Equivariant, strong and leading order characteristic classes associated to fibrations

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EQUIVARIANT, STRING AND LEADING ORDER CHARACTERISTIC CLASSES ASSOCIATED TO FIBRATIONS

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1. INTRODUCTION

Infinite rank principal or vector bundles appear frequently in mathematical physics, even before quantization. For example, string theory involves the tangent bundle to the space of maps $\mathrm{Maps}(\Sigma^2, M)$ from a Riemann surface to a manifold M, while any gauge theory relies on the principal bundle $\mathcal{A}^* \longrightarrow \mathcal{A}^*/\mathcal{G}$ of irreducible connections over the quotient by the gauge group. Finally, formal proofs of the Atiyah-Singer index theorem take place on the free loop space LM, and in particular use calculations on TLM. As explained below, many of these examples arise from pushing finite rank bundles on the total space of a fibration down to an infinite rank bundle on the base space.

For the correct choice of structure group, these infinite rank bundles can be topologically nontrivial. As for finite rank bundles, nontriviality is often detected by infinite dimensional analogs of the Chern-Weil construction of characteristic classes, as in [9], [11], [13], [16], [17], with a survey in [19]. The choice of structure group is determined by natural classes of connections on these bundles, which typically take values in either the Lie algebra of a gauge group or a Lie algebra of zeroth order pseudodifferential operators (ΨDOs). There are essentially three types of characteristic forms for these connections, one using the Wodzicki residue for ΨDOs, one using the zeroth order or leading order symbol, and one using integration over the fiber. The corresponding cohomology classes are called residue classes, leading order classes, and string classes, respectively. As shown in [10], the residue classes vanish, but nontrivial residue Chern-Simons classes exist [13].

In this paper, we focus on gauge group connections and produce examples of nontrivial leading order and string characteristic classes for some infinite rank bundles associated to loop spaces, Gromov-Witten theory and gauge theory. While the residue classes are inherently infinite dimensional objects and difficult to compute, the leading order and string classes for infinite rank bundles on the base space of a fibration are often related to characteristic classes of the finite rank bundle on the total space. This makes the leading order and string classes more computable. In particular, in some cases we can relate the leading order classes to the string classes.

In §2, we describe the basic setup, which is well known from local proofs of the families index theorem. To a fibration $Z \longrightarrow M \longrightarrow B$ of closed manifolds and a bundle with connection $(E, \nabla) \longrightarrow M$, one can associate an infinite rank bundle with connection $(\mathcal{E}, \nabla') \longrightarrow B$. This is a gauge connection if the fibration admits a flat connection, for example if the fibration is trivial. In this case, we can define the associated leading order Chern classes of $\mathcal{E}$. Even if the fibration is not flat, $\mathcal{E}$ has string classes, which are topological pushdowns of the Chern classes of E. The leading order and string classes do not live in the same degrees. Both classes have associated Chern-Simons or transgression forms.

In §3, we show that the S^1 Atiyah-Singer index theorem can be rewritten as an equality involving leading order classes on the loop space LM of a closed manifold M (Thm. 3.4). (More precisely, we work with the version of the S^1-index theorem called the Kirillov formula in [2].) This is an attempt to mimic the formal proofs of the ordinary index theorem on loop space [1], [3], but differs in significant ways. In particular, the statement involves integration of a leading order class over a finite cycle in LM, not over all of LM, so the nonrigorous localization step in the formal proof is sidestepped. It should be emphasized that this is only a restatement and not a loop space proof of the index theorem, as the S^1-index theorem is used in the restatement. Along the way, we construct equivariant characteristic forms on LM, such as the equivariant $\hat{A}$-genus and Chern character, which restrict to the corresponding forms on M sitting inside LM as constant loops (Thm. 3.3). It is unclear if the Chern character form we construct is the same as those constructed in [3] and [21].

In §4 we apply similar techniques to the moduli space of pseudoholomorphic maps from a Riemann surface Σ to a symplectic manifold M. We prove that certain Gromov-Witten invariants and gravitational descendants can be expressed in terms of leading order classes and string classes, and we recover the Dilaton Axiom. These techniques work when the GW invariants are really given by integrals over the smooth interior of the compactified moduli space, for which we rely on [22]. In particular, we have to restrict ourselves to genus zero GW invariants for semipositive manifolds. The main geometric observation is that the fibration of (interiors of) moduli spaces associated to forgetting a marked point is flat, so that leading order classes are defined. The main results (Thms. 4.1, 4.6) involve a mixture of string and leading order classes.

In §5 we prove that the real cohomology of a based loop group ΩG, for G compact, is generated by leading order Chern-Simons classes. This amounts to noting that the cohomology of G is generated by Chern-Simons classes, and then relating these finite rank classes to the leading order classes. We note that the generators of ΩG can also be written in terms of string classes, a known

result [9], and we specifically relate the string and leading order classes (Thm. 5.5). Related results are in [8].

In §6 we study leading order classes associated to the gauge theory fibration $\mathcal{A}^* \longrightarrow \mathcal{A}^*/\mathcal{G}$. This fibration has a natural gauge connection [7], [20], whose curvature involves nonlocal Green's operators. Leading order classes only deal with the locally defined symbol of these operators, so the calculation of these classes is relatively easy. In Prop. 6.2, we show that the canonical representative of Donaldson's ν-class [5, Ch. V] in the cohomology of the moduli space $ASD/\mathcal{G}$ of ASD connections on a 4-manifold is the restriction of a leading order form on all of $\mathcal{A}^*/\mathcal{G}$. Thus the ν-class gives information on the cohomology of $\mathcal{A}^*/\mathcal{G}$. It is desirable to extend this construction to cover the more important μ-classes, but this seems to require a theory of leading order currents. We give a preliminary result in this direction.

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2. Two types of characteristic classes

Perhaps the simplest type of infinite rank vector bundles come from fibrations. Let $Z \longrightarrow M \xrightarrow{\pi} B$ be a locally trivial fibration, with Z, M, B smooth, closed, oriented manifolds, and let $E \longrightarrow M$ be a smooth bundle. The pushdown bundle $\mathcal{E} = \pi_* E$ is a bundle over B with fiber $\Gamma(E|_{\pi^{-1}(b)})$ over $b \in B$. To specify the topology of $\mathcal{E}$, we can choose either a Sobolev class of H^s sections for the fibers or the Fréchet topology on smooth sections.

Using the transition functions of E, we can check that $\mathcal{E}$ is a smooth bundle with Banach spaces or Fréchet spaces as fibers in these two cases. For local triviality, take a connection ∇ on E, and fix a neighborhood U containing b over which the fibration is trivial. We can assume that U is filled out by radial curves centered at b. Fix a connection D for the fibration, i.e., a complement to the kernel of π_* in TM. For $m \in M_b = \pi^{-1}(b)$, each radial curve has a unique horizontal lift to a curve in M starting at m. For $s \in \Gamma(E|_{\pi^{-1}(b)})$, take the ∇-parallel translation of s along each horizontal lift at m. This gives a smooth isomorphism of $\mathcal{E}_b$ with $\mathcal{E}_{b'}$ for all $b' \in U$.

The connection ∇ pushes down to a connection $\pi_* \nabla = \nabla'$ on $\mathcal{E}$ by

$$\pi_* \nabla_X(s')(m) = \nabla_{X^h}(\tilde{s})(m), \tag{2.1}$$

where X^h is the D-horizontal lift of $X \in T_b B$ to $T_m M$, $s' \in \Gamma(\mathcal{E})$, and $\tilde{s} \in \Gamma(E)$ is defined by $\tilde{s}(m) = s'(\pi(m))(m)$. Thus ∇' acts as a first order operator on $\mathcal{E}_b$. The curvature Ω' of ∇', defined

by

$$
\begin{aligned}
\Omega'(X,Y) &= \pi_*\nabla_X\pi_*\nabla_Y - \pi_*\nabla_Y\pi_*\nabla_X - \pi_*\nabla_{[X,Y]} \\
&= \nabla_{X^h}\nabla_{Y^h} - \nabla_{Y^h}\nabla_{X^h} - \nabla_{[X,Y]^h} \\
&= \nabla_{X^h}\nabla_{Y^h} - \nabla_{Y^h}\nabla_{X^h} - \nabla_{[X^h,Y^h]} + \left(\nabla_{[X^h,Y^h]} - \nabla_{[X,Y]^h}\right) \\
&= \Omega(X^h,Y^h) + \left(\nabla_{[X^h,Y^h]} - \nabla_{[X,Y]^h}\right),
\end{aligned}
$$

satisfies $\Omega'(X,Y)(s')(m) = \Omega(X^h,Y^h)(\tilde{s})(m)$ iff D has vanishing curvature: $\Omega_D = [X,Y]^h - [X^h,Y^h] = 0$ [2, p. 20].[1] $\Omega(X^h,Y^h)$ is a zeroth order or multiplication operator, so in general, $\nabla_{[X^h,Y^h]} - \nabla_{[X,Y]^h}$ and hence Ω' acts on $\mathcal{E}_b$ as a first order differential operator: $\Omega \in \Lambda^2(B,\mathcal{D}^1)$ in the obvious notation. In summary:

Lemma 2.1. *Let* $(E,\nabla) \longrightarrow M$ *be a bundle with connection, and let* $(\mathcal{E} = \pi_*E, \nabla' = \pi_*\nabla) \longrightarrow B$ *be the pushdown bundle with connection associated to the fibration* $\pi : M \longrightarrow B$. *Let* D *be a connection on the fibration. Then the curvature two-form* Ω' *of* ∇' *takes values in multiplication operators on the fibers* $\mathcal{E}_b = \Gamma(E|_{\pi^{-1}(b)})$ *iff* D *is flat:* $\Omega_D = [X,Y]^h - [X^h,Y^h] = 0$. *In this case,* $\Omega'(X,Y) = \Omega(X^h,Y^h)$. *If* $\Omega_D \neq 0$, Ω' *takes values in first order differential operators on the fibers.*

For a finite rank bundle $F \longrightarrow B$ with connection, the curvature lies in $\Lambda^2(B,\mathrm{End}(F))$, and Chern classes are built from the usual matrix trace tr on $\mathrm{End}(F)$. There are no known nontrivial traces on $\mathcal{D}^1$, as the Wodzicki residue vanishes on differential operators. However, in the flat case $\Omega_D = 0$, we have $\Omega' \in \Lambda^2(B,\mathrm{End}_m(\mathcal{E}))$, where $\mathrm{End}_m(\mathcal{E})$ is the space of endomorphisms/multiplication operators $A \in \mathrm{End}(\mathcal{E})$ given by pointwise endormophisms of the fiber: $A(b) \in \Gamma(M_b,\mathrm{End}(E))$. There is a trace $\mathrm{Tr}^{\mathrm{lo}}$ on multiplication operators built from the matrix trace as follows. Choose smoothly varying volume forms dvol_b on each fiber M_b. For $\eta \in \Lambda^k(B,\mathrm{End}_m(\mathcal{E}))$ locally of the form $\eta = \sum_i a_i\omega^i \otimes A^i$ with $\omega^i \in \Lambda^k(B)$, $A^i(b) \in \Gamma(M_b,\mathrm{End}(E))$, set

$$
\mathrm{Tr}^{\mathrm{lo}}(\eta) = \sum_i a_i\omega^i \int_{M_b} \mathrm{tr}(A^i)\mathrm{dvol}_b \in \Lambda^k(B). \tag{2.2}
$$

In [17], this trace is called the leading order trace, as it extends to bundles whose transition functions are zeroth order pseudodifferential operators.

This trace is related to an averaging operator on forms that will be used in §5. To define this, again choose volume forms dvol_b and a connection on the fibration; for example, a choice of

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[1]This is equivalent to the usual condition for a flat connection, namely that D's distribution be integrable: since $\pi_*[X^h,Y^h] = \pi_*[X,Y]^h = [X,Y]$, the distribution is integrable iff $[X^h,Y^h] = [X,Y]^h$.

Riemannian metric on M determines the volume forms and the connection by setting the horizontal spaces to be $(\text{Ker } \pi_*)^\perp$. The averaging operator $\Upsilon : \Lambda^k(M) \longrightarrow \Lambda^k(B)$ is given by

$$\Upsilon(\alpha)(X_1, \ldots, X_k) = \int_{M_b} \alpha(X_1^h, \ldots, X_k^h) \text{dvol}_b. \tag{2.3}$$

It is easy to check that $d_B \Upsilon = \Upsilon d_M$, so $\Upsilon : H^*(M) \longrightarrow H^*(B)$ on de Rham cohomology is well defined. Υ extends to $\Upsilon' : \Lambda^k(M, \text{End}(E)) \longrightarrow \Lambda^k(B)$ by

$$\Upsilon'(\alpha)(X_1, \ldots, X_k)(b) = \int_{M_b} \text{tr}(\alpha(X_1^h, \ldots, X_k^h)) \text{dvol}_b. \tag{2.4}$$

Note that $\Upsilon \circ \pi^* = \text{Id}$, so $\text{Tr}^{\text{lo}}(\eta) = \Upsilon'(\pi^* \eta)$. In particular, for $\Omega_D = 0$ we have

$$\text{Tr}^{\text{lo}}((\Omega')^k) = \Upsilon'(\Omega^k). \tag{2.5}$$

From now on we denote Υ' by Υ.

Definition 2.1. *Let $Z \longrightarrow M \longrightarrow B$ be a flat fibration. In the notation of Lemma 2.1, the leading order Chern classes of $\mathcal{E}$ are*

$$c_k^{\text{lo}}(\mathcal{E}) = \left[\text{Tr}^{\text{lo}}((\Omega')^k)\right] = \left[\Upsilon'(\Omega^k)\right] \in H^{2k}(B, \mathbb{C}),$$

where the brackets denote the de Rham cohomology class. The leading order Chern character of $\mathcal{E}$ is

$$ch^{\text{lo}}(\mathcal{E}) = \left[\text{Tr}^{\text{lo}}(\exp(\Omega'))\right] \in H^{\text{ev}}(B, \mathbb{C}).$$

$\text{Tr}^{\text{lo}}(AB) = \text{Tr}^{\text{lo}}(BA)$ for $A, B \in \Lambda^{\text{even}}(B, \text{End}_m(\mathcal{E}))$, so the usual proof that the Chern form $\text{Tr}(\Omega^k)$ is closed with de Rham class independent of choice of connection on E carries over to $\text{Tr}^{\text{lo}}((\Omega')^k)$. In contrast, the leading order Chern classes are certainly not invariant under scalings of the volume form. However, assume that two families of volume forms satisfy $\int_{M_b} \text{dvol}_{g_1} = \int_{M_b} \text{dvol}_{g_2}$ for all b; this corresponds to deforming dvol_{g_1} perpendicularly to the scaling direction in the vector space of volume forms. In this case, the leading order Chern classes are the same. For $\text{dvol}_{g_1} = \text{dvol}_{g_2} + d_{M_b}\theta$ for $\theta = \theta_b \in \Lambda^{\dim(Z)-1}(M_b)$, and we have

$$\int_{M_b} \text{tr}(\Omega^k) \text{dvol}_{g_1} - \int_{M_b} \text{tr}(\Omega^k) \text{dvol}_{g_1}$$

$$= \int_{M_b} \text{tr}(\Omega^k) d\theta = \int_{M_b} d_{M_b} \text{tr}(\Omega^k) \wedge \theta = \int_{M_b} d_M \text{tr}(\Omega^k) \wedge \theta - \int_{M_b} d_B \text{tr}(\Omega^k) \wedge \theta$$

$$= \int_{M_b} \text{tr}(\nabla^E \Omega^k) \wedge \theta - d_B \int_{M_b} \text{tr}(\Omega^k) \wedge \theta = -d_B \int_{M_b} \text{tr}(\Omega^k) \wedge \theta,$$

by the Bianchi identity.

The integral in $\mathrm{Tr}^{\mathrm{lo}}(\eta)$ is an averaging of the endomorphism and leaves the degree of ω^i unchanged. In contrast, we can integrate ω^i over the fiber as well, which we denote by $\int_Z \omega^i$ or $\pi_* \omega^i$. This leads to a second type of Chern class, called string classes [16] or caloron classes [9]. Let $z = \dim Z$.

Definition 2.2. *The string Chern classes of $\mathcal{E}$ are*

$$c_k^{\mathrm{str}}(\mathcal{E}) = \left[\pi_* \mathrm{Tr}(\Omega^k)\right] = \pi_* \left[\mathrm{Tr}(\Omega^k)\right] \in H^{2k-z}(B, \mathbb{C}).$$

The string Chern character of $\mathcal{E}$ is

$$ch^{\mathrm{str}}(\mathcal{E}) = \left[\pi_* \mathrm{Tr}(\exp(\Omega))\right] \in H^{\mathrm{ev}-z}(B, \mathbb{C}).$$

The π_* outside the brackets denotes the induced pushforward on de Rham cohomology. Thus string classes satisfy a naturality condition:

$$c_k^{\mathrm{str}}(\pi_* E) = \pi_* c_k(E). \tag{2.6}$$

The string classes are the topological pushforward, and so can be defined for any coefficient ring, most easily if the total space M of the fibration is compact. Specifically, for Poincaré duality $\mathrm{PD}_M : H_*(M, \mathbb{Z}) \longrightarrow H^{\dim M - *}(M, \mathbb{Z})$, we have

$$c_k^{\mathrm{str}}(\pi_* E) = \mathrm{PD}_B \circ \pi_* \circ \mathrm{PD}_M^{-1} c_k(E), \tag{2.7}$$

where π_* on the right hand side is the usual homology pushforward. Thus the string classes are novel only in that they are identified with characteristic classes of infinite rank bundles.

In contrast, the leading order classes have no obvious interpretation for $\mathbb{Z}$ coefficients. However, the leading order forms for a fibration contain more information than the string forms: the string forms average only terms with all fiber variables and discard the rest, while the leading order forms average all terms as in (2.2).

Both classes have associated Chern-Simons classes; see §5. As with ordinary CS classes, the leading order and string CS classes are geometric objects.

3. Equivariant characteristic classes on loop spaces

In this section we construct equivariant characteristic classes on LM of leading order type and relate them to the corresponding characteristic classes on M. We use these constructions to restate the S^1 index theorem in terms of data on LM.

Let M be a closed, oriented, Riemannian n-manifold. Fix a parameter $s \gg 0$, and let $LM = L^{(s)}M$ be the space of maps $f : S^1 \longrightarrow M$ of Sobolev class $s \gg 0$. LM is a Banach manifold. The

space $L^{(\infty)}M$ of smooth loops is only a Fréchet manifold; the techniques of this section work in this case as well.

The tangent space $T_\gamma LM$ at a loop γ consists of vector fields along γ, *i.e.*, sections of $\gamma^* TM \longrightarrow S^1$ of either Sobolev class s or smooth. In sheaf theory terms, $TL^{(\infty)}M = \pi_* \, \text{ev}^* \, TM$, for

$\text{ev} : LM \times S^1 \longrightarrow M$ the evaluation map $\text{ev}(\gamma, \theta) = \gamma(\theta)$ and $\pi : LM \times S^1 \longrightarrow LM$ the projection. The following diagram encapsulates the setup.

$$
\begin{array}{ccc}
\text{ev}^* TM & \longrightarrow & TM \\
\downarrow & & \downarrow \\
LM \times S^1 & \xrightarrow{\ \text{ev}\ } & M \\
\pi \downarrow & & \\
TLM = \pi_* \, \text{ev}^* \, TM \longrightarrow & & LM
\end{array}
\qquad (3.1)
$$

Since $T_\gamma LM$ is noncanonically isomorphic to the space of sections of the trivial bundle $\mathbb{R}^n = S^1 \times \mathbb{R}^n \longrightarrow S^1$, the structure group of TLM is the gauge group of $\mathbb{R}^n$.

LM has the L^2 Riemannian metric

$$
\langle X, Y \rangle_\gamma = \frac{1}{2\pi} \int_{S^1} \langle X(\theta), Y(\theta) \rangle_{\gamma(\theta)} d\theta.
$$

Let ∇^{LM} be the L^2 connection on LM associated to the Levi-Civita connection ∇^M on M. This is given by "pulling back and pushing down ∇^M to LM." To define this carefully, particularly at self-intersections of γ, pick $X \in T_\gamma LM$, $Y \in \Gamma(TLM)$, define $\tilde{\gamma} : (-\epsilon, \epsilon) \times S^1 \longrightarrow M$ by $\tilde{\gamma}(t, s) = \exp_{\gamma(s)} tX(s)$, and define the vector field $Y(t, s)$ on $(-\epsilon, \epsilon) \times S^1$ by $Y(t, s) = Y_{\tilde{\gamma}(t,s)}$. Then

$$
(\nabla^{LM}_X Y)_\gamma(s) = [(\text{ev}^* \, \nabla^M)_{(X,0)} \widetilde{Y}]_{(\gamma,s)}, \qquad (3.2)
$$

where $\widetilde{Y} \in \Gamma(\text{ev}^* \, TM)$ is $\widetilde{Y}_{(\gamma,s)} = Y_{\gamma(s)}$.

Similarly, if $(E, \nabla^E, h) \longrightarrow M$ is a hermitian vector bundle with connection, we can form $\mathcal{E} = \pi_* \, \text{ev}^* \, E \longrightarrow LM$ with fiber $\mathcal{E}_\gamma = \Gamma(\gamma^* E \longrightarrow S^1)$ in the Fréchet case. The structure group of $\mathcal{E}$ is the gauge group of $\mathbb{R}^k \longrightarrow S^1$, $k = \text{rk}(E)$. $\mathcal{E}$ has an L^2 hermitian metric $\langle e_1, e_2 \rangle_\gamma = (2\pi)^{-1} \int_{S^1} h(e_1, e_2) d\theta$. As above, ∇^E pushes down to a hermitian connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$.

LM has a canonical S^1 action $k_s : LM \longrightarrow LM, s \in [0, 2\pi]$, given by rotation of loops: $k_s(\gamma)(\theta) = \gamma(s + \theta)$. k_s is an isometry of LM. $\mathcal{E}$ is an equivariant bundle for this action:

$$
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\ k'_s\ } & \mathcal{E} \\
\downarrow & & \downarrow \\
LM & \xrightarrow[k_s]{} & LM
\end{array}
$$

where $k'_s : \mathcal{E}_\gamma \longrightarrow \mathcal{E}_{k_s\gamma}, k'_s(e)(\theta) = e(s + \theta)$.

As pointed out in [3], ∇^{LM} is not an S^1-invariant connection. Recall that an invariant connection would satisfy [2, p. 26]

$$\nabla^{LM} k_{s,*} = k_{s,*} \nabla^{LM}. \tag{3.3}$$

More explicitly, let $W = W_\gamma(s)$ be a vector field on LM and $X \in T_\gamma LM$. Then (3.3) means

$$(\nabla^{LM}{}_X k_{s,*}[W_{k_{-s}(\gamma)}])(s_0) = k_{s,*} \left([\nabla^{LM}{}_{k_{-s,*}X} W]_{k_{-s}(\gamma)}\right)(s_0) \tag{3.4}$$

at γ. To compute $k_{s,*} : T_{k_{-s}(\gamma)}LM \longrightarrow T_\gamma LM$, set $\gamma_t(s) = \exp(tW(s))$. Then

$$(k_{s,*} W_\gamma)(s_0) = \frac{d}{dt}\Big|_{t=0} (k_s(\gamma_t))(s_0) = \frac{d}{dt}\Big|_{t=0} \exp(tW(s + s_0)) = W_\gamma(s + s_0) = W_{k_s(\gamma)}(s_0), \tag{3.5}$$

which is indeed a vector field along $k_s(\gamma)$. By (3.2), the left hand side of (3.4) equals

$$[(\text{ev}^* \nabla^M{}_{(X,0)})\widetilde{W}]_{(\gamma,s_0)} \in \text{ev}^* TM\big|_{(\gamma,s_0)} = T_{\gamma(s_0)}M,$$

while the right hand side equals

$$k_{s,*} \left[(\text{ev}^* \nabla^M)_{(k_{-s,*}X,0)}\widetilde{W}\right]_{(k_{-s}(\gamma),s_0)} \in k_{s,*} \left(\text{ev}^* TM\big|_{(k_{-s}(\gamma),s_0)}\right) = T_{\gamma(s_0)}M.$$

Even though both sides of (3.4) are vectors in $T_{\gamma(s_0)}M$, the two sides differentiate W at the different points $\gamma(s_0), \gamma(-s + s_0)$, and so ∇^{LM} is not S^1-invariant.

As in [2, (1.10)], we can average ∇^{LM} over the action to produce

$$\widetilde{\nabla}^{LM} = \int_0^{2\pi} \left(k_s^{T^*LM \otimes TLM}\right)^{-1} \nabla^{LM} k_{s,*} \, \bar{d}s, \tag{3.6}$$

where $(k_s^{T^*LM \otimes TLM})^{-1}\nabla^{LM}_X Y = \nabla^{LM}_{k_{-s,*}X} k_{-s,*}Y$ and $\bar{d}s = \frac{1}{2\pi}ds$. $\widetilde{\nabla}^{LM}$ is S^1-equivariant, since (simplifying the notation)

$$
\begin{aligned}
k_{s_0}\widetilde{\nabla}^{LM} &= k_{s_0}\left(\int_0^{2\pi} k_s^{-1}\nabla^{LM}k_s \, \bar{d}s\right) = \int_0^{2\pi} k_{s_0-s}\nabla^{LM}k_s \, \bar{d}s \\
&= \int_0^{2\pi} k_{-s}\nabla^{LM}k_{s+s_0} \, \bar{d}s = \widetilde{\nabla}^{LM}k_{s_0}.
\end{aligned}
$$

We can similarly average $\nabla^{\mathcal{E}}$ to obtain an S^1-invariant connection $\widetilde{\nabla}^{\mathcal{E}}$.

We now follow [2, §7.1]. Let $(\mathbb{C}[u] \otimes \Lambda^*(LM))^{S^1}$ be the space of equivariant forms on LM with values in polynomials on $\mathfrak{u} = \mathfrak{u}(1)$, the Lie algebra of S^1. Equivalently, this is the space of equivariant polynomial maps from $\mathfrak{u}$ to $\Lambda^*(LM)$. For $\deg(u) = 2$, this space becomes a complex for the degree one equivariant differential $(d_{\mathfrak{g}}\alpha)(u) = d(\alpha(u)) - \iota_u\alpha(u), u \in \mathfrak{u}$, where ι_u is the interior product of the vector field on LM associated to u. In particular, if u is the usual generator of $\mathfrak{u}$

corresponding to $i \in i\mathbb{R} \simeq \mathfrak{u}$, then $\iota_u = \iota_{\dot\gamma}$ at the loop γ (so $\dot\gamma \in T_\gamma LM$)). The cohomology of this complex is the Cartan model for the equivariant cohomology $H_{S^1}(LM, \mathbb{C})$.

$\widetilde\nabla^\mathcal{E}$ has the associated so-called equivariant connection $\widetilde\nabla^\mathcal{E}_u$ acting on $(\mathbb{C}[u] \otimes \Lambda^*(LM, \mathcal{E}))^{S^1}$:

$$(\widetilde\nabla^\mathcal{E}_u \alpha)(X) = (\widetilde\nabla^\mathcal{E} - \iota_X)(\alpha(X)), \quad X \in \mathfrak{u}.$$

With $\mathcal{E}$ and α understood, we also denote the left hand side of this equation by $\widetilde\nabla_u(X)$. The equivariant curvature is by definition

$$\widetilde\Omega^\mathcal{E}_u = \widetilde\nabla^2_u(X) + L^\mathcal{E}_X.$$

Here $L^\mathcal{E}_X$ is the Lie derivative along the vector field on the S^1-manifold $\mathcal{E}$ determined by $X \in \mathfrak{u}$. By [2, Prop. 7.4], $\widetilde\Omega^\mathcal{E}_u \in (\mathbb{C}[u] \otimes \Lambda^*(LM, \text{End}(\mathcal{E})))^{S^1}$.

Assume that M has an S^1 action $a : S^1 \times M \longrightarrow M$. The case of the trivial action $a(\theta, m) = m$ is not uninteresting. By averaging the metric over S^1, we may assume that the action is via isometries. Let $a_s : M \longrightarrow M$ be $a_s(m) = a(s, m)$. The action induces an embedding

$$a' : M \longrightarrow LM, \quad a'(m) = (s \mapsto a(s, m)).$$

The following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{a_s} & M \\ a' \downarrow & & \downarrow a' \\ LM & \xrightarrow{k_s} & LM \end{array} \tag{3.7}$$

Let Y be the vector field for the flow $\{a_s\}$ on M, i.e., Y is the vector field corresponding to the standard generator of $\mathfrak{u}$. Since $\{k_s\}$ is the flow of $\dot\gamma$ on LM, it follows from (3.7) that for a vector field V on M,

$$L_{\dot\gamma}(a'_* V) = \frac{d}{ds}\Big|_{s=0} (k_{-s} \circ a')_* V = \frac{d}{ds}\Big|_{s=0} (a' \circ a_{-s})_* V = a'_* L_Y V. \tag{3.8}$$

For $V_0 \in T_m M$, we have

$$(a'_* V_0)(s) = (a_s)_*(V_0). \tag{3.9}$$

From now on, we denote a' just by a, so $(a_* V_0)(s) = (a_s)_*(V_0)$. Let $i : M \longrightarrow LM$ be the isometric embedding taking a point to a constant loop. On the fixed point set of a, we have $a = i$ and $ia_s = k_s i$. Let T be the rank n subbundle of $TLM|_{a(M)}$ of "rotated vectors": the fiber is $T_\gamma = \{s \mapsto a_{s,*}(V_0), V_0 \in T_{\gamma(0)} M\}$. Thus $V \in \Gamma(TM)$ implies $a_* V \in \Gamma(T)$. Clearly $i^*T \simeq TM$ and $\beta : a^*T \xrightarrow{\simeq} TM$ via $\beta(s \mapsto k_{s,*}(V_0)) = V_0$. β induces an isomorphism $\Lambda^*(M, a^*T) \xrightarrow{\simeq} \Lambda^*(M, TM)$.

We will need the analog of (3.8) for the Levi-Civita connection.

Lemma 3.1. *Under the isomorphism* $\beta : a^*T \simeq TM$, *we have*

$$a^*(\nabla_\cdot^{LM}\dot\gamma) = \nabla_\cdot^M Y \in \Lambda^1(M, TM). \tag{3.10}$$

Proof. Extend a fixed $V_0 \in T_{m_0}M$ to a vector field V on M, so a_*V is an extension of $a_*V_0 \in T_{a(m_0)}LM$ to a vector field on $a(M) \subset LM$. Then

$$
\begin{aligned}
a^*(\nabla_\cdot^{LM}\dot\gamma)(V_0) &= \nabla_{a_*V_0}^{LM}\dot\gamma = \nabla_{\dot\gamma}^{LM} a_*V + [a_*V, \dot\gamma] = \nabla_{\dot\gamma}^{LM} a_*V + [a_*V, a_*Y] \\
&= \nabla_{\dot\gamma}^{LM} a_*V + a_*[V,Y] = \nabla_{\dot\gamma}^{LM} a_*V - a_*L_Y V.
\end{aligned}
$$

In local coordinates,

$$(\nabla_{\dot\gamma}^{LM} a_*V)^i = (s \mapsto \dot\gamma(a_{s,*}V)^i + \Gamma_{jk}^i \dot\gamma^j (a_{s,*}V)^k) = (s \mapsto \nabla_Y^M a_{s,*}V = a_{s,*}\nabla_Y^M V),$$

since $Y = a_{s,*}Y$ is the velocity vector field for the orbit γ and a acts via isometries. Since $a_*L_Y V = (s \mapsto a_{s,*}L_Y V)$, we have

$$a^*(\nabla_\cdot^{LM}\dot\gamma)(V_0) = (s \mapsto a_{s_*}(\nabla_V^M V - L_Y V)) = (s \mapsto a_{s,*}(\nabla_V^M Y)) \in T|_\gamma = a^*T|_{m_0}$$

Thus using $\beta : a^*T \xrightarrow{\simeq} TM$, we have $a^*(\nabla^{LM}\dot\gamma) = \nabla_\cdot^M Y$. $\qquad\square$

We now focus on the Riemannian case with $E = TM$.

Let $\Omega_u = \Omega_u^M$ be the equivariant curvature of the S^1-invariant Levi-Civita connection ∇^M, and let $\widetilde\Omega_u$ be the equivariant curvature of $\widetilde\nabla^{LM}$. Since $\widetilde\Omega_u^k \in (\mathbb{C}[u] \otimes \Lambda^*(LM, \mathrm{End}(TLM)))^{S^1}$ takes values in pointwise endomorphisms, its powers have the trace

$$\mathrm{Tr}(\widetilde\Omega_u^k) = \int_{S^1} \mathrm{tr}(\widetilde\Omega_u^k(s))\bar{d}s \in (\mathbb{C}[u] \otimes \Lambda^*(LM))^{S^1}. \tag{3.11}$$

Referring back to (3.1), we can interpret this equation as the leading order trace of a form on $LM \times S^1$, so we also call (3.11) a leading order trace.

Lemma 3.2. $a^* \mathrm{Tr}(\widetilde\Omega_u^k) = \mathrm{tr}((\Omega_u^M)^k)$.

Since the curvature form is skew-symmetric, both sides are zero if k is odd.

Proof. Denote the orbit $a(m)$ by γ. Because the action is via isometries, we have [2, p. 209-210],

$$\widetilde\Omega_u = \widetilde\Omega^{LM} - \widetilde\nabla_\cdot^{LM}\dot\gamma, \tag{3.12}$$

where $\widetilde\Omega^{LM}$ is the curvature of $\widetilde\nabla^{LM}$. On the right hand side of (3.12), $\widetilde\Omega^{LM} \in \Lambda^2(LM, \mathrm{Hom}(TLM))$, and $\widetilde\nabla_\cdot^{LM}\dot\gamma \in \Lambda^0(LM, \mathrm{Hom}(TLM))$. Thus for $Y_1, Y_2, Z \in T_\gamma LM$, $\widetilde\Omega_u(Y_1, Y_2)Z = \widetilde\Omega^{LM}(Y_1, Y_2)Z - \widetilde\nabla_Z^{LM}\dot\gamma \in T_\gamma LM$. Because of the pointwise nature of the L^2 connection, we have

$$\Omega^{LM}(a_*Y_1, a_*Y_2)(s) = \Omega^M(a_*Y_1(s), a_*Y_2(s)), \quad (\nabla_{a_*Z}^{LM}\dot\gamma)(s) = \nabla_{a_*Z(s)}^M\dot\gamma, \tag{3.13}$$

and for $Z(0) \in T_m M$,

$$\Omega^M(a_* Y_1(s), a_* Y_2(s)) a_{s,*} Z = a_{s,*}[\Omega^M(a_{s,*}(Y_1(0)), a_{s,*}(Y_2(0)) Z(0)] = a_{s,*}[\Omega^M((Y_1(0), Y_2(0) Z(0)],$$

$$\nabla^M_{a_{s,*} Z(0)} \dot\gamma = a_{s,*} \nabla^M_{Z(0)} \dot\gamma. \tag{3.14}$$

Let $\{e_i\}$ be an orthonormal frame of $T_m M$, so $\{a_{s,*} e_i = e_i(s)\}$ is an orthonormal frame at $T_{a(m,s)} M$. It follows from (3.12) – (3.14) that

$$\operatorname{tr}((a^*(\Omega^{LM} - \nabla^{LM}_. \dot\gamma)^k(Y_1, \dots, Y_{2k})) = \langle (\Omega^{LM} - \nabla^{LM}_. \dot\gamma)^k (a_{s,*} Y_1, \dots, a_{s,*} Y_{2k})(e_i(s)), e_i(s) \rangle_{a(m)(s)}$$

is independent of s. Trivially averaging this expression over s to obtain $\widetilde\Omega_u$, we see that $\widetilde\Omega_u$ acts pointwise in s. Therefore

$$\begin{aligned}
\operatorname{tr}((a^* \widetilde\Omega_u)^k)(Y_1, \dots, Y_{2k})_m &= \langle (\widetilde\Omega^{LM} - \widetilde\nabla^{LM}_. \dot\gamma)^k (a_{s,*} Y_1, \dots, a_{s,*} Y_{2k})(e_i(s)), e_i(s) \rangle_{a(m)(s)} \Big|_{s=0} \\
&= \langle (\Omega^M - \nabla^M_. Y)^k (Y_1, \dots, Y_{2k})(e_i), e_i \rangle_m \tag{3.15} \\
&= \langle \Omega^k_u(Y_1, \dots, Y_{2k})(e_i), e_i \rangle_m \\
&= \operatorname{tr}(\Omega^k_u)(Y_1, \dots, Y_{2k})_m.
\end{aligned}$$

where the second line follows from (3.10) and (3.13) (and noting that ∇^M is already equivariant).

There is one final average in (3.11) to obtain

$$a^* \operatorname{Tr}(\widetilde\Omega^k_u)_m = \int_{S^1} \operatorname{tr}((a^*[(\widetilde\Omega_u)_{a(m)(s)}])^k) \bar{d}s.$$

However, (3.15) shows that

$$a^* \operatorname{Tr}(\widetilde\Omega^k_u)_m = \int_{S^1} \operatorname{tr}((a^*[(\widetilde\Omega_u)_{a(m)(0)}])^k) \bar{d}s = \operatorname{tr}((a^*[(\widetilde\Omega_u)_s])^k) = \operatorname{tr}(\Omega^k_u)_m.$$

$\square$

By this Lemma, we can extend the $\hat A$-polynomial as a characteristic form in the curvature on M to an equivariantly closed form on LM. The $\hat A$-polynomial of a curvature form Ω can be expressed as a polymonial in $\operatorname{tr}(\Omega^{2k})$. In particular, $\hat A(\widetilde\Omega_u)$ is defined using the leading order trace Tr in (3.11).

Let $T|_M$ be the restriction of $T \subset TLM$ to the constant loops $i(M) \subset LM$.

Theorem 3.3. *(i)* $a^* \hat A(\widetilde\Omega_u) = \hat A(\Omega^M_u)$.

(ii) $\hat A(\widetilde\Omega_u)$ *is an equivariant extension of the $\hat A$-form on constant loops, i.e. $\hat A(\widetilde\Omega_u) = \hat A(\Omega^M)$ when restricted to $T|_M$.*

As above, these equalities use the isomorphisms $i^*T \simeq a^*T \simeq TM$ to identify $a^*(\nabla^{LM}.\dot\gamma), a^*\Omega^{LM}$, with $\nabla^M_. Y, \Omega^M$, respectively. (i) follows immediately from the previous lemma, standard Chern-Weil theory, and the fact that the equivariant curvature satisfies the Bianchi identity [2, §7.1]. (ii) follows from $\dot\gamma = 0$ on $i(M)$.

Remark 3.1. (i) This discussion extends to equivariant hermitian bundles $E \longrightarrow M$ and their loopifications $\mathcal{E} = \pi_* \operatorname{ev}^* E \longrightarrow LM$. Namely, we can take an invariant connection ∇^E on E, form the L^2/pointwise connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, average it to $\widetilde\nabla^{\mathcal{E}}$, and prove that the corresponding equivariant curvatures satisfy

$$a^*(\widetilde\Omega^{\mathcal{E}}_\mathfrak{u}) = \Omega^E_\mathfrak{g}.$$

In particular, the Chern characters satisfy

$$a^*\operatorname{ch}(\widetilde\Omega^{\mathcal{E}}_\mathfrak{u}) = \operatorname{ch}(\Omega^E_\mathfrak{g}),$$

and the Chern character restricts to the ordinary Chern character on the constant loops. See [21] for an alternative construction of an equivariant Chern character on LM. By the localization formula for equivariantly closed forms, the top degree rational equivariant cohomology classes of these forms are determined by their values on constant loops, where they agree. It remains to be seen if the actual forms agree.

(ii) If $\Omega^{LM}_\mathfrak{u} = \Omega^{LM} - \nabla^{LM}_. \dot\gamma$ is built from the L^2 connection and curvature on LM, the proof of Lemma 3.2 (without the final S^1 average) implies that $a^* \operatorname{Tr}(\Omega^k_\mathfrak{u}) = \operatorname{tr}((\Omega^M_\mathfrak{u})^k)$. Thus $a^*\hat A(\Omega^{LM}_\mathfrak{u}) = \hat A(\Omega^M_\mathfrak{u})$. This is somewhat more natural than Theorem 3.3(i), but $\Omega^{LM}_\mathfrak{u}$ and hence $\hat A(\Omega^{LM}_\mathfrak{u})$ are not equivariantly closed.

Recall that $a : M \longrightarrow LM$ by abuse of notation.

Definition 3.1. *For an S^1 action $a : S^1 \times M \longrightarrow M$, set $[a] = a_*[M^n] \in H_n(LM, \mathbb{Z})$.*

Since $a : M \longrightarrow LM$ is injective, we also denote its image by $[a]$, an n-dimensional submanifold of LM.

We now review the S^1-index theorem. Assume M is spin and has an S^1 action via isometries. S^1 is also assumed to act on (E, ∇^E) covering its action on M, so the kernel and cokernel of $\partial\!\!\!/^E_\nabla$ are representations of S^1. The S^1-index of $\partial\!\!\!/^E_\nabla$ is the corresponding element of the representation ring $R(S^1)$:

$$\operatorname{ind}_{S^1}(\partial\!\!\!/_{\nabla^E}) = \sum (a^+_k - a^k_-)u^k \in \mathbb{Z}[u, u^{-1}] = R(S^1),$$

where u^k denotes the representation $e^{i\theta} \mapsto e^{ik\theta}$ of S^1 on $\mathbb{C}$, and $a^k_\pm$ are the multiplicities of u^k in the kernel and cokernel of $\partial\!\!\!/_{\nabla^E}$. For a general compact group G, the G-index theorem identifies

this analytically defined element of $R(G)$ with a topologically defined element. This is difficult to compute in general, but there is a formula to compute the character of the action of a fixed $g \in G$ on the index space $[\ker \partial_{\nabla^E}] - [\operatorname{coker} \partial_{\nabla^E}] \in K_G(\mathrm{pt}) = R(G)$; this is often also called the G-index theorem. For $g = e^{i\theta} \in S^1$, the character is just $\operatorname{ind}_{S^1}(e^{ik\theta}, \partial_{\nabla}^E) = \sum (a_k^+ - a_-^k)e^{ik\theta}$.

The Atiyah-Segal-Singer fixed point formula for the S^1-index computes $\operatorname{ind}_{S^1}(\partial_{\nabla^E})$ in terms of data on the fixed point set of a particular $e^{i\theta}$ [2, Thm. 6.16]. Using the localization theorem for integration in equivariant cohomology [2, Thm. 7.13], this can be rewritten as

$$\operatorname{ind}_{S^1}(e^{-ik\theta}, \partial_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_M \hat{A}_u(\theta, \Omega_u)\mathrm{ch}(\theta, \Omega_u^E), \qquad (3.16)$$

[2, Thm. 8.2]. Here $\hat{A}_u(\theta, \Omega_u) = \hat{A}(\Omega_u)(\theta) \in \Lambda^*(M)$ is the evaluation of $\hat{A}(\Omega_u) \in (\mathbb{C}[u] \otimes \Lambda^* M)^{S^1}$ at $\theta \in \mathfrak{u}(1)$. (For general compact groups G, this theorem only holds for group elements close to the identity. For the S^1-index theorem, both sides are analytic for θ small, and so the equality extends to all θ.) For notational ease, we rewrite (3.16) as

$$\overline{\operatorname{ind}}_{S^1}(\partial_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_M^{S^1} \hat{A}_u(\Omega_u)\mathrm{ch}(\Omega_u^E), \qquad (3.17)$$

with the left hand side evaluated at $e^{-ik\theta}$ and the right hand side evaluated at θ.

We can now restate the S^1-index theorem for the Dirac operator as a result involving the equivariant curvature of LM. Unlike the usual statement, in this version the action information is contained precisely in $[a]$, while the integrand depends only on the (action-compatible) Riemannian metric on M.

Theorem 3.4. *Let M be a spin manifold with an isometric S^1-action, and let E be an equivariant hermitian bundle with connection ∇^E over M. Then*

$$\overline{\operatorname{ind}}_{S^1}(\partial_{\nabla^E}) = (2\pi i)^{-\dim(M)/2} \int_{[a]}^{S^1} \hat{A}(\widetilde{\Omega}_u)\mathrm{ch}(\widetilde{\Omega}_u^{\mathcal{E}}).$$

Proof. By Thm. 3.3,

$$\int_{[a]}^{S^1} \hat{A}(\widetilde{\Omega}_u)\mathrm{ch}(\widetilde{\Omega}_u^{\mathcal{E}}) = \int_{a_*[M]}^{S^1} \hat{A}(\widetilde{\Omega}_u)\mathrm{ch}(\widetilde{\Omega}_u^{\mathcal{E}}) = \int_{[M]}^{S^1} a^*(\hat{A}(\widetilde{\Omega}_u)\mathrm{ch}(\widetilde{\Omega}_u^{\mathcal{E}})) = \int_M^{S^1} \hat{A}(\Omega_u)\mathrm{ch}(\Omega_u^E).$$

The S^1-index theorem in the form (3.17) finishes the proof. $\square$

Remark 3.2. (i) By the localization theorem, we have

$$\overline{\operatorname{ind}}_{S^1}(\partial_{\nabla^E}) = \int_{[a]}^{S^1} \hat{A}(\widetilde{\Omega}_u)\mathrm{ch}(\widetilde{\Omega}_u^{\mathcal{E}}) = \int_{[a] \cap M}^{S^1} \frac{\hat{A}(\widetilde{\Omega}_u)\mathrm{ch}(\widetilde{\Omega}_u^{\mathcal{E}})}{\chi_u(\nu_a^{LM})},$$

where ν_a^{LM} is the normal bundle of the fixed point set $\mathrm{Fix}(a) = [a] \cap M$ in $[a]$, and χ_u is the equivariant Euler form.

(ii) If we do not assume that the action a is via isometries, then as in [21], we should replace the rotational action k_s on LM by parallel translation along loops. The averaging procedure again produces an equivariantly closed form extending a given characteristic form on M. However, Thm. 3.4 does not extend.

(iii) The integrand in Thm. 3.4 is not really independent of the action, since it depends on the action-dependent metric. However, we can push this metric dependence out of the integrand as follows. Let $\mathcal{B}$ be the space of metrics on M. $\mathcal{B}$ comes with a natural Riemannian metric $g^{\mathcal{B}}$, the so-called L^2 metric, given at $T_{g_0}\mathcal{B}$ by

$$g^{\mathcal{B}}(X,Y) = \int_M g_0^{ab} g_0^{cd} X_{ac} Y_{bd} \mathrm{dvol}_{g_0}.$$

Thus $LM \times \mathcal{B}$ has a metric h which at (γ, g_0) is the product metric of the L^2 metric on $T_\gamma LM$ determined by g_0 and $g^{\mathcal{B}}$ on $T_{g_0}\mathcal{B}$. This is not a global product metric, but it is not difficult to compute the Levi-Civita connection and curvature F of h. We extend the rotational action on LM trivially to $LM \times \mathcal{B}$, so one obtains an equivariant curvature $\widetilde{F}_u$. One directly computes that $\widetilde{F}_u^{(1)} = P^{TLM} \widetilde{F}_u P^{TLM}$ equals $\widetilde{\Omega}_u$, where P^{TLM} is the h-orthogonal projection of $T(LM \times \mathcal{B})$ to LM. One obtains

Proposition 3.5. *Let* $i_{g_0} : LM \times \{g_0\} \longrightarrow LM \times \mathcal{B}$ *be the inclusion. If a is a g_0-invariant S^1 action on M, then*

$$\overline{\mathrm{ind}}_{S^1}(\partial) = \int_{i_{g_0,*}[a]}^{S^1} \hat{A}(\widetilde{F}_u^{(1)}).$$

Thus the integrand is a universal form on $LM \times \mathcal{B}$, and the choice of action and compatible metric are encoded in the cycle of integration.

4. GROMOV-WITTEN THEORY

In this section we relate string classes and leading order Chern classes to genus zero Gromov-Witten invariants and gravitational descendants associated to characteristic classes. We also investigate when the integrals that often denote GW invariants are rigorous expressions. In particular, we want to realize GW invariants as integrals of forms over the moduli space $\mathcal{M}_{0,k}(A)$ defined below, without using the compactification $\overline{\mathcal{M}}_{0,k}(A)$. Thus we want to avoid both the construction of the virtual fundamental class and discussions of non-smooth Poincaré duality spaces. This is certainly not possible in general, so we restrict ourselves mainly to the semipositive case, where the moduli space of pseudoholomorphic curves has an especially nice compactification.

Recall that GW invariants are built from cohomology classes α_i on the target manifold M, while gravitational descendants [4, Ch. 10] also involve ψ classes, which are first Chern classes of line

bundles on the moduli space of marked curves. We first relate gravitational descendants to string classes (Thm. 4.1), and then relate GW invariants built from even classes on the target manifold to string and leading order classes (Thm. 4.6). For most of this section, we work in the symplectic setting. At the end, we make some comments about the algebraic case and the role of the virtual fundamental class.

4.1. **Notation.** Following the notation in [14], let M be a closed symplectic manifold with a generic compatible almost complex structure. For $A \in H_2(M, \mathbb{Z})$, set $C_0^\infty(A) = \{f : \mathbb{P}^1 \longrightarrow M | f \in C^\infty, f_*[\mathbb{P}^1] = A\}$. Let $G = \mathrm{Aut}(\mathbb{P}^1) \simeq PSL(2, \mathbb{C})$ be the group of complex automorphisms of $\mathbb{P}^1$. Set $\mathbb{P}_k^1 = \{(x_1, \ldots, x_k) \in (\mathbb{P}^1)^k : x_i \neq x_j \text{ for } i \neq j\}$. For fixed $k \in \mathbb{Z}_{\geq 0}$, set

$$C_{0,k}^\infty(A) = (C_0^\infty(A) \times \mathbb{P}_k^1)/G,$$

where the action of G on $C_0^\infty(A)$ is $\phi \cdot f = f \circ \phi^{-1}$ and ϕ acts diagonally on $\mathbb{P}_k^1$. Denoting an element of $C_{0,k}^\infty(A)$ by $[f, x_1, \ldots, x_k]$, we set the moduli space of pseudoholomorphic maps to be $\mathcal{M}_{0,k}(A) = \{[f, x_1, \ldots, x_k] : f \text{ is pseudoholomorphic}\}$. $\mathcal{M}_{0,k}(A)$ is a smooth, finite dimensional, noncompact manifold.

We impose the condition that all maps f are simple, *i.e.* f does not factor through a branched covering map from $\mathbb{P}^1$ to $\mathbb{P}^1$. In this case, the action of G on $C_0^\infty(A) \times \mathbb{P}_k^1$ is free, and $C_{0,k}^\infty(A)$ is an infinite dimensional manifold of either Banach or Fréchet type.

The forgetful map $\pi = \pi_k : C_{0,k}^\infty(A) \longrightarrow C_{0,k-1}^\infty(A)$ given by $[f, x_1, \ldots, x_{k-1}, x_k] \mapsto [f, x_1, \ldots, x_{k-1}]$ is a locally trivial smooth fibration, since for disjoint neighborhoods $U_1, \ldots, U_{k-1}$ around a fixed $x_1, \ldots, x_{k-1}$, we have

$$\pi^{-1}[(C_0^\infty(A) \times \prod U_i)/G] \approx [(C_0^\infty(A) \times \prod U_i)/G] \times \mathbb{P}_-^1,$$

i.e., the fiber $\mathbb{P}_-^1 = \mathbb{P}^1 \setminus \{x_1, \ldots, x_{k-1}\}$ consists of all choices for the kth point. This fibration restricts to a fibration on the moduli spaces, but does not extend to compactifications of the moduli spaces.

Let L_i be the line bundle over $C_{0,k}^\infty(A)$ with fiber $T_{x_i}^* \mathbb{P}^1$ over $[f, x_1, \ldots, x_k]$. This bundle is well defined, since an automorphism ϕ gives an identification of tangent spaces $d\phi_{x_i}^* : T_{\phi(x_i)}^* \mathbb{P}^1 \longrightarrow T_{x_i}^* \mathbb{P}^1$. We set $\mathcal{L}_i = \pi_* L_i$ for $i = 1, \ldots, k$. The fibers of $\mathcal{L}_i$ are given by

$$\mathcal{L}_i|_{[f, x_1, \ldots, x_{k-1}]} = \{s : \mathbb{P}_-^1 \longrightarrow T_{x_i}^* \mathbb{P}^1\}, \quad i = 1, \ldots, k-1,$$

$$\mathcal{L}_k|_{[f, x_1, \ldots, x_{k-1}]} = \Gamma(T^* \mathbb{P}_-).$$

If we put a Sobolev topology on $C_{0,k}^\infty(A)$ (*i.e.* we consider two maps close if their first s partial derivatives are close for a fixed $s \gg 0$), then $C_{0,k}^\infty(A)$ is a Banach manifold and so admits partitions

of unity. Thus the L_i have connections. In any case, we are interested in L_i restricted to the finite dimensional manifold $\mathcal{M}_{0,k}(A)$, so the existence of connections is not an issue.

We want to choose connections that do not blow up as $x_i \longrightarrow x_j$ in $\mathbb{P}_k^1$. Since the line bundle L_i' with fiber $T_{x_i}^* \mathbb{P}^1$ is a well defined line bundle on $(\mathbb{P}^1)^k/G$, it restricts to a line bundle on $\mathbb{P}_k^1$, which then pulls back to the bundle L_i on $C_{0,k}^\infty(A)$. We always take connections on L_i', as these restrict to connections on L_i which are well behaved on $\mathbb{P}_k^1$.

The $\mathcal{L}_i$ have leading order first Chern classes, but it is better to consider the associated string classes. Definition 2.2 in the current context is as follows.

Definition 4.1. *The string class $c_1^{\mathrm{str},r}(\mathcal{L}_i) \in H^{2r-2}(\mathcal{M}_{0,k-1}(A))$ or in $H^{2r-2}(C_{0,k}^\infty(A))$ is the de Rham class of*

$$\int_{\mathbb{P}'} [\mathrm{Tr}(\Omega_i)]^r,$$

where Ω_i is the curvature of a restricted connection on L_i. Thus $c_1^{\mathrm{str},r}(\mathcal{L}_i) = \pi_*(c_1(L_i)^r)$, where π_* is the pushforward map given by integration over the fiber.

Since $[\mathrm{Tr}(\Omega_i)]^r$ is a closed form, the right hand side of the definition is closed. Here we use the fact that for restricted connections on L_i, the integral over the fiber exists and extends to the compact space $\mathbb{P}^1$. The usual arguments that $c_1^{\mathrm{str},r}$ is closed (which uses Stokes' Theorem on $\mathbb{P}^1$) with de Rham class independent of the connection carry over.

Note that for $r = 1$, the string class $c_1^{\mathrm{str}}(\mathcal{L}_i) \in H^0$ is the (constant function) $-2 + k$, since $\int_{\mathbb{P}'} \mathrm{Tr}(\Omega)$ equals $\chi(T^*\mathbb{P}') = -\chi(T\mathbb{P}')$.

Let $\mathrm{ev}^k : C_{0,k}^\infty(A) \longrightarrow M^k$ be $\mathrm{ev}^k[f, x_1, \ldots, x_k] = (f(x_1), \ldots, f(x_k))$, let $p_i : M^k \longrightarrow M$ be the projection onto the ith factor, and set $\mathrm{ev}_i^k = p_i \circ \mathrm{ev}^k : C_{g,k}^\infty(A) \longrightarrow M$. Then $\mathrm{ev}_i^k = \mathrm{ev}_i^{k-1} \circ \pi$ for $i < k$. When the context is clear, we will denote ev^k by ev.

4.2. Semipositive manifolds.
We will give cases when GW invariants and gravitational descendants can be detected by integration of forms over the moduli space $\mathcal{M}_{0,k}(A)$, without using the compactification $\overline{\mathcal{M}}_{0,k}(A)$. This is expected to happen if the boundary strata have codimension at least two in $\overline{\mathcal{M}}_{0,k}(A)$, *e.g.*, if M is semipositive [15, §6.4]. The main results of this section justify this integration over just $\mathcal{M}_{0,k}(A)$.

For the rest of this section, except for the remarks at the end, we assume that M is semipositive. For motivation, we first pretend that $\overline{\mathcal{M}}_{0,k}(A)$ carries a fundamental class. Then for $\alpha_i \in H^{d_i}(M, \mathbb{C})$ of appropriate degree, the GW invariant associated to the α_i is

$$\langle \alpha_1 \ldots \alpha_k \rangle \overset{\mathrm{def}}{=} \mathrm{ev}_*[\overline{\mathcal{M}}_{0,k}(A)] \cdot a = \langle \mathrm{PD}\, \mathrm{ev}_*[\overline{\mathcal{M}}_{0,k}(A)] \cup \alpha, [M^k]\rangle \tag{4.1}$$

$$= \langle \alpha, \mathrm{ev}_*[\overline{\mathcal{M}}_{0,k}(A)]\rangle = \langle \mathrm{ev}^* \alpha, [\overline{\mathcal{M}}_{0,k}(A)]\rangle.$$

Here $\alpha = \alpha_1 \times \ldots \times \alpha_k \in H^*(M_k)$ and $a = \mathrm{PD}\, \alpha$ is the Poincaré dual of α. In (4.1), we use the characterization of Poincaré duality: for $a \in H_*(X, \mathbb{Z}), \beta \in H^*(X, \mathbb{Z})$ on an oriented compact manifold X,

$$\langle \alpha \cup \beta, [X] \rangle = \langle \beta, a \rangle. \tag{4.2}$$

To do this more precisely, we follow the careful exposition in [22]. For M semipositive,

$$\partial \, \mathrm{ev} \stackrel{\mathrm{def}}{=} \bigcap_{K \; \mathrm{compact}} \overline{\mathrm{ev}(\mathcal{M}_{0,k}(A) \setminus K)} \subset M^k$$

lies in the image of a map of a manifold of dimension at most $\dim \mathcal{M}_{0,k}(A) - 2 := r - 2$. Recall that $r = \dim M + 2c_1(A) + 2k - 6$, with $c_1(A) = \langle c_1(M), A \rangle$. Thus by definition $\mathcal{M}_{0,k}(A)$ defines a pseudocycle in M^k. By [22, Prop 2.2], there exists an open set $U = U_k$ in M^k with

$$\partial \, \mathrm{ev} \subset U \subset M^k, \quad H_r(M^k, U; \mathbb{Z}) \simeq H_r(M^k; \mathbb{Z}) \tag{4.3}$$

(see (4.7)). Let $\bar{V}$ be a compact manifold with boundary inside $\mathcal{M}_{0,k}(A)$ with

$$\mathcal{M}_{0,k}(A) \setminus \mathrm{ev}^{-1}(U) \subset \bar{V};$$

we think of $\bar{V}$ as "most of" $\mathcal{M}_{0,k}(A)$. Specifically, $\mathrm{ev}_*[\bar{V}] \in H_r(M^k, U; \mathbb{Z}) \simeq H_r(M^k; \mathbb{Z})$ is a substitute for the ill-defined $\mathrm{ev}_*[\overline{\mathcal{M}_{0,k}(A)}]$.

Definition 4.2. *The GW invariant associated to α is*

$$\langle \alpha_1 \ldots \alpha_k \rangle = \mathrm{ev}_*[\bar{V}] \cdot a \in \mathbb{Z},$$

provided $\sum_i |\alpha_i| = k \dim M - r$, for $|\alpha_i|$ the degree of α_i.

The GW invariant is independent of the choice of U and V. More generally, we can take positive integers ℓ_i with $\sum_i \ell_i |\alpha_i| = k \dim M - r$, take $a = \mathrm{PD}\, \alpha) = \mathrm{PD}(\alpha_1^{\ell_1} \times \ldots \times \alpha_k^{\ell_k})$, and similarly define $\langle \alpha_1^{\ell_1} \ldots \alpha_k^{\ell_k} \rangle$.

There is an integer q such that qa has a representative submanifold N; if N is unorientable, we have to pass to its oriented double cover. Of course, $\mathrm{PD}(N) = qa$, but we can represent α by a compactly supported closed form, the Thom class of the normal bundle of N in M^k, thought of as a tubular neighborhood of N. Then

$$\langle \alpha_1^{\ell_1} \ldots \alpha_k^{\ell_k} \rangle = \frac{1}{q} \mathrm{ev}_*[\bar{V}] \cdot N = \frac{1}{q} \langle \mathrm{PD}(N), \mathrm{ev}_*[\bar{V}] \rangle = \frac{1}{q} \langle \mathrm{ev}^* \mathrm{PD}(N), [\bar{V}] \rangle.$$

In the last term, $[\bar{V}] \in H_r(\mathcal{M}_{0,k}(A), \mathcal{M}_{0,k}(A) - V; \mathbb{Z})$ and $\mathrm{ev}^* \mathrm{PD}(N) \in H^r(\mathcal{M}_{0,k}(A), \mathbb{R})$. Since $\mathrm{ev}^* \mathrm{PD}(N)$ is a differential form, we can write

$$\langle \alpha_1^{\ell_1} \ldots \alpha_k^{\ell_k} \rangle = \frac{1}{q} \int_{\mathcal{M}_{0,k}(A)} \mathrm{ev}^* \mathrm{PD}(N) \quad (\mathrm{mod}\ \mathcal{M}_{0,k}(A) \setminus V), \tag{4.4}$$

where mod $\mathcal{M}_{0,k}(A) \setminus V$ imposes the relation $\int_K \theta = 0$ for a form θ and a submanifold, possibly with boundary, $K \subset \mathcal{M}_{0,k}(A) \setminus V$. The modding out ensures that (4.4) is independent of the representative of $PD(N)$. This justifies writing a GW invariant as the integral of a form over $\mathcal{M}_{0,k}(A)$.

From now on, we assume $q = 1$ for convenience and drop "mod $\mathcal{M}_{0,k}(A) \setminus V$".

To bring in the bundle $\mathcal{L}_i$, define the gravitational descendant(or gravitational correlator) associated to classes $\alpha_1 \ldots \alpha_k \in H^*(M, \mathbb{C})$ and multi-indices $(\ell_1, \ldots, \ell_k), (r_1, \ldots, r_k)$ by

$$\langle t_1^{r_1} \alpha_1^{\ell_1} \ldots t_k^{r_k} \alpha_k^{\ell_k} \rangle_{0,k} = \int_{\mathcal{M}_{0,k}(A)} c_1(L_1)^{r_1} \wedge \alpha_1^{\ell_1} \wedge \ldots \wedge c_1(L_k)^{r_k} \wedge \alpha_k^{\ell_k}.$$

Here $\sum_i 2r_i + \ell_i |\alpha_i| = \dim \mathcal{M}_{0,k}(A)$.

We set

$$\langle t_1^{r_1} \alpha_1^{\ell_1} \ldots t_{k-1}^{r_{k-1}} c_1^{\mathrm{str},r_k} \rangle_{0,k-1}$$

$$= \int_{\mathcal{M}_{0,k-1}(A)} c_1(L_1)^{r_1} \wedge \mathrm{ev}_1^* \alpha_1^{\ell_1} \wedge \ldots \wedge c_1(L_{k-1})^{r_{k-1}} \wedge \mathrm{ev}_{k-1}^* \alpha_{k-1}^{\ell_{k-1}} \wedge c_1^{\mathrm{str},r_k}(L_k).$$

Theorem 4.1. *For $\ell_k = 0$, the gravitational descendents satisfy*

$$\langle t_1^{r_1} \alpha_1^{\ell_1} \ldots \alpha_{k-1}^{\ell_{k-1}} t_k^{r_k} \rangle_{0,k} = \langle t_1^{r_1} \alpha_1^{\ell_1} \ldots t_{k-1}^{r_{k-1}} \alpha_{k-1}^{\ell_{k-1}} c_1^{\mathrm{str},r_k} \rangle_{0,k-1}.$$

Corollary 4.2. *(Dilaton Axiom)* [4, p. 306] *For $r_k = 1$,*

$$\langle t_1^{r_1} \alpha_1^{\ell_1} \ldots \alpha_{k-1}^{\ell_{k-1}} t_k \rangle_{0,k} = (-2 + k) \langle t_1^{r_1} \alpha_1^{\ell_1} \ldots t_{k-1}^{r_{k-1}} \alpha_{k-1}^{r_{k-1}} \rangle_{0,k-1}.$$

Proof of the Theorem. For a fibration $Z \longrightarrow M \xrightarrow{\pi} B$ of smooth compact manifolds, we have for $\omega \in \Lambda^*(B), \eta \in \Lambda^*(M)$,

$$\int_M \pi^* \omega \wedge \eta = \int_B \omega \wedge \pi_* \eta. \tag{4.5}$$

In particular, this holds for $\mathbb{P}^1_- \longrightarrow \mathcal{M}_{0,k}(A) \xrightarrow{\pi_k} \mathcal{M}_{0,k-1}(A)$ provided the forms extend to the closures of the moduli spaces.

For this fibration, we can extend (4.5) to integration mod $\mathcal{M}_{0,k}(A) \setminus V$ provided

$$\pi_k(\mathcal{M}_{0,k}(A) \setminus V_k) \subset \mathcal{M}_{0,k-1}(A) \setminus V_{k-1}, \tag{4.6}$$

where appropriate subscripts for V have been added.

We first sketch the proof of (4.6). By [22, Lemma 2.4], for a fixed generic triangulation T of M^k with simplices σ, the set $U = U_k$ in (4.3) is given by

$$U_k = \bigcup_{m=\dim M^k - \dim \mathcal{M}_{0,k}(A)}^{\dim M^k} \bigcup_{\sigma, |\sigma| = m} \mathrm{St}(b_\sigma, \mathrm{sd}\, T), \tag{4.7}$$

where b_σ is the barycenter of σ, sd is the first subdivision of T, and the star $\mathrm{St}(b_\sigma, \mathrm{sd}\ T)$ consists of the interior of all simplices in sd K containing b_σ. (We don't distinguish between the simplices in the triangulation and their images in M^k.) By the proof of [22, Lemma 2.4], we can restrict the simplices in U_k by any subset of $\{\sigma : |\sigma| \geq \dim M^k - \dim \mathcal{M}_{0,k}(A)\}$, provided we keep all the top simplices that meet $\partial\,\mathrm{ev}_k$. Given U_{k-1}, we will suitably restrict U_k so that $p(U_k) \subset U_{k-1}$, for $p : M^k \longrightarrow M^{k-1}$ the projection onto the first $k-1$ factors. Since π_k is a fibration, we will conclude that $x \notin V_k$ implies $\pi_k(x) \notin V_{k-1}$, which is (4.6).

To fill in the details of (4.6), we note that

$$
\begin{array}{ccc}
\mathcal{M}_{0,k}(A) & \xrightarrow{\ \mathrm{ev}^k\ } & M^k \\
\pi_k \downarrow & & \downarrow p \\
\mathcal{M}_{0,k-1}(A) & \xrightarrow[\mathrm{ev}^{k-1}]{} & M^{k-1}
\end{array}
\tag{4.8}
$$

commutes, with π_k and p surjective and open. It follows that for $X \subset M^{k-1}$,

$$
(\mathrm{ev}^{k-1})^{-1}(X) = \pi_k(\mathrm{ev}^k)^{-1}p^{-1}(X).
\tag{4.9}
$$

We claim that

$$
p(\partial\,\mathrm{ev}^k) \subset \partial\,\mathrm{ev}^{k-1}.
\tag{4.10}
$$

To see this, recall that $\partial\,\mathrm{ev}^k = \mathrm{ev}^k(\partial\mathcal{M}_{0,k}(A))$, and points in $\partial\mathcal{M}_{0,k}(A)$ are given by $Z = [h, y_1, \ldots, y_k]$, for h a pseudoholomorphic maps on smooth curves or cusp curves (*i.e.*, curves with bubbling), with the y_j possibly coincident. Choose such a Z and set $\mathrm{ev}^k(Z) = Y$. Take compact sets K_i exhausting M^k with $p(K_i)$ exhausting M^{k-1}, and choose $F^i = \mathrm{ev}^k[f^i, x_1^i, \ldots, x_k^i] \notin K_i$ with $\lim_i F^i = Y$. Then

$$
p(Y) = \lim_i p(F_i) = \lim_i p\,\mathrm{ev}^k[f^i, x_1^i, \ldots, x_k^i] = \lim_i \mathrm{ev}^{k-1}\pi_k[f^i, x_1^i, \ldots, x_k^i] \in \partial\,\mathrm{ev}^{k-1}.
$$

As mentioned above, choose U_{k-1} to contain only those top simplices σ^{top} which meet $\partial\,\mathrm{ev}^{k-1}$. Refine the triangulation of M^k to a new triangulation, also called T, so that each $p^{-1}(\sigma_{k-1}^{\mathrm{top}})$ is the sum of top simplices in T. Set U_k to contain only the simplices in $p^{-1}(U_{k-1})$; by (4.10), $U_k = p^{-1}(U_{k-1})$ is an open neighborhood of $\partial\,\mathrm{ev}_k$, and

$$
x \in U_k \Leftrightarrow p(x) \in U_{k-1}.
$$

By (4.9), $\pi_k\,\mathrm{ev}_k^{-1}(U_k) = \mathrm{ev}_{k-1}^{-1}(U_{k-1})$. Thus for a choice of compact manifold with boundary $\bar{V}_{k-1} \subset \mathcal{M}_{0,k-1}(A) \setminus \mathrm{ev}_{k-1}^{-1}(U_{k-1})$ and a slight perturbation of π_k, $\bar{V}_k = \pi_k^{-1}(\bar{V}_{k-1})$ is a manifold with boundary containing $\mathcal{M}_{0,k}(A) \setminus \mathrm{ev}_{k-1}^{-1}(U_{k-1})$. Finally, $\bar{V}_k$ misses a smaller open neighborhood of $\overline{\mathcal{M}}_{0,k}(A)$, so the closed set $\bar{V}_k$ is contained in a compact subset of $\mathcal{M}_{0,k}(A)$. Thus $\bar{V}_k$ is compact.

By this construction, (4.6) is satisfied, so we can apply (4.5) to the fibration $\mathbb{P}^1_- \longrightarrow \mathcal{M}_{0,k}(A) \xrightarrow{\pi_k} \mathcal{M}_{0,k-1}(A)$.

By (4.8), $\pi_k^* \operatorname{ev}_{k-1}^* = \operatorname{ev}_k^* \pi^*$, which we abbreviate by dropping π and denoting π_k by π: $\pi^* \operatorname{ev}_{k-1}^* = \operatorname{ev}_k^*$. Since $L_{i,k}$, that is, L_i as a bundle over $\mathcal{M}_{0,k}(A)$, satisfies $\pi^* L_{i,k-1} = L_{i,k}$ for $i < k$, by (4.5) we have (with even more subscripts omitted)

$$
\begin{aligned}
\langle t_1^{r_1} \alpha_1^{\ell_1} \dots \alpha_{k-1}^{\ell_{k-1}} t_k^{r_k} \rangle_{0,k} \\
= \int_{\mathcal{M}_{0,k}(A)} \pi^* \left(c_1(L_1)^{r_1} \wedge \operatorname{ev}_1^* \alpha_1^{\ell_1} \wedge \dots \wedge c_1(L_{k-1})^{r_{k-1}} \wedge \operatorname{ev}_{k-1}^* \alpha_{k-1}^{\ell_{k-1}} \right) \wedge c_1(L_k)^{r_k} \\
= \int_{\mathcal{M}_{0,k-1}(A)} c_1(L_1)^{r_1} \wedge \operatorname{ev}_1^* \alpha_1^{\ell_1} \wedge \dots \wedge c_1(L_{k-1})^{r_{k-1}} \wedge \operatorname{ev}_{k-1}^* \alpha_{k-1}^{\ell_{k-1}} \wedge \pi_* c_1(L_k)^{r_k} \\
= \langle t_1^{r_1} \alpha_1 \dots t_{k-1}^{r_{k-1}} \alpha_{k-1}^{\ell_{k-1}} c_1^{\mathrm{str}, r_k} \rangle_{0,k-1}.
\end{aligned}
$$

$\square$

For pure GW invariants, we investigate the geometry of the fibration π. The next series of lemmas hold for moduli spaces of genus g curves.

Lemma 4.3. π *is flat; i.e., for each* $[f, x_1, \dots, x_k] \in C_{0,k}^\infty(A)$, *there exists an integrable distribution* $H_{[f,x_1,\dots,x_k]} \subset T_{[f,x_1,\dots,x_k]} C_{0,k}^\infty(A)$ *such that*

$$
T_{[f,x_1,\dots,x_k]} C_{0,k}^\infty(A) \simeq \ker(\pi_*) \oplus H_{[f,x_1,\dots,x_k]}.
$$

Corollary 4.4. $\pi : \mathcal{M}_{0,k}(A) \longrightarrow \mathcal{M}_{0,k-1}(A)$ *is flat.*

Proof of the Lemma. Let $\gamma_0(t)$ be a curve in $C_{0,k}^\infty(A)$ with $\gamma_0(0) = f$, and let $\gamma(t)$ be curves in $\mathbb{P}^1$ with $\gamma_i(0) = x_i$ for $i = 1, \dots, k-1$. Set

$$
\begin{aligned}
H_{[f,x_1,\dots,x_k]} &= \{ (d/dt|_{t=0})[\gamma_0(t), \gamma_1(t), \dots, \gamma_{k-1}(t), x_k] : \gamma_0, \dots, \gamma_{k-1} \text{ as above} \} \\
&= \{ [\dot{\gamma}_0(0), \dot{\gamma}_1(0), \dots, \dot{\gamma}_{k-1}(0), 0] \}.
\end{aligned}
$$

Since

$$
(d/dt|_{t=0})[\gamma_0(t), \gamma_1(t), \dots, \gamma_{k-1}(t), x_k] = (d/dt|_{t=0})[\gamma_0(t) \circ \phi^{-1}, \phi \circ \gamma_1(t), \dots, \phi \circ \gamma_{k-1}(t), x_k],
$$

H is well defined.

Let X^h be the horizontal lift of a vector field X on $C_{0,k-1}^\infty(A)$, so X^h is of the form $[X_0, X_1, \dots, X_{k-1}, 0]$. The Lie bracket $\mathcal{L}_{X^h} Y^h$ is computed using the flow of X^h, which is locally of the form $[\eta_0(t), \eta_1(t), \dots, \eta_{k-1}(t), x_k]$. Thus $X = [\dot{\eta}_0(0), \dot{\eta}_1(0), \dots, \dot{\eta}_{k-1}(0)]$ and similarly for Y, so

$$
[X^h, Y^h] = \mathcal{L}_{X^h} Y^h = [\mathcal{L}_X Y, 0] = [X, Y]^h \in H. \tag{4.11}
$$

(The second bracket in (4.11) refers to a point in $\mathcal{M}_{0,k}(A)$.) $\qquad\qquad\square$

Let (F, ∇) be a finite rank hermitian bundle with connection over $C^\infty_{0,k}(A)$, so $\mathcal{F} = \pi_* F$ is an infinite rank bundle over $C^\infty_{0,k-1}(A)$. As in §2, $\pi_* F$ has the connection $\pi_* \nabla = \nabla'$ defined on $s \in \Gamma(\pi_* F)$ by

$$\nabla'_X s[f, x_1, \ldots, x_{k-1}] = \nabla_{X^h} \tilde{s}[f, x_1, \ldots, x_k],$$

where $\tilde{s}[f, x_1, \ldots, x_k] = s[f, x_1, \ldots, x_{k-1}, x_k]$. By Lemmas 2.1, 4.3, we have

Lemma 4.5. *Let* $(F, \nabla) \longrightarrow C^\infty_{g,k}(A)$ *be a bundle with connection with curvature* Ω. *In the notation of Lemma 4.3, the induced connection* ∇' *on* $\mathcal{F} = \pi_* F$ *has curvature* $\Omega'(X, Y) = \Omega(X^h, Y^h)$.

Let α_i be elements of the *even* cohomology of M. Since the Chern character $ch : K(M) \otimes \mathbb{C} \longrightarrow H^{\mathrm{ev}}(M, \mathbb{C})$ is an isomorphism, $\alpha_i = ch(E_i)$ for a virtual bundle E_i $(i = 1, \ldots, k-1)$, and $\alpha_k^{\ell_k} = ch(E_k)$. Pullbacks and pushdowns of the E_i are well defined virtual bundles.

Theorem 4.6. *Let* $\alpha_i \in H^{\mathrm{ev}}(M, \mathbb{C})$ *satisfy* $\alpha_i = ch(E_i)$, $i = 1, \ldots k-1$, *and let* $\alpha_k^{\ell_k} = ch(E_k)$ *for* $E_i \in K(M)$. *Set* $\mathcal{E}_i = \pi_* \mathrm{ev}_i^* E_i \longrightarrow \mathcal{M}_{0,k-1}(A)$. *Then*

$$\langle \alpha_1^{\ell_1} \ldots \alpha_k^{\ell_k} \rangle_{0,k} = \langle ch^{\mathrm{lo}}(\mathcal{E}_1) \cdots ch^{\mathrm{lo}}(\mathcal{E}_{k-1}) ch^{\mathrm{str}}(\mathcal{E}_k) \rangle_{0,k-1}.$$

Proof. Pick connections ∇_i on E_i with curvature Ω_i. Then

$$\langle \alpha_1^{\ell_1} \ldots \alpha_k^{\ell_k} \rangle_{0,k} \qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.12)$$

$$= \int_{\mathcal{M}_{0,k}(A)} \mathrm{ev}_1^*([\mathrm{Tr}(\exp(\Omega_1)]^{\ell_1}) \wedge \ldots \wedge \mathrm{ev}_{k-1}^*([\mathrm{Tr}(\exp \Omega_{k-1})]^{\ell_{k-1}}) \wedge \mathrm{ev}_k^*([\mathrm{Tr}(\exp \Omega_k)])$$

$$= \int_{\mathcal{M}_{0,k}(A)} [\mathrm{Tr}(\pi^* \mathrm{ev}_1^* \exp(\Omega_1))]^{\ell_1} \wedge \ldots \wedge [\mathrm{Tr}(\pi^* \mathrm{ev}_{k-1}^* \exp(\Omega_{k-1}))]^{\ell_{k-1}} \wedge [\mathrm{Tr}(\mathrm{ev}_k^* \exp \Omega_k)]$$

$$= \int_{\mathcal{M}_{0,k-1}(A)} [\mathrm{Tr}(\mathrm{ev}_1^* \exp(\Omega_1))]^{\ell_1} \wedge \ldots \wedge [\mathrm{Tr}(\mathrm{ev}_{k-1}^* \exp(\Omega_{k-1}))]^{\ell_{k-1}} \wedge \pi_*[\mathrm{Tr}(\mathrm{ev}_k^* \exp(\Omega_k))].$$

We have

$$\pi_* \left([\mathrm{Tr}(\mathrm{ev}_k^* \exp(\Omega_k))]^{\ell_k} \right) = ch^{\mathrm{str}}(\mathcal{E}_k),$$

where $\mathcal{E}_k = \pi_* \mathrm{ev}_k^* E_k$.

We claim that in the last line of (4.12), $[\mathrm{Tr}(\mathrm{ev}_i^* \exp(\Omega_i))]$ is the leading order Chern character for $\mathcal{E}_i = \pi_* E_i$, for $i < k$. Dropping the index i, let $\mathrm{ev}_u : \mathcal{M}_{0,k}(A) \longrightarrow M, \mathrm{ev}_d : \mathcal{M}_{0,k-1}(A) \longrightarrow M$ be the i-th evaluation maps. Then the leading order Chern character for $\mathcal{E}$ as a differential form is given by

$$ch^{\mathrm{lo}}(\mathcal{E})(X_1, \ldots, X_{2r}) = \int_{\mathbb{P}'} \mathrm{Tr}(\exp(\Omega^{\mathrm{ev}_u^* E}))(X_1^h, \ldots, X_{2r}^h) d\mathrm{vol}_{\mathbb{P}'},$$

by (2.2). Since

$$
\begin{aligned}
\mathrm{Tr}(\exp(\Omega^{\mathrm{ev}_u^* E}))_z(X_1^h,\ldots,X_{2r}^h) &= \mathrm{Tr}(\exp(\Omega^{\pi^* \mathrm{ev}_d^* E}))_z(X_1^h,\ldots,X_{2r}^h) \\
&= \mathrm{Tr}(\exp(\Omega^{\mathrm{ev}_u^* E}))_z(\pi_* X_1^h,\ldots,\pi_* X_{2r}^h) \\
&= \mathrm{Tr}(\exp(\Omega^{\mathrm{ev}_u^* E}))(X_1,\ldots,X_{2r})
\end{aligned}
$$

is independent of $z \in \mathbb{P}'$, we get

$$
ch^{\mathrm{lo}}(\mathcal{E}) = \int_{\mathbb{P}'} \mathrm{Tr}(\exp(\Omega^{\mathrm{ev}_u^* E}))\mathrm{dvol}_{\mathbb{P}'} = \mathrm{Vol}(\mathbb{P}')\,\mathrm{Tr}(\exp(\Omega^{\mathrm{ev}_u^* E})) = \mathrm{Vol}(\mathbb{P}')\,\mathrm{Tr}(\mathrm{ev}^* \exp(\Omega^E)).
$$

Setting the volume of $\mathbb{P}'$ equal to one finishes the claim and the proof.   □

We briefly discuss the algebraic setting. M is now a smooth projective variety, with $\mathcal{M}_{g,k}(A)$, $\overline{\mathcal{M}}_{g,k}(A)$ the moduli space/stack of stable maps of a fixed genus g curve into X representing $A \in H_2(M)$, and its compactification, respectively. The forgetful map $\pi : \mathcal{M}_{g,k}(A) \longrightarrow \mathcal{M}_{g,k-1}(A)$ exists as long as $n + 2g \geq 4$ or $A \neq 0$ and $n \geq 1$ [4, p. 183]. Provided the open moduli spaces are oriented manifolds, we can represent a suitable multiple of $a \in H_*(\mathcal{M}_{g,k-1}(A), \mathbb{Z})$ by a submanifold N, and then

$$
\int_N \omega \wedge \pi_* \eta = \int_{\pi^{-1}N} \pi^* \omega \wedge \eta,
$$

for $\omega \in \Lambda^*(\mathcal{M}_{g,k-1}(A)), \eta \in \Lambda^*(\mathcal{M}_{g,k}(A))$ compactly supported forms.

For a fibration $M \xrightarrow{\pi} B$ of oriented compact manifolds, define the homology pullback $\pi^* : H_*(B) \longrightarrow H_*(M)$ by $\pi^* = \mathrm{PD}_M^{-1} \circ \pi^* \circ \mathrm{PD}_B$, where π^* on the right hand side is the usual cohomology pullback. By (4.2), (4.5),

$$
\int_N \omega \wedge \pi_* \eta = \int_B \omega \wedge \pi_* \eta \wedge \mathrm{PD}_B(N) = \int_M \pi^* \omega \wedge \eta \wedge \pi^* \mathrm{PD}_B(N) = \int_{\pi^* N} \pi^* \omega \wedge \eta, \qquad (4.13)
$$

where we identify N with its homology class.

We would like to apply (4.13) to $[N] = [\overline{\mathcal{M}}_{g,k-1}(A)]^{\mathrm{virt}}$, the virtual fundamental class of $\overline{\mathcal{M}}_{g,k-1}(A)$. Provided $\overline{\mathcal{M}}_{g,k-1}(A), \overline{\mathcal{M}}_{g,k}(A)$ are orbifolds, we have

$$
\int_{[\overline{\mathcal{M}}_{g,k-1}(A)]^{\mathrm{virt}}} \omega \wedge \pi_* \eta = \int_{[\overline{\mathcal{M}}_{g,k}(A)]^{\mathrm{virt}}} \pi^* \omega \wedge \eta,
$$

since $\pi^*[\overline{\mathcal{M}}_{g,k-1}(A)]^{\mathrm{virt}} = [\overline{\mathcal{M}}_{g,k}(A)]^{\mathrm{virt}}$ [4, (7.22)]. The moduli spaces are orbifolds if e.g. $g = 0$ and $M = \mathbb{P}^n$ (or more generally if M is convex) [6]. In these cases, Theorem 4.1 continues to hold, since string classes are given by topological pushforwards. However, for Theorem 4.6, we would need to know in addition that $\mathrm{ev}_i^* E_i$ admit (suitable variants of) connections over the compactified moduli spaces, since the leading order classes are constructed from connections. It is not clear that this is possible.

5. Leading order Chern-Simons classes on loop groups

In this section we return to loop spaces in the special case of based loop groups ΩG. We define Chern-Simons analogues of the string and leading order characteristic forms of §2. The main result is that $H^*(\Omega G, \mathbb{R})$ is generated by Chern-Simons string classes (Thm. 5.4) or by a pushforward of of leading order Chern-Simons classes (Thm. 5.5). The relation between these different approaches is also given in Thm. 5.5.

5.1. Relative Chern-Simons forms on compact Lie groups.

We first express the generators of the cohomology ring of a compact Lie group in terms of relative Chern-Simons classes.

Let $E \longrightarrow M$ be a rank n hermitian vector bundle over a closed manifold M, and let ∇_0, ∇_1 be connections on E. Locally, we have $\nabla_i = d + \omega_i$, and $\omega_1 - \omega_0$ is globally defined. Let Ω_t be the curvature of the connection $\nabla_t = t\nabla_0 + (1-t)\nabla_1$. For a subgroup G' of $U(n)$ and an $\mathrm{Ad}_{G'}$-invariant polynomial f, the relative Chern-Simons form

$$CS_f(\nabla_0, \nabla_1) = \int_0^1 f(\omega_1 - \omega_0, \Omega_t, \dots, \Omega_t) dt \tag{5.1}$$

satisfies $f(\Omega_1) - f(\Omega_0) = dCS_f(\nabla_0, \nabla_1)$, provided the ∇_i are G'-connections.

Assume now that E is a trivialized bundle and $\nabla_0 = d$ is the canonical flat connection. A gauge transformation $h \in \mathrm{Aut}(E)$ yields the flat connection $\nabla_1 = h \cdot \nabla_0 = h^{-1}dh$. We have $\omega_1 - \omega_0 = h^{-1}dh$ and $\Omega_t = C_t[h^{-1}dh, h^{-1}dh]$ for $C_t > 0$. Note the confusing notation for $\nabla_1 = h^{-1}dh$ and the global connection one-form $h^{-1}dh$. In any case, CS_f is a closed form of odd degree and so determines a Chern-Simons cohomology class.

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$ and Maurer-Cartan form θ^G. Choose a finite dimensional faithful unitary representation $h : G \longrightarrow \mathrm{Aut}(V)$ with $\mathrm{Im}(h) = G'$. We may assume that h is the exponentiated version of a faithful Lie algebra representation $dh = h_* : \mathfrak{g} \longrightarrow \mathrm{End}(V)$. For $\underline{V} \longrightarrow G$ the trivial vector bundle $G \times V \longrightarrow G$, we can view h as a gauge transformation of $\underline{V}$. Let $\nabla_0 = d$ and $\nabla_1 = h^{-1}dh$ be connections on $\underline{V}$ as above. As before, $\omega_1 - \omega_0 = h^{-1}dh$, where

$$h^{-1}dh|_g(g_*X) = h^{-1}(g)\, dh_g(g_*X) = h^{-1}(g)h(g)h_*(X) = h_*(X),$$

for $X \in \mathfrak{g}$. Here $g_* = (L_g)_*$ is the differential of left multiplication by g. Thus $h^{-1}dh = h_*(\theta^G) \in \Lambda^1(G, \mathrm{End}(\underline{V}))$.

An Ad_G-invariant polynomial f on $\mathfrak{g}$ determines an $h(G)$-invariant polynomial h_*f on $\mathrm{Im}(h_*) \subset \mathrm{End}(V)$ by

$$(h_*f)(\alpha_1, \alpha_2, \dots, \alpha_k) = f(h_*^{-1}\alpha_1, \dots, h_*^{-1}\alpha_k).$$

In particular,

$$f(\theta^G, [\theta^G, \theta^G], \ldots, [\theta^G, \theta^G]) = (h_* f)(h^{-1} dh, [h^{-1} dh, h^{-1} dh], \ldots, [h^{-1} dh, h^{-1} dh]).$$

The corresponding Chern-Simons form is equal to

$$CS_f(\nabla_0, \nabla_1) = C(h_* f)(h^{-1} dh, [h^{-1} dh, h^{-1} dh], \ldots, [h^{-1} dh, h^{-1} dh]),$$

for some $C \neq 0$.

The following result is classical [18, §4.11].

Theorem 5.1. *Let $\{f_i\}$ be a set of generators of the algebra of Ad_G-invariant polynomials on $\mathfrak{g}$. Then $\{f_i(\theta^G, [\theta^G, \theta^G], \ldots, [\theta^G, \theta^G])\}$ is a set of ring generators of $H^*(G, \mathbb{R})$.*

Thus $H^*(G, \mathbb{R})$ is generated by Chern-Simons classes:

Corollary 5.2. *For $\{f_i\}$ as above, $CS_{f_i}(\nabla_0, \nabla_1)$ is a set of ring generators of $H^*(G, \mathbb{R})$*

For example, for $G = U(n)$ itself, the generators are given by $\mathrm{Tr}((h^{-1} dh)^k)$, although these generators vanish for $k = 1$, k even, and $k > n^2 - n$.

5.2. String Chern-Simons forms on loop groups.

Using [18], we show that generators of the real cohomology ring of a loop group can be written as Chern-Simons forms for a finite rank bundle.

From now on, G denotes a simply connected compact Lie group. Let ΩG be the group of smooth loops based at the identity. ΩG in the compact-open topology is an infinite dimensional Lie group with the homotopy type of a CW-complex. As in (3.1), the evaluation map ev : $\Omega G \times S^1 \longrightarrow G, \mathrm{ev}(\gamma, \theta) = \gamma(\theta)$, gives

$$
\begin{array}{ccc}
\mathrm{ev}^* \underline{V} & \longrightarrow & \underline{V} \\
\downarrow & & \downarrow \\
\Omega G \times S^1 & \xrightarrow{\ \mathrm{ev}\ } & G \\
\pi \downarrow & & \\
\mathcal{E} = \pi_* \mathrm{ev}^* \underline{V} \longrightarrow & \Omega G &
\end{array}
$$

Of course, $\Omega G \times S^1 \xrightarrow{\pi} \Omega G$ admits a flat connection. The $\mathfrak{g}$-valued one-form $\mathrm{ev}^* h_* \theta^G = \mathrm{ev}^*(h^{-1} dh)$ on $\Omega G \times S^1$ decomposes as

$$\mathrm{ev}^*(h^{-1} dh) = \xi + \eta,$$

where ξ, resp. η, are supported on ΩG, resp. S^1, directions.

It is easy to calculate ξ and η.

Lemma 5.3. *(i) At $(\gamma, t) \in \Omega G \times S^1$,*

$$\eta = \gamma^{-1}(t)\dot{\gamma}(t)\, dt. \tag{5.2}$$

(ii) For $(\gamma_* X, 0) \in T_{(\gamma, t_0)}(\Omega G \times S^1)$, we have

$$\xi|_{(\gamma, t_0)}(\gamma_* X, 0) = X(t_0). \tag{5.3}$$

Proof. (i) For $(0, \partial_t)$ tangent to $\{\gamma\} \times S^1$,

$$\eta|_{(\gamma, t_0)}(0, \partial_t) = \mathrm{ev}^*(h^{-1}dh)|_{(\gamma, t_0)}(0, \partial_t) = (h^{-1}dh)|_{\gamma(t_0)}(\mathrm{ev}_*(0, \partial_t)) = \gamma(t_0)^{-1}\dot{\gamma}(t_0).$$

(ii) We have

$$
\begin{aligned}
\xi|_{(\gamma, t_0)}(\gamma_* X, 0) &= \mathrm{ev}^*(h^{-1}dh)|_{(\gamma, t_0)}(\gamma_* X, 0) = (h^{-1}dh)|_{\gamma(t_0)}(\mathrm{ev}_*(\gamma_* X, 0)) \\
&= (h^{-1}dh)|_{\gamma(t_0)}(\gamma(t_0)_* X(t_0)) = X(t_0).
\end{aligned}
$$

$\square$

By Theorem 5.1, a set of generators for $H^*(\Omega G, \mathbb{R}) = H^{\mathrm{ev}}(\Omega G, \mathbb{R})$ is given by

$$\int_{S^1} f_i([\xi, \xi], \ldots, [\xi, \xi], \eta) \tag{5.4}$$

$$= \int_{S^1} f_i\left([\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)], \ldots, [\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)], \mathrm{ev}^*(h^{-1}dh)\right),$$

for $\{f_i\}$ a set of generators for the algebra of Ad-invariant polynomials on $\mathfrak{g}$. To go from the first to the second line in (5.4), we use $[\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)] = [\xi + \eta, \xi + \eta] = [\xi, \xi]$, and $\int_{S^1} f_i([\xi, \xi], \ldots, [\xi, \xi], \xi) = 0$.

We want to recognize the right hand side of (5.4) both as a string version of a Chern-Simons form and as a contraction of a leading order Chern-Simons form. To begin, we give the Chern-Simons analogues of the primary string and leading order forms of §2.

Definition 5.1. *(i) Let $Z \longrightarrow M \xrightarrow{\pi} B$ be a flat fibration of manifolds with Z oriented and closed. Let $E \longrightarrow M$ be a vector bundle with structure group G and with G-connections ∇_0, ∇_1, and let $\mathcal{E} = \pi_* E \longrightarrow B$ be the infinite rank pushdown bundle. The string CS form on $\mathcal{E}$ associated to a degree k invariant polynomial f on $\mathfrak{g}$ is*

$$CS_f^{\mathrm{str}, \mathcal{E}}(\pi_* \nabla_0, \pi_* \nabla_1) = \pi_* CS_f^E(\nabla_0, \nabla_1) \in \Lambda^{2k-1-z}(B),$$

where $z = \dim(Z)$.

(ii) Assume in addition that M is Riemannian. Let $\Upsilon : \Lambda^*(M) \longrightarrow \Lambda^*(B)$ be the averaging map (2.3). The leading order CS form associated to f is

$$CS_f^{\mathrm{lo},\mathcal{E}}(\pi_*\nabla_0, \pi_*\nabla_1) = \Upsilon(CS_f^E(\nabla_0, \nabla_1)) \in \Lambda^{2k-1}(B).$$

Note that there is a corresponding more natural definition for principal G-bundles, which is the setting for primary string classes in [9], [16].

We see that the generators in (5.4) satisfy

$$
\begin{aligned}
\int_{S^1} & f_i \left([\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)], \ldots, [\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)], \mathrm{ev}^*(h^{-1}dh) \right) \\
= & \ \pi_* f_i \left([\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)], \ldots, [\mathrm{ev}^*(h^{-1}dh), \mathrm{ev}^*(h^{-1}dh)], \mathrm{ev}^*(h^{-1}dh) \right) \quad (5.5) \\
= & \ \pi_* CS_{f_i}(\mathrm{ev}^* \nabla_0, \mathrm{ev}^* \nabla^1),
\end{aligned}
$$

where $\nabla_0 = d, \nabla_1 = h^{-1}dh$ are connections on $\underline{V}$ as before. Of course, $\mathrm{ev}^* \nabla_i$ is a connection on $\mathrm{ev}^* \underline{V}$, so we write $CS_{f_i}(\mathrm{ev}^* \nabla_0, \mathrm{ev}^* \nabla_1) = CS_{f_i}^{\mathrm{ev}^* \underline{V}}(\mathrm{ev}^* \nabla_0, \mathrm{ev}^* \nabla_1)$. More explicitly, $a = h \circ \mathrm{ev}$ is a gauge transformation on $\mathrm{ev}^*\underline{V}$ with $a^{-1}da = \mathrm{ev}^*(h^{-1}dh)$. Therefore, $\mathrm{ev}^* \nabla_0 = d, \mathrm{ev}^* \nabla_1 = a^{-1}da$, and

$$\pi_* CS_{f_i}^{\mathrm{ev}^* \underline{V}}(\mathrm{ev}^* \nabla_0, \mathrm{ev}^* \nabla_1) = \pi_* f_i \left([a^{-1}da, a^{-1}da], \ldots, [a^{-1}da, a^{-1}da], a^{-1}da \right). \quad (5.6)$$

Applying the definition of string CS forms to $E = \mathrm{ev}^* \underline{V}, M = \Omega G \times S^1, Z = S^1$ and using (5.4), (5.5), we obtain

Theorem 5.4. *Let G be a compact Lie group, let $\{f_i\}$ be a set of generators for the algebra of Ad-invariant polynomials on $\mathfrak{g}$, and let $h' : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ be a faithful finite dimensional representation with exponentiated representation $h : G \longrightarrow \mathrm{Aut}(V)$. Take connections $\nabla_0 = d, \nabla_1 = h^{-1}dh = h^{-1}h'$ on $\underline{V} = G \times V$. Then $H^*(\Omega G, \mathbb{R})$ is generated by*

$$CS_{f_i}^{\mathrm{str},\pi_* \mathrm{ev}^* \underline{V}}(\pi_* \mathrm{ev}^* \nabla_0, \pi_* \mathrm{ev}^* \nabla_1).$$

5.3. Leading order Chern-Simons forms on loop groups.

For the case of circle fibrations $S^1 \longrightarrow M \longrightarrow B$, there is a relation between the string classes and the leading order classes, both for Chern and Chern-Simons classes. We will only treat the CS case for the loop group fibration $S^1 \longrightarrow \Omega G \times S^1 \longrightarrow \Omega G$, but the results immediately extend to loop spaces and other flat circle fibrations. In particular, the construction below produces nontrivial examples of leading order Chern-Simons forms on loop groups.

Pushing down the trivial bundle $\mathrm{ev}^* \underline{V} \longrightarrow \Omega G \times S^1$ gives a trivial infinite rank bundle $\mathcal{V} = \pi_* \mathrm{ev}^*(\underline{V}) \longrightarrow \Omega G$ with fiber $C^\infty(S^1, V)$. Associated to the gauge transformation h of $\underline{V}$ is the

gauge transformation

$$\tilde{h} : \Omega G \longrightarrow \mathrm{Aut}(\mathcal{V}),$$

given by $\tilde{h}_\gamma(s)(t) = h_{\gamma(t)}(s(t))$ for $s \in C^\infty(S^1, V)$. By abuse of notation, $\tilde{h}^{-1} d\tilde{h} \in \Lambda^1(\Omega G, \Omega\mathfrak{g}) = \Lambda^1(\Omega G, C^\infty(S^1) \otimes \mathfrak{g})$ can be identified with the Maurer-Cartan form $\theta^{\Omega G}$ on ΩG, and we have connections $\widetilde{\nabla}_0 = d, \widetilde{\nabla}_1 = \tilde{h} \cdot \widetilde{\nabla}_0 = \tilde{h}^{-1} d\tilde{h}$ on $\mathcal{V}$. For an Ad_G-invariant polymonial f on $\mathfrak{g}$, it is convenient to introduce

$$CS_f^{\Omega G, \infty}(\widetilde{\nabla}_0, \widetilde{\nabla}_1) \overset{\mathrm{def}}{=} f(\theta^{\Omega G}, [\theta^{\Omega G}, \theta^{\Omega G}] \ldots, [\theta^{\Omega G}, \theta^{\Omega G}]) \tag{5.7}$$
$$= f(\tilde{h}^{-1} d\tilde{h}, [\tilde{h}^{-1} d\tilde{h}, \tilde{h}^{-1} d\tilde{h}] \ldots, [\tilde{h}^{-1} d\tilde{h}, \tilde{h}^{-1} d\tilde{h}]).$$

This is the analogue of taking the trace in the integral in (2.4). $CS_{f_i}^{\Omega G, \infty}(\widetilde{\nabla}_0, \widetilde{\nabla}_1) \in \Lambda^*(\Omega G, C^\infty(S^1))$, because f only acts on the $\mathfrak{g}$ part of $C^\infty(S^1) \otimes \mathfrak{g}$. In particular, the averaging map in Definition 5.1 satisfies

$$CS_f^{\mathrm{lo}, \mathcal{V}}(\pi_* \nabla_0, \pi_* \nabla_1) = \Upsilon(CS_f^{\mathrm{ev}^* \underline{V}}(\nabla_0, \nabla_1)) = \int_{S^1} CS_f^{\Omega G, \infty} \wedge dt. \tag{5.8}$$

Let $\chi(\gamma) = \dot{\gamma}$ be the fundamental vector field on ΩG with associated Lie algebra valued function $\chi(\gamma) = \theta^{\Omega G}(\dot{\gamma}) = \gamma_*^{-1} \dot{\gamma} \in \Omega\mathfrak{g}$. Note that for $\omega \in \Lambda^*(\Omega G, C^\infty(S^1))$, $\iota_\chi \int_{S^1} \omega \wedge dt = \int_{S^1} \iota_\chi \omega \wedge dt$, because ι_χ involves ΩG variables and the integral involves the S^1 variable.

We can now relate the string and leading order CS classes, and prove that the contraction of the leading order CS classes with χ generate $H^*(\Omega G)$.

Theorem 5.5. *Let f be an Ad_G-invariant polynomial on $\mathfrak{g}$. Then for the connections $\widetilde{\nabla}_0, \widetilde{\nabla}_1$, we have*

$$\iota_\chi(CS_f^{\mathrm{lo}, \mathcal{V}}(\widetilde{\nabla}_0, \widetilde{\nabla}_1)) = CS_f^{\mathrm{str}, \mathcal{V}}(\widetilde{\nabla}_0, \widetilde{\nabla}_1).$$

In particular, if $\{f_i\}$ generate the algebra of invariant polynomials on $\mathfrak{g}$, then the closed forms

$$\iota_\chi(CS_{f_i}^{\mathrm{lo}, \mathcal{V}}(\widetilde{\nabla}_0, \widetilde{\nabla}_1)) = \iota_\chi \int_{S^1} CS_{f_i}^{\Omega G, \infty}(\widetilde{\nabla}_0, \widetilde{\nabla}_1) \wedge dt$$

generate $H^*(\Omega G)$.

Proof. By (5.8), we have

$$\iota_\chi(CS_{f_i}^{\mathrm{lo}, \mathcal{V}}(\widetilde{\nabla}_0, \widetilde{\nabla}_1)) = \int_{S^1} \iota_\chi CS_{f_i}^{\Omega G, \infty}(\widetilde{\nabla}_0, \widetilde{\nabla}_1) \wedge dt.$$

Take $\gamma_* X_1, \ldots, \gamma_* X_{2k-2} \in T_\gamma \Omega G$, where $X_j \in \Omega\mathfrak{g}$ and $k = \deg(f)$. Then for permutations $\sigma \in \Sigma_{2k-2}$,

$$
\left(\int_{S^1} \iota_\chi CS_f^{\Omega G, \infty}(\widetilde{\nabla}_0, \widetilde{\nabla}_1) \wedge dt \right) (\gamma_* X_1, \ldots, \gamma_* X_{2k-2})
$$

$$
= \sum_\sigma |\sigma| \int_{S^1} f\left([X_{\sigma(1)}, X_{\sigma(2)}], \ldots, [X_{\sigma(k-3)}, X_{\sigma(2k-2)}], \gamma_*^{-1}(t)\dot\gamma(t) \right) dt
$$

$$
= \left(\int_{S^1} f([\xi,\xi], \ldots, [\xi,\xi], \eta) \right) (\gamma_* X_1, \ldots, \gamma_* X_{2k-2})
$$

$$
= CS_f^{\mathrm{str}, \mathcal{V}}(\widetilde{\nabla}_0, \widetilde{\nabla}_1)(\gamma_* X_1, \ldots, \gamma_* X_{2k-2}),
$$

where we use (5.2), (5.3), (5.7).   □

As an example, the even cohomology classes of the string CS forms $\int_{S^1} \mathrm{tr}(\iota_\chi(\tilde{h}^{-1}d\tilde{h})^{2k-1})$ generate $H^*(\Omega U(n), \mathbb{R})$, whereas the odd cohomology classes of the leading order CS forms $\int_{S^1} \mathrm{tr}((\tilde{h}^{-1}d\tilde{h})^{2k-1})$ vanish.

In summary, both string and leading order Chern-Simons forms give representatives of the generators of $H^*(\Omega G, \mathbb{R})$. The use of string CS classes is more natural, reflecting the fact that the primary string classes are topological objects. In fact, the relation between the string CS classes and the contracted leading order CS classes appears only because we have an S^1-fibration.

6. LEADING ORDER CLASSES AND CURRENTS IN GAUGE THEORY

Let $P \longrightarrow M$ be a principal G-bundle over a closed manifold M with compact semisimple group G. We denote by $\mathcal{A}^*$, resp. $\mathcal{G}$, the space of irreducible connections on P, resp. the gauge group of P. In this section we show that the leading order Chern classes of the canonical connection on the principal gauge bundle $\mathcal{A}^* \longrightarrow \mathcal{A}^*/\mathcal{G} = \mathcal{B}^*$ are related to Donaldson classes.

We put appropriate Sobolev norms on $\mathcal{A}^*$ and $\mathcal{G}$, so that the moduli space $\mathcal{B}^* = \mathcal{A}^*/\mathcal{G}$ is a Hilbert manifold. The right action of $\mathcal{G}$ on $\mathcal{A}$ is the usual $A \cdot g = \mathrm{Ad}_g(A)$, recalling that $A \in \Lambda^1(P, \mathfrak{g})$ with the adjoint action on $\mathfrak{g}$. Set $\mathrm{Ad}\, P = P \times_{\mathrm{Ad}} \mathfrak{g}$. The tangent space $T_A \mathcal{A}$ is canonically isomorphic to $\Lambda^1(M, \mathrm{Ad}\, P)$. A fixed metric $h = (h_{ij})$ on M induces a Riemannian or L^2 metric on $T\mathcal{A}$ by

$$
\langle X, Y \rangle_0 = \int_M h^{ij} \langle A_i, B_j \rangle \mathrm{dvol}_h,
$$

where $X = A_i dx^i, Y = B_j dx^j \in T_A \mathcal{A}$ and $\langle\ ,\ \rangle$ is an Ad_G-invariant positive definite inner product on $\mathrm{Ad}\, P$. Since the derivative of the gauge action (also denoted by $\cdot g$) is $X \cdot g = \mathrm{Ad}_g(X)$, the metric is gauge invariant.

The Lie algebra $\mathrm{Lie}(\mathcal{G})$ of $\mathcal{G}$ is $\Lambda^0(M, \mathrm{Ad}\, P)$, which has the L^2 metric $\langle f, g \rangle_0 = \int_M \langle f, g \rangle \mathrm{dvol}_h$. Let $d_A : \mathrm{Lie}(\mathcal{G}) = \Lambda^0(M, \mathrm{Ad}\, P) \longrightarrow \Lambda^1(M, \mathrm{Ad}\, P)$ be the covariant derivative associated to A.

Then the vertical space of $\mathcal{A}^* \longrightarrow \mathcal{B}^*$ at A is $\mathrm{Im}(d_A)$. It is straightforward to check that the orthogonal complement $\ker d_A^*$ forms the horizontal space of a connection on $\mathcal{A}^* \longrightarrow \mathcal{B}^*$. Let ω be the corresponding connection one-form.

Let Ω be the curvature of ω. Ω is horizontal. An explicit formula for Ω is known [7, 20], but to our knowledge the following proof has not appeared.

Lemma 6.1. *For X, Y horizontal tangent vectors at A, we have*

$$\Omega(X, Y) = -2G_A * [X, *Y] \in \mathrm{Lie}(\mathcal{G}).$$

Here $\Delta_A = d_A^* d_A$, $G_A = \Delta_A^{-1}$ is the Green's operator for Δ_A, and $*$ is the Hodge star associated to h.

Proof. For $X \in T_A \mathcal{A}^*$, let X^h, X^v denote the horizontal and vertical components of X. As a vertical vector, the curvature of ω at A is

$$
\begin{aligned}
\Omega(X, Y) &= d\omega(X^h, Y^h) = X^h(\omega(Y^h)) - Y^h(\omega(X^h)) - \omega([X^h, Y^h]) \\
&= -\omega([X^h, Y^h]) = -[X^h, Y^h]^v,
\end{aligned}
$$

for any extension of X, Y to vector fields near A. We have

$$X^v = d_A G_A d_A^* X, \quad X^h = X - d_A G_A d_A^* X.$$

In a local trivialization of $\mathcal{A}^* \longrightarrow \mathcal{B}^*$, we can write $[X^h, Y^h] = \delta_{X^h} Y^h - \delta_{Y^h} X^h$.

We may extend X, Y to constant vector fields near A with respect to this trivialization, so for any tangent vector Z, $\delta_Z Y = \delta_Z X = 0$ at A. Then

$$
\begin{aligned}
\delta_{X^h} Y^h &= \delta_{X - d_A G_A d_A^* X}(Y - d_A G_A d_A^* Y) = -\delta_{X - d_A G_A d_A^* X} d_A G_A d_A^* Y \\
&= -(\delta_X d_A) G_A d_A^* Y - d_A(\delta_X G_A) d_A^* Y - d_A G_A(\delta_X d_A^*) Y \\
&\quad + (\delta_{d_A G_A d_A^* X} d_A) G_A d_A^* Y + d_A(\delta_{d_A G_A d_A^* X} G_A) d_A^* Y + d_A G_A(\delta_{d_A G_A d_A^* X} d_A^*) Y \\
&= -d_A G_A(\delta_X d_A^*) Y + d_A G_A(\delta_{d_A G_A d_A^* X} d_A^*) Y,
\end{aligned}
$$

since $d_A^* Y = 0$ at A. Locally, $d_A^* = -* d_A * = *(d + [A, *\cdot])$, so

$$\delta_X d_A^* = -(d/dt)|_{t=0} * (d + [A + tX, *\cdot]) = -* [X, *\cdot].$$

Thus

$$\delta_{X^h} Y^h = d_A G_A * [X, *Y] - d_A G_A[d_A G_A d_A^* X, *Y] = d_A G_A * [X, *Y],$$

since $d_A^* X = 0$.

For $X = A_i e^i, Y = B_j e^j$ in a local orthonormal frame $\{e^i\}$,

$$[X, *Y] = [A_i, B_j] e^i \wedge *e^j = \sum_i [A_i, B_i] e^i \wedge *e^i = -\sum_i [B_i, A_i] e^i \wedge *e^i = -[Y, *X].$$

Therefore $\delta_{X^h} Y^h - \delta_{Y^h} X^h = 2 d_A G_A * [X, *Y]$. This gives

$$\Omega(X, Y) = -d_A G_A d_A^*(\delta_{X^h} Y^h - \delta_{Y^h} X^h) = -2 d_A G_A d_A^* d_A G_A * [X, *Y] = -2 d_A G_A * [X, *Y]$$

as a vertical vector. Since d_A^{-1} is takes vertical vectors isomorphically to Lie$(\mathcal{G})$, we get

$$\Omega(X, Y) = -2 G_A * [X, *Y] \in \text{Lie}(\mathcal{G}).$$

$\square$

The curvature takes values in Lie$(\mathcal{G}) = \Lambda^0(M, \text{Ad } P)$. Up until now, we have only considered connections on vector bundles, where the curvature takes values in an endomorphism bundle. If G is a matrix group, Lie$(\mathcal{G})$ has a global trace given by integrating the trace on $\mathfrak{g}$, the fiber of Ad P. In general, Lie$(\mathcal{G})$ can be thought of as an algebra of multiplication operators via the injective adjoint representation of $\mathfrak{g}$. Equivalently, we can pass to the vector bundle Ad $\mathcal{A}^* = \mathcal{A}^* \times_{\mathcal{G}} \text{Lie}(\mathcal{G})$ with fiber Lie$(\mathcal{G})$ and take the leading order classes of its associated connection $d\text{Ad}(\omega)$, whose curvature $[\Omega, \cdot]$ is usually denoted just by Ω. Either way, the leading order Chern form $c_k^{\text{lo}}(\Omega)$ of $\mathcal{A}^* \longrightarrow \mathcal{B}^*$ is a positive multiple of

$$\int_M \text{tr}(\Omega^k) \text{dvol}_h.$$

We claim that if $\dim(M) = 4$, we can identify $c_2^{\text{lo}}(\Omega)$ with Donaldson's ν-form. We briefly recall the construction of this form [5, Ch. 5]. Let $\widetilde{P} = \pi^* P \longrightarrow \mathcal{A}^* \times M$ be the pullback bundle for the projection $\pi : \mathcal{A}^* \times M \longrightarrow M$. $\widetilde{P} = \mathcal{A}^* \times P$ has the connection A on the slice $\{A\} \times M$. $\widetilde{P}$ descends to a $G^{\text{ad}} = G/Z(G)$-bundle, denoted P^{ad}, over $\mathcal{B}^* \times M$. $P^{\text{ad}} \longrightarrow \mathcal{B}^* \times M$ has a family of framed connections, denoted (A, ϕ), once a framing is fixed at some $m_0 \in M$. For example, if $G = SU(2)$, then P^{ad} is a $SO(3)$-bundle. ν is defined by

$$\nu = -\frac{1}{4} p_1(P^{\text{ad}}).$$

By the calculation in [5, §5.2], the form $\nu = p_1^{\text{lo}}(\mathcal{A}^*) = c_2^{\text{lo}}((\text{Ad } \mathcal{A}^*) \otimes \mathbb{C})$ is given by

$$\nu(X_1, X_2, X_3, X_4) = c \cdot \sum_{\sigma \in \Sigma_4} \text{sgn}(\sigma) \int_M \text{tr}(G_A * [X_{\sigma(1)}, *X_{\sigma(2)}] G_A * [X_{\sigma(3)}, *X_{\sigma(4)}]) \text{dvol},$$

for some constant c. By Lemma 6.1, we obtain.

Proposition 6.2. *As differential forms, ν equals $p_1^{\text{lo}}(\mathcal{A}^*)$ up to a constant.*

It is of course more interesting to relate leading order classes to $\mu(a)$ for $a \in H_2(M, \mathbb{Q})$, and Donaldson's map $\mu : H_*(M, \mathbb{Q}) \longrightarrow H^{4-*}(\mathcal{M}, \mathbb{Q})$. ($\mathcal{M}$ is the moduli space of ASD connections.) Recall that $\mu(a) = i^*(\nu/a)$, for the slant product $\nu/ : H_*(M, \mathbb{Q}) \longrightarrow H^{4-*}(\mathcal{B}^*, \mathbb{Q})$ and $i : \mathcal{M} \longrightarrow \mathcal{B}^*$ the inclusion. In particular, $\nu = \mu(1)$ for $1 \in H_0(M)$. The difficulty in showing that $\mathcal{M}$ has a fundamental class, so that the slant product is defined, is similar in spirit to the issues treated in §4.

There is a positive integer k such that ka has a representative submanifold, and therefore PD(ka) has a representative closed two-form ω. If $k \neq 1$, we replace ω by $k^{-1}\omega$. In general, let ω be a closed two-form on M. By [5, Prop. 5.2.18], the two-form $C_\omega \in \Lambda^2(\mathcal{M})$ representing $\nu/a = \nu/\mathrm{PD}^{-1}(\omega)$ and hence $\mu(a)$ is given at $[A] \in \mathcal{M}$ by

$$C_\omega(X, Y) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(X \wedge Y) \wedge \omega + \frac{1}{2\pi^2} \int_M \mathrm{tr}(\Omega_A(X, Y) F_A) \wedge \omega, \tag{6.1}$$

where F_A is the curvature of A. On the right hand side, we use any $A \in [A]$ and $X, Y \in T_A \mathcal{A}^*$ with $d_A^* X = d_A^* Y = 0$.

There is a leading order class associated to any distribution or zero current Λ on $C^\infty(M)$, given pointwise by

$$c_k^{\mathrm{lo},\Lambda} = \Lambda(\mathrm{tr}(\Omega^k)),$$

where Ω is the curvature of a connection taking values in the Lie algebra of a gauge group, as in this section. (More generally, there are leading order classes associated to distributions on the unit cosphere bundle of M for connections taking values in nonpositive order pseudodifferential operators [12], [17].) In particular, for a fixed $f \in C^\infty(M)$ we have the characteristic class

$$\int_M f \cdot \mathrm{tr}(\Omega^k).$$

We can turn this around and consider $\mathrm{tr}(\Omega^k)$ as a zero-current acting on f. Looking back at (6.1), we can consider the two-currents

$$\mathrm{tr}(X \wedge Y), \quad \mathrm{tr}(\Omega_A(X, Y) F_A), \tag{6.2}$$

for fixed X, Y. Thus we can consider C as an element of $\Lambda^2(\mathcal{M}, \mathcal{D}^2)$, the space of two-current valued two-forms on $\mathcal{M}$.

Because these two-currents are $\mathrm{Ad}_\mathcal{G}$-invariant, the usual Chern-Weil proof shows that $C(\omega) = C_\omega$ is closed. (Its class is of course independent of the connection on $\mathcal{A}^*$, but we have a preferred connection.) C is built from $\mathrm{Ad}_\mathcal{G}$-invariant functions but only the first term in (6.2) comes from an invariant polynomial in $\mathrm{Lie}(\mathcal{G})^\mathcal{G}$. Nevertheless, we interpret (6.1) as a sum of "leading order currents" evaluated on ω.

Proposition 6.3. *For $a \in H_2(M^4, \mathbb{Q})$, a representative two-form for Donaldson's μ-invariant $\mu(a)$ is given by evaluating the leading order two-current*

$$\frac{1}{8\pi^2} \int_M \mathrm{tr}(X \wedge Y) \wedge \cdot + \frac{1}{2\pi^2} \int_M \mathrm{tr}(\Omega_A(X,Y)F_A) \wedge \cdot$$

on any two-form Poincaré dual to a.

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