rho z} \leq |\rho|z$ for any $z > 0$, we have $\int_0^\infty(1-e^{\rho z})^2\nu(dz) \leq \int_0^1 \rho^2z^2\nu(dz) +$

$\int_1^\infty \nu(dz) < \infty$; and $\int_0^\infty (\rho z e^{\rho z} + 1 - e^{\rho z})\nu(dz) \le \int_0^\infty (1 - e^{\rho z})\nu(dz) \le \int_0^\infty |\rho| z \nu(dz) < \infty$. As for (3), setting $\mathcal{B}(t) := \frac{1}{R}(1 - e^{-Rt})$ for $t \in [0, T]$, we have

$$\begin{aligned} \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \Sigma_s^2 ds\right)\right] &= \mathbb{E}\left[\exp\left(\frac{1}{2}\Sigma_0^2 \mathcal{B}(T) + \frac{1}{2}\int_0^T \mathcal{B}(T-s) dH_s\right)\right] \\ &\le \exp\left(\frac{1}{2}\Sigma_0^2 \mathcal{B}(T)\right) \mathbb{E}\left[\exp\left(\frac{\mathcal{B}(T) H_T}{2}\right)\right]. \end{aligned}$$

By Proposition 3.14 of [6], Assumption 2.9 ensures $\mathbb{E}\left[\exp\left(\frac{\mathcal{B}(T)H_T}{2}\right)\right] < \infty$.

Step 2. Next, we see $\mathbb{E}[e^{2\widehat{M}_T}] < \infty$. We have

$$\begin{aligned} 2\widehat{M}_T &= 2\int_0^T \Sigma_s dW_s - \int_0^T \Sigma_s^2 ds + 2\rho \int_0^T \int_{\mathbb{R}_0} z \widetilde{N}(ds, dz) + 2T \int_{\mathbb{R}_0} [\rho z - e^{\rho z} + 1]\nu(dz) \\ &= Y_T + \mathcal{B}(T)\Sigma_0^2 + \int_0^T \int_{\mathbb{R}_0} [e^{g(s)z} - 2e^{\rho z} + 1]\nu(dz) ds, \end{aligned}$$

where $g(s) := \mathcal{B}(T-s) + 2\rho$ and

$$Y_t := 2\int_0^t \Sigma_s dW_s - 2\int_0^t \Sigma_s^2 ds + \int_0^t \int_{\mathbb{R}_0} g(s) z \widetilde{N}(ds, dz) + \int_0^t \int_{\mathbb{R}_0} [g(s)z - e^{g(s)z} + 1]\nu(dz) ds.$$

Because $2\rho \le g(s) \le \mathcal{B}(T) + 2\rho$ for any $s \in [0, T]$,

$$|1 - e^{g(s)z}| \le \begin{cases} z(e^{g(s)} - 1), & \text{if } g(s) \ge 0, z \in (0,1), \\ e^{g(s)z}, & \text{if } g(s) \ge 0, z \ge 1, \\ -g(s)z, & \text{if } g(s) < 0, z > 0, \end{cases}$$

and Assumption 2.9, we have $\int_0^T \int_{\mathbb{R}_0} |e^{g(s)z} - 1|\nu(dz) ds < \infty$. Moreover, we have $\int_0^\infty (1 - e^{\rho z})\nu(dz) < \infty$. We have then $\mathbb{E}[e^{2\widehat{M}_T}] < \infty$ if $\mathbb{E}[e^{Y_T}] = 1$.

Step 3. We show $\mathbb{E}[e^{Y_T}] = 1$. By Theorem 1.4 of [8], it suffices to see the following:

(4) $\int_0^T \int_0^\infty \left\{(1 - e^{g(s)z})^2 + g(s)^2 z^2 + |g(s) z e^{g(s)z} + 1 - e^{g(s)z}|\right\} \nu(dz) ds < \infty$,

(5) $\mathbb{E}\left[\exp\left(2\int_0^T \Sigma_s^2 ds\right)\right] < \infty$.

(4) is reduced by the same sort argument as Step 2 and

$$|g(s) z e^{g(s)z}| \le \begin{cases} g(s) z e^{g(s)}, & \text{if } g(s) \ge 0, z \in (0,1), \\ e^{2g(s)z}, & \text{if } g(s) \ge 0, z \ge 1, \\ -g(s)z, & \text{if } g(s) < 0, z > 0. \end{cases}$$

As for (5), Assumption 2.9 and the same argument as Step 1 yield $\mathbb{E}\left[\exp\left(2\int_0^T \Sigma_s^2 ds\right)\right] \le \exp(2\Sigma_0^2 \mathcal{B}(T))\mathbb{E}\left[\exp(2\mathcal{B}(T) H_T)\right] < \infty$.

Step 4. Since we have $2L_t = 2\mu t + 2\widehat{M}_t + 2t\int_{\mathbb{R}_0}(e^{\rho z}-1)\nu(dz) \leq 2(\mu\vee 0)T + 2\widehat{M}_t$, the Doob inequality yields

$$\mathbb{E}\left[\sup_{t\in[0,T]}|S_t|^2\right] = S_0^2\mathbb{E}\left[\sup_{t\in[0,T]} e^{2L_t}\right] \leq S_0^2 e^{2(\mu\vee 0)T}\mathbb{E}\left[\sup_{t\in[0,T]} e^{2\widehat{M}_t}\right]$$
$$\leq 4S_0^2 e^{2(\mu\vee 0)T}\mathbb{E}\left[e^{2\widehat{M}_T}\right] < \infty$$

by Steps 1-3. □

3 Representation results for LRM

In this section, we focus on representations of LRM ξ^F for claim F. First of all, we study it through the martingale representation theorem.

3.1 Approach based on the martingale representation theorem

Throughout this subsection, we assume Assumptions 2.1 and 2.6. Let $\mathbb{P}^*$ be a minimal martingale measure, that is, $d\mathbb{P}^* = Z_T d\mathbb{P}$ holds. The martingale representation theorem (see, e.g. Proposition 9.4 of [6]) provides

$$Z_T F = \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T g_t^0 dW_t + \int_0^T\int_{\mathbb{R}_0} g_{t,z}^1 \widetilde{N}(dt,dz)$$

for some predictable processes g_t^0 and $g_{t,z}^1$. By the same sort of calculations as the proof of Theorem 4.4 of [16], we have

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T \frac{g_t^0 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]u_t}{Z_{t-}} dW_t^{\mathbb{P}^*} \\ &\quad + \int_0^T\int_{\mathbb{R}_0} \frac{g_{t,z}^1 + \mathbb{E}[Z_T F|\mathcal{F}_{t-}]\theta_{t,z}}{Z_{t-}(1-\theta_{t,z})} \widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &=: \mathbb{E}_{\mathbb{P}^*}[F] + \int_0^T h_t^0 dW_t^{\mathbb{P}^*} + \int_0^T\int_{\mathbb{R}_0} h_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt,dz) \end{aligned}$$

where $u_t := \lambda_t S_{t-}\beta_t$, $\theta_{t,z} := \lambda_t S_{t-}\gamma_{t,z}$, $dW_t^{\mathbb{P}^*} := dW_t + u_t dt$ and $\widetilde{N}^{\mathbb{P}^*}(dt,dz) := \widetilde{N}(dt,dz) + \theta_{t,z}\nu(dz)dt$. Girsanov's theorem implies that $W^{\mathbb{P}^*}$ and $\widetilde{N}^{\mathbb{P}^*}$ are a Brownian motion and the compensated Poisson random measure of N under $\mathbb{P}^*$, respectively. Additionally, we assume that

$$\mathbb{E}\left[\int_0^T\left\{(h_t^0)^2 + \int_{\mathbb{R}_0}(h_{t,z}^1)^2\nu(dz)\right\}dt\right] < \infty. \tag{3.1}$$

Denoting $i_t^0 := h_t^0 - \xi_t S_{t-}\beta_t$, $i_{t,z}^1 := h_{t,z}^1 - \xi_t S_{t-}\gamma_{t,z}$, and

$$\xi_t := \frac{\lambda_t}{\alpha_t}\{h_t^0\beta_t + \int_{\mathbb{R}_0} h_{t,z}^1\gamma_{t,z}\nu(dz)\}, \tag{3.2}$$

we can see $i_t^0\beta_t + \int_{\mathbb{R}_0} i_{t,z}^1 \gamma_{t,z}\nu(dz) = 0$ for any $t \in [0,T]$, which implies $i_t^0 u_t + \int_{\mathbb{R}_0} i_{t,z}^1 \theta_{t,z}\nu(dz) = 0$. We have then

$$\begin{aligned} F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_t dS_t &= \int_0^T i_t^0 dW_t^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} i_{t,z}^1 \widetilde{N}^{\mathbb{P}^*}(dt,dz) \\ &= \int_0^T i_t^0 dW_t + \int_0^T \int_{\mathbb{R}_0} i_{t,z}^1 \widetilde{N}(dt,dz). \end{aligned}$$

The following lemma implies that $L_t^F := \mathbb{E}[F - \mathbb{E}_{\mathbb{P}^*}[F] - \int_0^T \xi_s dS_s | \mathcal{F}_t]$ is a square integrable martingale orthogonal to M with $L_0^F = 0$.

Lemma 3.1 *Under Assumptions 2.1 and 2.6, and (3.1), we have*

$$\mathbb{E}\left[\int_0^T (i_t^0)^2 dt + \int_0^T \int_{\mathbb{R}_0} (i_{t,z}^1)^2 \nu(dz)dt\right] < \infty.$$

Proof. Noting that $\frac{\beta_t^2}{\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,x}^2\nu(dx)}$ and $\frac{\int_{\mathbb{R}_0}\gamma_{t,x}^2\nu(dx)}{\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,x}^2\nu(dx)}$ are less than 1, we have

$$\begin{aligned} \mathbb{E}\left[\int_0^T \xi_t^2 S_{t-}^2 \beta_t^2 dt\right] &\leq 2\mathbb{E}\left[\int_0^T \frac{\beta_t^4 (h_t^0)^2 + \beta_t^2\left(\int_{\mathbb{R}_0} h_{t,x}^1 \gamma_{t,x}\nu(dx)\right)^2}{\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,x}^2\nu(dx)\right)^2} dt\right] \\ &\leq 2\mathbb{E}\left[\int_0^T \frac{\beta_t^4 (h_t^0)^2 + \beta_t^2 \int_{\mathbb{R}_0}(h_{t,x}^1)^2\nu(dx)\int_{\mathbb{R}_0}\gamma_{t,x}^2\nu(dx)}{\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,x}^2\nu(dx)\right)^2} dt\right] \\ &\leq 2\mathbb{E}\left[\int_0^T \left\{(h_t^0)^2 + \int_{\mathbb{R}_0}(h_{t,z}^1)^2\nu(dz)\right\} dt\right]. \end{aligned}$$

By the same way as the above, we can see $\mathbb{E}\left[\int_0^T \int_{\mathbb{R}_0} \xi_t^2 S_{t-}^2 \gamma_{t,z}^2 \nu(dz)dt\right]$. Together with (3.1), Lemma 3.1 follows. □

Consequently, we can conclude the following:

Theorem 3.2 *Assume that Assumptions 2.1, 2.6, and (3.1). We have then $\xi^F = \xi$ defined in (3.2).*

In the above theorem, a representation of LRM ξ^F is obtained under a mild setting. Since the processes h^0 and h^1 appeared in (3.2) are induced by the martingale representation theorem, it is almost impossible to calculate them explicitly, and confirm if (3.1) holds. In the rest of this section, we aim to get concrete expressions for h^0 and h^1 by using Malliavin calculus.

3.2 Malliavin calculus

In this subsection, we prepare some definitions and terminologies with respect to Malliavin calculus. In particular, we introduce a Clark-Ocone type formula under change of measure (under $\mathbb{P}^*$). The main results of this section will be given in the following subsection.

We adapt the canonical Lévy space framework undertaken by [15]. Remark that Malliavin calculus is discussed based on the underlying Lévy process X. First of all, we define measures q and Q on $[0,T]\times\mathbb{R}$ as

$$q(E) := \sigma^2 \int_E \delta_0(dz)dt + \int_E z^2\nu(dz)dt,$$

and

$$Q(E) := \sigma \int_E \delta_0(dz)dW_t + \int_E z\widetilde{N}(dt,dz),$$

where $E \in \mathcal{B}([0,T]\times\mathbb{R})$ and δ_0 is the Dirac measure at 0. Denote by $L^2_{T,q,n}$ the set of product measurable, deterministic functions $h : ([0,T]\times\mathbb{R})^n \to \mathbb{R}$ satisfying

$$\|h\|^2_{L^2_{T,q,n}} := \int_{([0,T]\times\mathbb{R})^n} |h((t_1,z_1),\cdots,(t_n,z_n))|^2 q(dt_1,dz_1)\cdots q(t_n,z_n) < \infty.$$

For $n\in\mathbb{N}$ and $h \in L^2_{T,q,n}$, we define

$$I_n(h) := \int_{([0,T]\times\mathbb{R})^n} h((t_1,z_1),\cdots,(t_n,z_n))Q(dt_1,dz_1)\cdots Q(dt_n,dz_n).$$

Formally, we denote $L^2_{T,q,0} := \mathbb{R}$ and $I_0(h) := h$ for $h\in\mathbb{R}$. Under this setting, any $F \in L^2(\mathbb{P})$ has the unique representation $F = \sum_{n=0}^\infty I_n(h_n)$ with functions $h_n \in L^2_{T,q,n}$ that are symmetric in the n pairs $(t_i,z_i), 1\le i\le n$, and we have $\mathbb{E}[F^2] = \sum_{n=0}^\infty n!\|h_n\|^2_{L^2_{T,q,n}}$. We prepare some notations.

Definition 3.3 *1. Let $\mathbb{D}^{1,2}$ denote the set of random variables $F\in L^2(\mathbb{P})$ with $F = \sum_{n=0}^\infty I_n(h_n)$ satisfying $\sum_{n=1}^\infty nn!\|h_n\|^2_{L^2_{T,q,n}} < \infty$.*

2. For any $F\in\mathbb{D}^{1,2}$, $DF : [0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ is defined by

$$D_{t,z}F = \sum_{n=1}^\infty nI_{n-1}(h_n((t,z),\cdot)).$$

3. $\mathbb{L}^{1,2}_0$ denotes the space of $G : [0,T]\times\Omega\to\mathbb{R}$ satisfying

(a) $G_s \in \mathbb{D}^{1,2}$ for a.e. $s\in[0,T]$,

(b) $\mathbb{E}\left[\int_{[0,T]} |G_s|^2 ds\right] < \infty$,

(c) $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_0^T |D_{t,z}G_s|^2 ds q(dt,dz)\right] < \infty$.

4. $\mathbb{L}_1^{1,2}$ *is defined as the space of* $G : [0,T]\times\mathbb{R}_0\times\Omega\to\mathbb{R}$ *such that*

   (a) $G_{s,x}\in\mathbb{D}^{1,2}$ *for q-a.e.* $(s,x)\in[0,T]\times\mathbb{R}_0$,

   (b) $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}_0}|G_{s,x}|^2\nu(dx)ds\right] < \infty$,

   (c) $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}G_{s,x}|^2\nu(dx)ds q(dt,dz)\right] < \infty$.

5. $\widetilde{\mathbb{L}}_1^{1,2}$ *is defined as the space of* $G\in\mathbb{L}_1^{1,2}$ *such that*

   (a) $\mathbb{E}\left[\left(\int_{[0,T]\times\mathbb{R}_0}|G_{s,x}|\nu(dx)ds\right)^2\right] < \infty$,

   (b) $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\left(\int_{[0,T]\times\mathbb{R}_0}|D_{t,z}G_{s,x}|\nu(dx)ds\right)^2 q(dt,dz)\right] < \infty$.

Theorem 3.5 below is a Clark-Ocone type formula under $\mathbb{P}^*$, which is concerned about an integral representation of $F\in L^2(\mathbb{P})$. The assumptions needed to see it are given in Assumption 3.4.

Assumption 3.4 *1.* $u, u^2\in\mathbb{L}_0^{1,2}$*; and* $2u_sD_{t,z}u_s + z(D_{t,z}u_s)^2\in L^2(q\times\mathbb{P})$ *for a.e.* $s\in[0,T]$.

2. $\theta+\log(1-\theta)\in\widetilde{\mathbb{L}}_1^{1,2}$*, and* $\log(1-\theta)\in\mathbb{L}_1^{1,2}$.

3. *For q-a.e.* $(t,z)\in[0,T]\times\mathbb{R}_0$*, there is an* $\varepsilon_{t,z}\in(0,1)$ *such that* $\theta_{t,z}<1-\varepsilon_{t,z}$.

4. $Z_T\left\{D_{t,0}\log Z_T\mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z}\log Z_T}-1}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\}\in L^2(q\times\mathbb{P})$.

5. $F\in\mathbb{D}^{1,2}$*; and* $Z_TD_{t,z}F + FD_{t,z}Z_T + zD_{t,z}F\cdot D_{t,z}Z_T\in L^2(q\times\mathbb{P})$.

6. $FH^*_{t,z}$, $H^*_{t,z}D_{t,z}F\in L^1(\mathbb{P}^*)$ *for q-a.e.* $(t,z)\in[0,T]\times\mathbb{R}_0$*, where* $H^*_{t,z} := \exp\{zD_{t,z}\log Z_T - \log(1-\theta_{t,z})\}$.

Theorem 3.5 (Theorem 3.4 of [17]) *Under Assumptions 2.6 and 3.4, we have, for any* $F\in L^2(\mathbb{P})$,

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{P}^*}[F] + \sigma\int_0^T\mathbb{E}_{\mathbb{P}^*}\Bigg[D_{t,0}F - F\Big[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} \\ &\quad + \int_0^T\int_{\mathbb{R}_0}\frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}}\widetilde{N}^{\mathbb{P}^*}(ds,dx)\Big]\Big|\mathcal{F}_{t-}\Bigg]dW_t^{\mathbb{P}^*} \\ &\quad + \int_0^T\int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[F(H^*_{t,z}-1) + zH^*_{t,z}D_{t,z}F|\mathcal{F}_{t-}]\widetilde{N}^{\mathbb{P}^*}(dt,dz). \end{aligned}$$

Remark 3.6 *1. Assumption 2.1 has nothing to do with the above theorem.*

2. *The original version of the above theorem is shown in [16]. Theorem 3.5 is its canonical Lévy space version introduced in [17] as a special case.*

3.3 Main results

Under the above preparations, we calculate h^0 and h^1 by using Theorem 3.5. Together with Theorem 3.2, we obtain the following:

Theorem 3.7 *Under Assumptions 2.1, 2.6 and 3.4, h^0 and h^1 are described as*

$$h_t^0 = \sigma \mathbb{E}_{\mathbb{P}^*}\left[D_{t,0}F - F\left[\int_0^T D_{t,0}u_s dW_s^{\mathbb{P}^*} + \int_0^T \int_{\mathbb{R}_0} \frac{D_{t,0}\theta_{s,x}}{1-\theta_{s,x}} \widetilde{N}^{\mathbb{P}^*}(ds,dx) \right] \Big| \mathcal{F}_{t-} \right], \tag{3.3}$$

$$h_{t,z}^1 = \mathbb{E}_{\mathbb{P}^*}[F(H_{t,z}^* - 1) + zH_{t,z}^* D_{t,z}F | \mathcal{F}_{t-}]. \tag{3.4}$$

Moreover, LRM ξ^F is given by substituting (3.3) and (3.4) for h^0 and h^1 in (3.2) respectively, if (3.1) holds.

Remark 3.8

1. *LRM for Lévy markets has been also discussed in Vandaele and Vanmaele [18] without Malliavin calculus. They considered the case where all coefficients in (2.3) are deterministic; and studied LRM for unit-linked life insurance contracts.*
2. *Benth et al [3] also concerned a similar issue by using Malliavin calculus. They however studied minimal variance portfolio which is different from LRM, and considered only the case where the underlying asset price process is a martingale.*

In order to calculate LRM concretely through Theorem 3.7, we need to confirm if all the assumptions in Theorem 3.7 are satisfied for a given model. But, it seems to be a hard work. So that, we introduce a simple framework satisfying all the assumptions.

Example 3.9 *We consider the case where α, β and γ in (2.3) are deterministic functions satisfying the three conditions in Example 2.8. Additionally, we assume that*

$$Z_T F \in L^2(\mathbb{P}), \text{ and condition 5 in Assumption 3.4.} \tag{3.5}$$

Now, we confirm if this model satisfies all the conditions in Theorem 3.7. Remark that we discuss this framework in sections 4 and 5 again for the case where F is a call option or an Asian option.

As seen in Example 2.8, Assumption 2.1 is satisfied; and Z is a positive square integrable martingale. Thus, together with the above additional condition, Assumption 2.6 is satisfied. Since u is bounded and deterministic, condition 1 of Assumption 3.4 is satisfied. Since θ is deterministic, the third condition in Example 2.8 ensures that condition 3 holds with $\varepsilon \in (0,1)$ independent of $(t,z) \in [0,T] \times \mathbb{R}_0$. Note that $|x + \log(1-x)| \leq \frac{1}{2\varepsilon}|x|^2$, and $|\log(1-x)| \leq \frac{-\log\varepsilon}{1-\varepsilon}|x|$ hold for any $x < 1-\varepsilon$. Then, $\int_0^T \int_{\mathbb{R}_0} |\theta_{t,z}|^2 \nu(dz)dt < \infty$ implies condition 2. As for condition 4, noting that Lemmas 3.2 and 3.3 in [7]; and Proposition 3.5 in [16], we can see that $\log Z_T \in \mathbb{D}^{1,2}$, *and* $D_{t,z}\log Z_T = -\sigma^{-1}u_t\mathbf{1}_{\{0\}}(z) + z^{-1}\log(1-\theta_{t,z})\mathbf{1}_{\mathbb{R}_0}(z)$*. In addition, we have*

$$\int_0^T\int_{\mathbb{R}}\left\{D_{t,0}\log Z_T\mathbf{1}_{\{0\}}(z) + \frac{e^{zD_{t,z}\log Z_T}-1}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\}^2 q(dt,dz)$$
$$= \int_0^T u_t^2 dt + \int_0^T\int_{\mathbb{R}_0}\theta_{t,z}^2\nu(dz)dt < \infty,$$

from which condition 4 follows. Since $H^* = 1$ *identically,* $F \in \mathbb{D}^{1,2}$ *and* $Z_T \in L^2(\mathbb{P})$*, we have condition 6. It remains to make sure of (3.1). Note that* $h^0 = \sigma\mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}]$*, and* $h^1_{t,z} = \mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]$*. Since* $K_T \in L^\infty$*, we can see that* Z *satisfies the reverse Hölder inequality by Proposition 3.7 of Choulli, Krawczyk and Stricker [5]. We have then* $(\mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}])^2 \le C\mathbb{E}[(D_{t,0}F)^2|\mathcal{F}_{t-}]$ *for some* $C>0$*. By Fubini's theorem, (3.1) is satisfied.*

Consequently, all the conditions in Theorem 3.7 are satisfied; and ξ^F *is given by*

$$\xi_t^F = \frac{\sigma\beta_t\mathbb{E}_{\mathbb{P}^*}[D_{t,0}F|\mathcal{F}_{t-}] + \int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[zD_{t,z}F|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}. \tag{3.6}$$

4 Call options

In this section, we deal with call options as a common example of contingent claims. The payoff of the call option with strike price $K>0$ is expressed by $(S_T-K)^+$ where $x^+ = x\vee 0$. First of all, we calculate the Malliavin derivatives of $(F-K)^+$ for $F\in\mathbb{D}^{1,2}$ and $K\in\mathbb{R}$. After that, we shall give an explicit representation of LRM for the deterministic coefficients case discussed in Example 3.9.

Regarding $(F-K)^+$ as a functional of F, it is continuous, but not smooth. Thus, we cannot use the chain rule (Proposition 2.5 in [17]). Instead, the mollifier approximation is very useful.

Theorem 4.1 *For any* $F\in\mathbb{D}^{1,2}$*,* $K\in\mathbb{R}$ *and q-a.e.* $(t,z)\in[0,T]\times\mathbb{R}$*, we have* $(F-K)^+\in\mathbb{D}^{1,2}$ *and*

$$D_{t,z}(F-K)^+ = \mathbf{1}_{\{F>K\}}D_{t,0}F\cdot\mathbf{1}_{\{0\}}(z) + \frac{(F+zD_{t,z}F-K)^+-(F-K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z).$$

Proof. We take a mollifier function φ which is a C^∞-function from $\mathbb{R}$ to $[0,\infty)$ with $\mathrm{supp}(\varphi)\subset[-1,1]$ and $\int_{-\infty}^\infty\varphi(x)dx = 1$. We denote $\varphi_n(x) := n\varphi(nx)$ and $f_n(x) := \int_{-\infty}^\infty(y-K)^+\varphi_n(x-y)dy$ for any $n\ge 1$. Noting that

$$f_n(x) = \int_{-\infty}^\infty\left(x-\frac{y}{n}-K\right)^+\varphi(y)dy = \int_{-\infty}^{n(x-K)}\left(x-\frac{y}{n}-K\right)\varphi(y)dy,$$

we have $f_n'(x) = \int_{-\infty}^{n(x-K)}\varphi(y)dy$, so that $f_n\in C^1$ and $|f_n'|\le 1$, that is, f_n is Lipschitz continuous with constant 1. Thus, Proposition 2.5 in [17] implies that, for any $n \geq 1$, $f_n(F) \in \mathbb{D}^{1,2}$ and

$$D_{t,z}f_n(F) = f_n'(F)D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{f_n(F + zD_{t,z}F) - f_n(F)}{z}\mathbf{1}_{\mathbb{R}_0}(z). \tag{4.1}$$

In addition, noting that

$$\begin{aligned} |f_n(x) - (x-K)^+| &= \left| \int_{-1}^1 \left\{ \left(x - \frac{y}{n} - K \right)^+ - (x-K)^+ \right\} \varphi(y)dy \right| \\ &\leq \frac{1}{n}\int_{-1}^1 |y|\varphi(y)dy \leq \frac{1}{n} \end{aligned} \tag{4.2}$$

for any $x \in \mathbb{R}$, we have $\lim_{n\to\infty} \mathbb{E}[|f_n(F) - (F-K)^+|^2] = 0$. Thus, from the view of Proposition 2.4 of [16], all we have to do is to make sure that $D_{t,z}f_n(F)$ converges to

$$\mathbf{1}_{\{F>K\}}D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) + \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z) =: I_\infty$$

in $L^2(q \times \mathbb{P})$ as n tends to ∞.

First of all, we have

$$\lim_{n\to\infty} f_n'(x) = \begin{cases} \int_{-\infty}^0 \varphi(y)dy & \text{if } x = K, \\ 1 & \text{if } x > K, \\ 0 & \text{if } x < K, \end{cases}$$

from which we obtain $\lim_{n\to\infty} f_n'(F) = \mathbf{1}_{\{F>K\}} + \mathbf{1}_{\{F=K\}}\int_{-\infty}^0 \varphi(y)dy$. By (4.1), (4.2) and Lemma 4.2 below, we have $\lim_{n\to\infty} D_{t,z}f_n(F) = I_\infty$ in $q \times \mathbb{P}$-a.e., and

$$\begin{aligned} &|D_{t,z}f_n(F) - I_\infty| \\ &\leq |f_n'(F)D_{t,0}F - \mathbf{1}_{\{F>K\}}D_{t,0}F|\mathbf{1}_{\{0\}}(z) \\ &\quad + \left| \frac{f_n(F + zD_{t,z}F) - f_n(F)}{z} - \frac{(F + zD_{t,z}F - K)^+ - (F-K)^+}{z} \right| \mathbf{1}_{\mathbb{R}_0}(z) \\ &\leq 2|D_{t,z}F| \in L^2(q \times \mathbb{P}). \end{aligned}$$

Thus, the dominated convergence theorem provides that $D_{t,z}f_n(F) \to I_\infty$ in $L^2(q \times \mathbb{P})$. □

Lemma 4.2 *For any $F \in \mathbb{D}^{1,2}$, we have $\mathbf{1}_{\{F=0\}}D_{t,0}F = 0$, (t,ω)-a.e.*

Proof. *Step 1.* We take the same mollifier function φ as Theorem 4.1. Additionally, we assume that $\varphi(0) = 1$. We denote, for any $n \geq 1$, $\varphi_n(x) := \varphi(nx)$ and $\Phi_n(x) := \int_{-\infty}^x \varphi_n(y)dy$. Remark that $\Phi_n \in C^1$; and $\Phi_n'(x) = \varphi_n(x)$ is bounded. Proposition 2.5 of [17] implies

$$D_{t,0}\Phi_n(F) = \varphi_n(F)D_{t,0}F. \tag{4.3}$$

Since $\varphi_n(x) \to \mathbf{1}_{\{0\}}(x)\varphi(0) = \mathbf{1}_{\{0\}}(x)$ for any $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} D_{t,0}\Phi_n(F) = \mathbf{1}_{\{F=0\}} D_{t,0}F. \tag{4.4}$$

Step 2. Recall that any function $u \in L^2(q \times \mathbb{P})$ has a chaotic representation

$$u(t,z) = \sum_{n=0}^{\infty} I_n(h_n(\cdot,(t,z))),$$

where $h_n \in L^2_{T,q,n+1}$ is symmetric in the first n pairs of variables. Denoting by $\hat{h}_n$ the symmetrization of h_n with respect to all $n+1$ pairs of variables, we define

$$\mathrm{Dom}_\delta := \left\{ u \in L^2(q \times \mathbb{P}) \Big| \sum_{n=0}^{\infty} (n+1)! \|\hat{h}_n\|^2_{L^2_{T,q,n+1}} < \infty \right\}.$$

We shall show that Dom_δ is dense in $L^2(q \times \mathbb{P})$. Now, we prepare a subclass of Dom_δ as

$$\mathrm{Dom}_f := \left\{ u \in L^2(q \times \mathbb{P}) \Big| u(t,z) = \sum_{n=0}^{N} I_n(h_n(\cdot,(t,z))) \text{ for some } N \geq 1 \right\}.$$

Taking a $u \in L^2(q \times \mathbb{P})$ with $u(t,z) = \sum_{n=0}^{\infty} I_n(h_n(\cdot,(t,z)))$ arbitrarily; and denoting $u_N(t,z) := \sum_{n=0}^{N} I_n(h_n(\cdot,(t,z))) \in \mathrm{Dom}_f$ for any $N \geq 1$, we have $u_N \to u$ in $L^2(q \times \mathbb{P})$. Thus, Dom_f is dense in $L^2(q \times \mathbb{P})$. So is Dom_δ.

Step 3. By the dense property of Dom_δ, we have only to see

$$\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} \mathbf{1}_{\{F=0\}} D_{t,0}F \cdot \mathbf{1}_{\{0\}}(z) u(t,z) q(dt,dz) \right] = 0 \tag{4.5}$$

for any $u \in \mathrm{Dom}_\delta$. Fix $u \in \mathrm{Dom}_\delta$ arbitrarily. By (4.4), we have

$$\mathbb{E}\left[\int_0^T \mathbf{1}_{\{F=0\}} D_{t,0}F \cdot u(t,0) dt \right] = \mathbb{E}\left[\int_0^T \lim_{n\to\infty} D_{t,0}\Phi_n(F) \cdot u(t,0) dt \right]. \tag{4.6}$$

Since we can find a $C_\varphi > 0$ such that $\varphi \leq C_\varphi$, (4.3) implies $|D_{t,0}\Phi_n(F)| \leq |\varphi_n(F)||D_{t,0}F| \leq C_\varphi |D_{t,0}F|$. In addition, we have

$$\mathbb{E}\left[\int_0^T |D_{t,0}F \cdot u(t,0)| dt \right] \leq \sqrt{\mathbb{E}\left[\int_0^T |D_{t,0}F|^2 dt \right]} \sqrt{\mathbb{E}\left[\int_0^T |u(t,0)|^2 dt \right]} < \infty.$$

Thus, the dominated convergence theorem yields

$$\mathbb{E}\left[\int_0^T \lim_{n\to\infty} D_{t,0}\Phi_n(F) \cdot u(t,0) dt \right] = \lim_{n\to\infty} \mathbb{E}\left[\int_0^T D_{t,0}\Phi_n(F) \cdot u(t,0) dt \right]. \tag{4.7}$$

Next, by the duality formula introduced in Section 6 of [15], there exists a constant $C > 0$ such that

$$\left| \mathbb{E}\left[\int_{[0,T]\times\mathbb{R}} D_{t,z}\Phi_n(F) \cdot u(t,z) q(dt,dz) \right] \right| \leq C \|\Phi_n(F)\|_{L^2(\mathbb{P})} \leq C\frac{1}{n},$$

which means

$$\mathbb{E}\left[\int_0^T D_{t,0}\Phi_n(F) \cdot u(t,0) dt \right] \to 0 \tag{4.8}$$

as $n \to \infty$. Consequently, (4.6), (4.7) and (4.8) imply (4.5). □

4.1 The deterministic coefficients case

Throughout this subsection, we consider the case where α, β and γ in (2.3) are deterministic functions satisfying the three conditions in Example 2.8. Additionally, we assume the following condition:

$$\int_{\mathbb{R}_0} \{\gamma_{t,z}^4 + |\log(1+\gamma_{t,z})|^2\} \nu(dz) < C \text{ for some } C > 0. \tag{4.9}$$

We aim to obtain a concrete representation of LRM for the call option $(S_T - K)^+$. As seen in Example 3.9, this model satisfies all the conditions in Theorem 3.7, if (3.5) is satisfied. First of all, we calculate the Malliavin derivatives of S_T.

Proposition 4.3 *We have $S_T \in \mathbb{D}^{1,2}$; and*

$$D_{t,z}S_T = \frac{S_T\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{S_T\gamma_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z) \tag{4.10}$$

for q-a.e. $(t,z) \in [0,T] \times \mathbb{R}$.

Proof. Noting that

$$\begin{aligned} \log(S_T/S_0) &= \int_0^T \left[\alpha_t - \frac{1}{2}\beta_t^2 + \int_{\mathbb{R}_0} \{\log(1+\gamma_{t,z}) - \gamma_{t,z}\} \nu(dz) \right] dt \\ &\quad + \int_0^T \beta_t dW_t + \int_0^T \int_{\mathbb{R}_0} \log(1+\gamma_{t,z}) \widetilde{N}(dt,dz), \end{aligned}$$

we have $\log(S_T/S_0) \in \mathbb{D}^{1,2}$ and $D_{t,z}\log(S_T/S_0) = \frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\log(1+\gamma_{t,z})}{z}\mathbf{1}_{\mathbb{R}_0}(z)$ for any $(t,z) \in [0,T] \times \mathbb{R}$ by (4.9) and Lemma 3.3 of [7]. Setting $F := \log(S_T/S_0)$ and $f(x) := S_0 e^x$, we have $S_T = f(F)$. Thus, we have $f'(F)D_{t,0}F = S_T\frac{\beta_t}{\sigma}$ for any $t \in [0,T]$; and

$$\frac{f(F + zD_{t,z}F) - f(F)}{z} = S_T \frac{\exp\{zD_{t,z}F\} - 1}{z} = \frac{S_T\gamma_{t,z}}{z}$$

for any $(t,z) \in [0,T] \times \mathbb{R}_0$. Hence, Proposition 2.5 of [17] implies $S_T \in \mathbb{D}^{1,2}$ and (4.10). □

Remark 4.4 *A similar argument with Proposition 4.3, together with Example 3.9, yields* $D_{t,z}Z_T = -Z_T\left(\frac{u_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\theta_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right)$.

Now, we confirm condition (3.5).

Lemma 4.5 *Condition (3.5) in Example 3.9 is satisfied.*

Proof. By simple calculations, we have

$$d(Z_tS_t) = S_{t-}Z_{t-}\left\{(\beta_t - u_t)dW_t + \int_{\mathbb{R}_0}(\gamma_{t,z} - \theta_{t,z} - \gamma_{t,z}\theta_{t,z})\widetilde{N}(dt,dz)\right\},$$

which implies $Z_TS_T \in L^2(\mathbb{P})$ by Theorem 117 of [14]. Therefore, $Z_T(S_T - K)^+ \in L^2(\mathbb{P})$ holds.

Since Theorem 4.1 and Proposition 4.3 imply that $(S_T - K)^+ \in \mathbb{D}^{1,2}$, and

$$D_{t,z}(S_T-K)^+ = \mathbf{1}_{\{S_T>K\}}\frac{S_T\beta_t}{\sigma}\cdot\mathbf{1}_{\{0\}}(z) + \frac{(S_T(1+\gamma_{t,z})-K)^+ - (S_T-K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z), \tag{4.11}$$

we have

$$\left\|Z_TD_{t,z}(S_T-K)^+\right\|^2_{L^2(q\times\mathbb{P})} \le \mathbb{E}[Z_T^2S_T^2]\left(\int_0^T\beta_t^2dt + \int_0^T\int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)dt\right) < \infty,$$

and

$$\left\|(S_T-K)^+D_{t,z}Z_T\right\|^2_{L^2(q\times\mathbb{P})} \le \mathbb{E}[S_T^2Z_T^2]\left(\int_0^T u_t^2dt + \int_0^T\int_{\mathbb{R}_0}\theta_{t,z}^2\nu(dz)dt\right) < \infty.$$

In addition, there is a $C > 0$ such that

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\int_{\mathbb{R}}|zD_{t,z}(S_T-K)^+D_{t,z}Z_T|^2q(dt,dz)\right]\\ &\le \mathbb{E}[Z_T^2S_T^2]\left(\int_0^T\int_{\mathbb{R}_0}\gamma_{t,z}^2\theta_{t,z}^2\nu(dz)dt\right) \le C\mathbb{E}[Z_T^2S_T^2]\left(\int_0^T\int_{\mathbb{R}_0}\gamma_{t,z}^4\nu(dz)dt\right),\end{aligned}$$

from which condition 5 in Example 3.4 follows by (4.9). This completes the proof. □

Next, by using the above proposition and lemma, we can calculate an explicit representation of LRM for call options as follows:

Proposition 4.6 *For any* $K > 0$ *and* $t \in [0,T]$, *we have*

$$\begin{aligned}\xi_t^{(S_T-K)^+} &= \frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}\left\{\beta_t^2\mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{S_T>K\}}S_T|\mathcal{F}_{t-}]\right.\\ &\left.\quad + \int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}[(S_T(1+\gamma_{t,z})-K)^+ - (S_T-K)^+|\mathcal{F}_{t-}]\gamma_{t,z}\nu(dz)\right\}.\end{aligned} \tag{4.12}$$

Proof. From the view of Lemma 4.5, (4.12) is given by (3.6) and (4.11). □

5 Asian Options

In this section, we study Asian options, whose payoff is depending on $\frac{1}{T}\int_0^T S_s ds$. First of all, Lemma 3.2 in [7] implies the following proposition:

Proposition 5.1 *Besides Assumption 2.1, we assume the following two conditions:*

1. $S_s \in \mathbb{D}^{1,2}$ *for a.e.* $s \in [0,T]$.
2. $\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_{[0,T]} |D_{t,z}S_s|^2 dsq(dt,dz)\right] < \infty$.

We have then $\frac{1}{T}\int_0^T S_s ds \in \mathbb{D}^{1,2}$ *and* $D_{t,z}\frac{1}{T}\int_0^T S_s ds = \frac{1}{T}\int_0^T D_{t,z}S_s ds$ *for q-a.e.* $(t,z) \in [0,T]\times\mathbb{R}$.

Next, we calculate Malliavin derivatives and LRM of Asian options for the same setting as subsection 4.1.

Proposition 5.2 *When* α*,* β *and* γ *are deterministic functions satisfying the three conditions in Example 2.8 and (4.9), we have* $\frac{1}{T}\int_0^T S_s ds \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z}\frac{1}{T}\int_0^T S_s ds = \frac{1}{T}\left\{\frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\gamma_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\}\int_t^T S_s ds$$

for q-a.e. $(t,z) \in [0,T]\times\mathbb{R}$.

Proof. By the same way as Proposition 4.3, we can see that condition 1 in Proposition 5.1 and

$$D_{t,z}S_s = S_s\mathbf{1}_{[0,s]}(t)\left\{\frac{\beta_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{\gamma_{t,z}}{z}\mathbf{1}_{\mathbb{R}_0}(z)\right\}$$

for q-a.e. $(t,z) \in [0,T]\times\mathbb{R}$ and any $s \in [0,T]$. As for condition 2, we have the following:

$$\begin{aligned}
&\mathbb{E}\left[\int_{[0,T]\times\mathbb{R}}\int_0^T |D_{t,z}S_s|^2 dsq(dt,dz)\right] \\
&\le T\mathbb{E}\left[\sup_{s\in[0,T]} S_s^2\right]\left(\int_0^T \beta_t^2 dt + \int_{[0,T]\times\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)dt\right) < \infty.
\end{aligned}$$

□

We illustrate LRM for Asian options with payoff $(\frac{1}{T}\int_0^T S_s ds - K)^+$.

Proposition 5.3 *Under the same setting as Proposition 5.2, we have, for any $K > 0$ and $t \in [0, T]$,*

$$\begin{aligned}\xi_t^{(V_0-K)^+} &= \frac{1}{S_{t-}\left(\beta_t^2 + \int_{\mathbb{R}_0}\gamma_{t,z}^2\nu(dz)\right)}\Bigg\{\beta_t^2\mathbb{E}_{\mathbb{P}^*}[\mathbf{1}_{\{V_0>K\}}V_t|\mathcal{F}_{t-}] \\ &\quad + \int_{\mathbb{R}_0}\mathbb{E}_{\mathbb{P}^*}\left[(V_0+\gamma_{t,z}V_t-K)^+ - (V_0-K)^+|\mathcal{F}_{t-}\right]\gamma_{t,z}\nu(dz)\Bigg\},\end{aligned}$$

where $V_t = \frac{1}{T}\int_t^T S_s ds$ for $t \in [0, T]$.

Proof. Theorem 4.1 and Propositions 5.2 imply that

$$D_{t,z}(V_0-K)^+ = \mathbf{1}_{\{V_0>K\}}\frac{\beta_t V_t}{\sigma}\mathbf{1}_{\{0\}}(z) + \frac{(V_0+\gamma_{t,z}V_t-K)^+ - (V_0-K)^+}{z}\mathbf{1}_{\mathbb{R}_0}(z).$$

Thus, this proposition is concluded by (3.6). □

6 Lookback Options

We focus on lookback options, that is, options whose payoff depends on the running maximum of the asset price process $M^S := \sup_{t\in[0,T]} S_t$. We treat only the exponential Lévy case in this section.

6.1 Malliavin derivatives of running maximum

First of all, we calculate Malliavin derivatives of the running maximum over $[0, T]$ of the following Lévy process: $L_t = \mu t + X_t$, where X is the underlying Lévy process defined in (2.2), and $\mu \in \mathbb{R}$. Note that $L_t \in \mathbb{D}^{1,2}$ for any $t \in [0, T]$. Before stating the main theorem, we need some preparations.

Lemma 6.1 *Let $F_1, F_2, \cdots \in \mathbb{D}^{1,2}$. We have then, for any $n \geq 1$, $M_n := \max_{1\leq k\leq n} F_k \in \mathbb{D}^{1,2}$ and*

$$D_{t,z}M_n = \sum_{k=1}^n \mathbf{1}_{A_{n,k}}D_{t,0}F_k\cdot\mathbf{1}_{\{0\}}(z) + \frac{\max_{1\leq k\leq n}(F_k+zD_{t,z}F_k) - M_n}{z}\mathbf{1}_{\mathbb{R}_0}(z), \tag{6.1}$$

where $A_{n,1} = \{M_n = F_1\}$ and $A_{n,k} = \{M_n \neq F_1, \cdots, M_n \neq F_{k-1}, M_n = F_k\}$ for $2 \leq k \leq n$.

Proof. Remark that $M_2 = F_1 \vee F_2 = (F_2 - F_1)^+ + F_1 \in \mathbb{D}^{1,2}$ by Theorem 4.1; and $M_n = F_n \vee M_{n-1}$. We have then $M_n \in \mathbb{D}^{1,2}$ for any $n \geq 1$.

Next, we calculate $D_{t,0}M_n$. Theorem 4.1 implies

$$\begin{aligned} D_{t,0}M_n &= D_{t,0}(F_n \vee M_{n-1}) = D_{t,0}(F_n - M_{n-1})^+ + D_{t,0}M_{n-1} \\ &= \mathbf{1}_{\{F_n > M_{n-1}\}} D_{t,0}(F_n - M_{n-1}) + D_{t,0}M_{n-1} \\ &= \mathbf{1}_{\{F_n > M_{n-1}\}} D_{t,0}F_n + \mathbf{1}_{\{F_n \le M_{n-1}\}} D_{t,0}M_{n-1} \\ &= \mathbf{1}_{A_{n,n}} D_{t,0}F_n + \mathbf{1}_{\{M_n = M_{n-1}\}} D_{t,0}M_{n-1}. \end{aligned}$$

Iterating this calculation shows

$$\begin{aligned} D_{t,0}M_n &= \mathbf{1}_{A_{n,n}} D_{t,0}F_n \\ &\quad + \mathbf{1}_{\{M_n = M_{n-1}\}} \left\{ \mathbf{1}_{A_{n-1,n-1}} D_{t,0}F_{n-1} + \mathbf{1}_{\{M_{n-1} = M_{n-2}\}} D_{t,0}M_{n-2} \right\} \\ &= \mathbf{1}_{A_{n,n}} D_{t,0}F_n + \mathbf{1}_{A_{n,n-1}} D_{t,0}F_{n-1} + \mathbf{1}_{\{M_n = M_{n-2}\}} D_{t,0}M_{n-2} \\ &= \cdots = \sum_{k=1}^{n} \mathbf{1}_{A_{n,k}} D_{t,0}F_k. \end{aligned} \tag{6.2}$$

For the case where $z \neq 0$, we have

$$\begin{aligned} D_{t,z}M_n &= D_{t,z}(F_n - M_{n-1})^+ + D_{t,z}M_{n-1} \\ &= \frac{(F_n - M_{n-1} + zD_{t,z}(F_n - M_{n-1}))^+ - (F_n - M_{n-1})^+}{z} + D_{t,z}M_{n-1} \\ &= \frac{1}{z}\Big[(F_n - M_{n-1} + zD_{t,z}(F_n - M_{n-1}))^+ + M_{n-1} + zD_{t,z}M_{n-1} \\ &\quad -\{(F_n - M_{n-1})^+ + M_{n-1}\}\Big] \\ &= \frac{(F_n + zD_{t,z}F_n) \vee (M_{n-1} + zD_{t,z}M_{n-1}) - F_n \vee M_{n-1}}{z}, \end{aligned}$$

that is, $M_n + zD_{t,z}M_n = (F_n + zD_{t,z}F_n) \vee (M_{n-1} + zD_{t,z}M_{n-1})$. Thus, we have

$$\begin{aligned} M_n + zD_{t,z}M_n &= (F_n + zD_{t,z}F_n) \vee (M_{n-1} + zD_{t,z}M_{n-1}) \\ &= (F_n + zD_{t,z}F_n) \vee (F_{n-1} + zD_{t,z}F_{n-1}) \vee (M_{n-2} + zD_{t,z}M_{n-2}) \\ &= \cdots = \max_{1 \le k \le n} (F_k + zD_{t,z}F_k), \end{aligned}$$

which means

$$D_{t,z}M_n = \frac{\max_{1 \le k \le n}(F_k + zD_{t,z}F_k) - M_n}{z}. \tag{6.3}$$

By (6.2) and (6.3), we obtain (6.1). □

We need to show more two lemmas. We take a countable dense subset $\mathcal{U} := \{u_1, u_2, \cdots\} \subset [0, T]$ with $T \in \mathcal{U}$.

Lemma 6.2 *Let $\{Y_t\}_{t \in [0,T]}$ be an RCLL process. Denoting $M_n^Y := \max_{1 \le k \le n} Y_{u_k}$ for any $n \ge 1$; and $M^Y := \sup_{t \in [0,T]} Y_t$, we have $M_n^Y \to M^Y$ as $n \to \infty$.*

Proof. Since $M_n^Y \leq M^Y$ for any $n \geq 1$, it suffices to show that $\mathbb{P}(\lim_{n\to\infty} M_n^Y < M^Y) = 0$. Now, suppose that $\mathbb{P}\left(\lim_{n\to\infty} M_n^Y < M^Y\right) > 0$. Denoting $A_k := \{M^Y - \lim_{n\to\infty} M_n^Y \geq 1/k\}$ for $k \geq 1$, we have $0 < \mathbb{P}(\lim_{n\to\infty} M_n^Y < M^Y) = \mathbb{P}\left(\cup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mathbb{P}(A_k)$. Thus, $\mathbb{P}(A_k) > 0$ holds for any sufficiently large k. Now, fix such a k arbitrarily. Note that there exists a $[0,T)$-valued random time ζ such that $Y_\zeta \geq M^Y - \frac{1}{2k}$ on A_k, since we can find a $[0,T]$-valued random time $\hat{\zeta}$ such that $Y_{\hat{\zeta}} \geq M^Y - \frac{1}{2k}$ a.s., but $Y_T \leq M^Y - \frac{1}{k}$ on A_k because $T \in \mathcal{U}$. By the dense property of $\mathcal{U}$ and the RCLL property of Y, we can find a $\mathcal{U}$-valued random time η such that $Y_\eta > M^Y - \frac{1}{k}$ on A_k. This is a contradiction to the definition of A_k. □

To see Lemma 6.3 below, we denote $M_n^L := \max_{1\leq k\leq n} L_{u_k}$ for any $n \geq 1$, $M^L := \sup_{t\in[0,T]} L_t$, and $\tau := \inf\{t \in [0,T] | L_t \vee L_{t-} = M^L\}$. Note that $M^L = \sup_{t\in[0,T]}(L_t \vee L_{t-}) = L_\tau \vee L_{\tau-}$; and τ is a unique random time satisfying $M^L = L_\tau \vee L_{\tau-}$ by Lemma 49.4 of Sato [10].

Lemma 6.3 $\mathbb{P}(\tau = t) = 0$ *for any* $t \in [0,T]$.

Proof. Taking a $t \in [0,T)$ arbitrarily, we have

$$\mathbb{P}\left(\limsup_{s\downarrow 0} \frac{L_{t+s} - L_t}{s} = +\infty\right) = 1$$

by Theorem 47.1 and Proposition 10.7 of [10]. Thus, $\mathbb{P}(L_{t+s} \leq L_t \text{ for any } s \in (0, T-t]) = 0$ holds, from which $\mathbb{P}(L_t = M^L) = 0$ follows. On the other hand, $\mathbb{P}(L_{t-} = L_t) = 1$ by Proposition I.7 of Bertoin [4]. As a result, we obtain $\mathbb{P}(\tau = t) = 0$ for any $t \in [0,T)$. As for the case of $t = T$, Theorem 47.1 of [10] together with Lemma II.2 of [4] provides

$$\mathbb{P}\left(\limsup_{s\downarrow 0} \frac{L_{(T-s)-} - L_T}{s} = +\infty\right) = \mathbb{P}\left(\liminf_{s\downarrow 0} \frac{L_s}{s} = -\infty\right) = 1,$$

which implies $\mathbb{P}(L_s \leq L_T \text{ for any } s \in [0,T)) = 0$. By the same argument as the above, $\mathbb{P}(\tau = T) = 0$ follows. □

At last, we introduce the main theorem of this subsection.

Theorem 6.4 $M^L \in \mathbb{D}^{1,2}$ *and*

$$D_{t,z} M^L = \mathbf{1}_{\{\tau \geq t\}} \mathbf{1}_{\{0\}}(z) + \frac{\sup_{s\in[0,T]}(L_s + z\mathbf{1}_{\{t\leq s\}}) - M^L}{z} \mathbf{1}_{\mathbb{R}_0}(z). \tag{6.4}$$

Proof. Noting that $M^L \in L^2(\mathbb{P})$ by the square integrability of X, $M_n^L \in \mathbb{D}^{1,2}$ for any $n \geq 1$ by Lemma 6.1; and $M_n^L \to M^L$ in $L^2(\mathbb{P})$ by Lemma 6.2, we have only to see that $D_{t,z}M_n^L$ converges to the RHS of (6.4) in $L^2(q\times\mathbb{P})$ in view of Proposition 2.4 of [16].

Step 1. Firstly, we consider the case of $z\neq 0$. Lemma 6.1 implies

$$D_{t,z}M_n^L=\frac{\max_{1\le k\le n}(L_{u_k}+zD_{t,z}L_{u_k})-M_n^L}{z}.$$

Remark that $D_{t,z}L_s=\mathbf{1}_{\{s\ge t\}}$, which is RCLL on s. Thus, Lemma 6.2 yields

$$\lim_{n\to\infty}D_{t,z}M_n^L=\frac{\sup_{s\in[0,T]}(L_s+zD_{t,z}L_s)-M^L}{z}. \tag{6.5}$$

Moreover, noting that $|\max_{1\le k\le n}(a_k+b_k)-\max_{1\le k\le n}a_k|\le\max_{1\le k\le n}|b_k|$ for any $\{a_k\}_{1\le k\le n}$, $\{b_k\}_{1\le k\le n}\subset\mathbb{R}$, we obtain

$$\left|\max_{1\le k\le n}(L_{u_k}+zD_{t,z}L_{u_k})-M_n^L\right|\le\max_{1\le k\le n}|zD_{t,z}L_{u_k}|.$$

Thus, for any $z\in\mathbb{R}_0$,

$$\begin{aligned}
&\left|D_{t,z}M_n^L-\frac{\sup_{u\in[0,T]}(L_u+zD_{t,z}L_u)-M^L}{z}\right|^2\\
&\le\ 2\left\{|D_{t,z}M_n^L|^2+\frac{|\sup_{s\in[0,T]}(L_s+zD_{t,z}L_s)-M^L|^2}{|z|^2}\right\}\\
&\le\ \frac{2}{|z|^2}\left\{\left|\max_{1\le k\le n}(L_{u_k}+zD_{t,z}L_{u_k})-M_n^L\right|^2+\left|\sup_{s\in[0,T]}(L_s+zD_{t,z}L_s)-M^L\right|^2\right\}\\
&\le\ 2\left\{\max_{1\le k\le n}|D_{t,z}L_{u_k}|^2+\sup_{s\in[0,T]}|D_{t,z}L_s|^2\right\}\le 4\sup_{s\in[0,T]}|D_{t,z}L_s|^2=4.
\end{aligned}$$

The dominated convergence theorem implies that the convergence in (6.5) also holds in $L^2(q\times\mathbb{P})$.

Step 2. Next, we see that $D_{t,0}M_n^L\cdot\mathbf{1}_{\{0\}}(z)$ converges to $\mathbf{1}_{\{\tau\ge t\}}\mathbf{1}_{\{0\}}(z)$ in $L^2(q\times\mathbb{P})$. Similarly with Lemma 6.1, we denote $A_{n,1}^L=\{M_n^L=L_{u_1}\}$, and $A_{n,k}^L=\{M_n^L\neq L_{u_1},\cdots,M_n^L\neq L_{u_{k-1}},M_n^L=L_{u_k}\}$ for $2\le k\le n$. In addition, defining $\tau_n:=\sum_{k=1}^n u_k\mathbf{1}_{A_{n,k}^L}$ for any $n\ge 1$, we have

$$D_{t,0}M_n^L=\sum_{k=1}^n\mathbf{1}_{A_{n,k}^L}D_{t,0}L_{u_k}=\sum_{k=1}^n\mathbf{1}_{A_{n,k}^L}\mathbf{1}_{\{u_k\ge t\}}=\mathbf{1}_{\{\tau_n\ge t\}}$$

by Lemma 6.1. Recall that $\sup_{s\in[t,T]}(L_s\vee L_{s-})<L_\tau\vee L_{\tau-}$ on $\{\tau<t\}$ by Lemma 49.4 of [10]. Then, on $\{\tau<t\}$, we can find a $k\in\mathbb{N}$ such that $L_{u_k}>\sup_{s\in[t,T]}(L_s\vee L_{s-})$. Note that k depends on ω. As a result, $\tau_n<t$ holds for

any $n \geq k$. Similarly, we can see that, on $\{\tau > t\}$, we have $\tau_n > t$ for any sufficiently large n. Since $\mathbb{P}(\tau = t) = 0$ by Lemma 6.3, we can conclude that $\lim_{n\to\infty} \mathbf{1}_{\{\tau_n \geq t\}} = \mathbf{1}_{\{\tau \geq t\}}$ a.s., from which Theorem 6.4 follows. □

6.2 LRM for lookback options

We consider the case where S_t is given as an exponential Lévy process $S_t = S_0 \exp\{L_t\}$, where $S_0 > 0$; and denote $M^S := \sup_{t\in[0,T]} S_t$. In this subsection, we calculate Malliavin derivatives and LRM of lookback options whose payoffs are given as $M^S - S_T$ and $(M^S - K)^+$ for $K > 0$. Here we assume that $\int_{\mathbb{R}_0} \left\{z^2 + (e^z - 1)^4\right\} \nu(dz) < \infty$; and there exists an $\varepsilon \in (0, 1)$ such that

$$\frac{\left\{\mu + \frac{\sigma^2}{2} + \int_{\mathbb{R}_0} (x - e^x + 1)\nu(dx)\right\} (e^z - 1)}{\sigma^2 + \int_{\mathbb{R}_0} (e^x - 1)^2 \nu(dx)} < 1 - \varepsilon$$

for ν-a.e. $z \in \mathbb{R}_0$. These conditions are corresponding to (4.9) and condition 3 in Example 2.8, respectively. Note that the other two conditions in Example 2.8 are also satisfied. In addition, $\int_{\mathbb{R}_0} (z - e^z + 1)\nu(dz)$ is well-defined since $e^z - 1 - z \leq (e - 1)z^2$ for any $z \in [-1, 1]$. The following lemma is also given in a similar way with subsection 4.1.

Lemma 6.5 *(1) We have $M^S \in \mathbb{D}^{1,2}$; and*

$$D_{t,z} M^S = M^S \mathbf{1}_{\{\tau \geq t\}} \mathbf{1}_{\{0\}}(z) + \frac{\sup_{s\in[0,T]} \left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - M^S}{z} \mathbf{1}_{\mathbb{R}_0}(z).$$

(2) Condition (3.5) holds for both $M^S - S_T$ and $(M^S - K)^+$.

Proof. (1) Proposition 2.5 of [17], together with Theorem 6.4 and $\int_1^\infty (e^z - 1)^4 \nu(dz) < \infty$, implies that $M^S \in \mathbb{D}^{1,2}$, $D_{t,0} M^S = S_0 D_{t,0} e^{M^L} = S_0 e^{M^L} D_{t,0} M^L = M^S \mathbf{1}_{\{\tau\geq t\}}$; and, for $z \in \mathbb{R}_0$,

$$\begin{aligned}
D_{t,z} M^S &= S_0 D_{t,z} e^{M^L} = S_0 \frac{\exp\{M^L + z D_{t,z} M^L\} - e^{M^L}}{z} \\
&= S_0 \frac{\exp\left\{\sup_{s\in[0,T]} \left(L_s + z\mathbf{1}_{\{t\leq s\}}\right)\right\} - e^{M^L}}{z} \\
&= \frac{\sup_{s\in[0,T]} \left(S_s e^{z\mathbf{1}_{\{t\leq s\}}}\right) - M^S}{z}.
\end{aligned}$$

(2) We can see this assertion by Lemma 4.5. □

Now, we calculate Malliavin derivatives and LRM for lookback options by using Lemma 6.5, Theorem 4.1 and (3.6).

Proposition 6.6 *(1)*

$$\begin{aligned} D_{t,z}(M^S - S_T) &= (M^S \mathbf{1}_{\{\tau \geq t\}} - S_T)\mathbf{1}_{\{0\}}(z) \\ &\quad + \left(\frac{\sup_{s\in[0,T]} \left(S_s e^{z\mathbf{1}_{\{t\leq s\}}} \right) - M^S}{z} - S_T \frac{e^z - 1}{z} \right) \mathbf{1}_{\mathbb{R}_0}(z). \end{aligned}$$

(2) For any $K > 0$, we have

$$\begin{aligned} D_{t,z}(M^S - K)^+ &= M^S \mathbf{1}_{\{M^L > \log(K/S_0)\}} \mathbf{1}_{\{\tau \geq t\}} \mathbf{1}_{\{0\}}(z) \\ &\quad + \frac{\left(\sup_{s\in[0,T]} \left(S_s e^{z\mathbf{1}_{\{t\leq s\}}} \right) - K \right)^+ - (M^S - K)^+}{z} \mathbf{1}_{\mathbb{R}_0}(z). \end{aligned}$$

Corollary 6.7 *For any $K > 0$ and $t \in [0, T]$, we have*

$$\begin{aligned} \xi_t^{M^S - S_T} &= \frac{1}{CS_{t-}} \left\{ \sigma^2 \mathbb{E}_{\mathbb{P}^*}[M^S \mathbf{1}_{\{\tau \geq t\}} - S_T | \mathcal{F}_{t-}] \right. \\ &\quad \left. + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*} \left[\sup_{u\in[0,T]} \left(S_u e^{z\mathbf{1}_{\{t\leq u\}}} \right) - M^S - S_T \gamma_z | \mathcal{F}_{t-} \right] \gamma_z \nu(dz) \right\}, \end{aligned}$$

and

$$\begin{aligned} \xi_t^{(M^S - K)^+} &= \frac{1}{CS_{t-}} \left\{ \sigma^2 \mathbb{E}_{\mathbb{P}^*}[M^S \mathbf{1}_{\{M^L > \log(K/S_0)\}} \mathbf{1}_{\{\tau \geq t\}} | \mathcal{F}_{t-}] \right. \\ &\quad \left. + \int_{\mathbb{R}_0} \mathbb{E}_{\mathbb{P}^*} \left[\left(\sup_{u\in[0,T]} \left(S_u e^{z\mathbf{1}_{\{t\leq u\}}} \right) - K \right)^+ - (M^S - K)^+ | \mathcal{F}_{t-} \right] \gamma_z \nu(dz) \right\}, \end{aligned}$$

where $\gamma_z := e^z - 1$ and $C := \left(\sigma^2 + \int_{\mathbb{R}_0} \gamma_z^2 \nu(dz) \right)$.

Remark 6.8 *There are lookback options whose payoff is described by the running minimum of the asset price process, instead of the running maximum. Thus, we should mention about how to calculate Malliavin derivatives for the running minimum of exponential Lévy processes S.*

We denote $m^Y := \inf_{t\in[0,T]} Y_t$ for any stochastic process Y; and $S'_t := 1/S_t = S_0^{-1} e^{-L_t}$. Since S' is also an exponential Lévy process, we can calculate $M^{S'}$ through Theorem 6.4. Noting that $M^{S'} \geq S_0^{-1} > 0$, we take a C^1-function f on $\mathbb{R}$ such that $f(x) = 1/x$ if $x \geq S_0^{-1}$. Then, by $m^S = 1/M^{S'}$ and Proposition 2.5 of [17], we have

$$D_{t,z} m^S = D_{t,z} \frac{1}{M^{S'}} = -\frac{1}{(M^{S'})^2} D_{t,z} M^{S'}.$$

Remark that we can calculate $D_{t,z}(S_T - m^S)$ and $D_{t,z}(m^S - K)^+$ by the same way as Proposition 6.6.

7 Concluding remarks

Throughout this paper, we consider an incomplete financial market model whose asset price process is given as a solution to the SDE (2.3). Under some assumptions, we obtain representation results (Theorem 3.7 and Example 3.9) of LRM by using Malliavin calculus for Lévy processes based on the canonical Lévy space framework. So that, representations of LRM given in this paper include Malliavin derivatives of the claim to be hedged.

As typical examples of claims, we treat call options, Asian options and lookback options. As for call options, we formulate their Malliavin derivatives in a general form; and calculate their LRM explicitly for the case where the coefficients of the SDE are deterministic. Next, we illustrate how to calculate Malliavin derivatives of Asian options; and give expressions of their LRM for the deterministic coefficients case. Thirdly, we study lookback options for the exponential Lévy case.

As said above, we calculate LRM for only the deterministic coefficients case. It is because Malliavin derivatives of deterministic functions are given by 0, thereby we can comparatively easily make sure of Assumption 3.4 under some mild conditions as seen in subsection 4.1. Besides, LRM is expressed simply from the view of Example 3.9. On the other hand, in the random coefficients case, we need very complicated calculations to confirm if Assumption 3.4 holds. Furthermore, we need to calculate exactly H^* and Malliavin derivatives of u and θ. That's why, although we introduce the Barndorff-Nielsen and Shephard model as an typical example of models with random coefficients, we do not discuss its LRM in this paper. We shall postpone it to future research.

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