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On the theory of generalized Hilbert transforms
Chapter IV
The generalized harmonic analysis in the complex domain
(2)

by

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ABSTRACT

Proceeding to the previous paper, we shall treat the theory of G.H.A. in the strip domain by the use of our theory of G.H.T.. The works of Professors R.E.A.C.Paley and N.Wiener are very ingenious, but those of ours are elementary and orthodox.

11. Generalized Harmonic Analysis in the strip domain.

In this section, we shall prove Theorems $P-W_2$ and $P-W_3$ by our method that had been expanded in the preceding sections.

Theorem D_5. Let us suppose that $f(z),(z=x+iy)$ is analytic in the strip domain $a<y<b$ and let us suppose that

$$\int_{-\infty}^{\infty} \frac{|f(x+iy)|^2}{1+x^2}dx = O(1), \quad unif., \quad (a<y<b).$$

Then we have the following properties.

(i) There exist boundary functions at $y=a$ and $y=b$. If we denote these $f(x,a)$ and $f(x,b)$ respectively, then we have

$$\lim_{y\to a+} f(x+iy) = f(x,a)$$

and

$$\lim_{y\to b-} f(x+iy) = f(x,b)$$

respectively

(ii) The $f(x,a)$ and $f(x,b)$ are both belong to the class W^2 and we have

$$\lim_{y\to a+}\int_{-\infty}^{\infty} \frac{|f(x+iy)-f(x,a)|^2}{1+x^2}dx = 0$$

and

$$\lim_{y\to b-}\int_{-\infty}^{\infty}\frac{|f(x+iy)-f(x,b)|^2}{1+x^2}dx=0$$

respectively.

(iii) The $f(z)$ can be represented as the difference of analytic functions in the upper half-plane and lower half-plane respectively. That is as follows

$$f(z)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t,a)}{t+i(a-c)}\frac{dt}{t+ia-z}-\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t,b)}{t+i(b-c)}\frac{dt}{t+ib-z}$$
$$=f^{+}(z,a)-f^{-}(z,b)$$

where $-c<a<b<c$ and that $f^{+}(z,a)$ belongs to the class H_1^2 in $y>a$ and $f^{-}(z,b)$ does to the class H_1^2 in $y<b$ respectively.

We shall call these the Generalized Cauchy Integral of order 1 (G.C.I. of order 1) and denote these as follows

$$f^{+}(z,a)=C_1(z;f^{+}),\quad f^{+}=f(t,a)$$

and

$$f^{-}(z,b)=C_1(z;f^{-}),\quad f^{-}=f(t,b)$$

respectively.

(c.f. S.Koizumi[9],Theorem12,pp112~114).

It should be remarked that we suppose the hypothesis of analytic function in the open strip domain $a<y<b$. Then we apply the results of Paley-Wienner in the closed strip domain $a+\varepsilon\le y\le b-\varepsilon$, with ε to be an arbitrary small positive number and then we apply the F.Riesz theorem(c.f. S.Banach[6], p.135) to the formula when tending ε to 0.

Now we shall prove the following theorems of spectral decomposition.

Theorem D_6. Let $f^{+}(z,a),(z=x+iy)$ be analytic in the upper half-plane $y>a$ and belongs to the class H_1^2. Let us denote by $f^{+}=f(x,a)$ its boundary $y=a$. Then we have for any given positive number ε

(i) if $|u|>\varepsilon$

$$s(u+\varepsilon;f^{+}(z,a))-s(u-\varepsilon;f^{+}(z,a))$$
$$=\frac{(1+sign\,u)}{2}e^{-(y-a)u}\left(\left\{s(u+\varepsilon;f^{+})-s(u-\varepsilon;f^{+})\right\}+r_0^{+}(u,y-a,\varepsilon;f^{+})\right)$$

and

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|r_0^{+}(u,y-a,\varepsilon;f^{+})|^2\,du=0$$

for all $y > a$ and

(ii) if $|u| \le \varepsilon$

$$s(u+\varepsilon; f^+(z,a)) - s(u-\varepsilon; f^+(z,a))$$

$$= ir_1^+(u+\varepsilon; f^+) + ir_2^+(u+\varepsilon; f^+) + r_3^+(u+\varepsilon; f^+)$$

where

$$r_1^+(u+\varepsilon; f^+) = \underset{B\to\infty}{l.i.m.} \frac{(a-c)}{\sqrt{2\pi}} \int_{-B}^{B} \frac{f(s,a)}{s+i(a-c)} \frac{e^{-i(u+\varepsilon)s}-1}{-is} ds,$$

$$r_2^+(u+\varepsilon; f^+) = \underset{B\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} \frac{f(s,a)}{s+i(a-c)} e^{-i(u+\varepsilon)s} ds$$

$$ir_1^+(u+\varepsilon; f^+) + ir_2^+(u+\varepsilon; f^+) = s(u+\varepsilon; f^+) - s(u-\varepsilon; f^+)$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_{|u|\le\varepsilon} | r_3^+(u+\varepsilon, y-a; f^+)|^2 du = 0$$

for all $y > a$

(c.f. S.Koizumi[9],Theorem13, pp.114~115).

The proof of Theorem D_6. It can be done by running on the same lines as that of Theorem D_3 and so we shall cease it to sketch only.

We have

$$f^+(z,a) = \frac{z-ic}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t,a)}{t+i(a-c)} \frac{dt}{t+ia-z}$$

and

$$s(u+\varepsilon; f^+(z,a)) - s(u-\varepsilon; f^+(z,a)) = \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f^+(z,a) \frac{2\sin\varepsilon t}{t} e^{-iut} dt,$$

where $z = x+iy,\ y > a$.

Let us set

$$f_B(t,a) = \begin{cases} f(t,a), & |t| \le B \\ 0, & |t| > B \end{cases}$$

and

$$f_B^+(z,a) = \frac{z-ic}{2\pi i} \int_{-\infty}^{\infty} \frac{f_B(t,a)}{t+i(a-c)} \frac{dt}{t+ia-z},$$

where $z = x+iy,\ y > a$.

Then we have

$$s(u+\varepsilon;f^+(z,a))-s(u-\varepsilon;f^+(z,a))=\underset{B\to\infty}{l.i.m.}\left\{s(u+\varepsilon;f_B^+(z,a))-s(u-\varepsilon;f_B^+(z,a))\right\}$$

and

$$s(u+\varepsilon;f_B^+(z,a))-s(u-\varepsilon;f_B^+(z,a))=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f_B^+(z,a)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt$$

$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\left(\frac{z-ic}{2\pi i}\int_{-B}^{B}\frac{f(s,a)}{s+i(a-c)}\frac{ds}{s+ia-z}\right)dt$$

where $z=x+iy,\ y>a$ and

$$\frac{1}{s+i(a-c)}\frac{z-ic}{s+ia-z}=\frac{1}{s+ia-z}-\frac{1}{s+i(a-c)}.$$

Then we have

$$s(u+\varepsilon;f^+(z,a))-s(u-\varepsilon;f^+(z,a))$$

$$=\underset{A\to\infty}{l.i.m.}\frac{1}{2\pi i}\int_{-B}^{B}f(s,a)\left(\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s+ia-z}dt\right)ds-\underset{A\to\infty}{l.i.m.}\frac{1}{2\pi i}\int_{-B}^{B}\frac{f(s,a)}{s+i(a-c)}\left(\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt\right)ds$$

$$=\frac{1}{2\pi i}\int_{-B}^{B}f(s,a)\left(\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s+ia-z}dt\right)ds-\frac{1}{2\pi i}\int_{-B}^{B}\frac{f(s,a)}{s+i(a-c)}\left(\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt\right)ds$$

By the Lemma A_9, we have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s+ia-z}dt$$

$$=\begin{cases}\sqrt{2\pi}\,i\dfrac{(1+\mathrm{sign}\,u)}{2}e^{-i(s-i(y-a))u}\dfrac{e^{i(s-i(y-a))\varepsilon}-e^{-i(s-i(y-a))\varepsilon}}{i(s-i(y-a))}, & (|u|>\varepsilon)\\[2ex] \sqrt{2\pi}\,ie^{-i(s-i(y-a))u}\dfrac{e^{i(s-i(y-a))u}-e^{-i(s-i(y-a))\varepsilon}}{i(s-i(y-a))}, & (-\varepsilon\le u\le\varepsilon)\end{cases}$$

and by the Lemma A_2, we have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt=\sqrt{2\pi}\chi_\varepsilon(u),$$

where $\chi_\varepsilon(u)$ is the characteristic function of interval $(-\varepsilon,\varepsilon)$.

Then we have the following estimations

(i) $|u|>\varepsilon$

$$s(u+\varepsilon;f^+(z,a))-s(u-\varepsilon;f^+(z,a))$$

$$= \frac{(1+signu)}{2} e^{-(y-a)u} \underset{B\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} f(s,a) \frac{e^{i(s-i(y-a))\varepsilon} - e^{-i(s-i(y-a))\varepsilon}}{i(s-i(y-a))} e^{-ius}\, ds,$$

where

$$\frac{e^{i(s-i(y-a))\varepsilon} - e^{-i(s-i(y-a))\varepsilon}}{i(s-i(y-a))}$$

$$= \frac{2\sin\varepsilon s}{2} + \frac{i(y-a)}{s-i(y-a)} \frac{2\sin\varepsilon s}{s} + \frac{e^{i\varepsilon s}(e^{\varepsilon(y-a)}-1)}{i(s-i(y-a))} + \frac{ie^{-i\varepsilon s}(e^{-\varepsilon(y-a)}-1)}{s-i(y-a)}$$

$$= \frac{2\sin\varepsilon s}{2} + K_{01}^{+}(s,y-a,\varepsilon) + K_{02}^{+}(s,y-a,\varepsilon) + K_{03}^{+}(s,y-a,\varepsilon)$$

Let us set

$$r_{0i}^{+}(u,y-a,\varepsilon;f^{+}) = \underset{B\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} f(s,a) K_{0i}^{+}(s,y-a,\varepsilon) e^{-ius}\, ds, \quad (i=1,2,3)$$

and

$$r_{0}^{+}(u,y-a,\varepsilon;f^{+}) = \sum_{i=1}^{3} r_{0i}^{+}(u,y-a,\varepsilon;f^{+}).$$

Then we have

$$\frac{1}{2\varepsilon} \int_{|u|\ge\varepsilon} |r_{0i}^{+}(u,y-a,\varepsilon;f^{+})|^2\, du = O((y-a)^2\varepsilon) \int_{-\infty}^{\infty} \frac{|f(s,a)|^2}{s^2+(y-a)^2} ds = o(1),$$

$(\varepsilon \to 0, y > a)$, $(i=1,2,3)$

Therefore we have

$$s(u+\varepsilon; f^{+}(z,a)) - s(u-\varepsilon; f^{+}(z,a))$$

$$= \frac{(1+signu)}{2} e^{-(y-a)u} \left(\left\{ s(u+\varepsilon;f^{+}) - s(u-\varepsilon;f^{+}) \right\} + r_{0}^{+}(u,y-a,\varepsilon;f^{+}) \right)$$

and

$$\frac{1}{2\varepsilon} \int_{|u|\ge\varepsilon} |r_{0}^{+}(u,y-a,\varepsilon;f^{+})|^2\, du = O((y-a)^2\varepsilon) \int_{-\infty}^{\infty} \frac{|f(s,a)|^2}{s^2+(y-a)^2} ds = o(1), \quad (\varepsilon \to 0)$$

for all $y > a$.

(ii) $|u| \le \varepsilon$

$$s(u+\varepsilon; f^{+}(z,a)) - s(u-\varepsilon; f^{+}(z,a))$$

$$= \underset{B\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} f(s,a) \left\{ \frac{e^{i(s-i(y-a))u} - e^{-i(s-i(y-a))\varepsilon}}{i(s-i(y-a))} e^{-i(s-i(y-a))u} + \frac{i}{(s-i(a-c))} \right\} ds$$

where

$$\{''\} = \frac{i(a-c)}{s+i(a-c)}\frac{e^{-i(u+\varepsilon)s}-1}{-is} + \frac{i}{s+i(a-c)}e^{-i(u+\varepsilon)s} + \left\{\frac{e^{-i(s-i(y-a))(u+\varepsilon)}-1}{-i(s-i(y-a))} - \frac{e^{-is(u+\varepsilon)}-1}{-is}\right\}$$

$$= iK_1^+(s,u+\varepsilon) + iK_2^+(s,u+\varepsilon) + K_3^+(s,u+\varepsilon,y-a), \quad say.$$

Let us set

$$r_1^+(u+\varepsilon;f^+) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} K_1^+(s,u+\varepsilon)f(s,a)ds = \underset{B\to\infty}{l.i.m.}\frac{(a-c)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,a)}{s+i(a-c)}\frac{e^{-i(u+\varepsilon)s}-1}{-is}ds$$

$$r_2^+(u+\varepsilon;f^+) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} K_2^+(s,u+\varepsilon)f(s,a)ds = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,a)}{s+i(a-c)}e^{-i(u+\varepsilon)s}ds$$

and

$$r_3^+(u+\varepsilon,y-a;f^+) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} K_3^+(s,u+\varepsilon,y-a)f(s,a)ds.$$

Then we have

$$s(u+\varepsilon;f^+(z,a)) - s(u-\varepsilon;f^+(z,a))$$

$$= ir_1^+(u+\varepsilon;f^+) + ir_2^+(u+\varepsilon;f^+) + r_3^+(u+\varepsilon,y-a;f^+).$$

and then we have to prove

$$ir_1^+(u+\varepsilon;f^+) + ir_2^+(u+\varepsilon;f^+) = s(u+\varepsilon;f^+) - s(u-\varepsilon;f^+).$$

Since $f^+ = f(x,a)$ is the boundary function of $f^+(z,a)$, we shall prove it by running on the same lines as the proof of Theorem D_3.

In the last we shall estimate the following remainder term

$$r_3^+(u+\varepsilon,y-a;f^+) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} K_3^+(s,u+\varepsilon,y-a)f(s,a)ds$$

where

$$K_3^+(s,u+\varepsilon,y-a) = \frac{e^{-i(s-i(y-a))(u+\varepsilon)}-1}{-i(s-i(y-a))}\frac{e^{-is(u+\varepsilon)}-1}{-is}$$

$$= \frac{(1-e^{-(y-a)(u+\varepsilon)})e^{-i(u+\varepsilon)s}}{i(s-i(y-a))} + \frac{(y-a)(e^{-(u+\varepsilon)s}-1)}{-is(s-i(y-a))}$$

$$= K_{31}^+(s,u+\varepsilon,y-a) + K_{32}^+(s,u+\varepsilon,y-a), \quad say.$$

Let us set

$$r_{3i}^+(u+\varepsilon,y-a;f^+) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B} K_{3i}^+(s,u+\varepsilon,y-a)f(s,a)ds, \quad (i=1,2).$$

Then we have

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon} |r_{31}^{+}(u+\varepsilon,y-a;f^{+})|^2\,du \le \frac{(1-e^{-(y-a)\varepsilon})}{2\varepsilon}\int_{-\infty}^{\infty}\frac{|f(s,a)|^2}{s^2+(y-a)^2}ds$$

$$=O((y-a)^2\varepsilon)\int_{-\infty}^{\infty}\frac{|f(s,a)|^2}{s^2+(y-a)^2}ds=o(1),\quad (\varepsilon\to 0,\, y>a).$$

Next let us set

$$r_{32}^{+}(u+\varepsilon,y-a;f^{+}) = \underset{B\to\infty}{l.i.m.}\frac{(y-a)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{e^{-i(u+\varepsilon)s}-1}{-s(s-i(y-a))}f(s,a)\,ds$$

$$= i(y-a)\int_{0}^{u+\varepsilon}dv\left(\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,a)}{s-i(y-a)}e^{-ivs}\,ds\right),$$

and

$$\hat{f}(v,a) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,a)}{s-i(y-a)}e^{-ivs}\,ds\,.$$

Then we shall have

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon} |r_{32}^{+}(u+\varepsilon,y-a;f^{+})|^2\,du \le \frac{(y-a)^2}{2\varepsilon}\int_{|u|\le\varepsilon}\left|\int_{0}^{u+\varepsilon}\hat{f}(v,a)\,dv\right|^2 du$$

$$\le \frac{(y-a)^2}{2\varepsilon}\int_{|u|\le\varepsilon}\left(\int_{0}^{u+\varepsilon}|\hat{f}(v,a)|^2\,dv\right)\left(\int_{0}^{u+\varepsilon}dv\right)du$$

$$=O((y-a)^2\varepsilon\int_{-\infty}^{\infty}\frac{|f(s,a)|^2}{s^2+(y-a)^2}ds=o(1),\quad (\varepsilon\to 0),$$

for all $y>a$.
Thus we have proved Theorem D_6.

Theorem D_7. Let $f^{-}(z,b),(z=x+iy)$ be analytic in the lower half-plane $y<b$ and belongs to the class H_1^2. Let us denote by $f^{-}=f(x,b)$ its boundary function at $x=b$. Then we have for any given positive number ε

(i) if $|u|>\varepsilon$

$$s(u+\varepsilon;f^{-}(z,b))-s(u-\varepsilon;f^{-}(z,b))$$
$$=(-1)\frac{(1-sign\,u)}{2}e^{-(y-b)u}\left(\{s(u+\varepsilon;f^{-})-s(u-\varepsilon;f^{-})\}+r_0^{-}(u,y-b,\varepsilon;f^{-})\right)$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\geq\varepsilon} |r_0^-(u,y-b,\varepsilon;f^-)|^2\,du=0$$

for all $y<b$.

(ii) if $|u|\leq\varepsilon$

$$s(u+\varepsilon;f^-(z,b))-s(u-\varepsilon;f^-(z,b))$$

$$=ir_1^-(u-\varepsilon;f^-)+ir_2^-(u-\varepsilon;f^-)+r_3^-(u-\varepsilon,y-b;f^-)$$

where

$$r_1^-(u-\varepsilon;f^-)=\underset{B\to\infty}{l.i.m.}\frac{(b-c)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s+i(b-c)}\frac{e^{-i(u-\varepsilon)s}-1}{-is}ds$$

$$r_2^-(u-\varepsilon;f^-)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s+i(b-c)}e^{-i(u-\varepsilon)s}\,ds$$

$$ir_1^-(u-\varepsilon;f^-)+ir_2^-(u-\varepsilon;f^-)=s(u+\varepsilon;f^-)-s(u-\varepsilon;f^-)$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{|u|\leq\varepsilon} |r_3^-(u-\varepsilon,y-b;f^-)|^2\,du=0$$

for all $y<b$.

(c.f. S.Koizumi[9], Theorem14, pp.115~116).

Proof of Theorem D_7. We shall prove it by running on the same lines as that of Theorem D_6, but we have to consider it in the lower half-plane. Therefore we shall prove it for the sake of completeness.

Let us set

$$f_B(t,b)=\begin{cases} f(t,b), & |t|\leq B \\ 0, & |t|>B \end{cases}$$

and

$$f_B^-(z,b)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f_B(t,b)}{t+i(b-c)}\frac{dt}{t+ib-z}.$$

Then we have

$$s(u+\varepsilon;f^-(z,b))-s(u-\varepsilon;f^-(z,b))=\underset{B\to\infty}{l.i.m.}\left\{s(u+\varepsilon;f_B^-(z,b))-s(u-\varepsilon;f_B^-(z,b))\right\}$$

and

$$s(u+\varepsilon; f_B^-(z,b)) - s(u-\varepsilon; f_B^-(z,b)) = \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f_B^-(z,b)\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt$$

$$= \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\left(\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f_B(t,b)}{t+i(b-c)}\frac{dt}{t+ib-z}\right)dt$$

where $z = t+iy,\ y<b$.

From the formula

$$\frac{z-ic}{s+ib-z}+1 = \frac{s+i(b-c)}{s+ib-z}$$

It follows that

$$\frac{z-ic}{s+ib-z} = \frac{s+i(b-c)}{s+ib-z}-1$$

and so we have

$$\frac{1}{s+i(b-c)}\frac{z-ic}{s+ib-z} = \frac{1}{s+ib-z}-\frac{1}{s+i(b-c)}.$$

Therefor we have

$$s(u+\varepsilon; f_B^-(z,b)) - s(u-\varepsilon; f_B^-(z,b))$$

$$= \underset{A\to\infty}{l.i.m.}\frac{1}{2\pi i}\int_{-B}^{B} f(s,b)ds\left(\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s+ib-z}dt\right) - \underset{A\to\infty}{l.i.m.}\frac{1}{2\pi i}\int_{-B}^{B}\frac{f(s,b)}{s+i(b-c)}ds\left(\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt\right)$$

$$= \frac{1}{2\pi i}\int_{-B}^{B} f(s,b)ds\left(\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s+ib-z}dt\right) - \frac{1}{2\pi i}\int_{-B}^{B}\frac{f(s,b)}{s+i(b-c)}ds\left(\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}\,dt\right)$$

For estimations of the inner integral of first part of above formula, we shall quote the Lemma A_9 as for lower half-plane so we shall state it as follows.

Lemma A_9'. We have

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt, \quad (z = t+iy,\ y<0)$$

$$= \begin{cases} 0, & (u>\varepsilon) \\ \sqrt{2\pi}\, ie^{-i(s-iy)u}\dfrac{e^{i(s-iy)u}-e^{i(s-iy)\varepsilon}}{i(s-iy)}, & (-\varepsilon<u<\varepsilon) \\ -\sqrt{2\pi}\, ie^{-i(s-iy)u}\dfrac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, & (u<-\varepsilon) \end{cases}$$

Now we shall apply Lemma A_9' to the following integral

$$I = \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} \frac{e^{-iut}}{s+ib-z} dt, \quad (z = t+iy,\ y<b)$$

then we have

$$I = \begin{cases} 0, & (u>\varepsilon) \\ \sqrt{2\pi}\, ie^{-i(s-i(y-b))u} \dfrac{e^{i(s-i(y-b))u} - e^{i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}, & (-\varepsilon<u<\varepsilon) \\ -\sqrt{2\pi}\, ie^{-i(s-i(y-b))u} \dfrac{e^{i(s-i(y-b))\varepsilon} - e^{-i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}, & (u<-\varepsilon) \end{cases}$$

and by the Lemma A_2, we have

$$J = \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} e^{-iut}\, dt = \sqrt{2\pi}\chi_\varepsilon(u),$$

where $\chi_\varepsilon(u)$ to be the characteristic function of interval $(-\varepsilon, \varepsilon)$.

(i) $|u| \geq \varepsilon$

We have

$$I = -\sqrt{2\pi}\, i \frac{(1-\operatorname{sign} u)}{2} e^{-i(s-i(y-b))u} \frac{e^{i)s-i(y-b))\varepsilon} - e^{-i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}$$

and

$$J = 0.$$

Therefor we have

$$s(u+\varepsilon; f^-(z,b)) - s(u-\varepsilon; f^-(z,b))$$
$$= -\frac{(1-\operatorname{sign} u)}{2} e^{-(y-b)u} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} f(s,b) \frac{e^{i(s-i(y-b))\varepsilon} - e^{-i(s-i(y-b))\varepsilon}}{i(s-i(y-b))} e^{-us}\, ds$$

where

$$\frac{e^{i(s-i(y-b))\varepsilon} - e^{-i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}$$
$$= \frac{2\sin\varepsilon s}{s} + \frac{i(y-b)}{s-i(y-b} \frac{2\sin\varepsilon s}{s} + \frac{e^{i\varepsilon s}(e^{\varepsilon(y-b)}-1)}{i(s-i(y-b))} + \frac{ie^{-i\varepsilon s}(e^{-\varepsilon(y-b)}-1)}{s-i(y-b)}$$
$$= \frac{2\sin\varepsilon s}{s} + K_{01}^-(s, y-b, \varepsilon) + K_{02}^-(s, y-b, \varepsilon) + K_{03}^-(s, y-b, \varepsilon), \quad say.$$

Let us set

$$r_{0i}^{-}(u,y-b,\varepsilon;f^{-})=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s,b)K_{0i}^{-}(s,y-b,\varepsilon)e^{-ius}\,ds,\quad (i=1,2,3),$$

then we have

$$\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|r_{0i}^{-}(u,y-b,\varepsilon;f^{-})|^{2}\,du=O((y-b)^{2}\varepsilon)\int_{-\infty}^{\infty}\frac{|f(s,b)|^{2}}{s^{2}+(y-b)^{2}}ds=o(1),\quad (\varepsilon\to 0),$$

$(i=1,2,3)$ and let us write

$$r_{0}^{-}(u,y-b,\varepsilon;f^{-})=\sum_{i=1}^{3}r_{0i}^{-}(u,y-b,\varepsilon;f^{-})\ .$$

Therefore we have

$$s(u+\varepsilon;f^{-}(z,b))-s(u-\varepsilon;f^{-}(z,b))$$

$$=-\frac{(1-signu)}{2}e^{-(y-b)u}\left(\left\{s(u+\varepsilon;f^{-})-s(u-\varepsilon;f^{-})\right\}+r_{0}^{-}(u,y-b,\varepsilon;f^{-})\right)$$

and

$$\frac{1}{2\varepsilon}\int_{|u|>\varepsilon}|r_{0}^{-}(u,y-b,\varepsilon;f^{-})|^{2}\,du=O((y-b)^{2}\varepsilon)\int_{-\infty}^{\infty}\frac{|f(s,b)|^{2}}{s^{2}+(y-b)^{2}}ds=o(1),\quad (\varepsilon\to 0),$$

for all $y<b$.

(ii) $|u|\le\varepsilon$

We have

$$I=\sqrt{2\pi}\,ie^{-i(s-i(y-b))u}\frac{e^{i(s-i(y-b))u}-e^{i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}$$

and

$$J=\sqrt{2\pi}\chi_{\varepsilon}(u).$$

Therefore we have

$$s(u+\varepsilon;f_{B}^{-}(z,b))-s(u-\varepsilon;f_{B}^{-}(z,b))$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s,b)\frac{e^{i(s-i(y-b))u}-e^{i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}e^{-i(s-i(y-b))u}\,ds-\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s,b)\frac{ds}{i(s+i(b-c))}$$

where

$$\frac{e^{i(s-i(y-b))u}-e^{i(s-i(y-b))\varepsilon}}{i(s-i(y-b))}e^{-i(s-i(y-b))u}-\frac{1}{i(s+i(b-c))}$$

$$=\frac{i(b-c)}{s+i(b-c)}\frac{e^{-i(u-\varepsilon)s}-1}{-is}+\frac{i}{s+i(b-c)}e^{-i(u-\varepsilon)s}+\left\{\frac{e^{-i(s-i(y-b))(u-\varepsilon)}-1}{-i(s-i(y-b))}-\frac{e^{-is(u-\varepsilon)}-1}{-is}\right\}$$

$$=iK_{1}^{-}(s,u-\varepsilon)+iK_{2}^{-}(s,u-\varepsilon)+K_{3}^{-}(s,u-\varepsilon,y-b),\quad say.$$

Let us set

$$r_1^-(u-\varepsilon;f^-)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}K_1^-(s,u-\varepsilon)f(s,b)ds$$

$$=\underset{B\to\infty}{l.i.m.}\frac{(b-c)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s+i(b-c)}\frac{e^{-i(u-\varepsilon)s}-1}{-is}ds$$

$$r_2^-(u-\varepsilon;f^-)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}K_2^-(s,u-\varepsilon)f(s,b)ds$$

$$=\underset{B\to\infty}{l.i.m.}\frac{(b-c)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s+i(b-c)}e^{-i(u-\varepsilon)s}ds$$

and

$$r_3^-(u-\varepsilon,y-b;f^-)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}K_3^-(s,u-\varepsilon,y-b)f(s,b)ds\,.$$

Then we have

$$s(u+\varepsilon;f^-(z,b))-s(u-\varepsilon;f^-(z,b))$$

$$=ir_1^-(u-\varepsilon;f^-)+ir_2^-(u-\varepsilon;f^-)+r_3^-(u-\varepsilon;y-b;f^-)\,.$$

By just the same arguments as Theorem D_6, we have

$$ir_1^-(u-\varepsilon;f^-)+ir_2^-(u-\varepsilon;f^-)=s(u+\varepsilon;f^-)-s(u-\varepsilon;f^-)\,.$$

Next we shall estimate the term $r_3^-(u-\varepsilon;y-b;f^-)$. We have

$$K_3^-(s,u-\varepsilon,y-b)=\frac{e^{-i(s-i(y-b))(u-\varepsilon)}-1}{-i(s-i(y-b))}-\frac{e^{-is(u-\varepsilon)}-1}{-is}$$

$$=\frac{(1-e^{-(y-b)(u-\varepsilon)})}{i(s-i(y-b))}+\frac{(y-b(e^{-is(u-\varepsilon)}-1)}{-s(s-i(y-b))}$$

$$=K_{31}^-(s,u-\varepsilon,y-b)+K_{32}^-(s,u-\varepsilon,y-b),\quad say.$$

Let us set

$$r_{3i}^-(u-\varepsilon,y-b;f^-)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}K_{3i}^-(s,u-\varepsilon,y-b)f(s,b)ds,\quad (i=1,2)\,.$$

We have

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_{31}^-(u-\varepsilon,y-b;f^-)|^2du\le\frac{(1-e^{(y-b)\varepsilon})^2}{2\varepsilon}\int_{-\infty}^{\infty}\frac{|f(s,b)|^2}{s^2+(y-b)^2}ds$$

$$=O((y-b)^2\varepsilon)\int_{-\infty}^{\infty}\frac{|f(s,b)|^2}{s^2+(y-b)^2}ds=o(1),\quad (\varepsilon\to 0)\,,$$

for all $y<b$.

We have also

$$r_{32}^{-}(u-\varepsilon, y-b; f^{-}) = \underset{B\to\infty}{l.i.m.}\frac{i(y-b)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s-i(y-b)}\frac{e^{-i(u-\varepsilon)s}-1}{-is}ds$$

$$= \underset{B\to\infty}{l.i.m.}\frac{i(y-b)}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s-i(y-b)}\left(\int_{0}^{u-\varepsilon}e^{-ivs}\,dv\right)ds$$

$$= i(y-b)\int_{0}^{u-\varepsilon}\left(\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s-i(y-b)}e^{-ivs}\,ds\right)dv$$

and so we have

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_{32}^{-}(u-\varepsilon, y-b; f^{-})|^2\,du \le \frac{(y-b)^2}{2\varepsilon}\int_{|u|\le\varepsilon}|\int_{0}^{u-\varepsilon}\hat{f}(v,b)\,dv|^2\,du$$

where

$$\hat{f}(v,b) = \underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s,b)}{s-i(y-b)}e^{-ivs}\,ds$$

and

$$|\int_{0}^{u-\varepsilon}\hat{f}(v,b)\,dv|^2 \le |\int_{0}^{u-\varepsilon}|\hat{f}(v,b)|^2\,dv\int_{0}^{u-\varepsilon}dv| \le 2\varepsilon\int_{|u|\le\varepsilon}|\hat{f}(v,b)|^2\,dv.$$

Therefore we have

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_{32}^{-}(u-\varepsilon, y-b; f^{-})|^2\,du$$

$$\le (y-b)^2\int_{|u|\le\varepsilon}\left(\int_{|u|\le\varepsilon}|\hat{f}(v,b)|^2\,dv\right)du = O((y-b)^2\varepsilon)\int_{-\infty}^{\infty}|\hat{f}(v,b)|^2\,dv$$

$$= O((y-b)^2\varepsilon)\int_{-\infty}^{\infty}\frac{|f(s,b)|^2}{s^2+(y-b)^2}ds = o(1), \quad (\varepsilon\to 0)$$

for all $y<b$.

Thus we have proved

$$\frac{1}{2\varepsilon}\int_{|u|\le\varepsilon}|r_{3}^{-}(u-\varepsilon, y-b; f^{-})|^2\,du$$

$$= O((y-b)^2\varepsilon)\int_{-\infty}^{\infty}\frac{|f(s,b)|^2}{s^2+(y-b)^2}ds = o(1), \quad (\varepsilon\to 0)$$

for all $y<b$.

Now let us combine the results of Theorem D_6 and Theorem D_7, then we shall obtain the theorem of spectral decomposition of $f(z)=f^{+}(z,a)-f^{-}(z,b)$ in the strip domain $z=x+iy,\ a<y<b$.

Theorem D_8. Let us suppose that $f(z),(z=x+iy,\ a<y<b)$ satisfy the same hypothesis as Theorem D_5. Then we have the spectral decomposition of $f(z)$ as follows.

(i) $|u|>\varepsilon$

We have

$$s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))$$
$$=\frac{(1+sign u)}{2}e^{-(y-a)u}\left(\left\{s(u+\varepsilon;f^{+})-s(u-\varepsilon;f^{+})\right\}+r_0^{+}(u,y-a,\varepsilon;f^{+})\right)$$
$$+\frac{(1-sign u)}{2}e^{-(y-b)u}\left(\left\{s(u+\varepsilon;f^{-})-s(u-\varepsilon;f^{-})\right\}+r_0^{-}(u,y-b,\varepsilon;f^{-})\right)$$

where

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{|u|\geq\varepsilon}|r_0^{+}(u,y-a,\varepsilon;f^{+})|^2\,du=0$$

and

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{|u|\geq\varepsilon}|r_0^{-}(u,y-b,\varepsilon;f^{-})|^2\,du=0$$

for all $y\ (a<y<b,\ -c<a<b<c)$.

(ii) $|u|\leq\varepsilon$

We have

$$s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))$$
$$=\left\{s(u+\varepsilon;f^{+})-s(u-\varepsilon;f^{+})\right\}-\left\{s(u+\varepsilon;f^{-})-s(u-\varepsilon;f^{-})\right\}$$
$$+r_3^{+}(u,y-a,\varepsilon;f^{+})-r_3^{-}(u,y-b,\varepsilon;f^{-})$$

where

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{|u|\leq\varepsilon}|r_3^{+}(u,y-a,\varepsilon;f^{+})|^2\,du=0$$

and

$$\lim_{\varepsilon\to0}\frac{1}{2\varepsilon}\int_{|u|\leq\varepsilon}|r_3^{-}(u,y-b,\varepsilon;f^{-})|^2\,du=0$$

for all $y\ (a<y<b,\ -c<a<b<c)$.

Now we shall intend to prove the Paley-Wiener theorem [2] (c.f. III, Theorem $P-W_2$), but at present with some additional conditions.

Theorem D_9 (Paley-Wiener). Let $f(z)$ be analytic function of complex variable $z = x+iy,\ a<y<b$ and let

$$\int_{-A}^{A} |f(x+iy)|^2 dx = o(A), unif., \quad (a<y<b).$$

Let its boundary function $f(x,a)$ and $f(x,b)$ both belong to the class S as a function of x for all y in $a<y<b$.

Let us suppose that following limits

$$\begin{cases} (C_0^+) & \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x,a)dx = c_0^+ \\ (C_0^-) & \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x,b)dx = c_0^- \end{cases}$$

exist respectively. Let us also suppose that the following relations between f and its auto-correlation φ

$$\begin{cases} (L_0^+) & \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x,a)dx = \left| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x,a)dx \right|^2 \\ (L_0^-) & \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x,b)dx = \left| \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x,b)dx \right|^2 \end{cases}$$

are satisfied respectively.

Then $f(z)$ belongs to the class S' as a function of x in the strip domain $z = x+iy$, $a<y<b$.

Proof of Theorem D_9. First of all, we should remark that R.Paley-N.Wiener, they proved this theorem without any additional condition. But at present we need the additional conditions for some reasons. Now, we shall denote its boundary functions for the sake of simplicity as follows

$$f^+ = f(x,a) \quad and \quad f^- = f(x,b)$$

and also

$$\varphi^+ = \varphi(x,a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t,a)\overline{f(t,a)}dt$$

$$\varphi^- = \varphi(x,b) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t,b)\overline{f(t,b)}dt$$

respectively.

Then we have by the condition $(C_0^{\pm})$ and the so-called One- Sided Wiener formula similarly as before

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f^{\pm}(x)\,dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right\}du=c_0^{\pm}$$

and so we have

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left|\left\{s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right\}-\sqrt{2\pi}c_0^{\pm}\right|^2du$$

$$=\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left|s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right|^2du$$

$$-\frac{\sqrt{2\pi}\,\overline{c_0^{\pm}}}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right\}du$$

$$-\frac{\sqrt{2\pi}\,c_0^{\pm}}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\overline{\left\{s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right\}}du+\sqrt{2\pi}|c_0^{\pm}|^2,$$

and then we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left|\left\{s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right\}-\sqrt{2\pi}\,c_0^{\pm}\right|^2du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left|s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right|^2du-\sqrt{2\pi}|c_0^{\pm}|^2.$$

Next we shall denote

$$\varphi^{\pm}(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f^{\pm}(x+t)\overline{f^{\pm}(t)}\,dt$$

and by $\sigma^{\pm}(u)$ its Generalized Fourier Transform respectively. Let us remark that hypothesis $(L_0^{\pm})$ guarantee the existence of limits of stated formulas and so we have by the One- Sided Wiener formula too

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varphi^{\pm}(x)\,dx=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{\sigma^{\pm}(u+\varepsilon)-\sigma^{\pm}(u-\varepsilon)\right\}du$$

Since $\sigma^{\pm}(u)$ is a function of bounded and monotone increasing, we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left\{\sigma^{\pm}(u+\varepsilon)-\sigma^{\pm}(u-\varepsilon)\right\}du=\frac{\sigma^{\pm}(0+)-\sigma^{\pm}(0-)}{\sqrt{2\pi}}.$$

Then we have by the condition $(L_0^{\pm})$

$$\sigma^{\pm}(0+)-\sigma^{\pm}(0-)=\sqrt{2\pi}\,|\,c_0^{\pm}\,|^2.$$

Here by the Lemma D_4 (c.f. III, p. 60) we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}|\,s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\,|^2\,du=\sigma^{\pm}(0+)-\sigma^{\pm}(0-).$$

Therefore we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\left|\left\{s(u+\varepsilon;f^{\pm})-s(u-\varepsilon;f^{\pm})\right\}-\sqrt{2\pi}\,c_0^{\pm}\right|^2du=0.$$

Then we have by the part (ii) $|\,u\,|\le\varepsilon$ of Theorem D_8

$$s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))$$

$$=\left\{s(u+\varepsilon;f^{+})-s(u-\varepsilon;f^{+})-\sqrt{2\pi}\,c_0^{+}\right\}-\left\{s(u+\varepsilon;f^{-})-s(u-\varepsilon;f^{-})-\sqrt{2\pi}\,c_0^{-}\right\}$$

$$+\sqrt{2\pi}\,(c_0^{+}-c_0^{-})+r_3^{+}(u,y-a,\varepsilon;f^{+})-r_3^{-}(u,y-b,\varepsilon;f^{-})$$

and applying the Minkowski inequality (c.f. II, p. 26), we have

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}|\,s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))\,|^2\,du=\sqrt{2\pi}\,|\,c_0^{+}-c_0^{-}\,|^2+o(1),\quad(\varepsilon\to 0).$$

Since $e^{iux}-1=O(\varepsilon x),\ (|\,u\,|\le\varepsilon,\forall x)$, we have

$$\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}e^{iux}\,|\,s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))\,|^2\,du$$

$$=\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}|\,s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))\,|^2\,du+o(1),\ (\varepsilon\to 0).$$

Therefore we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}e^{iux}\,|\,s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))\,|^2\,du=\sqrt{2\pi}\,|\,c_0^{+}-c_0^{-}\,|^2,$$

where $z = x + iy, \quad a < y < b$.

Next we shall estimate $s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))$ on the interval (ε, ∞). From the part (i) $|u| > \varepsilon$ of Theorem D_8, we have

$$\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux} \, |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2 \, du$$

$$\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux} \left| e^{-(y-a)u} \left\{ s(u+\varepsilon; f^+) - s(u-\varepsilon; f^+) \right\} \right|^2 du + o(1), \quad (\varepsilon \to 0).$$

We shall follow the same lines as the proof of Theorem D_4 (c.f. III, Theorem D_3, pp. 60~62). Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux} \, |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2 \, du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux} \left| e^{-(y-a)u} \left\{ s(u+\varepsilon; f^+) - s(u-\varepsilon; f^+) \right\} \right|^2 du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\left(\int_{\varepsilon}^{A} ('')\,du + \int_{A}^{\infty} ('')\,du \right) = \lim_{\varepsilon\to 0}(I_1^+ + I_2^+), \quad say.$$

We have

$$|I_2^+| \le \frac{e^{-(y-a)A}}{4\pi\varepsilon}\int_{A}^{\infty} |s(u+\varepsilon; f^+) - s(u-\varepsilon; f^+)|^2 \, du$$

and

$$\overline{\lim_{\varepsilon\to 0}} |I_2^+| \le e^{-2(y-a)A} \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon; f^+) - s(u-\varepsilon; f^+)|^2 \, du$$

$$= e^{-2(y-a)A} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(x,a)|^2 \, dx \to 0, \quad (A \to \infty).$$

Now, for the A sufficiently large and to be fixed, we have by the integration by part

$$I_1^+ = \frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{A} e^{iux} \left| e^{-(y-a)u} \left\{ s(u+\varepsilon; f^+) - s(u-\varepsilon; f^+) \right\} \right|^2 du$$

$$= \left[\frac{e^{-(2(y-a)-ix)u}}{4\pi\varepsilon}\int_{\varepsilon}^{u} |s(v+\varepsilon; f^+) - s(v-\varepsilon; f^+)|^2 \, dv \right]_{u=\varepsilon}^{u=A}$$

$$-\frac{-(2(y-a)-ix)}{4\pi\varepsilon}\int_{\varepsilon}^{A} e^{-(2(y-a)-ix)u}\left(\int_{\varepsilon}^{u} | s(v+\varepsilon; f^{+})-s(v-\varepsilon; f^{+})|^{2}\, dv\right)du$$

$$=\frac{e^{-(2(y-a)-ix)A}}{4\pi\varepsilon}\int_{\varepsilon}^{A} | s(v+\varepsilon; f^{+})-s(v-\varepsilon; f^{+})|^{2}\, dv$$

$$+\frac{2(y-a)-ix}{2\pi}\int_{\varepsilon}^{A} e^{-(2(y-a)-ix)u}\left(\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\varepsilon}^{u} | s(v+\varepsilon; f^{+})-s(v-\varepsilon; f^{+})|^{2}\, dv\right)du .$$

Since we have from Lemma D_4,

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\varepsilon}^{u} | s(v+\varepsilon; f^{+})-s(v-\varepsilon; f^{+})|^{2}\, dv=\sigma^{+}(u)-\sigma^{+}(0+), \quad a,e.\, u$$

and its bounded convergence is guaranteed on any finite range of u. Thus we have

$$\lim_{\varepsilon\to 0} I_1^{+}=\frac{e^{-(2(y-a)-ix)A}}{\sqrt{2\pi}}\left(\sigma^{+}(A)-\sigma^{+}(0+)\right)$$

$$+\frac{2(y-a)-ix}{\sqrt{2\pi}}\int_{0}^{A}\left(\sigma^{+}(u)-\sigma^{+}(0+)\right)e^{-(2(y-a)-ix)u}\, du$$

$$\to\frac{2(y-a)-ix}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\sigma^{+}(u)-\sigma^{+}(0+)\right)e^{-(2(y-a)-ix)u}\, du, \quad (A\to\infty)$$

Therefore we shall conclude that there exists the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} e^{iux}\, | s(u+\varepsilon; f(z))-s(u-\varepsilon; f(z))|^{2}\, du$$

$$=\frac{2(y-a)-ix}{4\pi\varepsilon}\int_{0}^{\infty}\left(\sigma^{+}(u)-\sigma^{+}(0+)\right)e^{-(2(y-a)-ix)u}\, du$$

for all x. In particular, if we put $x=0$ in the above formula, we shall obtain the following formula

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\varepsilon}^{\infty} | s(u+\varepsilon; f(z))-s(u-\varepsilon; f(z))|^{2}\, du .$$

$$=\frac{2(y-a)}{4\pi\varepsilon}\int_{0}^{\infty}\left(\sigma^{+}(u)-\sigma^{+}(0+)\right)e^{-2(y-a)u}\, du .$$

Similarly we hall estimate the part of $s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))$ on the interval $(-\infty,\ \varepsilon)$. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{-\varepsilon} e^{iux}\,|\,s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))\,|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{-\varepsilon} e^{iux}\left|e^{-(y-b)u}\left\{s(u+\varepsilon;f^-)-s(u-\varepsilon;f^-)\right\}\right|^2 du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\left(\int_{-\infty}^{-A}('')\,du+\int_{-A}^{-\varepsilon}('')\,du\right)=\lim_{\varepsilon\to 0}(I_1^-+I_2^-),\quad say.$$

We have

$$|\,I_1^-\,|\le\frac{e^{2(y-b)A}}{4\pi\varepsilon}\int_{-\infty}^{-A}|\,s(u+\varepsilon;f^-)-s(u-\varepsilon;f^-)\,|^2\,du$$

and then

$$\overline{\lim_{\varepsilon\to 0}}\,|\,I_1^-\,|\le e^{2(y-b)A}\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{-\infty}|\,s(u+\varepsilon;f^-)-s(u-\varepsilon;f^-)\,|^2\,du$$

$$=e^{2(y-b)A}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f(x,b)\,|^2\,dx\to 0,\quad (A\to\infty).$$

Now, for the A sufficiently large and to be fixed, we have by the integration by part

$$I_2^-=\frac{1}{4\pi\varepsilon}\int_{-A}^{-\varepsilon} e^{iux}\left|e^{-(y-b)u}\left\{s(u+\varepsilon;f^-)-s(u-\varepsilon;f^-)\right\}\right|^2 du$$

$$=\left[-\frac{e^{-(2(y-b)-ix)u}}{4\pi\varepsilon}\int_{u}^{-\varepsilon}|\,s(v+\varepsilon;f^-)-s(v-\varepsilon;f^-)\,|^2\,dv\right]_{u=-A}^{u=-\varepsilon}$$

$$+\frac{-2(y-b)+ix}{4\pi\varepsilon}\int_{-A}^{-\varepsilon} e^{-(2(y-b)-ix)u}\left(\int_{u}^{-\varepsilon}|\,s(v+\varepsilon;f^-)-s(v-\varepsilon;f^-)\,|^2\,dv\right)du$$

$$=\frac{e^{(2(y-b)-ix)A}}{\sqrt{2\pi}}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-A}^{-\varepsilon}|\,s(v+\varepsilon;f^-)-s(v-\varepsilon;f^-)\,|^2\,dv$$

$$+\frac{-2(y-b)+ix}{\sqrt{2\pi}}\int_{-A}^{-\varepsilon} e^{-(2(y-b)-ix)u}\left(\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{u}^{-\varepsilon}|\,s(v+\varepsilon;f^-)-s(v-\varepsilon;f^-)\,|^2\,dv\right)du$$

By the Lemma D_4, we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{u}^{-\varepsilon}|\,s(v+\varepsilon;f^-)-s(v-\varepsilon;f^-)\,|^2\,dv=-\left(\sigma^-(u)-\sigma^-(0-)\right),\quad a.e.\,u$$

and its bounded convergence is guaranteed on any finite range of u. Thus we have

$$\lim_{\varepsilon\to 0} I_2^- = -\frac{e^{(2(y-b)-ix)A}}{\sqrt{2\pi}}\left(\sigma^-(-A)-\sigma^-(0-)\right)$$
$$-\frac{-2(y-b)+ix}{\sqrt{2\pi}}\int_{-A}^{-\varepsilon}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-(2(y-b)-ix)u}\,du$$
$$\to \frac{2(y-b)-ix}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-(2(y-b)-ix)u}\,du, \quad (A\to\infty),$$

for all x. Therefore we shall conclude that the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{-\varepsilon} e^{iux}\,|\,s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))\,|^2\,du$$
$$=\frac{2(y-b)-ix}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-(2(y-b)-ix)u}\,du$$

exists for all x. In particular, if we put $x=0$ in this formula, then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{-\varepsilon} |\,s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))\,|^2\,du$$
$$=\frac{2(y-b)}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-2(y-b)u}\,du.$$

Summing up these estimations above, we shall prove that the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|\,s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))\,|^2\,du$$

exists and equals to

$$\frac{2(y-a)-ix}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\sigma^+(u)-\sigma^+(0+)\right)e^{-(2(y-a)-ix)u}\,du$$
$$+\frac{2(y-b)-ix}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-(2(y-b)-ix)u}\,du+\sqrt{2\pi}\,|\,c_0^+-c_0^-\,|^2,$$

for all x and y in $a<y<b$. Thus we have proved that $f(z)$ belongs to the class S. In particular if we put $x=0$ in this formula above,then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |\,s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))\,|^2\,du$$
$$=\frac{2(y-a)}{\sqrt{2\pi}}\int_{0}^{\infty}\left(\sigma^+(u)-\sigma^+(0+)\right)e^{-2(y-a)u}\,du$$
$$+\frac{2(y-b)}{\sqrt{2\pi}}\int_{-\infty}^{0}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-2(y-b)u}\,du+\sqrt{2\pi}\,|\,c_0^+-c_0^-\,|^2.$$

Since we have

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))|^2\,du$$

$$\le \lim_{A\to\infty}\frac{2(y-a)}{\sqrt{2\pi}}\int_{A}^{\infty}\left(\sigma^+(u)-\sigma^+(0+)\right)e^{-2(y-a)u}\,du$$

$$+\lim_{A\to\infty}\frac{2(y-b)}{\sqrt{2\pi}}\int_{-\infty}^{-A}\left(\sigma^-(u)-\sigma^-(0-)\right)e^{-2(y-b)u}\,du$$

$$=0.$$

Therefore we have proved $f(z)$ belongs to the class S' as a function of x for all y in $a<y<b$, by the N.Wiener Theorem (c.f. [1], p. 160, or II, Theorem W_3, pp.28~29).

12. Application to the almost periodic functions.

As well as III, we shall give an example .

Let $f(z)$ be an analytic function of complex variable $z=x+iy,\ a<y<b$ and let

$$\int_{-A}^{A}|f(x+iy)|^2\,dx=O(A),\quad unif.\,(a<y<b).$$

Let us suppose that its boundary functions $f(x,a)$ and $f(x,b)$ are both almost periodic in the sense of W. Stepanoff of order 2.

Then we shall conclude that $f(z)$ is almost periodic in the sense of H. Bohr.

By the Theorem D_5,(iii), we have that $f(z)$ can be represented as the difference of analytic functions in the upper half-plane and lower half-plane respectively. That is as follows

$$f(z)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t,a)}{t+i(a-c)}\frac{dt}{t+ia-z}-\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t,b)}{t+i(b-c)}\frac{dt}{t+ib-z}$$

$$=f^+(z,a)-f^-(z,b),\quad say.$$

where $-c<a<b<c$ and that $f^+(z,a)$ belongs to the H_1^2 in $y>a$, $f^-(z,b)$ does to H_1^2 in $y<b$ respectively.

We shall call these the Generalized Cauchy Integral (G. C. I.) of order 1 and denote these as follows

$$f^+(z,a)=C_1(z;f^+),\quad f^+=f(t,a)$$

and

$$f^-(z,b)=C_1(z;f^-),\quad f^-=f(t,b)$$

respectively. (c.f. S. Koizumi[9], Theorem 12, pp. 112~114). The former belongs to the category of Theorem D_6 and the latter does to Theorem D_7 respectively.

The proof can be done by running on the same lines as the example in III and so we shall cease to sketch main results and abbreviate to give detailed estimations.

In the first , we shall consider the analytic function in the upper-half plane

$$f^+(z,a)=C_1(z;f^+)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f^+(t)}{t+i(a-c)}\frac{dt}{t+ia-z},$$

where $f^+(t)=f(t,a),\quad z=x+iy,\, y>a.$

Let us separate the Cauchy kernel into the Poisson and its conjugate kernel as follows.

$$\frac{1}{2i}\frac{1}{t+ia-z}=\frac{1}{2i}\frac{1}{(t-x)-i(y-a)}=\frac{1}{2}\frac{y-a}{(t-x)^2+(y-a)^2}-\frac{i}{2}\frac{t-x}{(t-x)^2+(y-a)^2}$$

and let us define the harmonic function and its conjugate harmonic function in the upper-half plane as follows

$$U_1(x,y-a;f^+)=\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^+(t)}{t+i(a-c)}\frac{y-a}{(t-x)^2+(y-a)^2}\,dt$$

and

$$\widetilde{U}_1(x,y-a;f^+)=-\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^+(t)}{t+i(a-c)}\frac{t-x}{(t-x)^2+(y-a)^2}\,dt.$$

Then we have

$$f^+(z,a)=C_1(z;f^+)=\frac{1}{2}U_1(x,y-a;f^+)+\frac{i}{2}\widetilde{U}_1(x,y-a;f^+)$$

Now, we shall intend to prove that

$$f^+(z,a)=C_1(z;f^+)=U_1(x,y-a;f^+).$$

Let us remark that if we put $Re(f^+)=g^+$, then $Im(f^+)=(g^+)_1^{\sim}$ by the Theorem D_1 and Theorem D_2 in III and so we can write

$$f^+(x)=g^+(x)+i(g^+)_1^{\sim}(x)$$

where

$$(g^+)_1^{\sim}(x)=P.V.\frac{x+i(a-c)}{\pi}\int_{-\infty}^{\infty}\frac{g^+(t)}{t+i(a-c)}\frac{dt}{x-t}.$$

As well as the ordinary Hilbert Transform, the Generalized Hilbert Transform satisfy the skew reciprocal formula

$$((g^+)_1^{\sim})_1^{\sim}(x)=-g^+(x),\quad a.e.\ x$$

This is proved as follows. The formula $(g^+)_1^{\sim}(x)$ is equivalent to

$$\frac{(g^+)_1^{\sim}(x)}{x+i(a-c)}=P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g^+(t)}{t+i(a-c)}\frac{dt}{x-t}.$$

Iterating this process, we have

$$\frac{((g^+)_1^{\sim})_1^{\sim}(x)}{x+i(a-c)} = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(g^+)_1^{\sim}(t)}{t+i(a-c)}\frac{dt}{x-t} = -\frac{g^+(x)}{x+i(a-c)}$$

by the property of the ordinary Hilbert Transform (c.f. I, [3], E.C.Titchmarsh, Theorem 91, (5.3.1) and (5.3.2), pp. 122). Therefore we have

$$((g^+)_1^{\sim})_1^{\sim} = -g^+(x), \quad a.e.\ x.$$

Next we shall prove the following formula

$$\widetilde{U}_1(x,y-a;g^+) = U_1(x,y-a;(g^+)_1^{\sim}).$$

This is proved as follows . We shall quote also the property of ordinary Hilbert Transform (c.f. I, [3], E.C.Titchmarsh, (5.3.8) and (5.3.9), p.124)

$$\widetilde{U}(x,y;g) = U(x,y;\tilde{g})$$

where

$$U(x,y;\tilde{g}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \tilde{g}(t)\frac{y}{t^2+y^2}\,dt \quad \text{and} \quad \widetilde{U}(x,y;g) = -\frac{1}{\pi}\int_{-\infty}^{\infty} g(t)\frac{t-x}{t^2+y^2}\,dt.$$

Then, on the formula $\widetilde{U}(x,y,;g)$, if we replace $y \to y-a,\ g(s) \to g^+(s)/s+i(a-c)$,

then we obtain on the formula $U(x,y;\tilde{g})$,

$$\tilde{g}(t) \to P.V.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g^+(s)}{s+i(a-c)}\frac{ds}{t-s} = \frac{(g^+)_1^{\sim}(t)}{t+i(a-c)}.$$

correspondingly. Therefore we have

$$\widetilde{U}_1(x,y-a;g^+) = (z-ic)\widetilde{U}\left(x,y-a;g^+(t)/t+i(a-c)\right)$$

$$= (z-ic)U\left(x,y-a;(g^+)_1^{\sim}(t)/t+i(a-c)\right) = U_1(x,y-a;(g^+)_1^{\sim}).$$

We have also the following formula

$$C_1 z;g^+) = iC_1(z;(g^+)_1^{\sim}).$$

This is proved as follows. Since we have

$$C_1 z;g^+) = \frac{1}{2}U_1(x,y-a;g^+) + \frac{i}{2}\widetilde{U}_1(x,y-a;g^+)$$

where

$$U_1(x,y-a;g^+) = U_1(x,y-a;-((g^+)_1^{\sim})_1^{\sim}) = -\widetilde{U}_1(x,y-a;(g^+)_1^{\sim})$$

and

$$\widetilde{U}_1(x,y-a;g^+) = U_1(x,y-a;(g^+)_1^{\sim}).$$

$$C_1(z;g^+)=-\frac{1}{2}\widetilde{U}_1(x,y-a;(g^+)_1^{\sim})+\frac{i}{2}U_1(x,y-a;(g^+)_1^{\sim})=iC_1(z;(g^+)_1^{\sim}).$$

Now since we can represent $f^+=g^++i(g^+)_1^{\sim}$ by the Theorems D_1 and D_2 in III, we have

$$C_1(z;f^+)=C_1(z;g^+)+iC_1(z;(g^+)_1^{\sim})=2C_1(z;g^+)$$

$$=U_1(x,y-a;g^+)+i\widetilde{U}_1(x,y-a;g^+)=U_1(x,y-a;g^+)+iU_1(x,y-a;(g^+)_1^{\sim})$$

$$=U_1(x,y-a;g^++i(g^+)_1^{\sim})=U_1(x,y-a;f^+).$$

Thus we have proved the desired formula

$$C_1(z;f^+)=U_1(x,y-a;f^+).$$

Under these preparations we shall intend to prove $f^+(z)$ to be the almost periodic function in the sense of H.Bohr

The Boundedness of $f^+(z)$.

Now we can write $f^+(z)$ as follows

$$f^+(z)=U_1(x,y-a;f^+)=\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^+(t)}{t+i(a-c)}\frac{y-a}{(t-x)^2+(y-a)^2}dt$$

where $f^+(t)=f(t,a)$ and $z=x+iy,\ y>a$.

Then we have

$$|f^+(z)|^2\le\frac{|z-ic|^2}{\pi}\int_{-\infty}^{\infty}\frac{|f^+(t)|^2}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt$$

$$\le|z-ic|^2\left(\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^+(t)|^2\,dt\right)$$

where

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt=\frac{1}{c-a}\frac{y+c-2a}{x^2+(y+c-2a)^2}=\frac{1}{c-a}\frac{y+c-2a}{|(z-ic)+2i(c-a)|^2}$$

Therefore we have

$$|f^+(z)|\le O\left(\sqrt{\frac{(y+c-2a)l}{c-a}}\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^+(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Uniform Continuity of $f^+(z)$.

For any $z=x+iy$ and $z'=x'+iy$ $(y>a)$, we have

$$f^+(z)-f^+(z')$$

$$=\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^+(t)}{t+i(a-c)}\frac{y-a}{(t-x)^2+(y-a)^2}dt-\frac{z'-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^+(t)}{t+i(a-c)}\frac{y-a}{(t-x')^2+(y-a)^2}dt$$

$$=\frac{z-z'}{\pi}\int_{-\infty}^{\infty}\frac{f^{+}(t)}{t+i(a-c)}\frac{y-a}{(t-x)^2+(y-a)^2}dt$$

$$+\frac{z'-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^{+}(t)}{t+i(a-c)}\left\{\frac{y-a}{(t-x)^2+(y-a)^2}-\frac{y-a}{(t-x')^2+(y-a)^2}\right\}dt=J_1+J_2,\quad say.$$

As for J_1, since $z-z'=(x+iy)-(x'+iy)=x-x'$, we have

$$|J_1|^2\leq\frac{|x-x'|^2}{\pi}\int_{-\infty}^{\infty}\frac{|f^{+}(t)|^2}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt$$

$$\leq|x-x'|^2\left(\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f^{+}(t)|^2\,dt\right)$$

$$=\frac{|x-x'|^2}{c-a}\frac{(y+c-2a)l}{x^2+(y+c-2a)^2}\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f^{+}(t)|^2\,dt\right).$$

Therefore we have

$$|J_1|\leq O\left(\sqrt{\frac{l}{(c-a)(y+c-2a)}}\,|x-x'|\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f^{+}(t)|^2\,dt\right)^{\frac{1}{2}}.$$

As for J_2, since

$$\frac{y-a}{(t-x)^2+(y-a)^2}-\frac{y-a}{(t-x')^2+(y-a)^2}=\frac{(y-a)(x-x')(2t-x-x')}{\left\{(t-x)^2+(y-a)^2\right\}\left\{(t-x')^2+(y-a)^2\right\}}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(y-a)(2t-x-x')^2}{\left\{(t-x)^2+(y-a)^2\right\}\left\{(t-x')^2+(y-a)^2\right\}^2}dt\leq\frac{1}{(y-a)^2\pi}\int_{-\infty}^{\infty}\frac{y-a}{(t-x)^2+(y-a)^2}dt=\frac{1}{(y-a)^2}$$

We have

$$|J_2|^2\leq\frac{|x-x'|^2|z'-ic|^2}{(y-a)^2}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|f^{+}(t)|^2}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt\right)$$

$$\cong\frac{|x-x'|^2|z'-ic|^2}{(y-a)^2}\left(\frac{l}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(a-c)^2}\frac{y-a}{(t-x)^2+(y-a)^2}dt\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f^{+}(t)|^2\,dt\right)$$

and therefore we have

$$|J_2|=O\left(\sqrt{\frac{(y+c-2a)l}{(c-a)(y-a)^2}}\,|x-x'|\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f^{+}(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Here we shall conclude that

$$|f^+(z)-f^+(z')|=O\left(2\sqrt{\frac{(y+c-2a)l}{(c-a)(y-a)^2}}\,|x-x'|\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^+(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Remark. The calculation of the integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(a-c)^2}\frac{y-a}{(x-t)^2+(y-a)^2}dt.$$

We shall consider the Fourier Transform of $e^{-y|x|}$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-y|x|}e^{itx}\,dx=\sqrt{\frac{2}{\pi}}\frac{y}{t^2+y^2}.$$

Then we have by the inverse Fourier Transform

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{t^2+y^2}e^{-ixt}\,dt=e^{-y|x|}.$$

In this formula, if we replace $y\to y-a$ and $y\to c-a$ respectively, we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y-a}{t^2+(y-a)^2}e^{-ixt}\,dt=e^{-(y-a)|x|}\quad\text{and}\quad\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(c-a)^2}e^{-ixt}\,dt=\frac{1}{c-a}e^{-(c-a)|x|}$$

respectively and then we have by the Plancherel theorem and the Parseval equality

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(a-c)^2}\frac{y-a}{(x-t)^2+(y-a)^2}dt$$

$$=\sqrt{\frac{\pi}{2}}\frac{1}{c-a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-(y+c-2a)|u|}e^{ixu}\,du=\frac{1}{c-a}\frac{y+c-2a}{x^2+(y+c-2a)^2}.$$

The Approximation by Trigonometric Polynomials.

Let us consider the following trigonometric polynomials

$$p^+(x)=\sum_{\lambda_n}c_ne^{i\lambda_nx},\quad(\lambda_0=0)$$

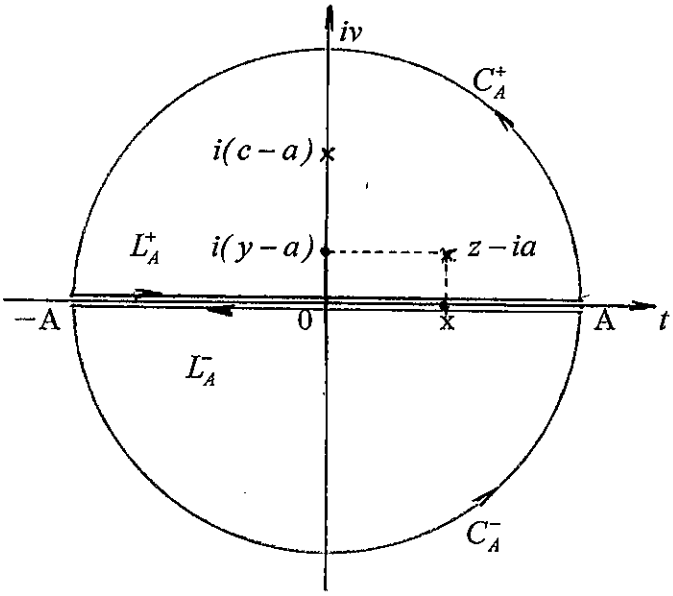
and

$$p^+(z)=C_1(z;p^+)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{p^+(t)}{t+i(a-c)}\frac{dt}{t+ia-z}\quad(z=x+iy,\,y>a).$$

Let us calculate it through the contour integral in the complex domain and the residue theorem. For this purpose , we shall consider the following contour integral.

(i) The case $\lambda\geq 0$. We have

$$I_\lambda=\frac{1}{2\pi i}\int_{\Gamma_A^+}\frac{e^{i\lambda w}}{(w+i(a-c))(w+ia-z)}dw=\frac{1}{2\pi i}\int_{C_A^+}('')\,dw+\frac{1}{2\pi i}\int_{L_A^+}('')\,dw\,.$$



$\Gamma_A^+=C_A^+\cup L_A^+$

$C_A^+=\{w=Ae^{i\theta},0\le\theta\le\pi\}$

$L_A^+=\{w=t,-A\le t\le A\}$

$\Gamma_A^-=C_A^-\cup L_A^-$

$C_A^-=\{w=Ae^{i\theta},\pi\le\theta\le 2\pi\}$

$L_A^-=\{w=t,-A\le t\le A\}$

Then if $w\in C_A^+$, since $w=Ae^{i\theta}$, $dw=iAe^{i\theta}d\theta$, we have

$$\frac{e^{i\lambda w}}{(w+i(a-c))(w+ia-z)}=\frac{e^{i\lambda A\cos\theta}e^{-\lambda A\sin\theta}}{(Ae^{i\theta}+i(a-c))(Ae^{i\theta}+ia-z)}=\begin{cases}O(A^{-2}e^{-\lambda A\sin\theta}), & (0<\theta<\pi,\lambda>0)\\ O(A^{-2}), & elsewhere\end{cases}$$

and so we have

$$\frac{1}{2\pi i}\int_{C_A^+}('')\,dw=\begin{cases}O(A^{-1}e^{-\lambda A\sin\theta}), & (0<\theta<\pi,\lambda>0)\\ O(A^{-1}), & elsewhere\end{cases}$$

$$=o(1),\quad (A\to\infty).$$

Next if $w\in L_A^+$, since $w=t$, $dw=dt$, we have

$$\frac{1}{2\pi i}\int_{L_A^+}('')\,dw=\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{i\lambda t}}{(t+i(a-c))(t+ia-z)}dt\to\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\lambda t}}{(t+i(a-c))(t+ia-z)}dt,\quad (A\to\infty)$$

On the other hand , by the theorem of residue we have

$$I_\lambda=\frac{1}{2\pi i}\int_{\Gamma_A^+}('')dw=\begin{cases}\dfrac{e^{-(y-a)\lambda}e^{i\lambda x}}{z-ic}-\dfrac{e^{-(c-a)\lambda}}{z-ic}, & (\lambda>0)\\ 0, & (\lambda=0)\end{cases}$$

Thus we have

$$\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\lambda t}}{(t+i(a-c))(t+ia-z)}dt=\begin{cases}e^{-(y-a)\lambda}e^{i\lambda x}-e^{-(c-a)\lambda}, & (\lambda>0)\\ 0, & (\lambda=0)\end{cases}$$

$$\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\lambda t}}{(t+i(a-c))(t+ia-z)}dt=\begin{cases}e^{-(y-a)\lambda}e^{i\lambda x}-e^{-(c-a)\lambda}, & (\lambda>0)\\ 0, & (\lambda=0)\end{cases}$$

(ii) The case $\lambda<0$. We have by running the same lines as above

$$\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\lambda t}}{(t+i(a-c))(t+ia-z)}dt=0$$

Therefore we have

$$p^{+}(z)=C_1(z;p^{+})=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{p^{+}(t)}{t+i(a-c)}\frac{dt}{t+ia-z}$$
$$=\sum_{\lambda_n>0}c_n e^{-(y-a)\lambda_n}e^{i\lambda_n x}-\sum_{\lambda_n>0}c_n e^{-(c-a)\lambda_n}.$$

Thus we have proved

$$|f^{+}(z)-p^{+}(z)|\le O\left(\sqrt{\frac{(y+c-2a)l}{c-a}}\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_{x}^{x+l}|f^{+}(t)-p^{+}(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Next, we shall consider the analytic function that belongs to the class H_1^2 in the lower-half plane as follows

$$f^{-}(z,b)=C_1(z;f^{-})=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{f^{-}(t)}{t+i(b-c)}\frac{dt}{t+ib-z}$$

where $f^{-}(t)=f(t,b)$ and $z=x+iy, y<b$.

Let us introduce as before the harmonic and its conjugate harmonic functions as follows

$$U_1(x,y-b;f^{-})=\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^{-}(t)}{t+i(b-c)}\frac{y-b}{(t-x)^2+(y-b)^2}dt$$

and

$$\widetilde{U}_1(x,y-b;f^{-})=-\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^{-}(t)}{t+i(b-c)}\frac{t-x}{(t-x)^2+(y-b)^2}dt.$$

Then we have

$$f^{-}(z,b)=C_1(z;f^{-})=\frac{1}{2}U_1(x,y-b;f^{-})+i\widetilde{U}_1(x,y-b;f^{-}).$$

Let us remark that if we put $Re(f^{-})=g^{-}$, then $Im(f^{-})=(g^{-})_1^{\sim}$ by the Theorem D_1 and Theorem D_2 in III and so we can write

$$f^{-}(x)=g^{-}(x)+i(g^{-})_1^{\sim}(x),$$

where

$$(g^-)_1^\sim(x) = P.V.\frac{x+i(b-c)}{\pi}\int_{-\infty}^{\infty}\frac{g^-(t)}{t+i(b-c)}\frac{dt}{x-t}$$

As well as the ordinary Hilbert Transform, the Generalized Hilbert Transform satisfy the skew reciprocal formula

$$((g^-)_1^\sim)_1^\sim(x) = -g^-(x), \quad a.e.\ x.$$

This is proved as follows . The formula $(g^-)_1^\sim(x)$ is equivalent to

$$\frac{(g^-)_1^\sim(x)}{x+i(b-c)} = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g^-(t)}{t+i(b-c)}\frac{dt}{x-t}.$$

Iterating this process, we have

$$\frac{((g^-)_1^\sim)_1^\sim(x)}{x+i(b-c)} = P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(g^-)_1^\sim(t)}{t+i(b-c)}\frac{dt}{x-t} = -\frac{g^-(x)}{x+i(b-c)}$$

by the property of the ordinary Hilbert Transform (c.f. I, [3], E.C.Titchmarsh, Theorem 91, pp.122~124). Therefore we have

$$((g^-)_1^\sim)_1^\sim(x) = -g^-(x), \quad a.e.\ x.$$

Next we shall prove the following formula

$$\widetilde{U}_1(x, y-b; g^-) = U_1(x, y-b; (g^-)_1^\sim)$$

and

$$C_1(z; g^-) = iC_1(z; (g^-)_1^\sim).$$

These formulas are proved by running on the same lines as the case of g^+ too.

Now since we can represent $f^- = g^- + i(g^-)_1^\sim$ by the Theorems D_1 *and* D_2 in III, we have

$$C_1(z; f^-) = C_1(z; g^-) + iC_1(z; (g^-)_1^\sim) = 2C_1(z; g^-)$$

$$= U_1(x, y-b; g^-) + i\widetilde{U}_1(x, y-b; g^-) = U_1(x, y-b; g^-) + iU_1(x, y-b; (g^-)_1^\sim)$$

$$= U_1(x, y-b; g^- + i(g^-)_1^\sim) = U_1(x, y-b; f^-).$$

Thus we have proved the desired formula

$$C_1(z; f^-) = U_1(x, y-b; f^-).$$

Under these preparations we shall intend to prove $f^-(z)$ to be almost periodic function in the sense of H.Bohr. It is carried out by running the same lines as the case of $f^+(z)$, so we cease it to state the estimation formulas without proofs.

The boundedness of $f^-(z)$.

Now we can write $f^-(z)$ as follows

$$f^-(z) = U_1(x, y-b; f^-) = \frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^-(t)}{t+i(b-c)}\frac{y-b}{(t-x)^2+(y-b)^2}dt$$

where $f^-(t) = f(t,b)$ and $z = x+iy,\ y < b$.

Then we have

$$|f^-(z)| \le O\left(\sqrt{\frac{(c-y)l}{c-b}}\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^-(t)|^2\,dt\right)^{\frac{1}{2}}$$

where we use the calculation of the following integral

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+(b-c)^2}\frac{b-y}{(t-x)^2+(b-y)^2}dt = \frac{1}{c-b}\frac{c-y}{|z-ic|^2}.$$

Uniform Continuity of $f^-(z)$.

We have

$$f^-(z)-f^-(z') \qquad z=x+iy,\quad z'=x'+iy \quad (y<b)$$

$$=\frac{z-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^-(t)}{t+i(b-c)}\frac{y-b}{(t-x)^2+(y-b)^2}dt-\frac{z'-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^-(t)}{t+i(b-c)}\frac{y-b}{(t-x)^2+(y-b)^2}dt$$

$$+\frac{z'-ic}{\pi}\int_{-\infty}^{\infty}\frac{f^-(t)}{t+i(b-c)}\left\{\frac{y-b}{(t-x)^2+(y-b)^2}-\frac{y-b}{(t-x')^2+(y-b)^2}\right\}dt = J_3+J_4,\quad say.$$

Then as for J_3, we have

$$|J_3| \le O\left(\sqrt{\frac{(c-y)l}{c-b}}\,|x-x'|\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^-(t)|^2\,dt\right)^{\frac{1}{2}}.$$

and as for J_4, we have

$$|J_4| \le O\left(\sqrt{\frac{l}{(b-y)(c-b)}}\,|x-x'|\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^-(t)|^2\,dt\right)^{\frac{1}{2}}.$$

Therefore we have

$$|f^-(z)-f^-(z')| \le O\left(\sqrt{\frac{(b+c-y)l}{'c-b)(b-y)}}\,|x-x'|\right)\left(\sup_{-\infty<x<\infty}\frac{1}{l}\int_x^{x+l}|f^-(t)|^2\,dt\right)^{\frac{1}{2}}.$$

The Approximation by Trigonometric Polynomials.

Let us consider the following trigonometric polynomials

$$p^-(x)=\sum_{\mu_n} d_n e^{i\mu_n x},\quad (\mu_0=0)$$

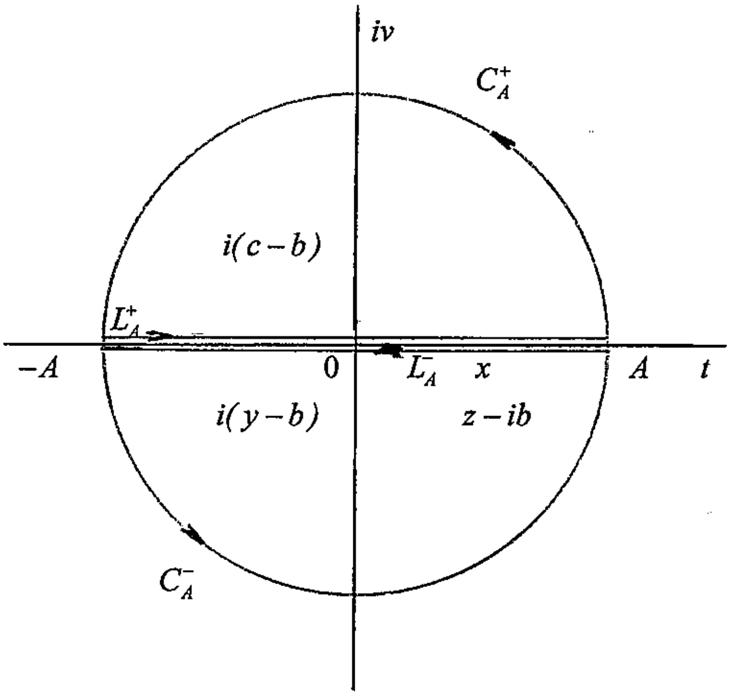
and

$$p^-(z)=C_1(z;p^-)=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{p^-(t)}{t+i(b-c)}\frac{dt}{t+ib-z}\quad (z=x+iy,\ y<b).$$

Let us calculate it through the contour integral in the complex domain and the residue theorem. For this purpose ,we shall consider the following contour integral.

(i) The case $\mu \le 0$. We have

$$I_\mu = \frac{1}{2\pi i}\int_{\Gamma_A^-} \frac{e^{i\mu t}}{(w+i(b-c))(w+ib-z)}dw = \frac{1}{2\pi i}\int_{C_A^-}('')dw + \frac{1}{2\pi i}\int_{L_A^-}('')dw$$



$$\Gamma_A^+ = C_A^+ \cup L_A^+$$

$$C_A^+ = \{w = Ae^{i\theta}, 0 \le \theta \le \pi\}$$

$$L_A^+ = \{w = t, -A \le t \le A\}$$

$$\Gamma_A^- = C_A^- \cup L_A^-$$

$$C_A^- = \{w = Ae^{i\theta}, \pi \le \theta \le 2\pi\}$$

$$L_A^- = \{w = t, -A \le t \le A\}$$

Then we have

$$\frac{1}{2\pi i}\int_{C_A^-}('')dw = \begin{cases} O(A^{-1}e^{-\mu \sin\theta}), & (\mu < 0, \pi < \theta < 2\pi) \\ O(A^{-1}), & (elsewhere) \end{cases}$$

$$= o(1), \quad (A \to \infty),$$

and

$$\frac{1}{2\pi i}\int_{L_A^-}('')dw \to -\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{i\mu t}}{(t+i(b-c))(t+ib-z)}dt, \quad (A \to \infty).$$

On the other hand, by the residue theorem, we have

$$I_\mu = \frac{1}{2\pi i}\int_{\Gamma_A^-} \frac{e^{i\mu w}}{(w+i(b-c))(w+ib-z)}dw = -\frac{e^{i\mu(z-ib)}}{z-ic}$$

Therefore we have

$$\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\mu t}}{(t+i(b-c))(t+ib-z)}dt=-e^{-(y-b)\mu}e^{i\mu x}.$$

(ii) The case $\mu>0$.

Similarly we have

$$I_\mu=\frac{1}{2\pi i}\int_{\Gamma_A^+}\frac{e^{i\mu w}}{(w+i(b-c))(w+ib-z)}dw=\frac{1}{2\pi i}\int_{C_A^+}('')\,dw+\frac{1}{2\pi i}\int_{L_A^+}('')\,dw.$$

By the residue theorem, we have

$$I_\mu=\frac{1}{2\pi i}\int_{\Gamma_A^+}\frac{e^{i\mu w}}{(w+i(b-c))(w+ib-z)}dw=-\frac{e^{\mu(b-c)}}{z-ic}.$$

On the other hand, we have

$$\frac{1}{2\pi i}\int_{C_A^+}('')\,dw=\begin{cases}O(A^{-1}e^{-\mu A\sin\theta}), & (0<\theta<\pi)\\ O(A^{-1}), & (elsewhere)\end{cases}$$

and

$$\frac{1}{2\pi i}\int_{L_A^+}('')\,dw\to\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\mu t}}{(t+i(b-c))(t+ib-z)}dt,\quad (A\to\infty).$$

Therefore we have

$$\frac{z-ic}{2\pi i}\int_{\Gamma_A^+}\frac{e^{i\mu t}}{(t+i(b-c))(t+ib-z)}dt=-e^{-(c-b)\mu}.$$

Let us write the trigonometric polynomials as follows

$$p^-(x)=\sum_{\mu_n}d_ne^{i\mu_nx},\quad(\mu_0=0)$$

then we have

$$\begin{aligned}p^-(z)=C_1(z;p^-)&=\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{p^-(t)}{t+i(b-c)}\frac{dt}{t+ib-z}\\&=\sum d_n\frac{z-ic}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\mu_nt}}{t+i(b-c)}\frac{dt}{t+ib-z}\\&=-\sum_{\mu_n\le0}d_ne^{-(y-b)\mu_n}e^{i\mu_nx}-\sum_{\mu_n>0}d_ne^{-(c-b)\mu_n},\end{aligned}$$

where $a<y<b$ and $-c<a<b<c$.

Now let us set as follows

$$f(z) = f^+(z) - f^-(z) \qquad \text{and} \qquad p(z) = p^+(z) - p^-(z)$$

Then we have

$$|f(z) - p(z)| \le |f^+(z) - p^+(z)| + |f^-(z) - p^-(z)|$$

$$\le O\left(\sqrt{\frac{y+c-2a)l}{c-a}}\right)\left(\sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f^+(x) - p^+(x)|^2\, dt\right)^{\frac{1}{2}}$$

$$+O\left(\sqrt{\frac{(c-y)l}{c-b}}\right)\left(\sup_{-\infty<x<\infty} \frac{1}{l}\int_x^{x+l} |f^-(x) - p^-(x)|^2\, dt\right)^{\frac{1}{2}}.$$

Thus we have proved that $f(z)$ is analytic and uniformly almost periodic function in the strip domain $z = x + iy, \quad a < y < b \ (-c < a < b < c)$.

13. Spectral Analysis on the N.Wiener class S.

Let us suppose that $f(x)$ belongs to the class S and $\varphi(x)$ denotes its correlation function. Let us also suppose that $s(u)$ and $\sigma(u)$ are the G.F.T. of $f(x)$ and $\varphi(x)$ respectively.

As well as Theorem D_9, we shall set the presupposed conditions as follows.

$$(C_\lambda) \qquad c_\lambda = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}\, dx \quad (\forall \text{real } \lambda)$$

and

$$(L_\lambda) \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \varphi(x)e^{-i\lambda x}\, dx = \left|\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}\, dx\right|^2 \quad (\forall \text{real } \lambda).$$

Then we shall define the spectrum of $f(x)$ as follows.

The point spectrum.

We say that $u = \lambda$ is the point spectrum of $f(x)$ if the following condition

$$\sigma(\lambda + 0) - \sigma(\lambda - 0) > 0$$

is satisfied and it is equivalent to the formula

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2\, du > 0.$$

This is proved by the same way as Lemma D_3 (c.f. III, p.60).

The continuous spectrum.

We say that $u = \lambda$ is the continuous spectrum of $f(x)$ if the following conditions

(i) $\qquad \sigma(\lambda + 0) - \sigma(\lambda - 0) = 0$

and

(ii) $\sigma(\lambda+\varepsilon)-\sigma(\lambda-\varepsilon)>0 \quad (\forall\varepsilon>0)$

are satisfied and these are equivalent to the formulas

$$\text{(i)} \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du=0$$

and

$$\text{(ii)} \qquad \lim_{\eta\to 0}\frac{1}{2\eta\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}|s(u+\eta;f)-s(u-\eta;f)|^2\,du>0 \quad (\forall\varepsilon>0).$$

These are proved by the same way as Lemma D_4 (c.f. III, p.60) too.

Now under these presupposed hypothesis, we shall intend to the spectral analysis of $f(x)$. Since we have

$$\sigma(\infty)-\sigma(-\infty)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\infty}^{\infty}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du,$$

the $\sigma(u)$ is a bounded and monotone increasing function (c.f. N.Wiener [1], (21.26), p. 162 ; III, Lemma D_4, p. 52) and so it has the first kind of discontinuity at most on the set of enumerable infinite points λ_n $(n=0,1,2,3,\ldots)$. Then by the condition (C_λ), (L_λ) and the One-sided Wiener formula we have

$$\sigma(\lambda_n+0)-\sigma(\lambda_n-0)=\sqrt{2\pi}\,|c_n|^2 \quad (n=0,1.2.3\ldots)$$

and

$$\sum_n |c_n|^2\le\sigma(\infty)-\sigma(-\infty),$$

where $\lambda_0=0$ and $c_0\ge 0$ (let us remark that $c_0=0$ is permitted).
The case $\lambda=\lambda_0$ is proved in this paper (c.f. Theorem D_9 ,pp.87~88) and other cases are proved by just the similar way, so we shall omit them.

Therefore by the theorem Riesz-Fisher (c.f. A.S.Besicovitch[5], pp.110~112) there exist the almost periodic function $f_0(x)$ in the sense of Besicovitch of order 2 (we shall denote it by B^2 almost periodic function) and its Fourier coefficients are $\{c_n\}$.

That is the Fourier Series of $f_0(x)$ is as follows

$$f_0(x)\sim\sum_n c_n e^{i\lambda_n x}.$$

Furthermore let us set its correlation function

$$\varphi_0(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_0(x+t)\overline{f_0(t)}\,dt$$

and its G.F.T.

$$\sigma_0(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\varphi_0(x)\frac{e^{-iux}-1}{-ix}dx+\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\varphi_0(x)\frac{e^{-iux}}{-ix}dx.$$

Then we have

$$\varphi_0(x)=\sum_n |c_n|^2 e^{i\lambda_n x}$$

and

$$\sigma_0(u)=\begin{cases}\sqrt{2\pi}\sum_{\lambda_n<u}|c_n|^2 & (u\neq\lambda_m)\\ \sqrt{2\pi}(\sum_{\lambda_n<u}|c_n|^2+\frac{1}{2}|c_m|^2) & (u=\lambda_m)\end{cases}.$$

Now let us set

$$f(x)-f_0(x)=f_1(x)$$

and let us define its correlation function $\varphi_1(x)$ and its G.H.T. $\sigma_1(u)$. We shall intend to prove

$$\sigma(u)-\sigma_0(u)=\sigma_1(u).$$

Then we have

$$\varphi_1(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(x+t)\overline{f_1(t)}dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f_0(t)}dt$$

$$-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_0(x+t)\overline{f(t)}dt+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_0(x+t)\overline{f_0(t)}dt.$$

Then we have for the Bochner-Fejer mean of Fourier series of $f_0(x)$

$$\left|\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f_0(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{\sigma_{B_m}(t;f_0)}dt\right|$$

$$\leq\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x+t)|^2dt\right)^{\frac{1}{2}}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_0(t)-\sigma_{B_m}(t;f_0)|^2dt\right)^{\frac{1}{2}}\to 0,\quad (T\to\infty, \text{in the first}; \text{next}, m\to\infty),$$

and

$$\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{\sigma_{B_m}(t;f_0)}dt=\sum_n d_n^{(m)}\,\overline{c_n}\,\frac{1}{2T}\int_{-T}^{T}f(x+t)e^{-i\lambda_n t}dt\to\sum_n d_n^{(m)}|c_n|^2e^{i\lambda_n x},\quad (T\to\infty).$$

Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f_0(t)}dt=\sum_n|c_n|^2e^{i\lambda_n x}.$$

Similarly we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_0(x+t)\overline{f(t)}dt=\sum_n |c_n|^2 e^{i\lambda_n x},\quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_0(x+t)\overline{f_0(t)}dt=\sum_n |c_n|^2 e^{i\lambda_n x}.$$

Thus we have proved

$$\varphi_1(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_1(x+t)\overline{f_1(t)}dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt-\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f_0(x+t)\overline{f_0(t)}dt=\varphi(x)-\varphi_0(x).$$

Therefore we shall conclude that $\sigma_1(u)=\sigma(u)-\sigma_0(u)$ and it is a bounded ,continuous and monotone increasing function. On the other hand, since $f(x)$ belongs to the class S and $f_0(x)$ is the B^2 almost periodic function, both $\varphi(x)$ and $\varphi_0(x)$ are to be bounded, and so $\varphi_1(x)$ does too. Therefore we shall conclude that

$$\frac{1}{2T}\int_{-T}^{T} |\varphi_1(x)|^2 dx$$

is bounded in T. Now we can apply the N.Wiener theorem (c.f. [1], Theorem 24, pp. 146~149) and we shall conclude that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\varphi_1(x)|^2 dx=0.$$

This means that the energy of system of which we considered is concentrate to (f_0,φ_0,σ_0) although (f_1,φ_1,σ_1) contribute to the turbulence apparently but its energy is a little. Therefore it is natural to consider that the behavior of the system is controlled by (f_0,φ_0,σ_0). It might be the observation of Prof. N.Wiener. After that he used these observations in his Prediction Theory of Time Series to determine the solution of a kind of integral equation.

In this section B^2 almost periodic function plays essential roles in the spectral analysis. It is well known that in the space L^2 of 2π -periodic functions, any function is best approximated by its Fourier Series oneself. We shall point out that the same property is satisfied on the space of B^2 almost periodic functions.

Let us suppose that $f(x)$ is a function of B^2 almost periodic and let us set

$$c(\lambda)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt \quad (\forall\ real\ \lambda).$$

The $c(\lambda)$ is not nought at most enumerable points. We denote it as $\lambda=\lambda_n$ and $c(\lambda_n)=c_n$ $(n=0,1,2,3,...)$, in particular $\lambda_0=0$ and the case of $c_0=0$ is permitted. We also denote Λ the set of points λ_n $(n=0,1,2,3,...)$. Then the Fourier Series of f

is represented as follows

$$f(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

Let us consider a trigonometric polynomial $q(x) = \sum_n d_n e^{i\mu_n x}$ that we shall intend to approximate $f(x)$ in the space of B^2 almost periodic functions. Let us denote Λ_0 the set of μ_n such as $\mu_n \in \Lambda$. Then we shall decompose the $q(x)$ into two parts as follows

$$q(x) = \sum_n d_n e^{i\mu_n x} = \sum_{\mu_n \in \Lambda_0} d_n e^{i\mu_n x} + \sum_{\mu_n \notin \Lambda_0} d_n e^{i\mu_n x}.$$

In the first part, let us replace the constant d_n by the Fourier coefficient c_n that correspond to the $\mu_n \in \Lambda$ and let us denote it as

$$p(x) = \sum_{\mu_n \in \Lambda_0} c_n e^{i\mu_n x}$$

Then we shall calculate the following formula

$$\frac{1}{2T}\int_{-T}^{T} |f(x) - q(x)|^2 dx$$

$$= \frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx - \sum_{\mu_n \in \Lambda_0} \overline{d_n} \frac{1}{2T}\int_{-T}^{T} f(t) e^{-i\mu_n t} dt - \sum_{\mu_n \in \Lambda_0} d_n \overline{\frac{1}{2T}\int_{-T}^{T} f(t) e^{-i\mu_n t} dt}$$

$$+ \sum_{m,n} d_m \overline{d_n} \frac{1}{2T}\int_{-T}^{T} e^{-i(\mu_n - \mu_m)t} dt + o(1), \quad (T \to \infty)$$

and we have

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x) - q(x)|^2 dx$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx - \sum_{\mu_n \in \Lambda_0} c_n \overline{d_n} - \sum_{\mu_n \in \Lambda_0} \overline{c_n} d_n + \sum_n |d_n|^2$$

On the other hand, we have similarly

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x) - p(x)|^2 dx = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2 dx - \sum_{\mu_n \in \Lambda_0} |c_n|^2.$$

Therefore we have by combining the above two formulas

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x) - q(x)|^2 dx$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)-p(x)|^2\, dx + \sum_{\mu_n \in \Lambda_0} |c_n - d_n|^2 + \sum_{\mu_n \notin \Lambda_0} |d_n|^2$$

$$\geq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)-p(x)|^2\, dx$$

where the equality occurs if and only if $d_n = c_n \quad (\mu_n \in \Lambda_0) \;\; and \;\; d_n = 0 \quad (\mu_n \notin \Lambda)$.

However we should remark that compared with the L^2 periodic case, in the B^2 almost periodic case, the order of summation of trigonometric polynomials is the problem. As for these circumferences N.Wiener remarked it on the end of his book(c.f. [1], pp.198~199). The dominating idea in proofs of the Bohr−de la Vallee Poussin type is that of arrangement of the terms $c_n e^{i\lambda_n x}$ in an order depending on the arithmetical properties of the λ_n. The dominating idea in the Weyl proof is that of the arrangement of these terms in the descending order of magnitude of the coefficients $|c_n|$. The dominating idea in the Wiener proof is that arrangement of these terms in the order of the exponents λ_n. This is the only order which is compatible with a unified treatment of almost periodic functions with continuous spectra.

We shall give an example. Let us suppose that $f(z), (z = x+iy,\, y>0)$ is analytic in the upper-half plane and belongs to the class H_1^2. Let us suppose that its boundary function $f(x)$ at $y=0$ is B^2 almost periodic function. Then we shall prove $f(z)$ to be the B^2 almost periodic function.

For this purpose we shall quote Theorem D_3 (c.f. III,pp.47~52), then we have

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(z)|^2\, dx$$

$$= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2\, du$$

$$= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\infty} |e^{-yu}\{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}|^2\, du$$

$$\leq \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2\, du$$

$$= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2\, dx \quad (z = x+iy,\, y>0).$$

Here, since $f(z)$ is represented by the Generalized Cauchy Integral of order 1 of its boundary function $f(x)$, we have

$$f(z)=C_1(z;f)=\frac{z+i}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{t-z}$$

(c.f. III, THeorem D_1, (iii), pp.46~47).

We shall also introduce the Bochner−Fejer mean of $f(x)$ as follows

$$\sigma_{B_m}(x;f)=\sum_n d_n^{(m)}c_n\,e^{i\lambda_n x}.$$

Then we have

$$\sigma_{B_m}(z;f)=C_1(z;\sigma_{B_m}(x;f))=\frac{z+i}{2\pi i}\int_{-\infty}^{\infty}\frac{\sigma_{B_m}(x;f)}{t+i}\frac{dt}{t-z}$$
$$=C_0+\sum_{\lambda_n\geq 0} d_n^{(m)}c_n\,e^{-\lambda_n y}\,e^{i\lambda_n x},$$

where $C_0=\sum_{\lambda_n<0} d_n^{(m)}c_n\,e^{\lambda_n}$. Therefore we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(z)-\sigma_{B_m}(z;f)|^2\,dx\leq\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)-\sigma_{B_m}(x;f)|^2\,dx\to 0,\quad (m\to\infty).$$

It should be remarked that the method of approximation by the Bochner−Fejer mean belongs to the category of Bohr−de la Valee Pousin.

For the sake of completeness, we shall prove that "*any* B^2 *almost periodic function* satisfy *hypotheses* (C_λ) *and* (L_λ)".

Let us set $f(x)$ as the B^2 almost periodic function and as its Fourier series

$$f(x)\sim\sum_n c_n\,e^{i\lambda_n x}.$$

Let us also set its Bochner-Fejer mean as follows

$$\sigma_{B_m}x;f)=\sum_n d_n^{(m)}c_n\,e^{i\lambda_n x}.$$

Then we have

$$\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt-\frac{1}{2T}\int_{-T}^{T}\sigma_{B_m}(x+t;f)\overline{\sigma_{B_m}(t;f)}dt$$
$$=\frac{1}{2T}\int_{-T}^{T}f(x+t)\left\{\overline{f(t)}-\overline{\sigma_{B_m}(t;f)}\right\}dt+\frac{1}{2T}\int_{-T}^{T}\left\{f(x+t)-\sigma_{B_m}(x+t;f)\right\}\overline{\sigma_{B_m}(t;f)}dt.$$

$=I_1+I_2$, *say*. As for I_1, we have for all x

$$|I_1|\leq\left(\frac{1}{2T}\int_{-T}^{T}|f(x+t)|^2\,dt\right)^{\frac{1}{2}}\left(\frac{1}{2T}\int_{-T}^{T}|f(t)-\sigma_{B_m}(t;f)|^2\,dt\right)\to 0,\quad (T\to\infty,\text{in the first};\text{next},m\to\infty),$$

and as for I_2, we have

$$|I_2| \le \left(\frac{1}{2T} \int_{-T}^{T} |f(x+t) - \sigma_{B_m}(x+t;f)|^2 \, dt \right)^{\frac{1}{2}} \left(\frac{1}{2T} \int_{-T}^{T} |\sigma_{B_m}(t;f)|^2 \, dt \right).$$

Then we shall use the following results for all x

$$\frac{1}{2T} \int_{-T}^{T} |f(x+t) - \sigma_{B_m}(x+t;f)|^2 \, dt \le \frac{T+|x|}{T} \frac{1}{2(T+|x|)} \int_{-(T+|x|)}^{T+|x|} |f(t) - \sigma_{B_m}(t;f)|^2 \, dt \to 0,$$

as $T \to \infty, in the\ first; next, m \to \infty$ and so we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\sigma_{B_m}(t;f)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2 \, dt, \quad (m \to \infty).$$

Therefore we have for all x

$$I_2 \to 0, \quad (T \to \infty, in the\ first; next, m \to \infty).$$

Thus we have proved

$$\varphi(x;f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(t)} \, dt = \lim_{m\to\infty} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_{B_m}(x+t)\overline{\sigma_{B_m}(t)} \, dt.$$

Now we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sigma_{B_m}(x+t)\overline{\sigma_{B_m}(t)} \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left(\sum_n d_n^{(m)} c_n e^{i\lambda_n(x+t)} \right) \left(\sum_n d_n^{(m)} \overline{c_n} e^{-i\lambda_n t} \right) dt$$

$$= \sum_{n,n'} d_n^{(m)} d_{n'}^{(m)} c_n \overline{c_{n'}} e^{i\lambda_n x} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{i(\lambda_n - \lambda_{n'})t} \, dt = \sum_n (d_n^{(m)})^2 |c_n|^2 e^{i\lambda_n x}.$$

Since $d_n^{(m)} \to 1, \quad (m \to \infty)$, we have

$$\varphi(x;f) = \sum_n |c_n|^2 e^{i\lambda_n x},$$

and then we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(x;f) e^{-i\lambda x} \, dx = \begin{cases} |c_n|^2, & \lambda = \lambda_n \\ 0, & \lambda \ne \lambda_n \end{cases}.$$

Thus we have proved that the hypothesis (L_λ) is satisfied. Since $f(x)$ is B^2 almost periodic function, so it is clear that the hypothesis (C_λ) is satisfied.

Next we shall give another proof for the approximation of $f(z)$ by trigonometric polynomials. Let us set the Fourier series of B^2 almost periodic function $f(x)$ as follows

$$f(x) \sim \sum_n c_n e^{i\lambda_n x}.$$

Then by the One-Sided Wiener formula it follows that

$$c(\lambda)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)e^{-i\lambda t}dt=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}e^{-i\lambda u}du \quad (\forall\ real\ \lambda).$$

Since we have proved that $f(z)\ (z=x+iy,\ y>0)$ is the B^2 almost periodic function as a function of x, so it satisfy *hypotheses* (C_λ) *and* (L_λ) and then we have by the One-Sided Wiener formula and the Theorem D_3 (c.f. III, pp.47~52)

$$c(\lambda,y)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(z)e^{-i\lambda x}\,dx \quad (z=x+iy,\ y>0)$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f(z))-s(u-\varepsilon;f(z))\}e^{-i\lambda u}du\,.$$

$$=\begin{cases}\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} e^{-yu}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}e^{-i\lambda u}du, & (\lambda\ge 0)\\ 0, & (\lambda<0)\end{cases}$$

$$=\begin{cases}\lim_{\varepsilon\to 0}\frac{e^{-y\lambda}}{2\varepsilon\sqrt{2\pi}}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}e^{-i\lambda u}\,du, & (\lambda\ge 0)\\ 0, & (\lambda<0)\end{cases}$$

and so we have

$$c(\lambda,y)=\begin{cases}c(\lambda)e^{-y\lambda}, & (\lambda\ge 0)\\ 0, & (\lambda<0)\end{cases} \qquad (\forall\ real\ \lambda)$$

Let us denote Λ the set of point λ such as $c(\lambda)$ not to be naught. Then we have

$$c(\lambda_n,y)=\begin{cases}c_n e^{-\lambda_n y}, & (\lambda_n\ge 0)\\ 0, & (\lambda_n<0)\end{cases} \qquad (\forall\ \lambda_n\in\Lambda\ ).$$

These are the relation of Fourier coefficients between analytic function $f(z)$ of the class H_1^2 and its boundary function $f(x)$ at $y=0$, as for point spectrum.

Now we shall intend to the approximation of $f(z)$ by trigonometric polynomials As for spectrum of $f(x)$, let us denote $\Lambda_N^+=\{\lambda_n\in\Lambda,\ 0\le\lambda_n\le N\}$ and let us define the trigonometric polynomial $p_N(x)=\sum_{\lambda_n\in\Lambda_N^+} c_n\,e^{i\lambda_n x}$ and $p_N(z)$ as its G.C.I. of order 1.

Then we shall obtain $p_N(z)=\sum_{\lambda_n\in\Lambda_N^+} c_n\,e^{-\lambda_n y}e^{i\lambda_n x}$ of which approximate the $f(z)$.

Let us define $p_N(x)$ and its correlation function $\varphi_N(x)=\varphi(x;p_N)$ and also its G.F.T. $\sigma_N(u)=\sigma(u;\varphi_N)$. Let us also define $f(x)$ and its correlation function $\varphi(x)=\varphi(x;f)$ and also its G.F.T. $\sigma(u)=\sigma(u;\varphi)$. Then we have by the Theorem D_4 (c.f. III, p. 61)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(z)-p_N(z)|^2\,dx$$
$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |s(u+\varepsilon;f(z)-p_N(z))-s(u-\varepsilon;f(z)-p_N(z))|^2\,du$$
$$=\frac{2y}{\sqrt{2\pi}}\int_0^{\infty}\left(\{\sigma(u;f)-\sigma(0-;f)\}-\{\sigma(u;p_N)-\sigma(0-;p_N)\}\right)e^{-2yu}\,du\,.$$

Here let us denote that $\Lambda^+=\{\lambda\ge 0,\ \lambda\in\Lambda\}$, and let us remark that $\sigma(0-;p_N)=0$.

Then we have

$$\{\sigma(u;f)-\sigma(0-;f)\}-\{\sigma(u;p_N)-\sigma(0-;p_N)\}$$
$$=\sum_{\lambda_n\in\Lambda^+,\lambda_n<u}|c_n|^2-\sum_{\lambda_n\in\Lambda_N^+,\lambda_n<u}|c_n|^2$$
$$=\begin{cases}0, & (0\le u<N)\\ \displaystyle\sum_{\lambda_n\in\Lambda^+,N\le\lambda_n<u}|c_n|^2, & (N\le u)\end{cases}.$$

Therefore we shall conclude that

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(z)-p_N(z)|^2\,dx=\frac{2y}{\sqrt{2\pi}}\int_N^{\infty}\left(\sum_{\lambda_n\in\Lambda^+,N\le\lambda_n<u}|c_n|^2\right)e^{-2yu}\,du$$

$$\le\frac{2y}{\sqrt{2\pi}}\left(\sum_{\lambda_n\in\Lambda^+,\lambda_n\ge N}|c_n|^2\right)\int_N^{\infty}e^{-2yu}\,du=\frac{1}{\sqrt{2\pi}}\left(\sum_{\lambda_n\in\Lambda^+,\lambda_n\ge N}|c_n|^2\right)e^{-2yN}\to 0,\quad (N\to\infty).$$

It should be remarked that the method of approximation belongs to the category of N.Wiener.

In the last, we shall also intend to notice some comments.

(1). Nevertheless we need the conditions $(C_0^{\pm})$ and $(L_0^{\pm})$ in the proof of the Theorem D_9, but Paley-Wiener, they proved it without these conditions. To eliminate these conditions in the proof of Theorem D_9, it is an interesting problem. Moreover the spectral analysis on the N.Wiener class S without condition (L_λ) does too.

(2). In III, section 10, we proved that if the boundary function $f(x)$ of analytic function $f(z) \in H_1^2$ is almost periodic in the sense W. Stepanoff of order 2, then $f(z)$ is almost periodic in the sense of H.Bohr. But in [9],Theorem 22, pp. 125~127, we proved it as for analytic function $f(z) \in H_1^1$ and its boundary function $f(x)$ to be almost periodic in the sense of W. Stepanoff of order 1. It seems to be more natural. As well as the theorem in III, section 10, the key point of the proof is as follows that the Cauchy Integral of order 1 is represented by the Poisson Integral of order 1 :

$$C_1(z;f) = U_1(x,y;f).$$

However we proved it by the use of conformal mapping of which transforms the upper half-plane into the interior of unite circle and of properties of Fourier series and its conjugate series (c.f. [8],Theorem 22,pp. 180~182). Then it is desirable to prove it in the upper half-plane directly. It is another important problem.

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