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On the theory of generalized Hilbert transforms
Chapter I
Theorem of spectral decomposition of G.H.T.

by

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ABSTRACT

The theory of G.H.T. had constructed by the author about fifty years ago by using N.Wiener's Generalized Harmonic Analysis. Nevertheless the theorem of spectral decomposition of G.H.T. is the core of this theory, we omitted the proofs of several lemmas with which related to calculations of simple poles by using of the residue theorem of the theory of functions of complex variables. Therefor we shall intend to give detailed proofs of this theorem for the sake of completeness. We shall also give new estimations of the remainder terms of this theory which enables us to present the more refined and advanced forms of the theory.

1. Generalized Fourier Transforms (G.F.T.)

Class of Functions W^2: It is defined by the set of function $f(x)$ which satisfies the following properties. It is a real valued measurable function f defined on the real line $-\infty < x < \infty$ and the integral

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx$$

exists.

For function $f(x) \in W^2$, the Generalized Fourier Transform (G.F.T.) was defined by N.Wiener as follows

$$s(u;f) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} f(x) \frac{e^{-iux}-1}{-ix} dx \; + \; \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \left[\int_{-A}^{-1} + \int_{1}^{A} \right] f(x) \frac{e^{-iux}}{-ix} dx$$

where the notation $l.i.m.$ means limit in the mean.

If we take the difference, we have

$$s(u+\varepsilon;f) - s(u-\varepsilon;f) = \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} f(x) \frac{2\sin \varepsilon x}{x} e^{-iux} dx$$

and we have by the theorem of Plancherel

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du=\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}|f(x)|^2\frac{\sin^2\varepsilon x}{x^2}dx.$$

It is also called the Parseval equality.

Class of Functions S_0 : It is defined by the set of functions $f(x)$ which satisfies the following properties. It is an element f of the class W^2 and the following limit

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx$$

exists.

Here we shall quote the so called the Wiener formula.

Theorem (The Wiener formula). Let us suppose that the function $f(x)$ belong to the class W^2.Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx = \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}|f(x)|^2\frac{\sin^2\varepsilon x}{x^2}dx$$

in the sense that if one of the limit on both side exist, then that of the other side does and have the same value.

Then applying the Wiener formula, we have the following theorem.

Theorem W_0. Let us suppose that the function $f(x)$ belongs to the class W^2.

Then we have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2dx = \lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du$$

in the sense that if one of the limit on both side exist, then that of the other side does and have the same value. Then the function $f(x)$ belongs to the class S_0.

This formula presents the total spectral intensity of system which behaves. As for the Generalized Fourier Transform, we shall refer to N.Wiener : The Fourier Integral and Certain of its Applications, Cambridge Univ. Press (1933).

2. Generalized Hilbert Transforms (G.H.T.)

Let $F(\varsigma)$ be defined on a Jordan curve C in the complex plane . Many authors treated the problem of representing $F(\varsigma)$ in the form $F_1(\varsigma)+F_2(\varsigma)$ where each $F_j(\varsigma)$ $(j=1,2)$ is the limit of analytic function $F_j(z)$ $(j=1,2)$ interior or exterior of the Jordan curve C respectively as $z=x+iy, z\to\varsigma\in C$.

H.Kober [6, 7] treated of the case $C=(-\infty,\infty)$ and $F(x)/1+x^2\in L(-\infty,\infty)$. For this purpose he introduced the modified Hilbert transform as follows

$$P.V.\frac{1}{\pi}\int_{-\infty}^{\infty} F(t)\left(\frac{1}{t-x}-\frac{t}{t^2+1}\right)dt.$$

Concerning to this problem, he proved that if $F(z)/(z+i)^2$ belongs to the Hardy class H then we have

$$F(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} F(t)\left(\frac{1}{t-z}-\frac{1}{t+i}\right)dt,\quad (y>0)\ \text{ and }\quad F(z)=0,\quad (y<0)$$

Let us remark that the above formula is just the generalized Cauchy integral of order 1 of us.

On the other hand N.I.Achiezer (c.f. [8], p.128~129) also introduced the generalized Hilbert transform as follows

$$\tilde{f}(x)=(x-i)\,\underset{A\to\infty}{l.i.m.}\int_{-A}^{A} (isignu)\psi(u)e^{ixu}\,du,.$$

where

$$\psi(u)=\underset{A\to\infty}{l.i.m.}\int_{-A}^{A}\frac{f(t)}{t-i}e^{-iut}\,dt.$$

The author also introduced the generalized Hilbert transform of oerder 1 as follows

$$\tilde{f}_1(x)=P.V.\frac{(x+i)}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t}.$$

In later, the author became aware of the two modified formula to be just equivalent each other and proved it (c.f. [10], Chap. 4, Theorem 41, pp.194~195).

Now we shall return to our generalized Hilbert transform and let us write the multiplier $x+i=(t+i)+(x-t)$ and let us transform $\tilde{f}_1(x)$ formally as follows

$$\tilde{f}_1(x)=P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}dt=\tilde{f}(x)+A(f).$$

Thus if the constant $A(f)$ is finitely determined, the ordinary Hilbert Transform $\tilde{f}(x)$ is well defined and $A(f)$ is the difference between G.H.T. $\tilde{f}_1(x)$ and $\tilde{f}(x)$.

Therefore when we shall discuss the spectral decomposition of G.H.T. $\tilde{f}_1(x)$ by the use of Generalized Harmonic Analysis(G.H.A.) due to N.Wiener ,we should consider the difference $s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)$ and so the influence of constant $A(f)$ will be vanish and we shall expect to be obtained the spectral decomposition of the ordinary H.T. $\tilde{f}(x)$.

About fifty years ago, the author established the following theorem (c.f.[10]Theorem 49,pp.201~205)

Theorem A. Let $f(x)$ belong to the class W^2. Then we have for any positive number ε,

(i) if $|u|>\varepsilon$ then

$$s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)=(-i\,sign\ u)\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}$$

and

(ii) if $|u|\le\varepsilon$ then

$$s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)=i\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+2r_1(u+\varepsilon;f)+2r_2(u+\varepsilon;f)$$

where

$$r_1(u;f)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}\frac{e^{-ius}-1}{-is}ds$$

and

$$r_2(u;f)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds.$$

3. Proofs of several Lemmas.

We shall prove these by the so called complex method which means the use of complex variable analysis. It is so elegant compared with the real method which means the use of real variable analysis.

The ordinary Hilbert Transform (H.T.) is defined by the formula

$$\widetilde{f}(x)=P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{x-t}dt$$

and behaves well on the space L^2. The G. H. T. could be transformed as follows

$$\frac{\widetilde{f}_1(x)}{x+i}=P.V.\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t}=\left(\frac{f}{t+i}\right)^{\sim}(x)$$

so it behaves as well as H.T. on the space W^2 and keeps properties of H.T.

The ordinary Cauchy integral (C.I.) and the Generalized Cauchy Integral (G.C.I.) are defined by the formulas respectively

$$C(z;f)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}f(t)\frac{dt}{t-z}$$

and

$$C_1(z;f)=\frac{z+i}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{t-z}$$

where z=x+iy, y>0. The G.C.I. could be transformed as follows

$$\frac{C_1(z;f)}{z+i}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{t-z}=C(z;\frac{f}{t+i})$$

so it behaves as well as C.I. on the space W^2 and keeps properties of C.I. As for the Hilbert Transform and the Cauchy Integral, we shall refer to E.C.Titchmarsh : Introduction to the theory of Fourier Integrals, Oxford Univ. Press (1937). We shall also refer to theorems (c.f. [10] Chap. 4,pp.192~200).

Now we shall start to prove several lemmas.

Lemma A_1 Let us suppose that f belong to W^2. Then we have

$$\underset{y\to 0}{l.i.m.}\ \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z;f)\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$

$$=\frac{1}{2}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+\frac{i}{2}\{s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\}$$

Proof. Because of

$$\underset{y\to 0}{l.i.m.}\ C_1(z;f)\frac{2\sin\varepsilon t}{t}=\frac{1}{2}(f+i\widetilde{f}_1)\frac{2\sin\varepsilon t}{t},$$

we shall apply the Plancherel theorem , then we have

$$\int_{-\infty}^{\infty}|\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z;f)\frac{2\sin\varepsilon t}{t}e^{-iut}dt-(s(u+\varepsilon;\frac{f+i\widetilde{f}_1}{2})-s(u-\varepsilon;\frac{f+i\widetilde{f}_1}{2}))|^2\,du$$

$$\le 2\int_{-\infty}^{\infty}|\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z;f)\frac{2\sin\varepsilon t}{t}e^{-iut}dt-\{s(u+\varepsilon;C_1(z;f))-s(u-\varepsilon;C_1(z;f))\}|^2\,du$$

$$+2\int_{-\infty}^{\infty}|\{s(u+\varepsilon;C_1(z;f))-s(u-\varepsilon;C_1(z;f))\}-(s(u+\varepsilon;\frac{f+i\widetilde{f}_1}{2})-s(u-\varepsilon;\frac{f+i\widetilde{f}_1}{2}))|^2\,du$$

$$\to 2\int_{-\infty}^{\infty}|\,s(u+\varepsilon;C_1(z;f)-\frac{f+i\widetilde{f}_1}{2})-s(u-\varepsilon;C_1(z;f)-\frac{f+i\widetilde{f}_1}{2})|^2\,du,\quad (A\to\infty)$$

$$=2\int_{-\infty}^{\infty}|\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}(C_1(z;f)-\frac{f+i\widetilde{f}_1}{2})\frac{2\sin\varepsilon t}{t}e^{-iut}dt\,|^2\,du$$

$$=2\int_{-\infty}^{\infty}|(C_1(z;f)-\frac{f+i\widetilde{f}_1}{2})\frac{2\sin\varepsilon t}{t}|^2\,dt\to 0,\quad (y\to 0).$$

As for the Plancherel theorem, we shall refer to E.C.Titchmarsh : Introduction to the theory of Fourier Integrals, Oxford Univ. Press (1937).

Lemma A_2 We have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}dt=\sqrt{2\pi}\chi_\varepsilon(u)$$

where $\chi_\varepsilon(u)$ is the characteristic function of interval $(-\varepsilon,\ \varepsilon)$.

Proof. Because of

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\sqrt{2\pi}\chi_\varepsilon(u)e^{iut}du=\int_{-\varepsilon}^{\varepsilon}e^{iut}du=\frac{2\sin\varepsilon t}{t},$$

the inverse Fourier transform of $\sqrt{2\pi}\chi_\varepsilon(u)$ is $2\sin\varepsilon t/t$ and so the Fourier transform of $2\sin\varepsilon t/t$ is $\sqrt{2\pi}\chi_\varepsilon(u)$.

To prove following two Lemmas, we have to apply the theorem of residues in the complex analysis. As for these, we shall refer to E.C.Titchmarsh : Theory of Functions, The Oxford Univ. Press (1932). The arguments to be required are very elementary, but complicate and delicate a little.

Lemma A_3 We have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt=\frac{(1+signu)}{2}\sqrt{2\pi}ie^{-i(s-iy)u},\quad a.e.\ u$$

where $z=t+iy,\ y>0$.

Proof. To prove this, we shall consider the following integral

$$P.V.\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{iut}}{s-z}dt=\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt$$

and translate it into the contour integral in the $w-plane, (w=t+iv)$ and calculate it by the use of residue theorem on the theory of functions of complex variable.

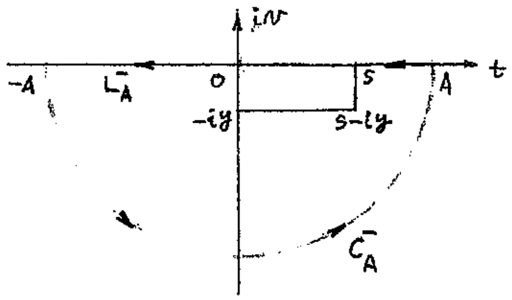
Since $1/s-z\in L^2$, the integral

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt$$

exists by the Plancherel theorem and so identifies with the above integral $a.e.\ u$.

We have to decompose into two cases according to the sign of u.

(i) the case of u > 0. We shall calculate along the following contour in the w−plane



$L_A^-=(-A,A)$

$C_A^-=\{w=Ae^{i\theta},\pi\le\theta\le2\pi\}$

$\Gamma_A^-=C_A^-\cup L_A^-$

Let us write

$$\frac{1}{2\pi i}\int_{\Gamma_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw$$

$$=\frac{1}{2\pi i}\int_{L_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw=\frac{1}{2\pi i}\int_{C_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw$$

If $w\in L_A^-$, then $w=t$ and we have

$$\frac{1}{2\pi i}\int_{L_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw=-\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt.$$

If $w\in C_A^-$, then $w=Ae^{i\theta}$, $(\pi<\theta<2\pi)$ and we have

$$\frac{1}{2\pi i}\int_{C_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw=\frac{1}{2\pi i}\int_{\pi}^{2\pi}\frac{e^{-iuw}}{(s-iy)-w}Aie^{i\theta}d\theta,$$

where

$$|(s-iy)-w|\sim|w|=A\text{, for sufficient large } A$$

and

$$|e^{-iuw}|\le 1,\quad e^{-iuw}=e^{-iuA\cos\theta}e^{uA\sin\theta}\to 0,\ \ a.e.\ u\ (\pi\le\theta\le 2\pi),(A\to\infty).$$

Then we have

$$\frac{1}{2\pi i}\int_{C_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw\to 0,\quad (A\to\infty).$$

On the other hand we have by the theorem of residues

$$\frac{1}{2\pi i}\int_{\Gamma_A^-}\frac{e^{-iuw}}{(s-iy)-w}dw=-e^{-iu(s-iy)},$$

for all sufficiently large A.

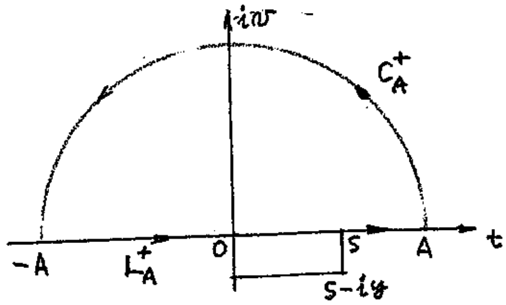
Therefore we have

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt=\sqrt{2\pi}ie^{-i(s-iy)}.$$

Furthermore we have $\frac{1}{s-z}=\frac{1}{(s-t)-iy}\in L^2$, as function of t for y>o, we shall conclude that

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt=\sqrt{2\pi}ie^{-i(s-iy)}.\quad \text{a.e. u.}$$

(ii) the case of u<0



$$L_A^+ = (-A, A)$$

$$C_A^+ = \{w = Ae^{i\theta}, 0 < \theta < \pi\}$$

$$\Gamma_A^+ = C_A^+ \cup L_A^+$$

Let us write

$$\cdot \ \frac{1}{2\pi i}\int_{\Gamma_A^-} \frac{e^{-iuw}}{(s-iy)-w}dw$$

$$= \frac{1}{2\pi \mathrm{i}}\int_{L_A^-} \frac{e^{-iuw}}{(s-iy)-w}dw = \frac{1}{2\pi \mathrm{i}}\int_{C_A^-} \frac{e^{-iuw}}{(s-iy)-w}dw$$

If $w \in L_A^+$, then $w = t$ and we have

$$\frac{1}{2\pi i}\int_{L_A^+} \frac{e^{-iuw}}{(s-iy)-w}dw = \frac{1}{2\pi i}\int_{-A}^{A} \frac{e^{-iut}}{s-z}dt.$$

If $w \in C_A^+$, then $w = Ae^{i\theta}$, $(0 < \theta < \pi)$ and we have

$$\frac{1}{2\pi i}\int_{C_A^+} \frac{e^{-iuw}}{(s-iy)-w}dw = \frac{1}{2\pi i}\int_0^{\pi} \frac{e^{-iuw}}{(s-iy)-w}Aie^{i\theta}d\theta,$$

where

$$|(s-iy)-w| \sim |w| = A, \quad \text{for sufficient large } A$$

and

$$|e^{-iuw}| \le 1, \quad e^{-iw} = e^{-iuA\cos\theta}e^{uA\sin\theta} \to 0, \quad (0 < \theta < \pi), \quad (A \to \infty).$$

Then we have

$$\frac{1}{2\pi i}\int_{C_A^+} \frac{e^{-iuw}}{(s-iy)-w}dw \to 0, \quad (A \to \infty).$$

On the other hand we have by the theorem of residues

$$\frac{1}{2\pi i}\int_{\Gamma_A^+} \frac{e^{-iuw}}{(s-iy)-w}dw = 0,$$

for all sufficiently large A.

Therefore we have

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt \ = 0,$$

and similarly as before we have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt \ = 0, \quad a.e. \ \ u$$

We shall state these results into one, we have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{e^{-iut}}{s-z}dt = \frac{(1+signu)}{2}\sqrt{2\pi}ie^{-i(s-iy)u}, \quad a.e. \ \ u,$$

where $z=t+iy, \ \ (y>0)$.

Lemma A_4 We have

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt, \quad (z=t+iy, \ \ y>0)$$

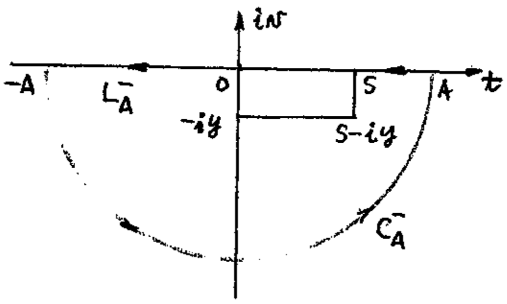
$$= \begin{cases} \sqrt{2\pi}ie^{-i(s-iy)u}\dfrac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, & (u>\varepsilon) \\ \sqrt{2\pi}ie^{-i(s-iy)u}\dfrac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, & (-\varepsilon<u<\varepsilon) \\ 0, & (u<-\varepsilon). \end{cases}$$

Proof. We shall estimate the integral along contour Γ on the $w-$plane

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw,$$

where $w=t+iv$ and the integrand has two singularities. The one is of simple pole at $w=s-iy$ and the other is that at $w=0$ of which we can exclude out. We shall separate into three cases and estimate it by the use of the theorem of residues too.

(i) the case of $u>\varepsilon$



$$L_A^- = (-A, A)$$

$$C_A^- = \{w = Ae^{i\theta}, \pi<\theta<2\pi\}$$

$$\Gamma_A^- = C_A^- \cup L_A^-$$

Let us write

$$\frac{1}{2\pi i}\int_{\Gamma_A^-}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw$$

$$=\frac{1}{2\pi i}\int_{C_A^-}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw+\frac{1}{2\pi i}\int_{L_A^-}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw$$

If $w\in C_A^-$, then $w=Ae^{i\theta}$ $(\pi<\theta<2\pi)$ and we have

$$\frac{1}{|w|}=\frac{1}{A},\quad \frac{1}{|(s-iy)-w|}\sim\frac{1}{|w|}=\frac{1}{A}\ \to 0,\quad (A\to\infty),$$

$$\left|\frac{2\sin\varepsilon w}{w}e^{-iuw}\right|\le\frac{1}{A}(e^{A(u-\varepsilon)\sin\theta}+e^{A(u+\varepsilon)\sin\theta})\to 0,\quad a.e.\ \ u\ \ (A\to\infty)$$

and

$$\left|\frac{2\sin\varepsilon w}{w}e^{-iuw}\right|\le\frac{2}{A},$$

for all sufficiently large A. Therefore we have

$$\left|\frac{1}{2\pi i}\int_{C_A^-}\frac{2\sin\varepsilon w}{w}\frac{e^{--iuw}}{(s-iy)-w}dw\right|$$

$$\le\frac{1}{2\pi A}\int_{\pi}^{2\pi}e^{A(u-\varepsilon)\sin\theta}d\theta+\frac{1}{2\pi A}\int_{\pi}^{2\pi}e^{A(u+\varepsilon)\sin\theta}d\theta\to 0,\quad (A\to\infty).$$

If $w\in L_A^-$, then $w=t$ and we have

$$\frac{1}{2\pi i}\int_{L_A^-}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw=-\frac{1}{2\pi i}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt,$$

where $z=t+iy,\ \ y>0$.

On the other hand we have by the theorem of residues

$$\frac{1}{2\pi i}\int_{\Gamma_A^-}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw=-e^{-iu(s-iy)}\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}.$$

Therefore we have

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt\ =\ \sqrt{2\pi}ie^{-i(s-iy)u}\frac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}$$

and so

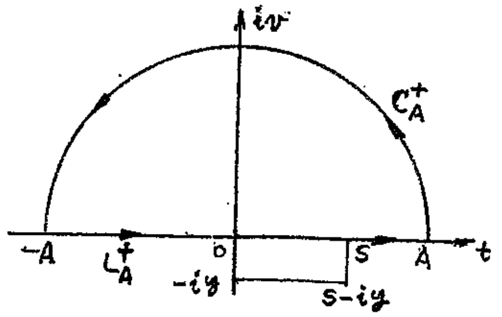
$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt = \sqrt{2\pi}ie^{-i(s-iy)u}\frac{e^{i(s-iy)\varepsilon}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, \quad a.e.\ \ u.$$

(ii) the case of $u<-\varepsilon$

$L_A^+=(-A,A)$

$C_A^+=\{w=Ae^{i\theta}, 0<\theta<\pi\}$

$\Gamma_A^+=C_A^+\cup L_A^+$



By the theorem of residue we have

$$\frac{1}{2\pi i}\int_{\Gamma_A^+}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw=0.$$

for all sufficient large A.

On the other hand ,let us write

$$\frac{1}{2\pi i}\int_{\Gamma_A^+}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw$$

$$=\frac{1}{2\pi i}\int_{C_A^+}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw+\frac{1}{2\pi i}\int_{L_A^+}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw.$$

If $w\in L_A^+$, then $w=t$, so we have

$$\frac{1}{2\pi i}\int_{L_A^+}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw=\frac{1}{2\pi i}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt,$$

where $z=t+iy,\ \ y>0$.

If $w\in C_A^+$, then $w=Ae^{i\theta}$ $(0<\theta<\pi)$, we have

$$\frac{1}{|w|}=\frac{1}{A},\quad \frac{1}{|(s-iy)-w|}\sim\frac{1}{|w|}=\frac{1}{A}\ \to 0,\quad (A\to\infty)$$

$$\left|\frac{2\sin\varepsilon w}{w}e^{-iuw}\right| \le \frac{1}{A}(e^{A(u-\varepsilon)\sin\theta}+e^{A(u+\varepsilon)\sin\theta})\to 0,\quad a.e.\ \ u\quad (A\to\infty)$$

and

$$\left|\frac{2\sin\varepsilon w}{w}e^{-iuw}\right|\le\frac{2}{A},$$

for all sufficiently large A. Then we have

$$\left|\frac{1}{2\pi i}\int_{C_A^+}\frac{2\sin\varepsilon w}{w}\frac{e^{-iuw}}{(s-iy)-w}dw\right|$$

$$\le\frac{1}{2\pi A}\int_0^{\pi}e^{A(u-\varepsilon)\sin\theta}d\theta+\frac{1}{2\pi A}\int_0^{\pi}e^{A(u+\varepsilon)\sin\theta}d\theta\to 0,\quad (A\to\infty).$$

Therefore we have

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt \;=\; 0$$

and so

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt \;=\; 0,\quad a.e.\ u.$$

(iii) the case of $|u|\le\varepsilon$

We have

$$I=\frac{1}{2\pi i}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt \;=\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-i(u-\varepsilon)t}}{t(s-z)}dt-\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-i(u+\varepsilon)t}}{t(s-z)}dt$$

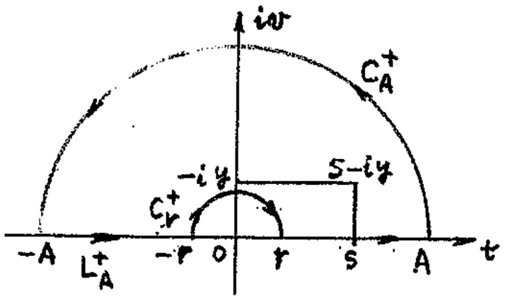
$$=I_1-I_2,\quad \text{say,}$$

where $z=t+iy, y>0$.

Estimation of I_1. We shall use the contour integral

$$\frac{1}{2\pi i}\int_{\Gamma_{A,r}^+}\frac{e^{-i(u-\varepsilon)w}}{w((s-iy)-w)}dw$$

where $w=t+iv$ and apply the theorem of residues.



$C_A^+=\{w=Ae^{i\theta}, 0<\theta<\pi\}$

$C_r^+=\{w=re^{i\theta}, 0<\theta<\pi\}$

$L_{A,r}^+=(-A,-r)\cup(r,A)$

$\Gamma_{A,r}^+=C_A^+\cup C_r^+\cup\Gamma_{A,r}^+$

We shall decompose the contour integral into three parts as follows

$$\frac{1}{2\pi i}\int_{\Gamma_{A,r}^+}\frac{e^{-i(u-\varepsilon)w}}{w((s-iy)-w)}dw=\frac{1}{2\pi i}\int_{C_A^+}(\text{"})dw-\frac{1}{2\pi i}\int_{C_r^+}(\text{"})dw+\frac{1}{2\pi i}\int_{L_{A,r}^+}(\text{"})dw$$

and estimate it as $r\to 0$ in the first and next $A\to\infty$ respectively.

We have by the theorem of residues

$$\frac{1}{2\pi i}\int_{\Gamma_{A,r}^{+}}\frac{e^{-i(u-\varepsilon)w}}{w((s-iy)-w)}dw=0,$$

for all r>0 and A>0 of $A>r>0$.

On the other hand, if $w\in C_A^+$, then we have $w=Ae^{i\theta}$ and $|w|=A$, $|(s-iy)-w|\sim|w|=A$ for all sufficiently large A. Since $|u|\le\varepsilon$, $0\le\theta\le\pi$, we have $-2\varepsilon\le u-\varepsilon\le 0$ and $\sin\theta\le 0$, then

$$e^{-i(u-\varepsilon)w}=e^{-i(u-\varepsilon)A\cos\theta}e^{(u-\varepsilon)A\sin\theta}\to 0,\qquad a.e.\ u\ \ (A\to\infty),$$

and

$$|e^{-iuw}|\le 1,$$

for all sufficiently large A. Therefore we have

$$|\int_{C_A^+}('')dw|\le\frac{1}{A}\int_0^{\pi}e^{(u-\varepsilon)A\sin\theta}d\theta\to 0,\qquad (A\to\infty).$$

If $w\in C_r^+$, then we have $w=re^{i\theta}$ and $|w|=r$, $(s-iy)-w\to s-iy$, $(r\to 0)$, then

$$e^{-i(u-\varepsilon)w}=e^{-i(u-\varepsilon)r\cos\theta}e^{(u-\varepsilon)r\sin\theta}\to 1,\qquad (r\to 0),$$

and

$$|e^{-iuw}|\le 1,$$

for all sufficiently large A. Therefore we have

$$\frac{1}{2\pi i}\int_{C_r^+}('')dw=\frac{1}{2\pi i}\int_{C_r^+}\frac{e^{-i(u-\varepsilon)w}}{w((s-iy)-w)}dw\to\frac{1}{2\pi(s-iy)}\int_0^{\pi}d\theta=\frac{1}{2(s-iy)},\qquad (r\to 0).$$

If $w\in L_{A,r}^+$ then we have $w=t$ and $(s-iy)-w=s-z$. .Therefore we have

$$\frac{1}{2\pi i}\int_{L_{A,r}^+}('')dw\to\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-i(u-\varepsilon)t}}{t(t-z)}dt,\qquad (r\to 0).$$

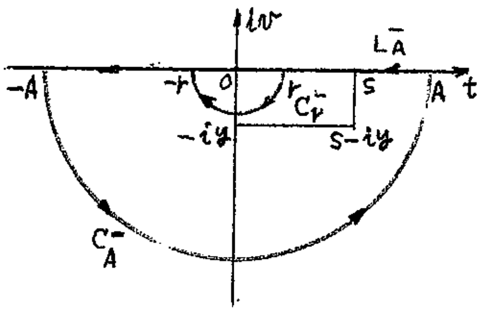
Now as for I_1 ,we shall tend $r\to 0$ in the first and next $A\to\infty$, we have

$$\lim_{A\to\infty}I_1=\lim_{A\to\infty}\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-i(u-\varepsilon)t}}{t(t-z)}dt=\frac{1}{2(s-iy)},\quad a.e.\ u.$$

Estimation of I_2. We shall use the contour integral

$$\frac{1}{2\pi i}\int_{\Gamma_{A,r}}\frac{e^{-i(u+\varepsilon)w}}{w((s-iy)-w)}dw,$$

where $w=t+iv$ and apply the residue theorem.



$C_A^- = \{w = Ae^{i\theta}, \pi < \theta < 2\pi\}$

$C_r^- = \{w = re^{i\theta}, \pi < \theta < 2\pi\}$

$L_{A,r}^- = (-A,-r) \cup (r, A)$

$\Gamma_{A,r}^- = C_A^- \in \cup C_r^- \cup L_{A,r}^-$

We shall decompose the contour integral into three parts as follows

$$\frac{1}{2\pi i}\int_{\Gamma_{A,r}^-} \frac{e^{-i(u+\varepsilon)w}}{w((s-iy)-w)}dw = \frac{1}{2\pi i}\int_{C_A^-}('')dw - \frac{1}{2\pi i}\int_{C_r^-}('')dw - \frac{1}{2\pi i}\int_{L_{A,r}^-}('')dw$$

and estimate it as $r \to 0$ in the first and next $A \to \infty$ respectively.

By the theorem of residues we have

$$\frac{1}{2\pi i}\int_{\Gamma_{A,r}^-} \frac{e^{-i(u+\varepsilon)w}}{w((s-iy)-w)}dw = -\frac{e^{-i(u+\varepsilon)(s-iy)}}{s-iy},$$

for all sufficiently small r and all sufficiently large A of $A > r > 0$.

If $w \in C_A^-$, we have $w = Ae^{i\theta}$ then $|w| = A$ and $|(s-iy)-w| \sim |w| = A$, for all sufficiently large A. Since $|u| \le \varepsilon$, $\pi \le \theta \le 2\pi$, we have $0 \le u+\varepsilon \le 2\varepsilon$ and $\sin\theta \le 0$,

then

$$e^{-i(u+\varepsilon)w} = e^{-i(u+\varepsilon)A\cos\theta}e^{(u+\varepsilon)A\sin\theta} \to 0, \qquad a.e.\ u \quad (A \to \infty)$$

and

$$|e^{-iuw}| \le 1,$$

for all sufficiently large A. Therefore we have

$$\left|\int_{C_A^-}('')dw\right| \le \frac{1}{A}\int_{\pi}^{2\pi} e^{(u+\varepsilon)A\sin\theta}d\theta \to 0, \qquad (A \to \infty).$$

If $w \in C_r^-$, we have $w = re^{i\theta}$ then $|w| = r$ and $r = (s-iy) - w \to s-iy, (r \to 0)$. Since $|u| \le \varepsilon, \pi \le \theta \le 2\pi$, we have $0 \le u+\varepsilon \le 2\varepsilon$ and $\sin\theta \le 0$, then

$$e^{-i(u+\varepsilon)w} = e^{-i(u+\varepsilon)r\cos\theta}e^{(u+\varepsilon)r\sin\theta} \to 1, \qquad a.e.\ u \quad (r \to 0)$$

and

$$|e^{-iuw}| \le 1,$$

for all sufficiently small r. Therefore we have

$$\frac{1}{2\pi i}\int_{C_r}(")dw \to \frac{1}{2\pi(s-iy)}\int_{\pi}^{2\pi}d\theta = \frac{1}{2(s-iy)}, \quad (r \to 0).$$

If $w \in L^-_{A,r}$,we have $w = t$ then $(s-iy)-w = (s-iy)-t = s-z$, therefore we have

$$\frac{1}{2\pi i}\int_{L^-_{A,r}}(")dw \to \frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-i(u+\varepsilon)t}}{t(t-z)}dt, \quad (r \to 0).$$

Now we shall tend $r \to 0$ in the first and next $A \to \infty$, we have

$$\lim_{A\to\infty} I_2 = \lim_{A\to\infty}\frac{1}{2\pi i}\int_{-A}^{A}\frac{e^{-i(u+\varepsilon)t}}{t(t-z)}dt = -\frac{1}{2(s-iy)} + \frac{e^{-i(u+\varepsilon)(s-iy)}}{s-iy}.$$

Combining both estimations into one, we have

$$\lim_{A\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt$$
$$= \sqrt{2\pi}\frac{1}{2(s-iy)} - \sqrt{2\pi}\left\{-\frac{1}{2(s-iy)} + \frac{e^{-i(u+\varepsilon)(s-iy)}}{s-iy}\right\}$$
$$= \sqrt{2\pi}\frac{1}{s-iy} - \sqrt{2\pi}\frac{e^{-i(u+\varepsilon)(s-iy)}}{s-iy} = \sqrt{2\pi}ie^{-i(s-iy)u}\frac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}.$$

Since $\frac{2\sin\varepsilon t}{t}\frac{1}{s-z} \in L^1 \cup L^2$, $(z = t+iy, y > 0)$, we shall identify its F.T. in the L^2 with its F.T. in the L^1. Thus we have proved

$$\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}\frac{e^{-iut}}{s-z}dt = \sqrt{2\pi}ie^{-i(s-iy)u}\frac{e^{i(s-iy)u}-e^{-i(s-iy)\varepsilon}}{i(s-iy)}, \quad a.e. \ u.$$

In the last ,we shall prove the following lemma.

Lemma A_5 For any y and ε $(0 < y < 1, 0 < \varepsilon < 1)$ there exist a constant A such that the following inequalities are satisfied.

$$\left|\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}\right| \le A\left\{\left|\frac{\sin\varepsilon s}{s}\right| + \frac{1}{1+|s|}\right\}.$$

Proof. It is transformed as follows

$$e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)} = (e^{i\varepsilon s}-e^{-i\varepsilon s}) + (e^{i\varepsilon(s-iy)}-e^{i\varepsilon s}) - (e^{-i\varepsilon(s-iy)}-e^{-i\varepsilon s})$$
$$= (e^{i\varepsilon s}-e^{-i\varepsilon s}) + e^{i\varepsilon s}(e^{\varepsilon y}-1) - e^{-i\varepsilon s}(e^{-\varepsilon y}-1)$$

and we have

$$\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)} = \frac{2\sin\varepsilon s}{(s-iy)} + \frac{e^{i\varepsilon s}(e^{\varepsilon y}-1)}{i(s-iy)} - \frac{e^{-i\varepsilon s}(e^{-\varepsilon y}-1)}{i(s-iy)}$$

$$= K_1 + K_2 + K_3, \text{ say.}$$

Let us remark the following inequality

$$\frac{y}{|s-iy|} \le \frac{1+y}{1+|s|}, \quad (0<y<1).$$

Then we have

$$|K_1| = \left|\frac{2\sin\varepsilon s}{s-iy}\right| \le \left|\frac{2\sin\varepsilon s}{s}\right| \frac{|s|}{|s-iy|} \le 2\left|\frac{\sin\varepsilon s}{s}\right|,$$

$$|K_2| \le \frac{e^{\varepsilon y}-1}{|s-iy|} \le \frac{A\varepsilon y}{|s-iy|} \le \frac{A\varepsilon(1+y)}{1+|s|} \le \frac{2A}{1+|s|},$$

$$|K_3| \le \frac{1-e^{-\varepsilon y}}{|s-iy|} \le \frac{A\varepsilon y}{|s-iy|} \le \frac{A\varepsilon(1+y)}{1+|s|} \le \frac{2A}{1+|s|}$$

where we shall use the same A at each occurrence as an absolute constant.

4. Proof of Theorem A.

We shall begin to introduce the function $f_B(x) = f(x)\chi_B(x)$, with which characteristic function χ_B on the interval $(-B,B)$. Then we have $f_B(x) \in L^1 \cap L^2$ and so it enable us to apply the Plancherel theorem.

Let us begin to consider the following formula

$$s(u+\varepsilon; C_1(z;f_B)) - s(u-\varepsilon; C_1(z;f_B))$$
$$= \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} C_1(z;f_B) \frac{2\sin\varepsilon t}{t} e^{-iut} dt, \quad (z = t+iy, y>0)$$
$$= \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} e^{-iut} dt \frac{z+i}{2\pi i} \int_{-B}^{B} \frac{f(s)}{s+i}\frac{ds}{s-z}$$

Since as

$$\frac{z+i}{s-z} = \frac{s+i}{s-z} - 1,$$

we have

$$s(u+\varepsilon; C_1(z;f_B)) - s(u-\varepsilon; C_1(z;f_B))$$
$$= \underset{A\to\infty}{l.i.m.} \frac{1}{2\pi i} \int_{-B}^{B} f(s) ds \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} \frac{e^{-iut}}{s-z} dt$$
$$- \underset{A\to\infty}{l.i.m.} \frac{1}{2\pi i} \int_{-B}^{B} \frac{f(s)}{s+i} ds \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} e^{-iut} dt$$
$$= \frac{1}{2\pi i} \int_{-B}^{B} f(s) ds \ \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \frac{2\sin\varepsilon t}{t} \frac{e^{-iut}}{s-z} dt$$

$$-\frac{1}{2\pi i}\int_{-B}^{B}\frac{f(s)}{s+i}ds\ \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$

The interchanges of integral and limit in mean are guaranteed as before by the theorem of Plancherel . By the reason of Lemma A_2, we shall estimate the above formula by decomposing into the case $|u|>\varepsilon$ and $|u|\le\varepsilon$ respectively.

(i) the case of $|u|>\varepsilon$. We have by Lemma A_4

$$s(u+\varepsilon;C_1(z;f_B))-s(u-\varepsilon;C_1(z;f_B))$$
$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}C_1(z;f_B)\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$
$$=\frac{(1+signu)}{2}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s)\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}e^{-yu}e^{-ius}ds.$$

Now we shall remark that

$$\underset{y\to o}{l.i.m.}\,C_1(z;f_B)\frac{2\sin\varepsilon t}{t}=\frac{1}{2}(f_B+i\tilde{f}_{B1})\frac{2\sin\varepsilon t}{t}$$

and

$$\underset{y\to o}{l.i.m.}\,f(s)\frac{e^{i\varepsilon(s-iy)}-e^{-i\varepsilon(s-iy)}}{i(s-iy)}e^{-yu}=f(s)\frac{2\sin\varepsilon s}{s}.$$

Then we have

$$s(u+\varepsilon;\frac{1}{2}(f_B+i\tilde{f}_{B1}))-s(u-\varepsilon;\frac{1}{2}(f_B+i\tilde{f}_{B1}))$$
$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(f_B+i\tilde{f}_{B1})\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$
$$=\frac{(1+signu)}{2}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s)\frac{2\sin\varepsilon s}{s}e^{-ius}ds.$$

Furthermore we shall remark that

$$\underset{B\to\infty}{l.i.m.}\ \underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(f_B+i\tilde{f}_{B1})\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$
$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(f+i\widetilde{f_1})\frac{2\sin\varepsilon t}{t}e^{-iut}dt.$$

Then we have

$$s(u+\varepsilon;\frac{1}{2}(f+i\tilde{f}_1))-s(u-\varepsilon;\frac{1}{2}(f+i\tilde{f}_1))$$
$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(f+i\widetilde{f_1})\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$

$$=\frac{(1+signu)}{2}\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s)\frac{2\sin\varepsilon s}{s}e^{-ius}ds$$

The last formula shows that

$$\frac{1}{2}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+\frac{i}{2}\{s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\}$$

$$=\frac{(1+signu)}{2}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}.$$

Thus we have proved that

$$s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)=(-isignu)\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}.$$

(ii) the case of $|u|\le\varepsilon$. We shall estimate it by the same method as before. By the use of Lemma A_2, Lemma A_4 and then by the interchange of the limit in the mean between A and y ,we have

$$s(u+\varepsilon;\frac{1}{2}(f_B+i\tilde{f}_{B1}))-s(u-\varepsilon;\frac{1}{2}(f_B+i\tilde{f}_{B1}))$$

$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(f_B+i\tilde{f}_{B1})\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}f(s)\frac{e^{-i(u+\varepsilon)s}-1}{-is}ds+\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}ds$$

and we have

$$=\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}\frac{e^{-i(u+\varepsilon)s}-1}{-is}ds+\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-i(u+\varepsilon)s}ds.$$

By the interchange of the limit in the mean between A and B, we have

$$s(u+\varepsilon;\frac{1}{2}(f+i\tilde{f}_1))-s(u-\varepsilon;\frac{1}{2}(f+i\tilde{f}_1))$$

$$=\underset{A\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}\frac{1}{2}(f+i\widetilde{f}_1)\frac{2\sin\varepsilon t}{t}e^{-iut}dt$$

$$=\underset{B\to\infty}{l.i.m.}\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}\frac{e^{-i(u+\varepsilon)s}-1}{-is}ds+\underset{B\to\infty}{l.i.m.}\frac{i}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-i(u+\varepsilon)s}ds$$

$$=ir_1(u+\varepsilon;f)+ir_2(u+\varepsilon;f),\quad \text{say.}$$

The last formula shows that

$$\frac{1}{2}\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+\frac{i}{2}\{s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)\}$$

$$=ir_1(u+\varepsilon;f)+ir_2(u+\varepsilon;f).$$

Thus we have proved

$$s(u+\varepsilon;\widetilde{f}_1)-s(u-\varepsilon;\widetilde{f}_1)=i\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}+2r_1(u+\varepsilon;f)+2r_2(u+\varepsilon;f),$$

where

$$r_1(u;f)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}\frac{e^{-ius}-1}{-is}ds$$

and

$$r_2(u;f)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds.$$

5 Estimation of the remainder terms.

(i) the case of $r_1(u;f)$. We have

$$r_1(u;f)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}\frac{e^{-ius}-1}{-is}ds$$

$$=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}ds\int_{0}^{u}e^{-ivs}dv=\underset{B\to\infty}{l.i.m.}\int_{0}^{u}dv\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ivs}ds$$

$$=\int_{0}^{u}dv\left(\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds\right)$$

By the Schwartz inequality and then the Plancherel theorem, we have

$$\int_{-\varepsilon}^{\varepsilon}|r_1(u+\varepsilon;f)|^2\,du=\int_{0}^{2\varepsilon}|r_1(u;f)|^2\,du$$

$$\le\int_{0}^{2\varepsilon}du\left(\int_{0}^{u}dv\right)\left(\int_{0}^{u}|\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds|^2\,dv\right)$$

$$=2\varepsilon^2\int_{-\infty}^{\infty}\frac{|f(s)|^2}{1+|s|^2}ds.$$

Therefore we have the following proposition

$$(R_1)\qquad\qquad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_1(u;f)|^2\,du=O(\varepsilon),\qquad(\varepsilon\to 0).$$

(ii) the case of $r_2(u;f)$.

$$r_2(u;f)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds$$

Let us set the following hypothesis.

There exist a constant $a(f)$ such that

$$(R_2)\qquad\qquad \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_2(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)|^2\,du\to 0,\quad(\varepsilon\to 0).$$

Let us examine this hypothesis as for some cases.

(a) The case of the constant $A(f)$ to be finitely determined. Let us suppose that $f(x)$ belong to W^2 and if the constant $A(f)$ is finitely determined, then the ordinary Hilbert Transform $\tilde{f}(x)$ exist and belong to W^2. Because there is the equality between them : $\tilde{f}_1(x) = \tilde{f}(x) + A(f)$ and $\tilde{f}_1(x)$ belong to W^2. Then we could define the G.F.T. of $\tilde{f}(x)$ and so the theorem of spectoral decomposition of $\tilde{f}_1(x)$ lead to that of $\tilde{f}(x)$ by the use of Theorem A and Lemma A_2. This is as follows

(i) if $|u| > \varepsilon$ then

$$s(u+\varepsilon;\tilde{f}) - s(u-\varepsilon;\tilde{f}) = (-isignu)\{s(u+\varepsilon;f) - s(u-\varepsilon;f)\}$$

and

(ii) if $|u| \leq \varepsilon$ then

$$s(u+\varepsilon;\tilde{f}) - s(u-\varepsilon;\tilde{f})$$

$$= i\{s(u+\varepsilon;f) - s(u-\varepsilon;f)\} + 2r_1(u+\varepsilon;f) + 2r_2(u+\varepsilon;f) - \sqrt{2\pi}A(f).$$

Now we shall desire to be satisfied the hypothesis with $A(f)$ as for $a(f)$. That is

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} | r_2(u+\varepsilon;f) - \sqrt{\frac{\pi}{2}} A(f) |^2 \, du = 0,$$

where

$$r_2(u+\varepsilon;f) - \sqrt{\frac{\pi}{2}} A(f) = \underset{B\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-B}^{B} \frac{f(s)}{s+i} e^{-i(u+\varepsilon)s} ds - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{f(s)}{s+i} ds$$

The general case is still open. But if we could suppose that the integral

$$\int_{-\infty}^{\infty} \left| \frac{f(s)}{s+i} \right| ds < \infty$$

exist, then the hypothesis is satisfied with the constant $A(f)$. Because Fourier transform of $f(s)/s+i$ in L^2 identifies with the one in L^1 $a.e.$ u and so continuous at $u = 0$ in the mean, that is

$$\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds=\lim_{B\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds\to\sqrt{\frac{\pi}{2}}A(f),\quad (u\to 0).$$

Therefore we have by the theorem of bounded convergence,

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{f(s)}{s+i}e^{-ius}ds-\sqrt{\frac{\pi}{2}}A(f)|^2\,du=0.$$

(b) The case of polynomial. Next if $f(x)$ is polynomial $p(x)=c_0+\sum_{\lambda_n\neq 0}c_n e^{i\lambda_n x}$, we shall examine what the remainder term (R_2) as for $p(x)$ behaves. By the Lemma A_3 we have

$$\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{e^{i\lambda s}}{s+i}e^{-ius}ds=-\frac{1+sign(u-\lambda)}{2}\sqrt{2\pi}ie^{-(u-\lambda)},\quad a.e.\ u$$

then we have

$$r_2(u+\varepsilon;p)=\underset{B\to\infty}{l.i.m.}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{p(s)}{s+i}e^{-i(u+\varepsilon)s}ds$$

$$=-\sqrt{2\pi}i\frac{1+sign(u+\varepsilon)}{2}c_0e^{-(u+\varepsilon)}-\sqrt{2\pi}i\sum_{\lambda_n\neq 0}\frac{1+sign(u+\varepsilon-\lambda_n)}{2}c_ne^{-(u+\varepsilon-\lambda_n)}.$$

Therefore we have for $|u|\leq\varepsilon$

$$r_2(u+\varepsilon;p)=-\sqrt{\frac{\pi}{2}}2i(c_0+\sum_{\lambda_n<0}c_ne^{\lambda_n})e^{-(u+\varepsilon)},\quad a.e.\ u$$

for all sufficient small ε. Therefore if we put

$$a(p)=-2i(c_0+\sum_{\lambda_n<0}c_ne^{\lambda_n})$$

then we have

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|r_2(u+\varepsilon;p)-\sqrt{\frac{\pi}{2}}a(p)|^2\,du=O\left(\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|e^{-(u+\varepsilon)}-1|^2\,du\right)=o(1),\quad (\varepsilon\to 0).$$

(iii) the case of $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2du$

Let us estimate the above formula. Then we have by the Lemma A_2 and the theorem of Plancherel

$$\int_{-\varepsilon}^{\varepsilon} | s(u+\varepsilon;f)-s(u-\varepsilon;f) |^2 du$$

$$= \int_{-\infty}^{\infty} | \{s(u+\varepsilon;f)-s(u-\varepsilon;f)\chi_\varepsilon(u)\} |^2 \, du = \int_{-\infty}^{\infty} | \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} f(t) \frac{2\sin \varepsilon t}{t} \frac{\sin \varepsilon(x-t)}{x-t} dt |^2 \, dx$$

$$= 8\int_{-\infty}^{\infty} | \frac{\sin \varepsilon x}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \varepsilon t \cos \varepsilon t}{t} \frac{dt}{x-t} - \frac{\cos \varepsilon x}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \varepsilon t \sin \varepsilon t}{t} \frac{dt}{x-t} |^2 \, dx .$$

Then we have by the Minkowski inequality and the theorem of Hilbert transform

$$\int_{-\varepsilon}^{\varepsilon} | s(u+\varepsilon;f)-s(u-\varepsilon;f) |^2 du$$

$$\le 16 \int_{-\infty}^{\infty} | \frac{\sin \varepsilon x}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \varepsilon t \cos \varepsilon t}{t} \frac{dt}{x-t} |^2 \, dx$$

$$+16 \int_{-\infty}^{\infty} | \frac{\cos \varepsilon x}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \varepsilon t \sin \varepsilon t}{t} \frac{dt}{x-t} |^2 \, dx$$

$$\le 32 \int_{-\infty}^{\infty} | f(x) |^2 \frac{\sin^2 \varepsilon x}{x^2} dx$$

Applying the Wiener formula , we have

$$\lim_{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} \int_{-\varepsilon}^{\varepsilon} | s(u+\varepsilon;f)-s(u-\varepsilon;f) |^2 du \le 8 \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon} \int_{-\infty}^{\infty} | f(x) |^2 \frac{\sin^2 \varepsilon x}{x^2} dx$$

$$= 8 \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} | f(x) |^2 dx .$$

Now let us estimate for trigonometric polynomial $p(x)$ instead of $f(x)$. Let us suppose that

$$p(x) = c_0 + \sum_{\lambda_n \ne 0} c_n e^{i\lambda_n x}$$

then, we have by the lemma A_2

$$s(u+\varepsilon;p)-s(u-\varepsilon;p) = \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} p(x) \frac{2\sin \varepsilon x}{x} e^{-iux} dx$$

$$= \sqrt{2\pi} c_0 \chi_\varepsilon(u) + \sqrt{2\pi} \sum_{\lambda_n \ne 0} c_n \chi_\varepsilon(u-\lambda_n) = \sqrt{2\pi} c_0 \chi_\varepsilon(u),$$

for all sufficient small ε .

Then we have

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} | \{s(u+\varepsilon;f)-s(u-\varepsilon;f)\} - \sqrt{2\pi} c_0 |^2 \, du$$

$$=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;f-p)-s(u-\varepsilon;f-p)|^2du$$

$$\leq 16\pi\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)-p(x)|^2dx.$$

Therefore it is natural to set the following hypothesis.

There exist a constant c_0 such that

$$(R_0)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\{s(u+\varepsilon;f)-s(u-\varepsilon;f)\}-\sqrt{2\pi}c_0|^2\,du=0.$$

If we put in the above formula, $p(x)=c_0$ then we shall conclude that the hypothesis (R_0) is equivalent to

$$(R_0)'\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;f-c_0)-s(u-\varepsilon;f-c_0)|^2\,du=0.$$

These mean that the spectrum of function f at the origin u=0 is isolate or concentrate to it with energy a few.

(iv) the case of $\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)|^2\,du$

By the part (ii) $|u|\leq\varepsilon$ of Theorem A, we have

$$\left\{s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)\right\}-\sqrt{2\pi}\tilde{c}_0$$

$$=\ i\left\{s(u+\varepsilon;f)-s(u-\varepsilon;f)-\sqrt{2\pi}c_0\right\}+2r_1(u+\varepsilon;f)+2\left\{r_2(u+\varepsilon;f)-\sqrt{\frac{\pi}{2}}a(f)\right\}$$

where

$$\tilde{c}_0=ic_0+a(f).$$

Then under the condition that hypothesis (R_0) is satisfied, (R_2) is equivalent to

$$(\tilde{R}_0)\qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|\left\{s(u+\varepsilon;\tilde{f}_1)-s(u-\varepsilon;\tilde{f}_1)\right\}-\sqrt{2\pi}\tilde{c}_0|^2\,du=0.$$

This is easily verified by the Minkowski inequality and the proposition (R_1)

These remarks will be useful to the application for almost periodic functions. Furthermore we shall prove that if $f(x)$ belongs to the class S, then we have

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}|s(u+\varepsilon;f)-s(u-\varepsilon;f)|^2\,du=\sigma(0+)-\sigma(0-)$$

where $\sigma(u)$ is the G.F.T. of $\varphi(x)$ the auto-correlation function of $f(x)$.

These are defined as follows

$$\varphi(x) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\overline{f(t)dt}$$

and

$$\sigma(u) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} \varphi(x) \frac{e^{-iux} - 1}{-ix} dx + \underset{A\to\infty}{l.i.m.} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} \varphi(x) \frac{e^{-iux}}{-ix} dx.$$

We shall refer these to the forth-coming papers II and III in this series.

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