

Research Report

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**On the theory of generalized Hilbert transforms
Chapter III
The generalized harmonic analysis in the complex domain**

by

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正誤表

誤

正

イ)、ロ)、ハ) p.45, 下から 6 行目 ; p.46, 上から 3 行目、同、上から 8 行目。

$$\int_{-A}^A |f(x+iy)|^2 dx = O(A)$$

$$\int_{-A}^A |f(x+iy)|^2 dy = O(A)$$

ニ) p.58, 上から 1 3 行目。

$$0 < \underline{u} < \varepsilon,$$

$$0 < \underline{\varepsilon} < u,$$

ホ) p.58, 上から 1 4 行目。

$$\sigma_\varepsilon(-u) \leq \underline{\sigma_\varepsilon(0)} \leq \sigma_\varepsilon(u).$$

$$\sigma_\varepsilon(-u) \leq \underline{\sigma_\varepsilon(0)} \leq \sigma_\varepsilon(u).$$

ヘ) p.58, 下から 7 行目。

$$(0 < \underline{u} < \varepsilon),$$

$$(0 < \underline{\varepsilon} < u),$$

以上。

ON THE THEORY OF GENERALIZED HILBERT TRANSFORMS
CHAPTER III
THE GENERALIZED HARMONIC ANALYSIS IN THE COMPLEX DOMAIN

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ON THE THEORY OF GENERALIZED HILBERT TRANSFORMS III
THE GENERALIZED HARMONIC ANALYSIS IN THE COMPLEX DOMAIN

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ABSTRACT

Professors RAYMOND E.A.C.PALEY and NORBERT WIENER pointed out that every theory of harmonic analysis of functions of arguments in the real domain has an associated theory of functions of arguments in the complex domain. They had expanded the theory of the generalized harmonic analysis in the complex domain. Their proofs were very ingenious [2] (c.f. Chap.VIII. pp.128~139).

In the previous paper [9] (c.f. Chap.2, pp.102~127), the author expanded the theory of G.H.T. in the complex domain and gave direct proofs of their theorems by our method under setting the presupposed condition on the real part of boundary function.

But it is seemingly unnatural, so we shall set it on the boundary function itself as well as those of Paley—Wiener. Then we shall reconstruct the theory and represent theorems here as the more refined and advanced forms. They proved it in the vertical strip domain, but we shall prove it in the upper-half plane and next in the horizontal strip domain.

Introduction.

The Paley—Wiener, they proved the following theorems.

Theorem $P - W_1$. Let $f(z)$ be a function of the complex variable $z = x + iy$ that is analytic in and on the boundary of the strip $a \leq x \leq b$ and let

$$\int_{-A}^A |f(x + iy)|^2 dx = O(A), \text{ unif. } (a \leq x \leq b).$$

Then we have over $a < x < b$

$$f(x + iy) = \frac{x + iy - c}{2\pi} \int_{-\infty}^{\infty} \frac{f(b + i\eta)}{b + i\eta - c} \frac{d\eta}{b + i\eta - x - iy} - \frac{x + iy - c}{2\pi} \int_{-\infty}^{\infty} \frac{f(a + i\eta)}{a + i\eta - c} \frac{d\eta}{a + i\eta - x - iy},$$

where c is a constant such as $a < b < c$.

They proved also the following theorem.

Theorem $P-W_2$. Let $f(z)$ be an analytic function of complex variable $z = x + iy$ of $a \leq x \leq b$ and let

$$\int_{-A}^A |f(x + iy)|^2 dx = O(A), \text{ unif. } (a \leq x \leq b).$$

Let $f(a + iy)$ and $f(b + iy)$ both belong to the class S as a function of y . Then $f(x + iy)$ belongs to the class S' as a function of y in $a < x < b$.

Theorem $P-W_3$. Let $f(z)$ be an analytic function of complex variable $z = x + iy$ of $a \leq x \leq b$ and let

$$\int_{-A}^A |f(x + iy)|^2 dx = O(A), \text{ unif. } (a \leq x \leq b).$$

Let $f(a + iy)$ and $f(b + iy)$ both be uniform almost periodic as a function of y . Then $f(x + iy)$ also does as a function of y on $a < x < b$.

9. Generalized Harmonic Analysis in the upper-half plane.

We shall start it in the upper-half plane. We shall define a class of analytic functions and notations.

Generalized Hardy Class of order 1. We shall denote it by H_1^2 and it is defined by the set of functions that satisfies the following properties. It is an analytic function $f(z)$, ($z = x + iy$) in the upper-half plane $y > 0$ and the integral

$$\int_{-\infty}^{\infty} \frac{|f(z)|^2}{1+x^2} dx = O(1),$$

uniformly in $y > 0$.

Generalized Cauchy Integral of order 1. We shall write it by $C_1(z; f)$ and it is defined for function $f(x)$ of the class W^2 by the following integral

$$C_1(z; f) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t) dt}{t+i} \frac{1}{t-z}$$

where $z = x + iy$, $y > 0$.

Then we have

Theorem D_1 . Let $f(z)$ be analytic in the upper-half plane and belong to the class H_1^2 . Then we can find the boundary function at $y = 0$. If we denote it by $f(x)$.

(i) We have

$$\lim_{y \rightarrow 0} f(x + iy) = f(x), \text{ a.e.}$$

where if we write $Rf = g$, then $If = \tilde{g}_1$ and so $f = g + i\tilde{g}_1$ ($= f_1$ say).

(ii) It belongs to the class W^2 and we have

$$\lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \frac{|f(x+iy) - f(x)|^2}{1+x^2} dx = 0.$$

(iii) The $f(z)$ is represented as the Generalized Cauchy Integral of order 1, that is

$$f(z) = C_1(z; f) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t) dt}{t+i} \frac{1}{t-z},$$

where $z = x + iy$, $y > 0$.

We proved that the inverse theorem also true as follows.

Theorem D_2 . Let us suppose that $g(x)$ belongs to the class W^2 , and let us define $f(x) = g(x) + i\tilde{g}_1(x)$ and $f(z) = 2C_1(z; g)$. Then we have the following properties

(i) We have $f(z) = C_1(z; f)$ and $f(z)$ is analytic in the upper half plane ($z = x + iy$, $y > 0$) and belongs to the class H_1^2 .

(ii) The $f(x)$ is the boundary function of $f(z)$ in the following sense

$$\lim_{y \rightarrow 0} f(x+iy) = f(x), \quad a.e. x$$

and

$$\lim_{y \rightarrow 0} \int_{-\infty}^{\infty} \frac{|f(x+iy) - f(x)|^2}{1+x^2} dx = 0.$$

(c.f. [8], I, Chap.3, Theorem 43,44, p.197).

Now we shall prove the theorem of spectral decomposition of $f(z)$ which belongs to the Generalized Hardy Class of order 1: H_1^2 .

Theorem D_3 . Let $f(z)$, ($z = x + iy$) be analytic in the upper half plane $y > 0$ and belong to the class H_1^2 . Let us denote by $f(x)$ its boundary function at $y = 0$.

Then we have for any given positive number ε

(i) if $|u| \geq \varepsilon$, then

$$\begin{aligned} & s(u + \varepsilon; f(z)) - s(u - \varepsilon; f(z)) \\ &= \frac{(1 + \operatorname{sign} u)}{2} e^{-yu} \left(\{s(u + \varepsilon; f) - s(u - \varepsilon; f)\} + r_0(u, y, \varepsilon; f) \right) \end{aligned}$$

where

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{|u| \geq \varepsilon} |r_0(u, y, \varepsilon; f)|^2 du = 0,$$

for every $y > 0$.

(ii) if $|u| \leq \varepsilon$, then

$$s(u + \varepsilon; f(z)) - s(u - \varepsilon; f(z)) = ir_1(u + \varepsilon; f) + ir_2(u + \varepsilon; f) + r_3(u + \varepsilon, y; f)$$

where

$$ir_1(u + \varepsilon; f) + ir_2(u + \varepsilon; f) = s(u + \varepsilon; f) - s(u - \varepsilon; f)$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{|u| \leq \varepsilon} |r_3(u + \varepsilon, y; f)|^2 du = 0,$$

for every $y > 0$.

Proof of Theorem D_3 . This can be done by running the same lines of Theorem A , so we shall remain to sketch it for the first half part and give the detailed proof for the estimation of remainder terms.

Let us put

$$f_B(t) = \begin{cases} f(t), & \text{if } |t| \leq B \\ 0, & \text{if } |t| > B \end{cases}$$

Then we have for any given ε

(i) if $|u| \geq \varepsilon$, then

$$\begin{aligned} s(u + \varepsilon; f_B(z)) - s(u - \varepsilon; f_B(z)) &= l.i.m. \frac{1}{\sqrt{2\pi}} \int_{-A}^A C_1(z; f_B) \frac{2 \sin \varepsilon t}{t} e^{-iut} dt \\ &= \frac{(1 + \operatorname{sign} u)}{2} e^{-uy} \frac{1}{\sqrt{2\pi}} \int_{-B}^B f(s) \frac{e^{i\varepsilon(s-iy)} - e^{-i\varepsilon(s-iy)}}{i(s-iy)} e^{-ius} ds. \end{aligned}$$

Let us remark that

$$l.i.m. C_1(z; f_B) \frac{\sin \varepsilon t}{t} = C_1(z; f) \frac{\sin \varepsilon t}{t},$$

then we have

$$s(u + \varepsilon; f(z)) - s(u - \varepsilon; f(z)) = l.i.m. \left\{ s(u + \varepsilon; f_B(z)) - s(u - \varepsilon; f_B(z)) \right\}.$$

Therefore we have

$$\begin{aligned} s(u + \varepsilon; f(z)) - s(u - \varepsilon; f(z)) &= l.i.m. \frac{1}{\sqrt{2\pi}} \int_{-A}^A C_1(z; f) \frac{2 \sin \varepsilon t}{t} e^{-iut} dt \\ &= \frac{(1 + \operatorname{sign} u)}{2} e^{-uy} l.i.m. \frac{1}{\sqrt{2\pi}} \int_{-B}^B f(s) \frac{e^{i\varepsilon(s-iy)} - e^{-i\varepsilon(s-iy)}}{i(s-iy)} e^{-ius} ds. \end{aligned}$$

Now we shall decompose the kernel of integral on the right hand side as follows

$$\begin{aligned} \frac{e^{i\varepsilon(s-iy)} - e^{-i\varepsilon(s-iy)}}{i(s-iy)} &= \frac{2 \sin \varepsilon s}{s-iy} + \frac{e^{i\varepsilon s} (e^{\varepsilon y} - 1)}{i(s-iy)} - \frac{e^{-i\varepsilon s} (e^{-\varepsilon y} - 1)}{i(s-iy)} \\ &= \frac{2 \sin \varepsilon s}{s} + \frac{iy}{s-iy} \frac{2 \sin \varepsilon s}{s} + \frac{e^{i\varepsilon s} (e^{\varepsilon y} - 1)}{i(s-iy)} - \frac{e^{-i\varepsilon s} (e^{-\varepsilon y} - 1)}{i(s-iy)} \end{aligned}$$

$$= \frac{2 \sin \varepsilon s}{s} + K_{01}(s, y, \varepsilon) + K_{02}(s, y, \varepsilon) + K_{03}(s, y, \varepsilon), \text{ say.}$$

Let us put

$$r_{0i}(u, y, \varepsilon; f) = \text{l.i.m.}_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B f(s) K_{0i}(s, y, \varepsilon) e^{-ius} ds, \quad (i=1,2,3)$$

and

$$r_0(u, y, \varepsilon; f) = r_{01}(u, y, \varepsilon; f) + r_{02}(u, y, \varepsilon; f) + r_{03}(u, y, \varepsilon; f).$$

Then as for $r_{01}(u, y, \varepsilon; f)$, we have

$$r_{01}(u, y, \varepsilon; f) = (iy) \text{l.i.m.}_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s-iy} \frac{2 \sin \varepsilon s}{s} e^{-ius} ds$$

and applying the Plancherel theorem, we have

$$\frac{1}{\varepsilon} \int_{|u| \geq \varepsilon} |r_{01}(u, y, \varepsilon; f)|^2 du \leq 4\varepsilon y^2 \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = o(1), \quad (\varepsilon \rightarrow 0),$$

for every $y > 0$.

As for $r_{02}(u, y, \varepsilon; f)$, we have

$$r_{02}(u, y, \varepsilon; f) = -i(e^{\varepsilon y} - 1) \text{l.i.m.}_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s-iy} e^{-i(u-\varepsilon)s} ds,$$

and applying the Plancherel theorem too, we have

$$\begin{aligned} \frac{1}{\varepsilon} \int_{|u| \geq \varepsilon} |r_{02}(u, y, \varepsilon; f)|^2 du &\leq \frac{(e^{\varepsilon y} - 1)^2}{\varepsilon} \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds \\ &= O(\varepsilon y^2) \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = o(1), \quad (\varepsilon \rightarrow 0), \end{aligned}$$

for every $y > 0$. By the similar way, we have

$$\frac{1}{\varepsilon} \int_{|u| \geq \varepsilon} |r_{03}(u, y, \varepsilon; f)|^2 du = O(\varepsilon y^2) \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = o(1), \quad (\varepsilon \rightarrow 0),$$

for every $y > 0$. Therefore we have

$$s(u + \varepsilon; f(z)) - s(u - \varepsilon; f(z)) = \frac{(1 + \text{sign} u)}{2} e^{-yu} (\{s(u + \varepsilon; f) - s(u - \varepsilon; f)\} + r_0(u, y, \varepsilon; f))$$

and

$$\frac{1}{2\varepsilon} \int_{|u| \geq \varepsilon} |r_0(u, y, \varepsilon; f)|^2 du = O(\varepsilon y^2) \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = o(1), \quad (\varepsilon \rightarrow 0),$$

for every $y > 0$. Thus the first half part of Theorem D_3 has established.

Next we have

(ii) if $|u| \leq \varepsilon$, then

$$\begin{aligned} & \lim_{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^A C_1(z; f_B) \frac{2 \sin \varepsilon t}{t} e^{-iut} dt \\ &= \frac{1}{\sqrt{2\pi}} \int_{-B}^B f(s) \frac{e^{i(s-iy)u} - e^{-i(s-iy)\varepsilon}}{i(s-iy)} e^{-i(s-iy)u} ds + \frac{i}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s+i} ds \end{aligned}$$

Now we shall decompose the kernel of integral of right hand side as follows

$$\begin{aligned} & \frac{e^{i(s-iy)u} - e^{-i(s-iy)\varepsilon}}{i(s-iy)} e^{-i(s-iy)u} + \frac{i}{s+i} \\ &= \frac{e^{isu} - e^{-is\varepsilon}}{is} e^{-isu} + \frac{i}{s+i} + \left\{ \frac{e^{i(s-iy)u} - e^{-i(s-iy)\varepsilon}}{i(s-iy)} e^{-i(s-iy)u} - \frac{e^{isu} - e^{-is\varepsilon}}{is} e^{-isu} \right\} \\ &= \frac{i}{s+i} \frac{e^{-i(u+\varepsilon)s} - 1}{-is} + \frac{i}{s+i} e^{-i(u+\varepsilon)s} + \left\{ \frac{e^{-(s-iy)(u+\varepsilon)} - 1}{-i(s-iy)} - \frac{e^{-is(u+\varepsilon)} - 1}{-is} \right\} \\ &= iK_1(s, u + \varepsilon) + iK_2(s, u + \varepsilon) + K_3(s, u + \varepsilon, y), \quad \text{say.} \end{aligned}$$

Then we have

$$r_1(u + \varepsilon; f) = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B K_1(s, u + \varepsilon) f(s) ds = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s+i} \frac{e^{-i(u+\varepsilon)s} - 1}{-is} ds$$

and

$$r_2(u + \varepsilon; f) = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B K_2(s, u + \varepsilon) f(s) ds = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s+i} e^{-i(u+\varepsilon)s} ds.$$

Now we have to prove that

$$ir_1(u + \varepsilon; f) + ir_2(u + \varepsilon; f) = s(u + \varepsilon; f) - s(u - \varepsilon; f).$$

For this purpose, let us remark the Theorems D_1 and D_2 . There we proved that the necessary and sufficient condition for $f = f(x, a)$ to be the boundary function of an analytic function of the class H_1^2 is as follows. If the real part is $Re f = g$ then the imaginary part is $Im f = \tilde{g}_1$ and we can write it as $f = g + i\tilde{g}_1$.

We shall also quote the skew reciprocal formula of G.H.T.. That is as follows

$$\frac{\tilde{g}_1(x)}{x+i} = P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(t) dt}{t+i} \frac{1}{x-t} \quad \text{and} \quad P.V. \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\tilde{g}_1(t) dt}{t+i} \frac{1}{x-t} = -\frac{g(x)}{x+i}.$$

We can write it as

$$(\tilde{g}_1)_1^{\sim}(x) = -g(x).$$

By the part (ii) of Theorem A, we have

$s(u + \varepsilon; \tilde{g}_1) - s(u - \varepsilon; \tilde{g}_1) = i\{s(u + \varepsilon; g) - s(u - \varepsilon; g)\} + 2r_1(u + \varepsilon; g) + 2r_2(u + \varepsilon; g)$
and we have

$$2r_1(u + \varepsilon; g) + 2r_2(u + \varepsilon; g) = -i\{s(u + \varepsilon; f) - s(u - \varepsilon; f)\}.$$

Next by the property of skew-reciprocal formula of the G.H.T., we have

$$\tilde{g}_1(x) + i(\tilde{g}_1)_1^-(x) = \tilde{g}_1(x) + i(-g(x)) = -if(x).$$

Then by the same reason that derived the above formula, we have

$$\begin{aligned} 2r_1(u + \varepsilon; \tilde{g}_1) + 2r_2(u + \varepsilon; \tilde{g}_1) &= -i\{s(u + \varepsilon; -if) - s(u - \varepsilon; -if)\} \\ &= -\{s(u + \varepsilon + f) - s(u - \varepsilon; f)\}. \end{aligned}$$

Therefor we shall have the desired formula

$$\begin{aligned} ir_1(u + \varepsilon; f) + ir_2(u + \varepsilon; f) &= i\{r_1(u + \varepsilon; g) + r_2(u + \varepsilon; g)\} - \{r_1(u + \varepsilon; \tilde{g}_1) + r_2(u + \varepsilon; \tilde{g}_1)\} \\ &= s(u + \varepsilon; f) - s(u - \varepsilon; f). \end{aligned}$$

In the last, let us write

$$r_3(u + \varepsilon, y; f) = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B K_3(s, u + \varepsilon, y) f(s) ds.$$

then nothing remains but to prove is

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_{|u| \leq \varepsilon} |r_3(u + \varepsilon, y; f)|^2 du = 0.$$

For this purpose, we shall rewrite the kernel $K_3(s, u + \varepsilon, y)$ of the integral as follows,

$$\begin{aligned} K_3(s, u + \varepsilon, y) &= \frac{e^{-(s-iy)(u+\varepsilon)} - 1}{-i(s-iy)} - \frac{e^{-is(u+\varepsilon)} - 1}{-is} \\ &= \frac{s(e^{-i(s-iy)(u+\varepsilon)} - 1) - (s-iy)(e^{-is(u+\varepsilon)} - 1)}{-is(s-iy)} = \frac{se^{-i(u+\varepsilon)s} - se^{-i(u+\varepsilon)(s-iy)} + iy(e^{-i(u+\varepsilon)s} - 1)}{-is(s-iy)} \\ &= \frac{(1 - e^{-(u+\varepsilon)y})e^{-i(u+\varepsilon)s}}{i(s-iy)} + \frac{y(e^{-i(u+\varepsilon)s} - 1)}{-s(s-iy)} = K_{31}(s, u + \varepsilon, y) + K_{32}(s, u + \varepsilon, y) \quad \text{say.} \end{aligned}$$

and let us put

$$r_{3i}(u + \varepsilon, y; f) = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B K_{3i}(s, u + \varepsilon, y) f(s) ds, \quad (i = 1, 2).$$

Then applying the Plancherel theorem to $r_{31}(u + \varepsilon, y; f)$, we have

$$\frac{1}{\varepsilon} \int_{|u| \leq \varepsilon} |r_{31}(u + \varepsilon, y; f)|^2 du \leq \frac{(1 - e^{-\varepsilon y})^2}{\varepsilon} \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = O(\varepsilon y^2) \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = o(1), \quad (\varepsilon \rightarrow 0).$$

for every $y > 0$.

As for $r_{32}(u + \varepsilon, y; f)$, we can rewrite it as follows

$$r_{32}(u + \varepsilon, y; f) = \lim_{B \rightarrow \infty} \frac{iy}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s-iy} \left(\int_0^{u+\varepsilon} e^{-ivs} dv \right) ds = iy \int_0^{u+\varepsilon} dv \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s-iy} e^{-ivs} ds$$

and if we put

$$\widehat{f}(v) = \lim_{B \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-B}^B \frac{f(s)}{s - iy} e^{-ivs} ds$$

then we have

$$\begin{aligned} \frac{1}{\varepsilon} \int_{|u| \leq \varepsilon} |r_{32}(u + \varepsilon, y; f)|^2 du &\leq \frac{y^2}{\varepsilon} \int_{|u| \leq \varepsilon} \left| \int_0^{u+\varepsilon} \widehat{f}(v) dv \right|^2 du \leq 2y^2 \left(\int_0^{2\varepsilon} |\widehat{f}(v)| dv \right)^2 \\ &\leq 4\varepsilon y^2 \int_0^{2\varepsilon} |\widehat{f}(v)|^2 dv \leq 4\varepsilon y^2 \int_{-\infty}^{\infty} \frac{|f(s)|^2}{s^2 + y^2} ds = o(1), \quad (\varepsilon \rightarrow 0), \end{aligned}$$

for every $y > 0$. Thus the second half part of Theorem D_3 has established.

Next we shall prove the theorem of spectral synthesis of analytic function in the upper-half plane.

Theorem D_4 . Let $f(z)$, ($z = x + iy$) be analytic in the upper-half plane $y > 0$ and belongs to the class H_1^2 . Let us denote by $f(x)$ its boundary function at $y = 0$. Let us suppose that $f(x)$ belongs to the class S , then $f(z)$ does to the class S' .

We shall use the following result as essential roles for proof of the theorem.

Lemma D_4 . Let us suppose that $f(x)$ belongs to the class S , then the following limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^u |s(v + \varepsilon, f) - s(v - \varepsilon; f)|^2 dv$$

exists and equals to

$$\sigma(u) - \sigma(0-), \quad a.e. u,$$

over any finite range of u , where

$$\begin{aligned} \varphi(x) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x+t) \overline{f(t)} dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} |s(u + \varepsilon; f) - s(u - \varepsilon; f)|^2 du \end{aligned}$$

and

$$(21.21) \quad \sigma(u) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \varphi(x) \frac{e^{-iux} - 1}{-ix} dx + \lim_{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_1^A \right] \varphi(x) \frac{e^{-iux}}{-ix} dx.$$

Remark (1). N.Wiener [1] (c.f.(21. 175) at p.160 ;(21. 21), (21. 22) ,(21. 23) and (21. 24) at p.161) introduced the following functions.

$$(21. 175) \quad \varphi_\varepsilon(x) = \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 du$$

and

$$(21. 22) \quad \sigma_\varepsilon(u) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx + \text{l.i.m.}_{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_1^A \right] \varphi_\varepsilon(x) \frac{e^{-iux}}{-ix} dx.$$

Then he pointed out the following formula

$$(21. 23) \quad \text{l.i.m.}_{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^A \varphi_\varepsilon(x) e^{-iux} dx = \frac{1}{2\varepsilon\sqrt{2\pi}} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2,$$

and if we integrate both side, we get

$$(21. 24) \quad P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx = \frac{1}{2\varepsilon\sqrt{2\pi}} \int_0^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv$$

But we could not follow the formula (21. 23) and so we shall prove (21.24) by other method.

Remark (2). N.Wiener[1] (c.f.21. 25),(21. 255),(21. 257) and (21.26) at p.161~162)) proved that formula5s (21. 22) and (21. 24) yield us

$$(21. 25) \quad \sigma_\varepsilon(u) = \text{const.} + \int_0^u \frac{1}{2\varepsilon\sqrt{2\pi}} |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv.$$

Since $\varphi_\varepsilon(x)$ tends to $\varphi(x)$ boundedly as $\varepsilon \rightarrow 0$, $\varphi_\varepsilon(x)/(-ix)$ tends in the mean to $\varphi(x)/(-ix)$ over any range not containing the origin. From these facts we may readily conclude that over any finite range of u,

$$(21. 255) \quad \sigma(u) = \text{l.i.m.}_{\varepsilon \rightarrow 0} \sigma_\varepsilon(u).$$

It follows that

$$(21. 257) \quad \sigma(u) = \text{l.i.m.}_{\varepsilon \rightarrow 0} \left(\text{const.} + \frac{1}{2\varepsilon\sqrt{2\pi}} \int_0^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv \right).$$

The constant in this formula may readily be verified to be

$$-\left[\int_{-A}^{-1} + \int_1^A \right] \frac{\varphi_\varepsilon(x)}{ix} dx,$$

as the A tends to infinity. It will be seen that

$$(21.26) \quad \sigma(u) - \sigma(-u) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-u}^u |s(v + \varepsilon; f) - s(v - \varepsilon; f)|^2 dv.$$

As we have proved in the Theorem D_3 that $s(u + \varepsilon; f(z)) - s(u - \varepsilon; f(z))$ vanish over any range of $u < -\varepsilon$, so we should require that the existence of the following limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^u |s(v + \varepsilon; f) - s(v - \varepsilon; f)|^2 dv,$$

over any finite range of u .

To obtain the formula (21.23) due to N.Wiener, we shall quote the theorems of S.Bochner [3](c.f. Theorem 23 at p.95) and P.Levy[4](c.f.P.Levy[4] at pp.163~172 or S Bochner[3] Theorem17 at p.83)

The Function of positive definite :

We shall call the function $\phi(x)$ to be of positive definite if the following properties are satisfied.

(i) $\phi(x)$ is continuous for all real arguments.

(ii) Hermitian symmetric : $\overline{\phi(-x)} = \phi(x)$.

(iii) For any finite real number x_1, x_2, \dots, x_n and any complex number $\alpha_1, \alpha_2, \dots, \alpha_n$, the following formula

$$\sum_{i,j=1}^n \phi(x_i - x_j) \alpha_i \bar{\alpha}_j \geq 0$$

is satisfied.

Then we have

The Theorem of S. Bochner : The necessary and sufficient condition for a function $\phi(x)$ to be of positive definite is that the following properties are satisfied. That is there exist a bounded and monotone increasing function $\Lambda(u)$ and $\phi(x)$ is represented by the following formula

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\Lambda(u).$$

The Inversion Formula of the Fourier-Stieltjes Transform :

The theorem of P.Levy : If $\phi(x)$ is the function of positive definite, then we have the following formula

$$\Lambda(u) - \Lambda(0) = P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) \frac{e^{-iux} - 1}{-ix} dx.$$

The former is often called the S.Bochner representation theorem as for function of

positive definite and the latter the P.Levy inversion formula.

Now we shall going to prove Lemma D_4 and Theorem D_4 .

Proof of the Lemma D_4 . We shall write the formula (21. 175) as follows

$$\varphi_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux} d\Lambda_\varepsilon(u),$$

where

$$\Lambda_\varepsilon(u) = \frac{1}{2\varepsilon\sqrt{2\pi}} \int_0^u |s(v + \varepsilon; f) - s(v - \varepsilon; f)| dv.$$

Since the $\varphi_\varepsilon(x)$ is a function of positive definite and so we have the formula (21. 24) by the inversion formula of P.Levy. Therefore by running on the same lines as N.Wiener, we have

$$l.i.m._{\varepsilon \rightarrow 0} \sigma_\varepsilon(u) = \sigma(u).$$

Now we shall decompose the right-hand side of (21.24) as follows

$$\begin{aligned} P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx \\ = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx + l.i.m._{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_1^A \right] \varphi_\varepsilon(x) \frac{e^{-iux}}{-ix} dx \\ + P.V. \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} + \int_1^{\infty} \right] \frac{\varphi_\varepsilon(x)}{ix} dx \\ = \sigma_\varepsilon(u) + C_\varepsilon, \quad \text{a.e. u.} \end{aligned}$$

Here, we should read the last term of the above formula as follows: There exist a sequence $\{A_j\}$ tending to infinity such that

$$P.V. \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} + \int_1^{\infty} \right] \frac{\varphi_\varepsilon(x)}{ix} dx = \lim_{A_j \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A_j}^{-1} + \int_1^{A_j} \right] \frac{\varphi_\varepsilon(x)}{ix} dx$$

and it is finitely determined. We shall denote this constant by C_ε .

Because we have

$$P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx = \lim_{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^A \varphi_\varepsilon(x) \frac{e^{-iux} - 1}{-ix} dx$$

and furthermore, by H.Weyl's lemma to the Riesz - Fisher theorem as for the criterion of completeness of L^2 space (c.f. N.Wiener[1], Section 3 at pp.27~34), there exists a

sub-sequence $\{A_j\}$ of $\{A\}$ tending to infinity, we have

$$\text{l.i.m.}_{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_1^A \right] \varphi_\varepsilon(x) \frac{e^{-ix}}{-ix} dx = \lim_{A_j \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A_j}^{-1} + \int_1^{A_j} \right] \varphi_\varepsilon(x) \frac{e^{-ix}}{-ix} dx.$$

Thus we have the above formula in this sense and the constant C_ε is finitely determined.

Here we should remark that if $f(x)$ belongs to the class S then its auto-correlation function $\phi(x)$ is not necessarily the function of positive definite.

(i) Continuity of $\phi(x)$: Because $\phi(x)$ exist for all x by definition but it is not necessarily continuous. N.Wiener(c.f.[1],Theorem 26,pp.154~156) pointed out about this circumference as follows that if $\phi(x)$ is continuous at $x = 0$ then it is continuous for all real arguments and $f(x)$ belongs to the class S' .

We should remark also that if $f(x)$ belongs to the class S then $\phi(x)$ satisfies the properties (ii) and (iii) as follows.

(ii) Hermitian Symmetry: We have

$$\begin{aligned} \overline{\phi(-x)} &= \overline{\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(-x+t) \overline{f(t)} dt} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(t) \overline{f(-x+t)} dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T-x}^{T-x} f(x+s) \overline{f(s)} ds = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x+s) \overline{f(s)} ds \\ &= \phi(x). \end{aligned}$$

(iii) Positive Definiteness: We have for any finite real number x_1, x_2, \dots, x_n and any complex number $\alpha_1, \alpha_2, \dots, \alpha_n$, the following formula

$$\begin{aligned} &\sum_{i,j=1}^n \phi(x_i - x_j) \alpha_i \overline{\alpha_j} \geq 0 \\ &= \sum_{i,j=1}^n \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x_i - x_j + t) \overline{f(t)} dt \right) \alpha_i \overline{\alpha_j} = \sum_{i,j=1}^n \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T-x_j}^{T-x_j} f(x_i + s) \overline{f(x_j + s)} ds \right) \alpha_i \overline{\alpha_j} \\ &= \sum_{i,j=1}^n \left(\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x_i + s) \overline{f(x_j + s)} ds \right) \alpha_i \overline{\alpha_j} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left(\sum_{i,j=1}^n f(x_i + s) \overline{f(x_j + s)} \alpha_i \overline{\alpha_j} \right) ds \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \left| \sum_{i=1}^n f(x_i + s) \alpha_i \right|^2 ds \geq 0. \end{aligned}$$

Now if $f(x)$ belongs to the class S , then we have by the theorem N.Wiener(c.f.

[1], Theorems 35, 36 at p.183)

$$S(u) = P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) \frac{e^{-iux} - 1}{-ix} dx$$

exist for every u and we shall have

$$S(u) - \sigma(u) = C, \quad a.e. u.$$

Thus we have similarly

$$\begin{aligned} & P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(x) \frac{e^{-iux} - 1}{-ix} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^1 \phi(x) \frac{e^{-iux} - 1}{-ix} dx + l.i.m._{A \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_1^A \right] \phi(x) \frac{e^{-iux}}{-ix} dx \\ & \quad + P.V. \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{-1} + \int_1^{\infty} \right] \frac{\phi(x)}{ix} dx \\ &= \sigma(u) + C, \quad a.e. u. \end{aligned}$$

We shall return to the formula, we have

$$\begin{aligned} \Lambda_{\varepsilon}(u) &= \frac{1}{2\varepsilon\sqrt{2\pi}} \int_0^u |s(v + \varepsilon; f) - s(v - \varepsilon; f)|^2 dv \\ &= P.V. \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi_{\varepsilon}(x) \frac{e^{-iux} - 1}{-ix} dx = \sigma_{\varepsilon}(u) + C_{\varepsilon}, \quad a.e. u, \end{aligned}$$

and we shall conclude that

$$\frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^u |s(v + \varepsilon; f) - s(v - \varepsilon; f)|^2 dv = \sigma_{\varepsilon}(u) - \sigma_{\varepsilon}(-\varepsilon), \quad a.e. u,$$

Now, we shall quote the Paley - Wiener lemma (c.f. R.E.A.C. Paley - N. Wiener [2], pp.134 ~5).

Lemma $P - W$. Let $\{f_n(x)\}$ be a sequence of monotone functions and

$$l.i.m._{n \rightarrow \infty} f_n(x) = f(x), \quad (L^2).$$

Then we have

$$\lim_{n \rightarrow \infty} f_n(x) = f(x), \quad a.e. x.$$

Since we have

$$l.i.m._{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(u) = \sigma(u), \quad (L^2),$$

and $\sigma_{\varepsilon}(u)$ is a bounded, continuous and monotone increasing function. Then applying Paley - Wiener's lemma , we could conclude that

$$\lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(u) = \sigma(u), \quad a.e.u,$$

and $\sigma(u)$ is also a bounded and monotone increasing function.

Next we shall intend to calculate the limit $\lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(-\varepsilon)$ and prove that it equals to $\sigma(0-)$. For this purpose, we shall consider the normalized $\sigma(u)$ by changing the values of definition of it on the set of measure zero, if it is required. Thus $\sigma(u)$ is the function of bounded and monotone increasing and it has the first kind of discontinuity at most on the set of enumerable points, there it is satisfied

$$\sigma(u) = \frac{\sigma(u+0) + \sigma(u-0)}{2}.$$

In the first, let us suppose that $\sigma(u)$ is continuous at $u = 0$. Then for any given positive number ε and all of u such as $0 < u < \varepsilon$, we have

$$\sigma_{\varepsilon}(-u) \leq \sigma_{\varepsilon}(0) \leq \sigma_{\varepsilon}(u).$$

Let us tend ε to 0, we have

$$\sigma(-u) \leq \lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(0) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(0) \leq \sigma(u), \quad a.e.u,$$

and let us tend u to 0 through which belongs to the set of existence of $\sigma(\pm u)$, we have

$$\lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(0) = \sigma(0).$$

Next, we have for any given positive number ε ,

$$\sigma_{\varepsilon}(-u) \leq \sigma_{\varepsilon}(-\varepsilon) \leq \sigma_{\varepsilon}(0), \quad (0 < u < \varepsilon),$$

and it follows that tending ε to 0,

$$\sigma(-u) \leq \lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(-\varepsilon) \leq \overline{\lim}_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(-\varepsilon) \leq \sigma(0), \quad a.e.u.$$

Let us tend u to 0 through which belongs to the set of existence of $\sigma(-u)$, we have

$$\lim_{\varepsilon \rightarrow 0} \sigma_{\varepsilon}(-\varepsilon) = \sigma(0) .$$

In the second ,let us suppose that $\sigma(u)$ is not continuous at $u = 0$. Since we consider the $\sigma(u)$ to be normalized, so we have

$$\sigma(0) = \frac{\sigma(0+) + \sigma(0-)}{2}.$$

Let us put

$$\sigma(0+) - \sigma(0-) = d, \text{ say}$$

and let us consider

$$\sigma^*(u) = \sigma(u) - dh(u),$$

where $h(u)$ is the Heaviside operator function such as

$$h(u) = \begin{cases} 0, & u < 0 \\ 1/2, & u = 0 \\ 1, & u > 0 \end{cases}$$

Then the function $\sigma^*(u)$ is continuous at $u=0$. We shall consider

$$\sigma_\varepsilon^*(u) = \sigma_\varepsilon(u) - dh(u).$$

Now if we apply the result of the first part to the $\{\sigma_\varepsilon^*(u)\}$ and $\sigma^*(u)$, then we have

$$\lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon^*(-\varepsilon) = \sigma^*(0)$$

and so

$$\lim_{\varepsilon \rightarrow 0} (\sigma_\varepsilon(-\varepsilon) - dh(-\varepsilon)) = \sigma^*(0) = \sigma(0) - dh(0).$$

Thus we conclude that

$$\lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon(-\varepsilon) = \sigma(0-).$$

Thus we could conclude that the following relation

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv \\ & = \sigma(u) - \sigma(0-), \text{ a.e. } u, \end{aligned}$$

is established over any finite range of u . This is the formula of which we have been desired.

Similarly we conclude that

$$\lim_{\varepsilon \rightarrow 0} \sigma_\varepsilon(+\varepsilon) = \sigma(0+)$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{\varepsilon}^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv = \sigma(u) - \sigma(0+), \text{ a.e. } u$$

over any finite range of u , and so we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^{\varepsilon} |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv = \sigma(0+) - \sigma(0-).$$

Remark. The proof of the above formula can be done by the use of the following properties only. The sequence $\{\sigma_\varepsilon(u)\}$ is that of a bounded monotone functions and converges to the $\sigma(u)$, *a.e.u.*

Proof of Theorem D_4 . In the first we shall prove that $f(z) = C_1(z; f)$ belongs to the class of S . We have the following formula by the Theorem D_3

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2 du \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\varepsilon}^{\infty} e^{iux} |e^{-yu} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}|^2 du \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \left(\int_{-\varepsilon}^A (") du + \int_A^{\infty} (") du \right) = I_1 + I_2, \quad \text{say.} \end{aligned}$$

We have

$$|I_2| \leq \frac{e^{-2Ay}}{4\pi\varepsilon} \int_A^{\infty} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 du$$

and then

$$\begin{aligned} \overline{\lim}_{\varepsilon \rightarrow 0} |I_2| &\leq e^{-2Ay} \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 du \\ &= e^{-2Ay} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx \rightarrow 0, \quad (A \rightarrow \infty). \end{aligned}$$

Now, for the A sufficiently large and to be fixed, we have by the integration by part

$$\begin{aligned} I_1 &= \frac{1}{4\pi\varepsilon} \int_{-\varepsilon}^A e^{iux} |e^{-yu} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}|^2 du \\ &= \left[\frac{e^{(ix-2y)u}}{4\pi\varepsilon} \int_{-\varepsilon}^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv \right]_{u=-\varepsilon}^{u=A} \\ &\quad - \frac{(ix-2y)}{4\pi\varepsilon} \int_{-\varepsilon}^A e^{(ix-2y)u} \left(\int_{-\varepsilon}^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv \right) du \end{aligned}$$

$$= \frac{e^{(ix-2y)A}}{\sqrt{2\pi}} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^A |s(v+\varepsilon; f) - s(u-\varepsilon; f)|^2 dv$$

$$- \frac{(ix-2y)}{\sqrt{2\pi}} \int_{-\varepsilon}^A e^{(ix-2y)u} \left(\frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^u |s(v+\varepsilon; f) - s(v-\varepsilon; f)|^2 dv \right) du.$$

By the Lemma D_4 , we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon\sqrt{2\pi}} \int_{-\varepsilon}^u |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 dv = \sigma(u) - \sigma(0-), \quad a.e. u$$

and its bounded convergence is guaranteed over any finite range of u .

Thus we have

$$\lim_{\varepsilon \rightarrow 0} I_1 = \frac{e^{(ix-2y)A}}{\sqrt{2\pi}} (\sigma(u) - \sigma(0-)) - \frac{(ix-2y)}{\sqrt{2\pi}} \int_0^A (\sigma(u) - \sigma(0-)) e^{(ix-2y)u} du$$

$$\rightarrow - \frac{(ix-2y)}{\sqrt{2\pi}} \int_0^\infty (\sigma(u) - \sigma(0-)) e^{(ix-2y)u} du, \quad (A \rightarrow \infty).$$

for all x . Therefor we could conclude that there exists the following limit

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^\infty e^{iux} |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2 du$$

$$= - \frac{(ix-2y)}{\sqrt{2\pi}} \int_0^\infty (\sigma(u) - \sigma(0-)) e^{(ix-2y)u} du,$$

for all x . Thus we have proved that $f(z) = C_1(z; f)$ belongs to the class S . In particular, if we put $x = 0$, then we get the following formula

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^\infty |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2 du = \frac{2y}{\sqrt{2\pi}} \int_0^\infty (\sigma(u) - \sigma(0-)) e^{-2yu} du.$$

In the last, we shall quote the theorem of N.Wiener([1], Theorem 28, p.160) and we shall prove that $f(z) = C_1(z; f)$ belongs to the class S' . We have

$$\lim_{A \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \left[\int_{-\infty}^{-A} + \int_A^\infty \right] |s(u+\varepsilon; f(z)) - s(u-\varepsilon; f(z))|^2 du$$

$$= \lim_{A \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_A^\infty |e^{-yu} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}|^2 du$$

$$\begin{aligned} &\leq \lim_{A \rightarrow \infty} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{e^{-2yA}}{4\pi\varepsilon} \int_A^\infty |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 du \\ &= \lim_{A \rightarrow \infty} e^{-2yA} \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^\infty |s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2 du \\ &= \lim_{A \rightarrow \infty} e^{-2yA} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |f(x)|^2 dx = 0, \end{aligned}$$

for all $y > 0$. Thus we have proved that $f(z) = C_1(z; f)$ belongs to the class S' for all $y > 0$. Then we have

$$\begin{aligned} \varphi(x; f(z)) &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x+t+iy) \overline{f(t+iy)} dt \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi\varepsilon} \int_{-\varepsilon}^\infty e^{ix} |e^{-yu} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\}|^2 du \\ &= -\frac{(ix-2y)}{\sqrt{2\pi}} \int_0^\infty (\sigma(u) - \sigma(-0)) e^{-2yu} e^{ixu} du, \end{aligned}$$

where $z = x + iy$, $y > 0$.

10. Application to the almost periodic functions.

We shall intend to apply the theory of G.H.T. to the almost periodic functions. Let us start to define the almost periodic functions.

Uniformly almost periodic functions in the sense of H.Bohr.

Let us suppose that $f(x)$ is defined on the real line $-\infty < x < \infty$, is bounded and uniformly continuous. Let us suppose that for any given positive number ε , there exists

a trigonometric polynomial $p(x) = \sum c_n e^{i\lambda_n x}$ such that

$$\|f - p\|_{L^\infty} = \sup_{-\infty < x < \infty} |f(x) - p(x)| \leq \varepsilon.$$

Then we shall define the $f(x)$ to be uniformly almost periodic in the sense of H.Bohr.

Almost periodic functions in the sense of W. Stepanoff.

Let us suppose that $f(x)$ is measurable and belongs to the class L^2 locally with the norm

$$\|f\|_{S^2} = \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(x)|^2 dx \right)^{\frac{1}{2}} < \infty.$$

The l is a length of interval $(x, x+l)$ and to be fixed. Let us suppose that for any

given positive number ε , there exists a trigonometric polynomial $p(x) = \sum_n c_n e^{i\lambda_n x}$

such that $\|f - p\|_{\mathcal{B}^2} \leq \varepsilon$. Then we shall define $f(x)$ to be almost periodic in the sense of W. Stepanoff.

Now we shall intend to give an example. Let us suppose that the boundary function $f(x)$ of analytic function $f(z) \in H_1^2$ is almost periodic in the sense of W. Stepanoff. Then we shall prove that $f(z)$ is almost periodic in the sense of H. Bohr in the upper -half plane $z = x + iy, y > 0$ as a function of x for all fixed $y > 0$.

Let us suppose that $f(z)$ be analytic in the upper-half plane $z = x + iy, y > 0$ and belongs to the class H_1^2 . Then by the Theorem D_2 , there exists boundary function $f(x)$ of the class W^2 and it is represented by the G.C.I.

$$f(z) = C_1(z; f) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{dt}{t-z}.$$

As for boundary function, if we put $Rf = g$, then we have $If = \tilde{g}_1$ and we shall write

$$f(x) = g(x) + i\tilde{g}_1(x).$$

As for the kernel, we shall decompose into

$$\frac{1}{2i} \frac{1}{t-z} = \frac{1}{2} \frac{y}{(t-x)^2 + y^2} - \frac{i}{2} \frac{t-x}{(t-x)^2 + y^2}.$$

We shall write

$$U_1(x, y; f) = \frac{z+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{y}{(t-x)^2 + y^2} dt$$

and

$$\tilde{U}_1(x, y; f) = -\frac{z+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{t-x}{(t-x)^2 + y^2} dt,$$

then we have

$$C_1(z; f) = \frac{1}{2} U_1(x, y; f) + \frac{i}{2} \tilde{U}_1(x, y; f).$$

We proved also the following formulas

$$\tilde{U}_1(x, y; g) = U_1(x, y; \tilde{g}_1), \quad C_1(z; g) = iC_1(z, \tilde{g}_1)$$

(c.f. S.Koizumi[8], Theorems 36,37, pp192-3). Then we have

$$\begin{aligned} C_1(z; f) &= C_1(z; g + i\tilde{g}_1) = C_1(z; g) + iC_1(z; \tilde{g}_1) = 2C_1(z; g) \\ &= U_1(x, y; g) + i\tilde{U}_1(x, y; g) = U_1(x, y; g) + iU_1(x, y; \tilde{g}_1) \end{aligned}$$

$$= U_1(x, y; g + i\tilde{g}_1) = U_1(x, y; f).$$

Let us remark that the Poisson kernel $P(x, y)$ has the following properties. It is a positive kernel and mean value 1. That is

$$P(x, y) = \frac{y}{x^2 + y^2} > 0, \quad (y > 0) \quad \text{and} \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{x^2 + y^2} dx = 1.$$

The boundedness of $f(z)$.

Let us start the following formula of which we stated in the previous paper without proof. This is as follows.

We have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt = \frac{y+1}{x^2 + (y+1)^2} = \frac{y+1}{|z+1|^2},$$

where $z = x + iy, y > 0$.

(c.f. S.Koizumi [9], II, *Lemma 22*₁, pp.125 ~ 6).

We shall prove it for the sake of completeness

Since

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y|x|} e^{itx} dx = \sqrt{\frac{2}{\pi}} \frac{y}{t^2 + y^2}$$

we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{t^2 + y^2} e^{-ixt} dt = e^{-y|x|},$$

and in particular if we put $y = 1$, we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2 + 1} e^{-ixt} dt = e^{-|x|}.$$

Taking the Fourier transform of convolution product between two functions, we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{y}{(x-t)^2 + y^2} dt \right) e^{-iux} dx \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} e^{-iu(x-t)} dx \right) \frac{1}{1+t^2} e^{-iut} dt \\ &= \sqrt{\frac{\pi}{2}} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{x^2 + y^2} e^{-iux} dx \right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-iut} dt \right) \\ &= \sqrt{\frac{\pi}{2}} e^{-y|u|} e^{-|u|} = e^{-(y+1)|u|}. \end{aligned}$$

Taking the invers Fourier transform on both side in the above formula, we have

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{y}{(x-t)^2 + y^2} dt \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(y+1)|u|} e^{ixu} du = \frac{y+1}{x^2 + (y+1)^2} = \frac{y+1}{|z+1|^2}. \end{aligned}$$

Since $f(z) = U_1(z; f)$, we have by the Schwaltz inequality

$$\begin{aligned} |f(z)|^2 &\leq \left(\frac{|z+i|}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|}{\sqrt{1+t^2}} \frac{y}{(t-x)^2 + y^2} dt \right)^2 \\ &= |z+i|^2 \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt \right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(t-x)^2 + y^2} dt \right) \\ &= \frac{|z+i|^2}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt = \frac{|z+i|^2}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \int_{nl}^{(n+1)l} \frac{|f(t)|^2}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt. \end{aligned}$$

Here if $nl \leq t \leq (n+1)l$, ($n = 0, \pm 1, \pm 2, \dots$), we have

$$\frac{1}{1+t^2} \cong \frac{1}{1+(nl)^2} \quad \text{and} \quad \frac{y}{(t-x)^2 + y^2} \cong \frac{y}{(x-nl)^2 + y^2},$$

where we shall use the notation between two functions $a(t), b(t)$ such as

$$a(t) \cong b(t),$$

if there exists absolute constants C_1, C_2 such that the following inequality

$$C_1 b(t) \leq a(t) \leq C_2 b(t)$$

is satisfied between them.

Then we have

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{1}{\pi} \int_{nl}^{(n+1)l} \frac{|f(t)|^2}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt \\ &\cong \left(\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{1+(nl)^2} \frac{y}{(nl-x)^2 + y^2} \right) \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right) \\ &\cong \left(\frac{l}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt \right) \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right) \end{aligned}$$

$$= \frac{(y+1)l}{|z+i|^2} \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right).$$

Therefore we have

$$\sup_{-\infty < x < \infty} |f(z)| \leq O(\sqrt{(y+1)l}) \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dx \right)^{\frac{1}{2}}$$

The uniform continuity of $f(z)$.

We shall prove that if $f(x)$ with the Stepanoff norm $\|f\|_{S^2} < \infty$, then $f(z)$ is uniformly continuous as a function of x for all fixed $y > 0$.

Let us put $z = x + iy$ and $z' = x' + iy$ ($y > 0$), then we have

$$\begin{aligned} & f(z) - f(z') \\ &= \frac{z+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{y}{(t-x)^2 + y^2} dt - \frac{z'+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{y}{(t-x)^2 + y^2} dt \\ &+ \frac{z'+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \left\{ \frac{y}{(t-x)^2 + y^2} - \frac{y}{(t-x')^2 + y^2} \right\} dt = J_1 + J_2, \quad \text{say.} \end{aligned}$$

Since $z - z' = (x + iy) - (x' + iy) = x - x'$, we have as for J_1

$$J_1 = \frac{x-x'}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{y}{(x-t)^2 + y^2} dt$$

and applying the Schwartz inequality

$$\begin{aligned} |J_1|^2 &\leq |x-x'|^2 \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left| \frac{f(t)}{t+i} \right|^2 \frac{y}{(x-t)^2 + y^2} dt \right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2 + y^2} dt \right) \\ &= \frac{|x-x'|^2}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2} \frac{y}{(x-t)^2 + y^2} dt \\ &\cong \frac{|x-x'|^2}{\pi} \sum_{n=-\infty}^{\infty} \frac{l}{1+(nl)^2} \frac{y}{(nl-x)^2 + y^2} \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right) \\ &\cong \frac{|x-x'|^2}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+t^2} \frac{y}{(t-x)^2 + y^2} dt \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right) \\ &= |x-x'|^2 \frac{(1+y)l}{x^2 + (1+y)^2} \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right) \end{aligned}$$

$$\leq \frac{l|x-x'|^2}{1+y} \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right).$$

Therefore we have

$$|J_1| \leq \frac{\sqrt{l}|x-x'|}{\sqrt{1+y}} \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Next we have as for J_2

$$J_2 = \frac{z'+i}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \left\{ \frac{y}{(t-x)^2+y^2} - \frac{y}{(t-x')^2+y^2} \right\} dt$$

and where

$$\begin{aligned} & \frac{y}{(t-x)^2+y^2} - \frac{y}{(t-x')^2+y^2} \\ &= \frac{y\{(t-x')^2 - (t-x)^2\}}{\{(t-x)^2+y^2\}\{(t-x')^2+y^2\}} = \frac{y(x-x')(2t-x-x')}{\{(t-x)^2+y^2\}\{(t-x')^2+y^2\}}. \end{aligned}$$

Then we have

$$J_2 = \frac{(x-x')(z'+i)}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{t+i} \frac{y(2t-x-x')}{\{(t-x)^2+y^2\}\{(t-x')^2+y^2\}} dt$$

and applying the Schwartz inequality we have

$$|J_2|^2 \leq |x-x'|^2 |z'+i|^2 \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2} \frac{y}{(x-t)^2+y^2} dt \right) \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y(2t-x-x')^2}{\{(t-x)^2+y^2\}\{(t-x')^2+y^2\}^2} dt \right)$$

where

$$|2t-x-x'|^2 = |(t-x) + (t-x')|^2 \leq 2\{|t-x|^2 + |t-x'|^2\} \cong 4|t-x'|^2$$

and

$$(t-x')^2 + y^2 \geq (t-x')^2$$

then we have

$$\frac{(2t-x-x')^2}{\{(t-x')^2+y^2\}^2} \leq \frac{4(t-x')^2}{\{(t-x')^2+y^2\}(t-x')^2} = \frac{4}{(t-x')^2+y^2}.$$

Then we have

$$\begin{aligned} & \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y(2t-x-x')^2}{\{(t-x)^2+y^2\}\{(t-x')^2+y^2\}^2} dt \\ & \leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{\{(t-x)^2+y^2\}\{(t-x')^2+y^2\}} dt \\ & = \frac{1}{\pi y^2} \int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} dt = \frac{1}{y^2}. \end{aligned}$$

and then we have

$$\begin{aligned} |J_2|^2 & \leq \frac{|x-x'|^2 |z'+i|^2}{y^2} \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|f(t)|^2}{1+t^2} \frac{y}{(x-t)^2+y^2} dt \right) \\ & \leq \frac{|x-x'|^2 |z'+i|^2 (1+y)l}{|z+i|^2 y^2} \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right). \end{aligned}$$

Therefore we have

$$|J_2| \leq \frac{\sqrt{(1+y)l}}{y} |x-x'| \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Thus we could conclude that the following estimations

$$|f(z) - f(z')| \leq \frac{2\sqrt{(1+y)l}}{y} |x-x'| \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

are satisfied. This shows that analytic function $f(z)$ of H_1^2 is bounded and uniformly continuous as a function of x on the upper-half plane $y > 0$, if its boundary function $f(x)$ has finite norm $\|f\|_{S^2} < \infty$.

Approximation of $f(z)$ by trigonometric polynomials.

Let us put $p(x) = \sum_{n=0}^N c_n e^{i\lambda_n x}$ where $\lambda_0 = 0$ and calculate the G.C.I. of $p(x)$

$$C_1(z; p) = \frac{(z+i)}{2\pi i} \int_{-\infty}^{\infty} \frac{p(t)}{t+i} \frac{dt}{t-z}, \quad (z = x+iy, y > 0).$$

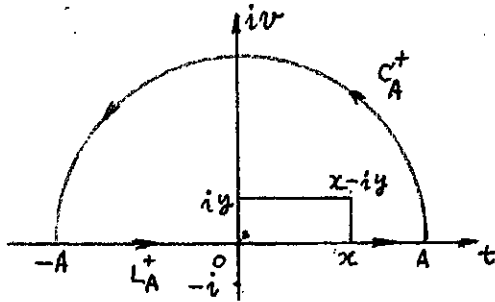
We shall prove the following formula

$$\frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{t+i} \frac{dt}{t-z} = e^{i\lambda z}, \quad (\lambda \geq 0); = e^\lambda, \quad (\lambda \leq 0),$$

where $z = x + iy, y > 0$. For the sake of completeness, we shall prove it. Let us start to calculate the following contour integral

$$\frac{1}{2\pi i} \int_{\Gamma_A^+} \frac{e^{i\lambda w}}{(w+i)(w-z)} dw, \quad (w = t + iv).$$

(i) $\lambda \geq 0$



$$\Gamma_A^+ = C_A^+ \cup L_A^+,$$

where

$$C_A^+ = \{w = Ae^{i\theta}, 0 < \theta < \pi\}$$

and

$$L_A^+ = \{w = t, -A \leq t \leq A\}$$

Let us decompose the contour integral into

$$\frac{1}{2\pi i} \int_{\Gamma_A^+} \frac{e^{i\lambda w}}{(w+i)(w-z)} dw = -\frac{1}{2\pi i} \int_{C_A^+} (f''') dw + \frac{1}{2\pi i} \int_{L_A^+} (f''') dw.$$

If $w \in C_A^+$, we have $w = Ae^{i\theta}$ and $dw = iAe^{i\theta} d\theta = iwd\theta$, so

$$\frac{e^{i\lambda w}}{(w+i)(w-z)} = \frac{e^{i\lambda A \cos \theta} e^{-\lambda A \sin \theta}}{(w+i)(w-z)} = O(A^{-2} e^{-\lambda A \sin \theta})$$

and then

$$\frac{1}{2\pi i} \int_{C_A^+} (f''') dw = O(A^{-1} e^{-\lambda A \sin \theta}) \rightarrow 0, \quad (A \rightarrow \infty).$$

If $w \in L_A^+$, we have $w = t$ and $dw = dt$, so

$$\frac{1}{2\pi i} \int_{L_A^+} (f''') dw = \frac{1}{2\pi i} \int_{-A}^A \frac{e^{i\lambda t}}{(t+i)(t-z)} dt \rightarrow \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{(t+i)(t-z)} dt, \quad (A \rightarrow \infty).$$

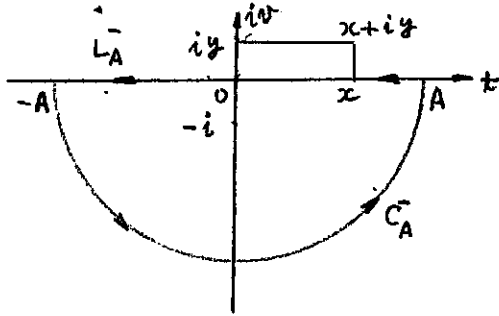
On the other hand, we have by the theorem residue

$$\frac{1}{2\pi i} \int_{\Gamma_A^+} \frac{e^{i\lambda w}}{(w+i)(w-z)} dw = \frac{e^{i\lambda z}}{z+i}.$$

Therefore we have

$$\frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{t+i} \frac{dt}{t-z} = e^{i\lambda z}.$$

(ii) $\lambda \leq 0$



$$\Gamma_A^- = C_A^- \cup L_A^-$$

where

$$C_A^- = \{w = Ae^{i\theta}, \pi < \theta < 2\pi\}$$

and

$$L_A^- = \{w = t, -A \leq t \leq A\}$$

Let us decompose it into

$$\frac{1}{2\pi i} \int_{\Gamma_A^-} \frac{e^{i\lambda w}}{(w+i)(w-z)} dw = \frac{1}{2\pi i} \int_{C_A^-} (") dw - \frac{1}{2\pi i} \int_{L_A^-} (") dw.$$

If $w \in C_A^-$, we have $w = Ae^{i\theta}$ and $dw = iAe^{i\theta} d\theta = iwd\theta$ so

$$\frac{e^{i\lambda w}}{(w+i)(w-z)} = \frac{e^{i\lambda A \cos \theta} e^{-\lambda A \sin \theta}}{(w+i)(w-z)} = O(A^{-2} e^{-\lambda A \sin \theta})$$

and then

$$\frac{1}{2\pi i} \int_{C_A^-} (") dw = O(A^{-1} e^{-\lambda A \sin \theta}) \rightarrow 0, \quad (A \rightarrow \infty).$$

If $w \in L_A^-$, we have $w = t$ and $dw = dt$, so

$$\frac{1}{2\pi i} \int_{L_A^-} (") dw = \frac{1}{2\pi i} \int_{-A}^A \frac{e^{i\lambda t}}{(t+i)(t-z)} dt \rightarrow \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{(t+i)(t-z)} dt, \quad (A \rightarrow \infty).$$

On the other hand we have by the residue theorem

$$\frac{1}{2\pi i} \int_{\Gamma_A^-} \frac{e^{i\lambda w}}{(w+i)(w-z)} dw = -\frac{e^\lambda}{z+i}.$$

Therefore we have

$$\frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\lambda t}}{t+i} \frac{dt}{t-z} = e^\lambda.$$

Now we have proved following formula

$$C_1(z; p) = \frac{z+i}{2\pi i} \int_{-\infty}^{\infty} \frac{p(t)}{t+i} \frac{dt}{t-z} = C_0 + \sum_{\lambda_n > 0} c_n e^{-\lambda_n y} e^{i\lambda_n x}, \quad C_0 = \sum_{\lambda_n \leq 0} c_n e^{\lambda_n}.$$

Since $f(x)$ is almost periodic in the sense W. Stepanoff, for any given positive

number ε , there exists a polynomial $p(x) = \sum c_n e^{i\lambda_n x}$ such that

$$\left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t) - p(t)|^2 dt \right)^{\frac{1}{2}} < \varepsilon / \sqrt{(1+y)l}.$$

Since we have the following formula

$$f(z) - p(z) = C_1(z; f - p) = U_1(x, y; f - p),$$

we shall conclude that

$$\sup_{-\infty < x < \infty} |f(z) - p(z)| \leq O(\sqrt{(1+y)l}) \left(\sup_{-\infty < x < \infty} \frac{1}{l} \int_x^{x+l} |f(t) - p(t)|^2 dt \right)^{\frac{1}{2}} \leq O(\varepsilon).$$

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