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Chapter II
Theorems of spectral synthesis of G.H.T.

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ABSTRACT

Refined and advanced form of theorems as for spectral synthesis of G.H.T. are presented with applications to almost periodic functions according to the new estimations of remainder terms

6. Relevant theorems of Generalized Harmonic Analysis (G.H.A.) .

The class of functions of W^2 and S_0 are defined already. Then we shall introduce two more class of functions as follows.

The Class of Functions S : It is defined by the set of the functions $f(x)$ which satisfies the following properties. It belongs to the class of functions S_0 and the convolution product of itself

$$\phi(x;f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(t)}dt$$

exists for all x.

The Class of Functions S' : It is defined by the set of functions $f(x)$ which satisfies following properties. It belongs to the class of functions S and the convolution product of itself $\phi(x;f)$ is continuous everywhere.

Then combining the G.F.T. of $f(x)$ and the Wiener formula and running on the same lines of his proof and also inverse direction carefully, we could prove

Theorem W_1. Let us suppose that the function $f(x)$ belongs to the class W^2, then we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(t)}dt \quad = \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} \,|\, s(u+\varepsilon;f) - s(u-\varepsilon;f)\,|^2 \, du$$

in the sense that if one of the limit on both side exist for all x, then that of other side does for all x too and have the same value. Then we can conclude that the function $f(x)$ belongs to the class S.

The first half part of this theorem had been proved by N.Wiener. He used the the Plancherel theorem and Minkowski inequality ingeniously. This is as follows.

Let us denote by $\|f\|$, the L^2 - norm of function $f(x)$. Let us suppose that $f_n \to f$, $g_n \to 0$ in L^2, then we have $\|f_n + g_n\| \to \|f\|$, as $n \to \infty$.

For the sake of completeness, we shall prove it.

From the inequality $\|f_n + g_n\| \leq \|f_n\| + \|g_n\|$, we have

$$\overline{\lim_{n\to\infty}} \|f_n + g_n\| \leq \overline{\lim_{n\to\infty}}(\|f_n\| + \|g_n\|) = \lim_{n\to\infty}\|f_n\| + \lim_{n\to\infty}\|g_n\| = \|f\|.$$

Similarly from the inequality

$$\|f\| = \|(f - f_n) + (f_n + g_n) - g_n\| \quad \leq \|f - f_n\| + \|f_n + g_n\| + \|g_n\|,$$

we have

$$\|f\| \leq \underline{\lim}_{n\to\infty}(\|f_n + g_n\| + \|f - f_n\| + \|g_n\|).$$

$$= \underline{\lim}_{n\to\infty}\|f_n + g_n\| + \underline{\lim}_{n\to\infty}(\|f - f_n\| + \|g_n\|) = \underline{\lim}_{n\to\infty}\|f_n + g_n\|.$$

Therefore we have

$$\lim_{n\to\infty}\|f_n + g_n\| = \lim_{n\to\infty}\|f_n\| = \|f\|.$$

We shall not go into and refer to his book (c.f.[1],Theorem 27,pp.156~158) as for essential part of his proof.

The other half part of this theorem could be done by running along just the same lines of his proof but inverse direction. We shall give it since this part play the essential roles on the spectral synthesis of G.H.T.. However let us remember that all of its estimations are created by N.Wiener.

Let us suppose that the limit

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} e^{iux}\,|s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2\, du$$

exist for all x.

Then the following limits

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty} |(e^{iux} + w)(s(u+\varepsilon; f) - s(u-\varepsilon; f)|^2\, du$$

$$= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}(2+we^{-iux}+\overline{w}e^{iux})\,|(s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du$$

exist for all x and $w=\pm1,\pm i$.

Let us denote $s_x(u;f)$ the G.F.T. of $f(\xi+x)$ as function of ξ and fixed any x

$$s_x(u;f)=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]\frac{f(\xi+x)e^{-iu\xi}}{-i\xi}d\xi+\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}f(\xi+x)\frac{e^{-iu\xi}-1}{-i\xi}d\xi.$$

Then we have

$$s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)-e^{iux}(s(u+\varepsilon;f)-s(u-\varepsilon;f))$$

$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(\xi+x)\frac{2\sin\varepsilon\xi}{\xi}e^{-iu\xi}d\xi-\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(\xi)\frac{2\sin\varepsilon\xi}{\xi}e^{-iu(\xi-x)}d\xi$$

and so we have by the Plancherel theorem

$$\int_{-\infty}^{\infty}|s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)-e^{iux}(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\,du$$

$$=\int_{-\infty}^{\infty}|f(\xi)|^2\left[\frac{2\sin\varepsilon(\xi-x)}{\xi-x}-\frac{2\sin\varepsilon\xi}{\xi}\right]^2d\xi,$$

where

$$T=\left|\frac{2\sin\varepsilon(\xi-x)}{\xi-x}-\frac{2\sin\varepsilon\xi}{\xi}\right|\le\frac{16\varepsilon|x|}{|\xi|+|x|}.$$

Therefore we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)-e^{iux}(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\,du=0$$

Now let us consider the following formula

$$\int_{-\infty}^{\infty}|s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)+w(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\,du$$

$$=\int_{-\infty}^{\infty}|s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)-e^{iux}(s(u+\varepsilon;f)-s(u-\varepsilon;f))$$

$$+(e^{iux}+w)(s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du,$$

and let us apply the Minkowski inequality. Then we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s_x(u+\varepsilon;f)-s_x(u-\varepsilon;f)+w(s(u+\varepsilon;f)-s(u-\varepsilon;f))|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|(e^{iux}+w)(s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du$$

$$= \lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} (2 + we^{-iux} + \overline{w}e^{iux}) \,|\, (s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du$$

for all x and $w = \pm 1, \pm i$.

Therefore we have proved that the following limit

$$\lim_{B \to \infty} \frac{1}{2B} \int_{-B}^{B} |\, f(\xi + x) + wf(\xi) \,|^2 d\xi$$

exist and equal to

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} (2 + we^{-iux} + \overline{w}e^{iux}) \,|\, (s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du$$

for all x and $w = \pm 1, \pm i$.

Now we shall use the identity

$$|\, a + b \,|^2 - |\, a - b \,|^2 + i \,|\, a + ib \,|^2 - i \,|\, a - ib \,|^2 = 4a\overline{b}.$$

Let us put $a = f(\xi + x), b = f(\xi)$ on the identity. Then we have

$$|\, f(\xi + x) + f(\xi) \,|^2 - |\, f(\xi + x) - f(\xi) \,|^2 + i \,|\, f(\xi + x) + if(\xi) \,|^2 - i \,|\, f(\xi + x) - if(\xi) \,|^2$$
$$= 4 f(\xi + x)\overline{f(\xi)}.$$

Let us also remark that

$$(2 + e^{-iux} + e^{iux}) - (2 - e^{-iux} - e^{iux}) + i(2 + ie^{-iux} - ie^{iux}) - i(2 - ie^{-iux} + ie^{iux})$$
$$= 4e^{iux}.$$

Then we have proved that the following limit

$$\lim_{B \to \infty} \frac{1}{2B} \int_{-B}^{B} f(\xi + x)\overline{f(\xi)} d\xi$$

exist for all x and equal to

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\infty}^{\infty} e^{iux} \,|\, s(u+\varepsilon; f) - s(u-\varepsilon; f) \,|^2 \, du.$$

Thus we have proved the half part of Theorem W_1.

Theorem W_2. Let us suppose that $f(x)$ belong to the class S. If $\phi(x; f)$ is continuous at $x = 0$, then it is continuous everywhere and therefore $f(x)$ belongs to the class S'.

Theorem W_3. Let us suppose that $f(x)$ belong to the class S. Then the necessary and sufficient condition for $f(x)$ to belong to the class S' will be that the following property

$$\lim_{A\to\infty}\ \overline{\lim_{\varepsilon\to 0}}\ \frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du=0$$

is satisfied.

Let us remark that the condition means the spectrum of $f(x)$ to vanish at $x=\pm\infty$ or to concentrate there with energy a few.

There is also the notion as for functions of positive definite due to S.Bochner, but we shall not use it here and so we shall not go into. We shall refer to S.Bochner : Lectures on Fourier Integrals with an author's supplement on Monotonic Functons, Stieltjes Integrals, and Harmonic Analysis, Annals of Mathematics Studies, No. 42, Princeton Univ., (1959)(c.f. Chap.IV. § 20,pp.92~96;Auther's supplement, § 8,325~328).

7. Spectral Synthesis of Generalized Hilbert Transforms.

By the use of Theorem A as for spectral decomposition, we shall prove several theorems of spectral synthesis of G.H.T. according to those of G.F.T. due to N.Wiener. The following theorems are refined forms of which we proved in the previous paper [6] (c.f. Theorems 51,52,54,55,pp.206~211).

Theorem B_1. Let us suppose that $f\in S_0$. Let us suppose that hypothesis (R_0) and $(\widetilde{R}_0)$ are satisfied. Then we have $\widetilde{f}_1\in S_0$ and the following equality

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{f}_1(x)|\,dx=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx+|\widetilde{c_0}|^2-|c_0|^2.$$

Proof. We shall prove the following equality

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du+|\widetilde{c_0}|^2-|c_0|^2.$$

For this purpose we shall divide the integral of left-hand side into two parts

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du=\frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon}|('')|^2du+\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}|('')|^2du$$

$$=I_1+I_2,say.$$

Then by the part (i) of theorem A ,we have

$$I_1 = \frac{1}{4\pi\varepsilon}\int_{|u|\geq\varepsilon} |s(u+\varepsilon;\widetilde{f}_1) - s(u-\varepsilon;\widetilde{f}_1)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\geq\varepsilon} |s(u+\varepsilon;f) - s(u-\varepsilon;f)|^2$$

and by the part (ii) of Theorem A and applying the Minkowski inequality and hypothesis (R_0), $(\widetilde{R}_0)$ we have

$$\frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} |s(u+\varepsilon;\widetilde{f}_1) - s(u-\varepsilon;\widetilde{f}_1)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} |i\{s(u+\varepsilon;f) - s(u-\varepsilon;f)\} + 2r_1(u+\varepsilon;f) + 2r_2(u+\varepsilon;f)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} |i\left\{s(u+\varepsilon;f) - s(u-\varepsilon;f) - \sqrt{2\pi}c_0\right\} + 2r_1(u+\varepsilon;f) + 2(r_2(u+\varepsilon;f) - \sqrt{\frac{\pi}{2}}a(f)) + \sqrt{2\pi}\widetilde{c_0}|^2\, du$$

$$= |\widetilde{c_0}|^2 + o(1), \quad (\varepsilon \to 0),$$

and similarly

$$\frac{1}{4\pi\varepsilon}\int_{|u|\geq\varepsilon} |s(u+\varepsilon;f) - s(u-\varepsilon;f)|^2$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} |\left\{s(u+\varepsilon;f) - s(u-\varepsilon;f) - \sqrt{2\pi}c_0\right\} + \sqrt{2\pi}c_0|^2\, du$$

$$= |c_0|^2 + o(1), \quad (\varepsilon \to 0).$$

Then we have

$$I_2 = \frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} |s(u+\varepsilon;\widetilde{f}_1) - s(u-\varepsilon;\widetilde{f}_1)|^2\, du$$

$$= \frac{1}{4\pi\varepsilon}\int_{|u|\leq\varepsilon} |s(u+\varepsilon;f) - s(u-\varepsilon;f)|^2 du + |\widetilde{c_0}|^2 - |c_0|^2 + o(1), \quad (\varepsilon \to 0).$$

Therefore, combining the estimations of I_1 and I_2 we have the required equality and applying the Theorem W_0 to the equality, we have proved Theorem B_1.

Theorem B_2. Let us suppose that $f \in S$. Let us suppose that the hypothesis $(R_0), (\tilde{R}_0)$ are satisfied. Then we have $\widetilde{f}_1 \in S$ and the following equality

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\widetilde{f}_1(x+t)\overline{\widetilde{f}_1(t)}dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}dt + |\widetilde{c_0}|^2 - |c_0|^2 .$$

Proof. We shall prove the following equality

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{iux}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du + |\widetilde{c_0}|^2 - |c_0|^2 .$$

We shall decompose into two part as before

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}e^{-iux}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon}e^{iux}|('')|^2\,du + \frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}e^{iux}|('')|^2du.$$

As for first part of right hand-side, we have by the case of (i) of Theorem

$$\frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon}e^{iux}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du = \frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon}e^{iux}|(-isignu)\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\ge\varepsilon}e^{iux}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du .$$

As for second part of right hand side, we have

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}e^{iux}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}(e^{iux}-1)|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du$$

$$+\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du ,$$

and let us remark that $\widetilde{f}_1 \in S_0$ by Theorem B_1 and $e^{iux}-1=O(\varepsilon)$, as $\varepsilon \to 0$,then we have by the use of our new estimation in § 5,the case (iii)

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon}(e^{iux}-1)|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du = O(\varepsilon)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{f}_1(x)|^2\,dx ,$$

as $\varepsilon \to 0$.

Therefore we have by running on the same lines as the estimations of Theorem B_1

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux} \,|\, s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\,|^2\, du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\, s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\,|^2\, du+\ o(1),\ \ (\varepsilon\to 0)$$

$$=|\,\widetilde{c_0}\,|^2 + o(1),\ \ (\varepsilon\to 0),$$

and similarly we have

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux} \,|\, s(u+\varepsilon;f)-s(u-\eta;f)\,|^2\, du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} |\, s(u+\varepsilon;f)-s(u-\varepsilon;f)\,|^2\, du+\ o(1),\ \ (\varepsilon\to 0)$$

$$=|\,c_0\,|^2 + o(1),\ \ (\varepsilon\to 0).$$

Now we have

$$\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux} \,|\, s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\,|^2\, du$$

$$=\frac{1}{4\pi\varepsilon}\int_{|u|\le\varepsilon} e^{iux} \,|\, s(u+\varepsilon;f)-s(u-\eta;f)\,|^2\, du+|\,\widetilde{c_0}\,|^2-|\,c_0\,|^2+o(1),\ \ (\varepsilon\to 0).$$

Therefore we have the required equality and applying the Theorem W_1 to the equality we have proved Theorem B_2

Theorem B_3. Let us suppose that $f\in S'$. Let us suppose that the hypothesis $(R_0),(\widetilde{R}_0)$ are satisfied. Then we have $\widetilde{f}_1\in S'$.

Proof. Let us remark the equality of the Theorem B_2. This is as follows

$$\phi(x;\widetilde{f}_1)=\phi(x;f)+|\,\widetilde{c_0}\,|^2-|\,c_0\,|^2.$$

Then the continuity of $\phi(x;\widetilde{f}_1)$ will be obtained by that of $\phi(x;f)$.

According to the Theorems of W_2, W_3 due to N.Wiener, we have Theorems of B_4, B_5 by running on the same lines as his proofs respectively.

Theorem B_4. Let us suppose that $f \in S$. Let us suppose that the hypothesis $(R_0),(\widetilde{R}_0)$ are satisfied. If the function $\phi(x;f)$ is continuous at $x = 0$, then we have $\widetilde{f}_1 \in S'$.

Proof. By the Theorem B_2, we have $\widetilde{f}_1 \in S$ and $\phi(x;\widetilde{f}_1)$ is continuous at $x = 0$. Then if we apply the Theorem W_2 to the function $\widetilde{f}_1(x)$, we obtain the continuity of $\phi(x;\widetilde{f}_1)$ everywhere and so $\widetilde{f}_1 \in S'$.

Theorem B_5. Let us suppose that $f \in S$. Let us suppose that the hypothesis $(R_0),(\widetilde{R}_0)$ are satisfied. Then the necessary and sufficient condition for $\widetilde{f}_1 \in S'$ is that the following property

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du=0$$

is satisfied.

Proof. By the Theorem B_2, we have $\widetilde{f}_1 \in S$. Let us remark that the property of the Theorem identifies with

$$\lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)|^2\,du=0$$

by the case (i) of the Theorem A. Then if we apply the Theorem W_3, we can conclude

that $\widetilde{f}_1 \in S'$ when and only when the property is true.

8. Applications to the Almost Periodic Function.

One of typical examples of our theory is almost periodic functions. We shall treat the almost periodic function in the sense of Besicovitch and check the reminder terms for this class of functions. Then we shall obtain the new or refined theorems of which we have proved in the previous paper[6](c.f.Theorems 67,68.pp.216~219).

As for preparations, we shall quote the following theorem that is the so called One Sided Wiener Formula(c.f N.Wiener [4]).

Theorem(S.Bochner-G.H.Hardy-N.Wiener). Let us suppose that function $f(x)$ belong to the class W^2. Let us suppose that the limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(t)dt$$

exist. Then the following limit

$$\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty} f(t)\frac{\sin^2 \varepsilon t}{t^2}dt$$

exist too and has the same limiting value.

Let us remark that in the theorem it is not supposed the non - negativity of function. As for these circumferences we shall refer to the book of S Bochner[9](c.f.Chap.II, § 9, pp.35~38)

The class of functions of almost periodic in the sense Besicovitch of order 2 are denoted by B^2 and are defined as the set of function $f(x)$ that is approximated arbitrarily by trigonometric polynomials $P(x)$ in the following sense

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} | f(x)-P(x)|^2 dx \le \varepsilon ,$$

where ε is an arbitrarily positive small number and

$$P(x)=\sum_{n=1}^{N} A_n e^{i\Lambda_n x} .$$

The principal properties of B^2-almost periodic function are as follows. We shall refer to the book of A.S.Besicovitch [2].

(1) If $f \in B^2$, the following limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = c(\lambda)$$

exist and naught for at most enumerable infinite number λ. We shall denote these

$$\lambda_0(=0),\ \lambda_1,\lambda_2,\lambda_3,\ldots\ldots,\lambda_n,\ldots\ldots;\quad \text{and}\quad c(\lambda_n)=c_n\quad (n=0,1,2,3,\cdots)$$

then we shall write

$$f(x)\ \sim\ \sum_{n=0}^{\infty} c_n e^{i\lambda_n x}$$

and it is called the Fourier series of $f(x)$ as well as the case of purely periodic function.

Now we have the Parseval equality

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx=\sum_{n=0}^{\infty}|c_n|^2<\infty$$

and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+t)\overline{f(t)}dt=\sum_{n=0}^{\infty}|c_n|^2\,e^{i\lambda_n x}.$$

Thus we shall conclude that if $f(x)$ is B^2-almost periodic function, it is also belongs to the Wiener class S'.

(2) The Bochner-Fejer polynomial

The Fejer sums for the case of a purely periodic function

$$f(x)\sim\sum_{\nu=-\infty}^{\infty} c_\nu e^{i\nu\beta x},$$

are given by the formula

$$\sigma_n(x;f)=\sum_{|\nu|<n}(1-\frac{|\nu|}{n})c_\nu e^{i\nu\beta t},$$

where the β is certain positive real constant.

Since

$$c_\nu=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\nu\beta t}dt,$$

and we shall denote it as $M\{f(t)e^{-i\nu\beta t}\}$. It follows that

$$c_\nu e^{i\nu\beta t}=M_t\{f(t)e^{-i\nu\beta(t-x)}\}=M_t\{f(x+t)e^{-i\nu\beta t}\},$$

whence

$$\sigma_n(x;f)=M_t\{f(x+t)K_n(\beta t)\},$$

where the kernel $K_n(\beta t)$ is defined by the equation

$$K_n(t)=\sum_{|\nu|<n}(1-\frac{|\nu|}{n})e^{-i\nu t}=\frac{1}{n}\left(\frac{\sin\frac{n}{2}t}{\sin\frac{t}{2}}\right)^2 .$$

This kernel has two properties important for the summation. It is never negative and its the mean value is equal to 1.

If $f(x)$ is continuous, then we have

$$\sigma_n(x;f)\to f(x),\quad \text{unif.},\quad (n\to\infty).$$

On the otherhand ,if $f\in L^2$,then we have

$$M\{|f(x)-\sigma_n(x;f)|^2\}\to 0,\quad (n\to\infty).$$

S.Bochner replaced this simple kernel by a finite product of such kernls

$$K_B(t)=K_{n_1}(\beta_1 t)K_{n_2}(\beta_2 t).....K_{n_p}(\beta_p t)=\sum_{|\nu_1|<n_1,...,|\nu_p|<n_p}(1-\frac{|\nu_1|}{n_1})(1-\frac{|\nu_2|}{n_2}).....(1-\frac{|\nu_p|}{n_p})e^{-i(\nu\beta_1+\nu_2\beta_2+.....+\nu_p\beta_p)t}$$

where, $\beta_1,\beta_2,.....,\beta_p$ are certain real linearly independent numbers.

This composite kernel has the same characteristic properties as the Fejer kernel. It is never negative and its mean value is equals to 1. Since the constant term being 1 on account of the linear independence of the β' s. We call this kernel the Bochner-Fejer kernel and we form a Bochner-Fejer polynomial

$$\sigma_{\begin{pmatrix}n_1,n_2,\cdots, & n_p\\ \beta_1,\beta_2,\cdots, & \beta_p\end{pmatrix}}(x)=M_t\{f(x+t)K_{\begin{pmatrix}n_1,n_2,\cdots, & n_p\\ \beta_1,\beta_2,\cdots, & \beta_p\end{pmatrix}}(t)\}=$$

$$\sum_{|\nu_1|<n_1,|\nu_2|<n_2,...,|\nu_p|<n_p}(1-\frac{|\nu_1|}{n_1})(1-\frac{|\nu_2|}{n_2})\cdots(1-\frac{|\nu_p|}{n_p})c(\nu_1\beta_1+\nu_2\beta_2+\cdots+\nu_p\beta_p)e^{i(\nu_1\beta_1+\nu_2\beta_2+\cdots+\nu_p\beta_p)x}$$

where as usual

$$c(\lambda)=M\{f(t)e^{-i\lambda t}\}.$$

Thus only those terms of the Bochner-Fejer polynomial differ from zero whose exponentsare Fourier exponents of $f(x)$, When $\nu_1\beta_1+\nu_2\beta_2+\cdots+\nu_p\beta_p$ is a Fourier exponent of $f(x)$,we write

$$\nu_1\beta_1+\nu_2\beta_2+.....+\nu_p\beta_p=\lambda_n$$

and we have

$$\sigma_{\begin{pmatrix} n_1, n_2, \cdots, & n_p \\ \beta_1, \beta_2, \cdots, & \beta_p \end{pmatrix}}(x) = \sum_{|\nu_1|<n_1, |\nu_2|<n_2, \ldots, |\nu_p|<n_p} (1-\frac{|\nu_1|}{n_1})(1-\frac{|\nu_2|}{n_2})\cdots(1-\frac{|\nu_p|}{n_p})c_n e^{i\lambda_n x}.$$

We call the numbers $\beta_1, \beta_2, \ldots\ldots, \beta_p$ basic numbers, and the numbers $n_1, n_2, \ldots\ldots, n_p$ indices of the Bochner-Fejer kernel or of the polynomial.

We shall denote the symbol

$$\mathrm{B} = \begin{pmatrix} n_1, n_2, \cdots\cdots, & n_p \\ \beta_1, \beta_2, \cdots\cdots, & \beta_p \end{pmatrix},$$

and write the above formula in the form

$$\sigma_{\mathrm{B}}(x) = \sum d_n^{(B)} c_n e^{i\lambda_n x},$$

where all $d_n^{(B)}$ satisfy the inequality

$$0 \le d_n^{(B)} \le 1,$$

and only finite number of them are different of zero.

We shall quote the following theorems.

Polynomial Approximation Theorem.

(1) To any u.a.p. function in the sense of H.Bohr, there exist a sequence of Bochner-Fejer polynomials convergent uniformly to the function. That is

$$\sigma_{B_m}(x; f) - f(x) \to 0, \quad unif., \quad (m \to \infty).$$

(c.f. [2],Chap.I, § 9,pp.46~51)

(2) To any B^2 almost periodic function, there exist a sequence of Bochner-Fejer Polynomials convergent in the mean to the function. That is

$$M\{| f(x) - \sigma_{B_m}(x; f)|^2\} \to 0, \quad (m \to \infty).$$

(c.f. [2],Chap.II, § 8,pp.104~109)

Let us call it the Bochner sequence and denote it as $\sigma_{B_m}(x; f)$ and write it as follows

$$\sigma_{B_m}(x; f) = \sum_{n=1}^{\infty} d_n^{(m)} c_n e^{i\lambda_n x}, \quad (m = 1, 2, 3, \cdots\cdots).$$

where

$$d_0^{(m)} = 1, \quad 0 \le d_n^{(m)} \le 1, \quad \text{and} \quad d_n^{(m)} \to 1, \quad \text{as} \quad m \to \infty, \quad \text{for all n.}$$

Now we shall consider the following problem.

Problem. Let us suppose that $f(x)$ is the almost periodic function in the sense of Besicovitch, then is its G.H.T. $\widetilde{f}_1(x)$ too ?

In the previous paper [6](c.f.Theorems 67,68. pp.216~219) , we have had proved some result to the problem , we shall give the more advanced result with perfect proof.

Let us start the following Lemmas.

Lemma C_0. Let us suppose that $f(x) \in W^2$. Let us also suppose that the limit

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\lambda t} dt = c_\lambda$$

exist for a real number λ. Then we have for this λ

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} du = \frac{c_\lambda}{\sqrt{2\pi}}.$$

In particular, for the $\lambda = 0$, if we suppose that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t) dt = c_0 ,$$

then we have

$$\lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{-\varepsilon}^{\varepsilon} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} du = \frac{c_0}{\sqrt{2\pi}}.$$

Proof. We have

$$I_\lambda = \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} \{s(u+\varepsilon; f) - s(u-\varepsilon; f)\} du$$

$$= \lim_{\varepsilon\to 0} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(x) \frac{2\sin\varepsilon x}{x} e^{-iux} dx) du .$$

Since the strong convergence imply that of weakly, so the above formula equals to

$$= \lim_{\varepsilon\to 0} \lim_{A\to\infty} \frac{1}{4\pi\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} (\frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(x) \frac{2\sin\varepsilon x}{x} e^{-iux} dx) du ,$$

and we change the order of integrations by the theorem of Fubini as for double integral, so the above formula equals to

$$= \lim_{\varepsilon\to 0}\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A} f(x)\frac{2\sin\varepsilon x}{x}\Big(\frac{1}{4\pi\varepsilon}\int_{\lambda-\varepsilon}^{\lambda+\varepsilon} e^{-iux}du\Big)dx$$

$$= \frac{1}{\sqrt{2\pi}}\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty} f(x)\frac{\sin^2\varepsilon x}{x^2}e^{-i\lambda x}\,dx.$$

In the last we shall apply the one sided Wiener formula, I_λ equals to

$$= \frac{1}{\sqrt{2\pi}}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda x}dx = \frac{c_\lambda}{\sqrt{2\pi}}.$$

Thus we have proved the Lemma C_0.

Lemma C_1. Let us suppose that $f(x)$ is B^2 almost periodic function. Then the hypothesis (R_0) is satisfied with c_0 where it is the constant term of Fourier series expansion of f.

Proof. Let us write the Fourier series of $f(x)$ and its Bochner sequence of which approximate f in the mean are as follows

$$f(x) \sim c_0 + {\sum}' c_n e^{i\lambda_n x},$$

and

$$\sigma_{B_m}(x;f) = c_0 + {\sum}' d_n^{(m)} c_n e^{i\lambda_n x},$$

where ${\sum}'$ means that the term $\lambda_0 = 0$ is excluded in the summation.

We have by the Lemma A_2

$$s(u+\varepsilon;\sigma_{B_m}(x;f)) - s(u-\varepsilon;\sigma_{B_m}(x;f)) = \sqrt{2\pi}c_0\chi_\varepsilon(u),$$

and therefore we have

$$\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon} |\{s(u+\varepsilon;f) - s(u-\varepsilon;f)\} - \sqrt{2\pi}c_0|^2\,du$$

$$= \frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon} |s(u+\varepsilon;f-\sigma_{B_m}) - s(u-\varepsilon;f-\sigma_{B_m})|^2\,du + o(1), \quad (\varepsilon\to 0).$$

Now if we apply the results of estimation of the remainder terms in § 5 (iii), we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon} |\{s(u+\varepsilon;f) - s(u-\varepsilon;f)\} - \sqrt{2\pi}c_0|^2\,du$$

$$\le 16\pi\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(x) - \sigma_{B_m}(x;f)|^2 dx \to 0, \quad (m\to\infty).$$

Thus we have proved the Lemma C_1.

Then we shall prove the following refined and advanced theorem.

Theorem C. If f belongs to the class $B^2a.p.$, then $\widetilde{f}_1$ belongs also to the class $B^2a.p.$ when and only when the hypothesis $(\widetilde{R}_0)$ with $\widetilde{c_0}$ is satisfied. If the Fourier series of f is

$$f(x) \sim c_0 + \sum{}' c_n e^{i\lambda_n x},$$

then that of $\widetilde{f}_1$ is

$$\widetilde{f}_1(x) \sim \widetilde{c_0} + \sum{}' (-i\,sign\,\lambda_n) c_n e^{i\lambda_n x},$$

where $\sum{}'$ means that the term $\lambda_0 = 0$ is excluded in the summation.

Proof of necessity of the condition $(\widetilde{R}_0)$. The necessity of the hypothesis $(\widetilde{R}_0)$ with $\widetilde{c_0}$ is clear, because if we apply Lemma C_1 to the function $\widetilde{f}_1$ in stead of f, we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon} |\left\{s(u+\varepsilon;\widetilde{f}_1) - s(u-\varepsilon;\widetilde{f}_1)\right\} - \sqrt{2\pi}\,\widetilde{c_0}\,|^2\, du = 0.$$

Next, by the Theorem A (i),the case of $|u| \geq \varepsilon$ and Lemma C_0 with $\lambda = \lambda_n (\neq 0)$, we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \widetilde{f}_1(x) e^{-i\lambda_n x} dx$$

$$= \sqrt{2\pi}\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon} \left\{s(u+\varepsilon;\widetilde{f}_1) - s(u-\varepsilon;\widetilde{f}_1)\right\} du$$

$$= \sqrt{2\pi}(-i\,sign\,\lambda_n)\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{\lambda_n-\varepsilon}^{\lambda_n+\varepsilon} \left\{s(u+\varepsilon;f) - s(u-\varepsilon;f)\right\} du$$

$$= (-i\,sign\lambda_n)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-i\lambda_n x}dx = (-i\,sign\lambda_n)c_n .$$

In the last by the hypothesis $(\widetilde{R}_0)$ with $\widetilde{c_0}$, it is clear that

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\}du = \frac{\widetilde{c_0}}{\sqrt{2\pi}}$$

and so we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\widetilde{f}_1(x)dx = \sqrt{2\pi}\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\}du = \widetilde{c_0} .$$

Proof of sufficiency of the condition $(\widetilde{R}_0)$. Let us denote the Bochner sequence

$$\sigma_{B_m}(x;f) = c_0 + \sum{}' d_n^{(m)} c_n e^{i\lambda_n x}$$

and its conjugate polynomial by

$$\widetilde{\sigma}_{B_m}(x;f) = \widetilde{c_0} + \sum{}'(-i\,sign\lambda_n)d_n^{(m)}c_n e^{i\lambda_n x} .$$

We have by the *Lemma* A_2

$$s(u+\varepsilon;\sigma_{B_m}(x;f)) - s(u-\varepsilon;\sigma_{B_m}(x;f))$$

$$= \sqrt{2\pi}c_0\chi_\varepsilon(u) + \sqrt{2\pi}\sum{}' d_n^{(m)}c_n\chi_\varepsilon(u-\lambda_n)$$

and

$$s(u+\varepsilon;\widetilde{\sigma}_{B_m}(x;f)) - s(u-\varepsilon;\widetilde{\sigma}_{B_m}(x;f))$$

$$= \sqrt{2\pi}\widetilde{c_0}\chi_\varepsilon(u) + \sqrt{2\pi}\sum{}'(-i\,sign\lambda_n)d_n^{(m)}c_n\chi_\varepsilon(u-\lambda_n) .$$

We shall consider these by decomposing into the two cases as before.

(i) the case of $|u|\geq\varepsilon.$. Let us denote intervals and sets as follows

$$I_{\varepsilon,0}=(-\varepsilon,\varepsilon),\ \ I_{\varepsilon,n}=(\lambda_n-\varepsilon,\lambda_n+\varepsilon)\ \text{ and }\ C=\bigcup_{n\neq 0}I_{\varepsilon,n},\ \ D=(-\infty,+\infty)-C.$$

Let us notice that these intervals did not overlap each other for all sufficient small positive number ε.

If $u\in\{u;|u|\geq\varepsilon\}\bigcap C$, there exist an interval $I_{\varepsilon,n}=(\lambda_n-\varepsilon,\lambda_n+\varepsilon)$ and $u\in I_{\varepsilon,n}$, $sign\lambda_n=signu$. Therefore we have

$$s(u+\varepsilon;\widetilde{\sigma}_{B_m}(x;f))-s(u-\varepsilon;\widetilde{\sigma}_{B_m}(x;f))$$

$$=\sqrt{2\pi}(-isignu)d_n^{(m)}c_n=\sqrt{2\pi}(-isignu)\sum{}' d_n^{(m)}c_n\chi_\varepsilon(u-\lambda_n)$$

$$=(-isignu)\{s(u+\varepsilon;\sigma_{B_m}(x;f))-s(u-\varepsilon;\sigma_{B_m}(x;f)).$$

If $u\in\{u;|u|\geq\varepsilon\}\bigcap D$, wehave

$$s(u+\varepsilon;\widetilde{\sigma}_{B_m}(x;f))-s(u-\varepsilon;\widetilde{\sigma}_{B_m}(x;f))$$

$$=\ s(u+\varepsilon;\sigma_{B_m}(x;f))-s(u-\varepsilon;\sigma_{B_m}(x;f))\ =0.$$

Then we have

$$s(u+\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))-s(u-\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))$$

$$=(-isignu)\{s(u+\varepsilon;f-\sigma_{B_m}(f))-s(u-\varepsilon;f-\sigma_{B_m}(f))\}.$$

(ii) the case of $|u|\leq\varepsilon$. From

$$s(u+\varepsilon;\widetilde{\sigma}_{B_m}(x;f))-s(u-\varepsilon;\widetilde{\sigma}_{B_m}(x;f))=\sqrt{2\pi}\,\widetilde{c_0},$$

we have

$$s(u+\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))-s(u-\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))$$

$$= s(u+\varepsilon;\widetilde{f}_1) - s(u-\varepsilon;\widetilde{f}_1) - \sqrt{2\pi}\,\widetilde{c_0}.$$

Therefore by the hypothesis $(\widetilde{R}_0)$ we have

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))-s(u-\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))|^2\,du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\,s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)-\sqrt{2\pi}\,\widetilde{c}_0\,|^2\,du = 0.$$

Similarly we have by the hypothesis (R_0)

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;f-\sigma_{B_m}(f))-s(u-\varepsilon;f-\sigma_{B_m}(f))|^2\,du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\,s(u+\varepsilon;f)-s(u-\varepsilon;f)-\sqrt{2\pi}\,c_0\,|^2\,du = 0.$$

Summing up two cases, we shall conclude

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))-s(u-\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))|^2\,du$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;f-\sigma_{B_m}(f))-s(u-\varepsilon;f-\sigma_{B_m}(f))|^2\,du.$$

Next we have by the Theorem W_0,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,\widetilde{f}_1(x)-\widetilde{\sigma}_{B_m}(x;f)\,|^2\,dx$$

$$=\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))-s(u-\varepsilon;\widetilde{f}_1-\widetilde{\sigma}_{B_m}(f))|^2\,du$$

and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\,f(x)-\sigma_{B_m}(x;f)\,|^2 dx$$

$$= \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;f-\sigma_{B_m}(f))-s(u-\varepsilon;f-\sigma_{B_m}(f))|^2\,du.$$

On the other hand as we have supposed that $f(x)$ is $B^2 a.p.$ function, we have by the

Parseval equality

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)-\sigma_{B_m}(x;f)|^2dx=$$

$$\lim_{T\to\infty}\left(\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx-\frac{1}{2T}\int_{-T}^{T}f(x)\overline{\sigma_{B_m}(x;f)}dx-\frac{1}{2T}\int_{-T}^{T}\overline{f(x)}\sigma_{B_m}(x)dx+\frac{1}{2T}\int_{-T}^{T}|\sigma_{B_m}(x;f)|^2\,dx\right)$$

$$=(|c_0|^2+\sum{}'|c_n|^2)-2(|c_0|^2+\sum{}'d_n^{(m)}|c_n|^2)+(|c_0|^2+\sum{}'(d_n^{(m)})^2|c_n|^2)$$

$$\to 0,\quad (m\to\infty).$$

Because we have as for $n=0$, $d_0^{(m)}=1$ and for $n\neq 0$, $d_n^{(m)}\to 1$, $(m\to\infty)$.

Thus we have proved

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\widetilde{f}_1(x)-\widetilde{\sigma}_{B_m}(x;f)|^2\,dx=0,\quad (m\to\infty).$$

and at the same time we have proved that the conjugate polynomial $\widetilde{\sigma}_{B_m}(x;f)$ is the

Bochner-Fejer polynomoial $\sigma_{B_m}(x;\widetilde{f}_1)$ of Fourier series of $\widetilde{f}_1(x)$.

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