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Noncommutative Deformation of Spinor Zero Mode and ADHM Construction

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Abstract

A method to construct noncommutative instantons as deformations from commutative instantons was provided in [1]. Using this noncommutative deformed instanton, we investigate the spinor zero modes of the Dirac operator in a noncommutative instanton background on noncommutative \mathbb{R}^4 , and we modify the index of the Dirac operator on the noncommutative space slightly and show that the number of the zero mode of the Dirac operator is preserved under the noncommutative deformation. We prove the existence of the Green's function associated with instantons on noncommutative \mathbb{R}^4 , as a smooth deformation of the commutative case. The feature of the zero modes of the Dirac operator and the Green's function derives noncommutative ADHM (Atiyah-Drinfeld-Hitchin-Manin) equations which coincide with the ones introduced by Nekrasov and Schwarz. We show a one-to-one correspondence between the instantons on noncommutative \mathbb{R}^4 and ADHM data. An example of a noncommutative instanton and a spinor zero mode are also given.

Key words: Noncommutative geometry, Yang-Mills theory, Instanton
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1 Introduction

Deformation Quantization, introduced by Flato et al [2] provided an idea for the method of quantization, whose crucial point is not to employ the representation space, and treat it from purely algebraic point of view. It might be worth to apply this idea to gauge theories for geometry and physics. One of the important problems for the gauge theory is an instanton, which has been

tried by several people. Our approach presented in this paper is to develop it in the context of the deformation quantization.

Nekrasov and Schwarz [3] discovered noncommutative ADHM equations and constructed noncommutative instantons using the ADHM construction [4]. (In the following, we call these solutions briefly noncommutative ADHM instantons.) This work initiated the study of noncommutative ADHM instantons, and at present there is a large body of work on this problem [5]. Several noncommutative instantons have been discovered, by the ADHM method. However, some of them like $U(1)$ instantons do not smoothly connect to commutative instantons. (There are several noncommutative instanton solutions whose commutative limits have been studied, and some of them are constructed without the ADHM method [6].) In [7–11], the topological charge of a noncommutative ADHM instanton is studied, where it is shown that the topological charge is given by an integer and coincides with the dimension of a vector space appearing in the ADHM construction. (Strictly speaking, the proof of the equivalence between the topological charge defined as the integral of the second Chern class and the instanton number given by the dimension of the vector space in the ADHM construction is not completed. In [8], this identification is shown when the noncommutative parameter is self-dual for a $U(N)$ gauge theory. In [11], the equivalence is shown with no restrictions on the noncommutative parameters, but a noncommutative version of the Osborn's identity (Corrigan's identity) is assumed.) However, the relation between the topological number and the corresponding numbers in the commutative space had not be clarified. Moreover, the calculation in [7,8] shows that the origin of the instanton number is deeply related to the noncommutativity.

On the other hand, we have constructed previously new noncommutative deformations of solitons in gauge theories. These deformations smoothly connect a commutative soliton to a noncommutative soliton [1,12,13]. In the following, we call these smooth noncommutative deformed instantons SNCD instantons for short. The SNCD instantons have a formal power series expansion in the noncommutative parameter, and the leading terms are instantons in commutative space. In particular, this produces instanton solutions on noncommutative \mathbb{R}^4 which are deformations of instanton solutions on commutative \mathbb{R}^4 . We showed that the instanton numbers of these noncommutative instanton solutions coincide with the commutative instanton numbers on \mathbb{R}^4 . Thus, it is natural to ask if there is a correspondence between the SNCD instantons and the noncommutative ADHM construction. Answering this question is the main purpose of this paper.

In the section 3.3 in [3], the completeness of the noncommutative ADHM construction is already discussed without any proofs. In this paper, we give a complete proof for the completeness for the SNCD instanton. The procedure of the proof is basically followed by the commutative case. One of the cru-

cial points for the differences to the commutative case is to observe the decay properties for the instantons on the noncommutative 4-space which has been obtained in [1]. Moreover, we have to have the asymptotic behavior of the zero modes of the Dirac operator associated with the SNCD instanton, the index of the Dirac operator, and the Green's function with the SNCD instanton background. In this article, we first investigate zero modes of the Dirac operator associated with the SNCD instanton. We give a (modified) index of the Dirac operator on the noncommutative space. It is shown that this index is determined by the index associated with the commutative instanton. We show the existence of the Green's function with the background SNCD instanton. Using these properties, we derive the noncommutative ADHM equations from the SNCD instanton. The ADHM equations coincide with the ones discovered by Nekrasov and Schwarz [3,5]. We construct one example of a SNCD instanton deformed from a $k = 1$ BPST instanton in commutative \mathbb{R}^4 , and we check its consistency with the theorems in this article. In the Appendix, we show that there is one-to-one correspondence between the ADHM data and the SNCD instantons.

This paper is organized as follows. In Section 2, we set the notation and review basic facts about star products and SNCD instantons. In Section 3, we show that the (modified) index of the Dirac operator is constant under noncommutative deformations. In Section 4, we construct the Green's function for the noncommutative Laplacian. In Section 5, we prove the main result, that the ADHM equations derived from noncommutative instantons are the same as the equations constructed by Nekrasov and Schwarz. In Section 6, we give a worked example of a noncommutative instanton. Section 7 is the conclusion. In Appendix A, some extension of the completeness relation of the Dirac zero modes is derived. In Appendix B, we show the one-to-one correspondence between ADHM data and noncommutative instantons, and in Appendix C, we discuss constraints imposed by the choice of the $U(N)$ gauge group.

2 Notations, Definitions and Known Facts

Noncommutative Euclidean 4-space \mathbb{R}^4 is given by the following commutation relations of the coordinates:

$$[x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4, \quad (2.1)$$

where $(\theta^{\mu\nu})$ is a real, x -independent, skew-symmetric matrix, whose entries are called the noncommutative parameters. \star is known as the Moyal (or star) product [14]. To consider smooth noncommutative deformations, we introduce a parameter \hbar and a fixed real constant $-\infty < \theta_0^{\mu\nu} < \infty$ with

$$\theta^{\mu\nu} = \hbar\theta_0^{\mu\nu}. \quad (2.2)$$

We define the commutative limit by letting $\hbar \rightarrow 0$.

The Moyal product is defined on functions by

$$\begin{aligned} f(x) \star g(x) &:= f(x) \exp\left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right) g(x) \\ &= f(x)g(x) + \sum_{n=1}^{\infty} \frac{1}{n!} f(x) \left(\frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu\right)^n g(x). \end{aligned}$$

Here $\overleftarrow{\partial}_\mu$ and $\overrightarrow{\partial}_\nu$ are partial derivatives with respect to x^μ for $f(x)$ and to x^ν for $g(x)$, respectively.

Moreover, we consider \hbar -expansions functions $f(x)$ (formal power series in \hbar with the values in $C^\infty(\mathbb{R}^4)$) in the following:

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(x) \hbar^n, \quad (2.3)$$

where $f^{(n)}(x) \in C^\infty(\mathbb{R}^4)$. We mainly consider each $f^{(n)}(x) \in C^\infty(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$. We extend the Moyal product to the above fields (2.3) and also to other fields like spinors \hbar linealy. In the following, we consider all subjects by using this formal expansion and solve equations (Dirac equations, etc.) recursively in increasing orders of \hbar .

We often use order estimates in the radius $|x|$. If s is a function on \mathbb{R}^4 and $s = O(|x|^{-m})$, the ‘‘natural growth condition’’ is defined by $|\partial_\mu^k s| = O(|x|^{-m-k})$. In this article, this natural growth condition of gauge fields and spinor fields is always required. ($s = O'(|x|^{-m})$ is defined by $s = O(|x|^{-m})$ and $|\partial_\mu^k s| = O(|x|^{-m-k})$ in [15]. We do not use this symbol $O'(|x|^{-m})$ because it is not standard in physics.)

We define a Lie algebra structure by $[\mathbf{T}_a, \mathbf{T}_b] = f_{abc} \mathbf{T}_c$, where the generators \mathbf{T}_a are anti-Hermitian matrices. In this article, $U(N)$ ($N > 1$) gauge theory on noncommutative \mathbb{R}^4 is considered. The covariant derivative for a some fundamental representation field $f(x)$ is defined by

$$D_\mu \star f(x) := \partial_\mu f(x) + A_\mu \star f(x), \quad A_\mu = A_\mu^a \mathbf{T}_a. \quad (2.4)$$

A gauge transformation of A is given by $A \rightarrow A + g \star dg^{-1}$, where g is an element of the gauge group $G = \{g \mid g^\dagger \star g = I_{n \times n}\}$. Here g has a formal expansion $g = \sum_{l=0}^{\infty} g^{(l)} \hbar^l$. As we see in [12] and Appendix C, $g^\dagger \star g = I$ is

equivalent to an infinite hierarchy of algebraic equations which we can solve recursively starting with the \hbar^0 term. The Laplacian is defined by

$$\Delta_A \star f := D^\mu \star D_\mu \star f. \quad (2.5)$$

The curvature two-form F is defined by

$$F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA + A \wedge \star A, \quad (2.6)$$

where $\wedge \star$ is defined by

$$A \wedge \star A := \frac{1}{2} (A_\mu \star A_\nu) dx^\mu \wedge dx^\nu. \quad (2.7)$$

Let $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ be the spinor bundle of \mathbb{R}^4 . We define σ_μ and $\bar{\sigma}_\mu$ by

$$\begin{aligned} (\sigma_1, \sigma_2, \sigma_3, \sigma_4) &:= (-i\tau_1, -i\tau_2, -i\tau_3, I_{2 \times 2}), \\ (\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_3, \bar{\sigma}_4) &:= (i\tau_1, i\tau_2, i\tau_3, I_{2 \times 2}), \end{aligned} \quad (2.8)$$

where τ_i are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (2.9)$$

and $I_{2 \times 2}$ is the identity matrix of dimension 2. Note that $\sigma_\mu^\dagger = \bar{\sigma}_\mu$. σ_μ and $\bar{\sigma}_\mu$ are a 2-dimensional matrix representation of the quaternions such, i.e.

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_\nu \bar{\sigma}_\mu = \bar{\sigma}_\mu \sigma_\nu + \bar{\sigma}_\nu \sigma_\mu = 2\delta_{\mu\nu}. \quad (2.10)$$

We define $\sigma_{\mu\nu}$ and $\bar{\sigma}_{\mu\nu}$ as

$$\sigma_{\mu\nu} := \frac{1}{4} (\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu), \quad \bar{\sigma}_{\mu\nu} := \frac{1}{4} (\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu), \quad (2.11)$$

which are the components anti-selfdual and selfdual two-form, respectively. The Dirac(-Weyl) operators $\mathcal{D}_{A\star} : \Gamma(\mathcal{S}^+ \otimes E)[[\hbar]] \rightarrow \Gamma(\mathcal{S}^- \otimes E)[[\hbar]]$ and $\bar{\mathcal{D}}_{A\star} : \Gamma(\mathcal{S}^- \otimes E)[[\hbar]] \rightarrow \Gamma(\mathcal{S}^+ \otimes E)[[\hbar]]$ are defined by

$$\mathcal{D}_{A\star} := \sigma^\mu D_\mu \star \quad \text{and} \quad \bar{\mathcal{D}}_{A\star} := \bar{\sigma}^\mu D_\mu^\dagger \star, \quad (2.12)$$

respectively. It is worth to comment on the spinor in the context of the deformation quantization. Since noncommutativity violates Lorentz symmetry of \mathbb{R}^4 , spinor as a representation of usual Poincare symmetry does not exist. However, as shown in [16,17], there exists a twisted Poincare symmetry in noncommutative space-time.

Instanton solutions or anti-selfdual connections satisfy the (noncommutative) instanton equation

$$F^+ = \frac{1}{2}(1 + *)F = 0, \quad (2.13)$$

where $*$ is the Hodge star operator. Note that in this article instantons are anti-selfdual connections, not selfdual connections. Formally, we expand the connection as

$$A_\mu = \sum_{l=0}^{\infty} A_\mu^{(l)} \hbar^l. \quad (2.14)$$

Then,

$$A_\mu \star A_\nu = \sum_{l,m,n=0}^{\infty} \hbar^{l+m+n} \frac{1}{l!} A_\mu^{(m)} (\overleftarrow{\Delta})^l A_\nu^{(n)}, \quad (2.15)$$

where

$$\overleftarrow{\Delta} \equiv \frac{i}{2} \overleftarrow{\partial}_\mu \theta_0^{\mu\nu} \overrightarrow{\partial}_\nu.$$

Using the selfdual projection operator

$$P := \frac{1 + *}{2}; \quad P_{\mu\nu,\rho\tau} = \frac{1}{4}(\delta_{\mu\rho}\delta_{\nu\tau} - \delta_{\nu\rho}\delta_{\mu\tau} + \epsilon_{\mu\nu\rho\tau}), \quad (2.16)$$

the instanton equation is

$$P_{\mu\nu,\rho\tau} F^{\rho\tau} = 0. \quad (2.17)$$

In the noncommutative case, the l -th order equation of (2.17) is given by

$$P^{\mu\nu,\rho\tau} (\partial_\rho A_\tau^{(l)} - \partial_\tau A_\rho^{(l)} + [A_\rho^{(l)}, A_\tau^{(l)}] + C_{\rho\tau}^{(l)}) = 0, \quad (2.18)$$

where

$$C_{\rho\tau}^{(l)} := \sum_{(p; m, n) \in I(l)} \hbar^{p+m+n} \frac{1}{p!} \left(A_{\rho}^{(m)} (\overleftarrow{\Delta})^p A_{\tau}^{(n)} - A_{\tau}^{(m)} (\overrightarrow{\Delta})^p A_{\rho}^{(n)} \right),$$

$$I(l) \equiv \{(p; m, n) \in \mathbb{Z}^3 | p + m + n = l, p, m, n \geq 0, m \neq l, n \neq l\}.$$

Note that the zeroth order equation is the commutative instanton equation with solution $A_{\mu}^{(0)}$ a commutative instanton. The asymptotic behavior of the commutative instanton $A_{\mu}^{(0)}$ is given by

$$A_{\mu}^{(0)} = g^{(0)} \partial_{\mu} (g^{(0)})^{-1} + O(|x|^{-2}), \quad g^{(0)} d(g^{(0)})^{-1} = O(|x|^{-1}), \quad (2.19)$$

where $g^{(0)}$, the zeroth order term in the expansion of g , is an element of the gauge group in commutative space. We impose a boundary condition that is a natural extension of (2.19):

$$A_{\mu} = g \star \partial_{\mu} g^{-1} + O(|x|^{-2}), \quad g \star dg^{-1} = O(|x|^{-1}). \quad (2.20)$$

In [1], we found a solution of (2.18), which we call a SNCD instanton. The order of the SNCD instanton is given by

$$A_{\mu}^{(l)} = O(|x|^{-3+\epsilon}), \quad l = 1, 2, 3, \dots, \quad (2.21)$$

for arbitrarily small $\epsilon > 0$. We denote (2.21) by $A^{(l)} = O(|x|^{-3+\epsilon})$ for simplicity. We proved also that the instanton number of SNCD coincides with the instanton number of $A^{(0)}$:

$$\frac{1}{8\pi^2} \int \text{tr} F \wedge \star F = \frac{1}{8\pi^2} \int \text{tr} F^{(0)} \wedge F^{(0)}. \quad (2.22)$$

For a later convenience, we introduce covariant derivatives associated to the commutative instanton connection by

$$D_{\mu}^{(0)} f := \partial_{\mu} f + A_{\mu}^{(0)} f, \quad (2.23)$$

and the Laplacian associated with the commutative instanton connection by

$$\Delta_A^{(0)} f := D^{(0)\mu} D_{\mu}^{(0)} f. \quad (2.24)$$

Let us introduce a \hbar -valued pairing for formal expansions $f(x), g(x) \in (C^{\infty}(\mathbb{R}^4) \cap L^2(\mathbb{R}^4))[[\hbar]]$ as

$$\langle f, g \rangle_\star := \int_{\mathbb{R}^4} d^4x (f^\dagger(x), g(x))_\star. \quad (2.25)$$

Here $(\ , \)_\star$ is the \hbar -valued point wise product used in Euclidean scalar product with contraction of spinors or tensors, that is $(f^\dagger(x), g(x))_\star$ is defined by

$$(f^\dagger(x), g(x))_\star := f^{\dagger\mu_1, \dots, \mu_n} \star g_{\mu_1 \dots \mu_n}. \quad (2.26)$$

Since each $f^{(n)}$ and $g^{(n)}$ are in $C^\infty(\mathbb{R}^4) \cap L^2(\mathbb{R}^4)$, we obtain

$$\int_{\mathbb{R}^4} d^4x f^{(n)}(x) \overleftrightarrow{\Delta} g^{(m)}(x) = 0. \quad (2.27)$$

then

$$\begin{aligned} \langle f, g \rangle_\star &= \int_{\mathbb{R}^4} d^4x (f(x)^\dagger, g(x)) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}^4} d^4x \sum_{k+l=n} (f(x)^{\dagger(k)}, g^{(l)}(x)) \hbar^n, \end{aligned} \quad (2.28)$$

where $(f^\dagger(x), g(x))$ is defined by $(f^\dagger(x), g(x)) := f^{\dagger\mu_1, \dots, \mu_n} g_{\mu_1 \dots \mu_n}$. We also use the usual L^2 inner product, that is for \hbar independent function $f(x), g(x)$, we set

$$\langle f(x), g(x) \rangle := \int_{\mathbb{R}^4} d^4x f^\dagger(x) g(x). \quad (2.29)$$

If $f(x)$ and $g(x)$ are not scalar functions, we regard $f^\dagger(x)g(x)$ as a point wise production with contraction.

We note that our formal space $(C^\infty(\mathbb{R}^4) \cap L^2(\mathbb{R}^4))[[\hbar]]$ is considered only as a formal expansion space.

3 The Index of the Dirac Operator

In this section, we investigate zero modes of the Dirac operators acting on the formal expansion space. The index theorem for the Dirac operator in a noncommutative ADHM instanton background was studied in [18], where it was shown that the number of zero modes of the Dirac operator equals the instanton number of the background instanton in the ADHM construction. In our case, we start with a commutative instanton and deform it into a SNCD instanton. The relation between SNCD instantons and ADHM instantons will

be clarified by using the theorem 3 proved in this section. To construct the ADHM data from SNCD instantons, we have to investigate the spinor zero modes and the index. In Section 5 these results will be used to derive the ADHM equations.

In this article, we treat the index theorem in a formal deformation setting. The usual index is defined by the difference between the kernel and the cokernel of the Dirac operator. In our context, the differential operators like the Dirac operator act on $(C^\infty(\mathbb{R}^4) \cap L^2(\mathbb{R}^4))[[\hbar]]$ that is considered only as formal expansion space. We introduce a (\hbar -valued) inner product in the formal expansions, we employ the orthonormal bases as in (3.20), and then we define the modified index as (3.27).

We consider operators acting on the weighted Sobolev spaces

$$\widetilde{W}_\delta^{k,p} = \left\{ \vec{u} \mid \sum_{j < k} \|\partial^j \vec{u}\|_{p,\delta-j} =: \|\vec{u}\|_{k,p,\delta} < \infty \right\}, \quad (3.1)$$

where $j = j_1 + j_2 + j_3 + j_4$, $\partial^j = \partial_1^{j_1} \partial_2^{j_2} \partial_3^{j_3} \partial_4^{j_4}$, and

$$\|\vec{u}\|_{p,\delta} := \left| \int_{\mathbb{R}^4} \{(1 + |x|^2)^{\frac{1}{2}}\}^{-\delta p - 4} |\vec{u}(x)|^p dx \right|^{\frac{1}{p}} < \infty. \quad (3.2)$$

Here $|\vec{u}(x)| = \sqrt{\vec{u}^\dagger \vec{u}}$. See [19] for the properties of weighted Sobolev spaces used here. We do not introduce the norm from the pairing (2.25) usual to complete spaces of \hbar -expansions. We deal with the Hilbert spaces (Sobolev spaces) as usual L^2 -space step by step for \hbar -expansions.

Let $\mathcal{D}_{A^\star} : \Gamma(\mathcal{S}^+ \otimes E)[[\hbar]] \rightarrow \Gamma(\mathcal{S}^- \otimes E)[[\hbar]]$ and $\bar{\mathcal{D}}_{A^\star} : \Gamma(\mathcal{S}^- \otimes E)[[\hbar]] \rightarrow \Gamma(\mathcal{S}^+ \otimes E)[[\hbar]]$ be the Dirac operator defined by (2.12). By the Weitzenbock formula,

$$\bar{\mathcal{D}}_{A^\star} \star \mathcal{D}_A = \Delta_A + \sigma^+ F^+, \quad (3.3)$$

$$\mathcal{D}_A \star \bar{\mathcal{D}}_{A^\star} = \Delta_A + \sigma^- F^-, \quad (3.4)$$

where $\sigma^+ F^+ = 2\bar{\sigma}^{\mu\nu} F_{\mu\nu}^+$, $\sigma^- F^- = 2\sigma^{\mu\nu} F_{\mu\nu}^-$ and $\Delta_A = D^\mu \star D_\mu$. Assume that A is a noncommutative anti-selfdual connection, i.e. $F^+ = 0$. We consider the \hbar expansion of $\psi \in \Gamma(\mathcal{S}^+ \otimes E)[[\hbar]]$:

$$\psi = \sum_{n=0}^{\infty} \hbar^n \psi^{(n)}. \quad (3.5)$$

Set

$$\begin{aligned} \text{Ker}\mathcal{D}_{A^\star} := \left\{ \psi \in \Gamma(\mathcal{S}^+ \otimes E) \cap L^2(\mathcal{S}^+ \otimes E)[[\hbar]] \right. \\ \left. \mid \mathcal{D}_{A^\star} \psi = 0 \in \Gamma(\mathcal{S}^- \otimes E)[[\hbar]] \right\}. \end{aligned} \quad (3.6)$$

As in the commutative case, we obtain the following theorem.

Theorem 1 *Assume that A is a SNCD anti-selfdual connection. Then If $\mathcal{D}_{A^\star} \psi = 0$ for $\psi^{(n)} \in L^2$, we have $\psi^{(n)} = 0$ for all n , i.e. $\text{Ker}\mathcal{D}_{A^\star} = 0$.*

PROOF. We show this theorem by induction. The zeroth order term $\mathcal{D}_{A^\star} \psi = 0$ is $\mathcal{D}_A^{(0)} \psi^{(0)} = 0$, and this equation only has the solution $\psi^{(0)} = 0$. We assume that the $\psi^{(k)} = 0$ ($k \leq n$). The equation of order $n+1$ is

$$\begin{aligned} 0 &= \hbar^{n+1} \left\{ \mathcal{D}_A^{(0)} \psi^{(n+1)} + \sigma^\rho A_\rho^{(n+1)} \psi^{(0)} + \sum_{(p; l, m) \in I(n+1)} \frac{1}{p!} (\sigma^\rho A_\rho^{(l)} (\overleftrightarrow{\Delta})^p \psi^{(m)}) \right\} \\ &= \hbar^{n+1} \mathcal{D}_A^{(0)} \psi^{(n+1)}, \end{aligned}$$

so $\psi^{(n+1)} = 0$. □

We investigate the zero modes of $\bar{\mathcal{D}}_{A^\star}$. Set

$$\begin{aligned} \text{Ker}\bar{\mathcal{D}}_{A^\star} := \left\{ \bar{\psi} \in \Gamma(\mathcal{S}^- \otimes E) \cap L^2(\mathcal{S}^- \otimes E)[[\hbar]] \right. \\ \left. \mid \bar{\mathcal{D}}_{A^\star} \bar{\psi} = 0 \in \Gamma(\mathcal{S}^+ \otimes E)[[\hbar]] \right\}. \end{aligned} \quad (3.7)$$

By expanding $\bar{\psi} \in \Gamma(\mathcal{S}^- \otimes E)[[\hbar]]$ as

$$\bar{\psi} = \sum_{n=0}^{\infty} \hbar^n \bar{\psi}^{(n)}, \quad (3.8)$$

the zeroth order equation of $\bar{\mathcal{D}}_{A^\star} \bar{\psi} = 0$ is $\bar{\mathcal{D}}_A^{(0)} \bar{\psi}^{(0)} = 0$, and there are k linearly independent zero-modes for a commutative instanton $A^{(0)}$ whose instanton number is $-k$. We define $\bar{\psi}_i$ ($i = 1, \dots, k$) as

$$\bar{\psi}_i = \sum_{n=0}^{\infty} \hbar^n \bar{\psi}_i^{(n)}, \quad (3.9)$$

where $\bar{\psi}_i^{(0)}$ ($i = 1, \dots, k$) are a basis of the k independent zero modes of $\bar{\mathcal{D}}_A^{(0)}$.

The n -th order equation of $\bar{\mathcal{D}}_A \star \bar{\psi} = 0$ is

$$\begin{aligned} 0 &= \hbar^n \left\{ \bar{\mathcal{D}}_A^{(0)} \bar{\psi}_i^{(n)} + \bar{\sigma}^\rho A_\rho^{(n)} \bar{\psi}_i^{(0)} + \sum_{(p; l, m) \in I(n)} \frac{1}{p!} \left(\bar{\sigma}^\rho A_\rho^{(l)} (\overleftarrow{\Delta})^p \bar{\psi}_i^{(m)} \right) \right\} \\ &= \hbar^n \left\{ \bar{\mathcal{D}}_A^{(0)} \bar{\psi}_i^{(n)} + H_i^{(n)} \right\}, \end{aligned} \quad (3.10)$$

where $H_i^{(0)} = 0$ and

$$H_i^{(n)} = \bar{\sigma}^\rho A_\rho^{(n)} \bar{\psi}_i^{(0)} + \sum_{(p; l, m) \in I(n)} \frac{1}{p!} \left(\bar{\sigma}^\rho A_\rho^{(l)} (\overleftarrow{\Delta})^p \bar{\psi}_i^{(m)} \right) \quad \text{for } n \in \mathbb{N}. \quad (3.11)$$

We can solve these equations recursively in the order in \hbar , so $H_i^{(n)}$ is determined by Eq. (3.10). $\bar{\mathcal{D}}_A^{(0)} \bar{\psi}_i^{(n)}$ in Eq. (3.10) has k zero modes. We denote an orthonormal basis of $\text{Ker} \bar{\mathcal{D}}_A^{(0)}$ by η_i ($i = 1 \dots k$). Note that $\mathcal{D}_A^{(0)} H_i^{(n)}$ is orthogonal to $\text{Ker} \bar{\mathcal{D}}_A^{(0)}$ with respect to usual L^2 inner product :

$$\langle (\mathcal{D}_A^{(0)} H_j^{(n)}), \eta_i \rangle = \int_{\mathbb{R}^4} d^4x (\mathcal{D}_A^{(0)} H_j^{(n)})^\dagger \eta_i = -\langle H_j^{(n)}, \bar{\mathcal{D}}_A^{(0)} \eta_i \rangle = 0. \quad (3.12)$$

Then we get

$$\bar{\psi}_i^{(n)} = \sum_{j=1}^k a_{n,i}^j \eta_j - \frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H_i^{(n)}, \quad (3.13)$$

where $a_{n,i}^j$ are arbitrary constants. Here $\frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}}$ denotes integration over \mathbb{R}^4 against the Green's function of $\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}$. Note that the ambiguity in the $a_{n,i}^j$ are occur only in the coefficients of the zero modes of the commutative Dirac operator $\bar{\mathcal{D}}_A^{(0)}$. $a_{n,i}^j$ is a constant matrix in general, because the symmetries realized in matrix representations remain after noncommutative deformation.

We denote $K(x, y)$ by the kernel function of $(\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)})^{-1}$. We recall the Weitzenböck formula,

$$\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)} = \Delta_A^{(0)} + \sigma^- F^{-{(0)}}.$$

The following is known (cf. see [19], Theorem 1.7):

$$K(x, y) = \frac{C}{|x - y|^2} + O\left(\frac{1}{|x - y|^3}\right). \quad (3.14)$$

In the following, we consider $\bar{\psi} := (\bar{\psi}_1, \dots, \bar{\psi}_k)$, $H := (H_1, \dots, H_k)$ as matrices.

Theorem 2 Assume that A is a SNCD anti-selfdual connection. Let $\bar{\psi} = (\bar{\psi}_i) = \sum_{n=0}^{\infty} \psi^{(n)} \hbar^n$ be a zero mode of $\bar{\mathcal{D}}_{A^\star}$ as above. Then

$$\bar{\psi}_i^{(n)} = \sum_{j=1}^k a_{n,i}^j \eta_j - \frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H_i^{(n)}, \quad (3.15)$$

$$\eta_j = O(|x|^{-3}), \quad \frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H_i^{(n)} = O(|x|^{-5+\epsilon}), \quad (3.16)$$

and

$$\bar{\psi}_i = \sum_{n=0}^{\infty} \left(\sum_{j=1}^k a_{n,i}^j \eta_j \right) \hbar^n + O(|x|^{-5+\epsilon}), \quad \eta_j = O(|x|^{-3}). \quad (3.17)$$

PROOF. We prove this theorem by induction. (i) By (2.21), $A^{(k)} = O(|x|^{-3+\epsilon})$, so we obtain

$$H^{(1)} = \bar{\sigma}^\rho A_\rho^{(1)} \bar{\psi}^{(0)} + \left(\bar{\sigma}^\rho A_\rho^{(0)} (\overleftarrow{\Delta}) \bar{\psi}^{(0)} \right) = O(|x|^{-6+\epsilon}). \quad (3.18)$$

From $K(x, y) = \frac{C}{|x-y|^2}$, we have $\frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H^{(1)} = O(|x|^{-5+\epsilon})$. The detailed derivation of this equation is similar to the proof of Proposition 1 in [1]. $\eta_i = O(|x|^{-3})$ is a well-known fact (see for example [15,20] and Appendix B). Using $\eta_i = O(|x|^{-3})$ and (3.13), we obtain

$$\bar{\psi}^{(1)} = O(|x|^{-3}). \quad (3.19)$$

(ii) Assume

$$\bar{\psi}_i^{(l)} = \sum_{j=1}^k a_{l,i}^j \eta_j + O(|x|^{-5+\epsilon}), \quad (0 \leq l \leq n).$$

Then we obtain $H^{(n+1)} = O(|x|^{-6+\epsilon})$ and $\frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H^{(n+1)} = O(|x|^{-5+\epsilon})$.

Therefore, $\bar{\psi}_i^{(n)} = \sum_{j=1}^k a_{n,i}^j \eta_j + O(|x|^{-5+\epsilon})$. \square

Note that this theorem implies that each $\bar{\psi}^{(n)} \in L^2(\mathcal{S}^- \otimes E)$.

We give a canonical choice of zero modes of $\bar{\mathcal{D}}_{A^\star}$ by introducing a formal orthonormalization of the zero modes of $\bar{\mathcal{D}}_{A^\star}$. Let $\bar{\psi}_0$ be a zero mode of $\bar{\mathcal{D}}_{A^\star}$. Formal expansion of the pairing $\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle_\star$ (defined by (2.25)) is given by

$$\begin{aligned} \int_{\mathbb{R}^4} d^4x \bar{\psi}_0^\dagger \star \bar{\psi}_0 &= \int_{\mathbb{R}^4} d^4x \bar{\psi}_0^\dagger \bar{\psi}_0 = \sum_{n=0}^{\infty} \sum_{k+l=n} \int_{\mathbb{R}^4} d^4x \bar{\psi}_0^{(k)\dagger} \bar{\psi}_0^{(l)} \hbar^n \\ &= \sum_{n=0}^{\infty} [\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle]^{(n)} \hbar^n. \end{aligned} \quad (3.20)$$

Here we use the decay condition $\bar{\psi}_0 \rightarrow 0$ as $|x| \rightarrow \infty$. The inverse of the formal power series of $\sum_{n=0}^{\infty} a^{(n)} \hbar^n$ with $a^{(0)} \neq 0$ is defined by $\sum_{n=0}^{\infty} b^{(n)} \hbar^n$, where $b^{(0)} = \frac{1}{a^{(0)}}$ and $b^{(n)} = -\frac{1}{a^{(0)}} \sum_{i=0}^{n-1} a^{(n-i)} b^{(i)}$. Since $\langle \bar{\psi}_0^{(0)\dagger}, \bar{\psi}_0^{(0)} \rangle \neq 0$, its formal inverse is defined by

$$\left(\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle_\star \right)^{-1} := \sum_{n=0}^{\infty} \hbar^n [\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle_\star^{-1}]^{(n)}, \quad (3.21)$$

where $[\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle_\star^{-1}]^{(n)}$ is determined by

$$[\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle_\star^{-1}]^{(n)} = -\frac{1}{\langle \bar{\psi}_0^{(0)\dagger}, \bar{\psi}_0^{(0)} \rangle} \sum_{i=0}^{n-1} [\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle]^{(n-i)} [\langle \bar{\psi}_0^\dagger, \bar{\psi}_0 \rangle^{-1}]^{(i)}. \quad (3.22)$$

This construction allows us to construct an orthonormalization. Let the $2N \times k$ matrix $\bar{\psi}$ be a zero mode of $\bar{\mathcal{D}}_A \star$. We set the following orthonormal condition

$$\int_{\mathbb{R}^4} d^4x \bar{\psi}^\dagger \star \bar{\psi} = I_{k \times k}. \quad (3.23)$$

The l -th order equation in \hbar for (3.23) is

$$\sum_{n+m=l} \int_{\mathbb{R}^4} d^4x \left(\sum_{j=1}^k \eta_j^\dagger a_{n,i}^{j\dagger} - \mathcal{H}_i^{n\dagger} \right) \left(\sum_{j=1}^k a_{m,p}^j \eta_j - \mathcal{H}_p^m \right) = \delta_{ip} \delta_{l0}, \quad (3.24)$$

where

$$\mathcal{H}_i^n = \frac{1}{\mathcal{D}_A^{(0)} \bar{\mathcal{D}}_A^{(0)}} \mathcal{D}_A^{(0)} H_i^{(n)}. \quad (3.25)$$

Gram-Schmidt orthonormalization determines the constants $a_{n,i}^j$ recursively. We introduce a linear space that is expanded by these formal orthonormalized zero modes

$$\widehat{Ker}\bar{\mathcal{D}}_{A^\star} := \left\{ \bar{\psi} \mid \bar{\psi} = \sum_{i=1}^k c_i \bar{\psi}_i, \psi_i \in Ker\bar{\mathcal{D}}_{A^\star}, \bar{\psi}_i^{(0)} = \eta_i, \int_{\mathbb{R}^4} d^4x \bar{\psi}_i^\dagger \star \bar{\psi}_j = \delta_{ij}, c_i \in \mathbb{C} \right\}. \quad (3.26)$$

We recall the index for the \mathcal{D}_A^0 is defined by

$$\text{Ind } \mathcal{D}_A^0 := \dim \ker \mathcal{D}_A^{(0)} - \dim \ker \bar{\mathcal{D}}_A^{(0)}$$

as usual. We define the modified index for the \mathcal{D}_{A^\star} as

$$\widehat{\text{Ind}} \mathcal{D}_{A^\star} := \dim \text{Ker} \mathcal{D}_{A^\star} - \dim \widehat{Ker} \bar{\mathcal{D}}_{A^\star}. \quad (3.27)$$

Thus we have the following theorem.

Theorem 3 *If $\text{Ind } \mathcal{D}_A^0 = -k$, then $\widehat{\text{Ind}} \mathcal{D}_{A^\star} = -k$.*

Note that this $\widehat{\text{Ind}} \mathcal{D}_{A^\star}$ is not index in usual sense. One reason is that the \mathcal{D}_{A^\star} and the $\bar{\mathcal{D}}_{A^\star}$ are not Fredholm operators because we consider formal power series. Another reason is that $\widehat{Ker}\bar{\mathcal{D}}_{A^\star} \neq Ker\bar{\mathcal{D}}_{A^\star}$ ($\widehat{Ker}\bar{\mathcal{D}}_{A^\star} \subset Ker\bar{\mathcal{D}}_{A^\star}$). For example, if $\bar{\psi} = \sum_{n=0} \hbar^n \bar{\psi}^{(n)}$ is a zero mode of $\bar{\mathcal{D}}_{A^\star}$, then $\bar{\psi}' = \sum_{n=0} \hbar^{n+k} \bar{\psi}^{(n)}$ is also a zero mode for arbitrary integer k . We find that $\bar{\psi}' \in Ker\bar{\mathcal{D}}_{A^\star}$ but $\bar{\psi}' \notin \widehat{Ker}\bar{\mathcal{D}}_{A^\star}$. However, in our context, it is a natural extension of the index of usual commutative space, because the dimension of the \widehat{Ker} is essential for the construction of the ADHM data and the relation with the instanton number.

4 Green's Function

In this section, we construct the Green's function for Δ_A . The definition of the Green's function is

$$\Delta_A \star G_A(x, y) = \delta_\star(x - y), \quad (4.1)$$

where

$$\int d^4x \delta_\star(x - y) \star f(y) = f(x). \quad (4.2)$$

Note that if $f(x)$ is smooth,

$$\int d^4x \delta_\star(x-y) \star f(y) = \int d^4x \delta_\star(x-y) f(y). \quad (4.3)$$

Then, we do not distinguish $\delta_\star(x-y)$ and $\delta(x-y)$ in the following. (This discussion might be too naive. To avoid any risk of error, we should define $G_A(x, y)$ by $\Delta_A \star G_A(x, y) = \delta(x-y)$.)

We expand (4.1) in \hbar :

$$\hbar^0 : \Delta_A^{(0)} G_A^{(0)}(x, y) = \delta(x-y) \quad (4.4)$$

$$\hbar^1 : \Delta_A^{(0)} G_A^{(1)}(x, y) + [\Delta_A \star G_A^{(0)}(x, y)]^{(1)} = 0 \quad (4.5)$$

\vdots

$$\hbar^n : \Delta_A^{(0)} G_A^{(n)}(x, y) + [\Delta_A \star \sum_{0 \leq k < n} \hbar^k G_A^{(k)}(x, y)]^{(n)} = 0. \quad (4.6)$$

\vdots

We solve (4.4)-(4.6) recursively as

$$G_A^{(n)}(x, y) = \int d^4w G_A^{(0)}(x, w) [\Delta_A \star \sum_{0 \leq k < n} \hbar^k G_A^{(k)}(w, y)]^{(n)}. \quad (4.7)$$

Note that $G_A^{(0)}(x, w)$ was constructed in [21–23], and

$$G_A^{(0)}(x, y) = O(|x-y|^{-2}). \quad (4.8)$$

From (2.19) and (2.21) $A^{(l)} = O(|x|^{-3+\epsilon})$ we found that

$$[\Delta_A \star \sum_{0 \leq k < n} \hbar^k G_A^{(k)}(x, y)]^{(n)} = O(|x-y|^{-5}). \quad (4.9)$$

Therefore,

$$G_A^{(n)}(x, y) = O(|x-y|^{-3}). \quad (4.10)$$

5 From Instantons to The ADHM Equations

In this section we derive the ADHM equations from a noncommutative instanton.

We let \star_x denote \star with respect to the variable $x = (x_1, \dots, x_4)$. Let $\bar{\psi}_i$ ($i = 1, \dots, k$) be orthonormal zero modes of $\bar{\mathcal{D}}_A \star$ and set matrix $\bar{\psi} = (\bar{\psi}_i)$ as in Section 3. The concept of completeness in the Hilbert spaces is extended to the one in formal expansion spaces, and we obtain the following identity for arbitrary functions $f(x), g(y)$.

$$\begin{aligned} & \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y f(x) \star_x \bar{\psi}(x) \bar{\psi}^\dagger(y) \star_y g(y) \\ &= \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y \left\{ f(x) \star_x \delta(x-y) \star_y g(y) \right. \\ & \quad \left. - f(x) \star_x \mathcal{D}_A \star_x G_A(x, y) \star_y \overleftarrow{\mathcal{D}}_A \star_y g(y) \right\}, \end{aligned} \quad (5.1)$$

The proof for (5.1) is given in Appendix A. For consistency, we impose the commutation

$$[x^\mu, y^\nu]_\star = \begin{cases} i\theta^{\mu\nu}, & (x = y), \\ 0, & (x \neq y). \end{cases} \quad (5.2)$$

In the following derivation of the ADHM equations, we use the completeness condition and the asymptotic behavior of the zero modes of the $\bar{\mathcal{D}}_A \star$ given by Theorem 2.

We first define T^μ by

$$\begin{aligned} T^\mu &:= \int_{\mathbb{R}^4} d^4x \frac{1}{2} (x^\mu \star \bar{\psi}^\dagger \star \bar{\psi} + \bar{\psi}^\dagger \star \bar{\psi} \star x^\mu) \\ &= \int_{\mathbb{R}^4} d^4x (x^\mu \star \bar{\psi}^\dagger \star \bar{\psi}) = \int_{\mathbb{R}^4} d^4x (\bar{\psi}^\dagger \star \bar{\psi} \star x^\mu). \end{aligned} \quad (5.3)$$

Here we use $\int_{\mathbb{R}^4} d^4x \partial_\mu (\bar{\psi}^\dagger \star \bar{\psi}) = 0$ in the second and third equalities in (5.3), which follows from $\bar{\psi} = O(|x|^{-3})$ (see Theorem 2). Then,

$$T^\mu T^\nu = \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y (x^\mu \star_x \bar{\psi}^\dagger(x) \star_x \bar{\psi}(x)) (\bar{\psi}^\dagger(y) \star_y \bar{\psi}(y) \star_y y^\nu) \quad (5.4)$$

Using (5.1) and integration by parts, (5.4) becomes

$$\begin{aligned}
T^\mu T^\nu &= \int_{\mathbb{R}^4} d^4x x^\mu \star \bar{\psi}^\dagger \star \bar{\psi} \star x^\nu \\
&\quad + \int_{S^3} dS_x^\rho \int_{\mathbb{R}^4} d^4y (x^\mu \star_x \bar{\psi}^\dagger(x) \sigma_\rho) \star_x G_A(x, y) \star_y \overleftarrow{D}_A \star_y (\bar{\psi}(y) \star_y y^\nu) \\
&\quad - \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y (\bar{\psi}^\dagger(x) \sigma^\mu) \star_x G_A(x, y) \star_y \overleftarrow{D}_A \star_y (\bar{\psi}(y) \star_y y^\nu),
\end{aligned}$$

where $dS_x^\mu = |x|^2 x^\mu d\Omega$ and $d\Omega$ is the solid angle. The first term is deformed as follows.

$$\begin{aligned}
&\int_{\mathbb{R}^4} d^4x x^\mu \star \bar{\psi}^\dagger \star \bar{\psi} \star x^\nu \\
&= \int_{\mathbb{R}^4} d^4x (\bar{\psi}^\dagger \star \bar{\psi} \star x^\nu \star x^\mu + [x^\mu, \bar{\psi}^\dagger \star \bar{\psi}]_\star \star x^\nu + \bar{\psi}^\dagger \star \bar{\psi} \star [x^\mu, x^\nu]_\star) \\
&= \int_{\mathbb{R}^4} d^4x (\bar{\psi}^\dagger \star \bar{\psi} \star x^\nu \star x^\mu + i\theta^{\mu\rho} \partial_\rho (\bar{\psi}^\dagger \star \bar{\psi}) \star x^\nu + i\theta^{\mu\nu} \bar{\psi}^\dagger \star \bar{\psi}) \\
&= \int_{\mathbb{R}^4} d^4x \bar{\psi}^\dagger \star \bar{\psi} \star x^\nu \star x^\mu. \tag{5.5}
\end{aligned}$$

Here $\bar{\psi} = O(|x|^{-3})$ is used in the third equality. By integration by parts again, we get

$$T^\mu T^\nu = \int_{\mathbb{R}^4} d^4x \bar{\psi}^\dagger \star \bar{\psi} \star x^\nu \star x^\mu \tag{5.6}$$

$$+ \int_{S^3} dS_x^\rho \int_{S^3} dS_y^\tau (x^\mu \star_x \bar{\psi}^\dagger(x) \sigma_\rho) \star_x G_A(x, y) \star_y (\bar{\sigma}_\tau \bar{\psi}(y) \star_y y^\nu) \tag{5.7}$$

$$- \int_{S^3} dS_x^\rho \int_{\mathbb{R}^4} d^4y (x^\mu \star_x \bar{\psi}^\dagger(x) \sigma_\rho) \star_x G_A(x, y) \star_y (\bar{\sigma}^\nu \bar{\psi}(y)) \tag{5.8}$$

$$- \int_{\mathbb{R}^4} d^4x \int_{S^3} dS_y^\tau (\bar{\psi}^\dagger(x) \sigma^\mu) \star_x G_A(x, y) \star_y (\bar{\sigma}^\tau \bar{\psi}(y) \star_y y^\nu) \tag{5.9}$$

$$+ \int_{\mathbb{R}^4} d^4x \int_{\mathbb{R}^4} d^4y (\bar{\psi}^\dagger(x) \sigma^\mu) \star_x G_A(x, y) \star_y (\bar{\sigma}^\nu \bar{\psi}(y)). \tag{5.10}$$

(5.7) and (5.9) vanish when $R_y \rightarrow \infty$, where R_y is a radius of S_y^3 . (5.10) will vanish on the selfdual projection $[T^\mu, T^\nu]^+ := P^{\mu\nu, \rho\tau} [T_\rho, T_\tau]$ (see (2.16)), because $\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu$ is anti-selfdual with respect to the μ, ν . Thus only (5.6) and (5.8) remain.

We introduce an asymptotically parallel section $g^{-1}S$ of $\mathcal{S}^+ \otimes E$ by

$$\tilde{\psi} = -\frac{g^{-1}Sx^\dagger}{|x|^4} + O(|x|^{-4}), \tag{5.11}$$

where S is a constant matrix, $x^\dagger := \bar{\sigma}_\mu x^\mu$, $\tilde{\psi} := {}^t \bar{\psi} \sigma_2$, and t means transposing the spinor indices. (See also Appendix B.) Recall that A has asymptotic

behavior given by (2.20), and note that $D_\mu \star g^{-1} \rightarrow 0$ as $r \rightarrow \infty$. Using these facts and $\bar{D}_A \star \bar{\psi} = 0$, we can prove that $\tilde{\psi}$ has the expression (5.11) by direct calculations similar to the commutative case. Note that $\tilde{\psi}$ and $\bar{\psi}$ have one-to-one correspondence and $D_\mu \star \tilde{\psi} \sigma^\mu = 0$ iff $\bar{D}_A \star \bar{\psi} = 0$ (see Appendix B).

Let us introduce χ by

$$\chi(x) := 4\pi \int_{\mathbb{R}^4} d^4y G_A(x, y) \tilde{\psi}(y) = 4\pi \int_{\mathbb{R}^4} d^4y G_A(x, y) \star_y \tilde{\psi}(y). \quad (5.12)$$

Lemma 4 χ is given asymptotically by

$$\chi = -\frac{g^\dagger S x^\dagger}{|x|^2} + O(|x|^{-2}). \quad (5.13)$$

PROOF. Consider

$$\Delta_A \star (|x|^2 \tilde{\psi}) = 8\tilde{\psi} + 4x^\mu (D_\mu \star \tilde{\psi}) + |x|^2 (D^\mu \star D_\mu \star \tilde{\psi}) + O(|x|^{-4}). \quad (5.14)$$

Here

$$x^\mu (D_\mu \star O(|x|^{-4})) = O(|x|^{-4}), \quad (5.15)$$

and

$$x^\mu (D_\mu \star \frac{g^\dagger S x^\dagger}{|x|^4}) = -3 \frac{g^\dagger S x^\dagger}{|x|^4} + O(|x|^{-5}). \quad (5.16)$$

Using (5.15) and (5.16), we have

$$x^\mu D_\mu \star \tilde{\psi} = -3\tilde{\psi} + O(|x|^{-4}). \quad (5.17)$$

Note that

$$D^\mu \star D_\mu \star O(|x|^{-4}) = O(|x|^{-6}) \quad (5.18)$$

and

$$D^\mu \star D_\mu \star \frac{g^\dagger S x^\dagger}{|x|^4} = O(|x|^{-6}). \quad (5.19)$$

Thus, we get

$$D^\mu \star D_\mu \star \tilde{\psi} = O(|x|^{-6}). \quad (5.20)$$

From (5.17) and (5.20),

$$\Delta_A \star (|x|^2 \tilde{\psi}) = -4\tilde{\psi} + O(|x|^{-4}) \quad (5.21)$$

Applying the Green's function and using (4.8) and (4.10), we get the desired result. \square

Note that

$$\mathcal{D}_A^2 \star \chi = -4\pi\tilde{\psi}.$$

By this relation and the asymptotic behaviors of χ and $\bar{\psi}$, (5.8) becomes

$$\frac{1}{8} \text{tr}(S^\dagger S \bar{\sigma}^\mu \sigma^\nu), \quad (5.22)$$

where the trace tr is taken with respect to the spinor indices.

In the $[T^\mu, T^\nu]^+$ combination, (5.6) becomes $-i\theta^{\mu\nu+} = -iP^{\mu\nu, \rho\tau} \theta^{\rho\tau}$.

Then from (5.6)-(5.10) and the definition of $[T^\mu, T^\nu]^+$, we obtain the following theorem.

Theorem 5 *Let A^μ be a SNCD instanton, and $\bar{\psi}$ be the zero mode of $\bar{\mathcal{D}}_A \star$ determined by A^μ as in Section 3. Let T^μ, S be constant matrices defined by (5.3) and (5.11), respectively. Then, they satisfy the ADHM equations:*

$$[T^\mu, T^\nu]^+ = \frac{1}{2} \text{tr}(S^\dagger S \bar{\sigma}^{\mu\nu}) - i\theta^{\mu\nu+} I_{k \times k}. \quad (5.23)$$

These ADHM equations are the same as those given by Nekrasov and Schwarz [3].

In [3], it is shown that instantons can be constructed from ADHM data satisfying (5.23). The spinor zero modes of the Dirac operator in a background of noncommutative ADHM instantons are studied, and the index of the Dirac operator is given in [18]. The question of whether there is a one-to-one correspondence between ADHM data and instantons is answered affirmatively.

Theorem 6 *There is a one-to-one correspondence between ADHM data satisfying (5.23) and SNCD instantons in noncommutative \mathbb{R}^4 .*

The proof for this theorem is given in Appendix B.

It may be useful to note the relation between Theorem 2 and the term S in the ADHM data. S is given as the coefficient of the $O(|x|^{-3})$ term in $\bar{\psi}$, and Theorem 2 implies that the $O(|x|^{-3})$ term is a zero mode of $\mathcal{D}_A^{(0)}$. For example, when we consider $k = 1$, there is only one zero mode $\bar{\psi}$. One might think that the $O(|x|^{-3})$ term in each $\bar{\psi}^{(n)}$ is proportional to $\bar{\psi}^{(0)}$, and $S^{(n)}$ is also proportional to $S^{(0)}$, but this is not true in general, due to gauge symmetries and global symmetries. g^\dagger can also be expanded as a power series in \hbar (see Appendix C). For example, $\tilde{\psi}^{(1)}$ is given by

$$\tilde{\psi}^{(1)} = -\frac{\{(g^\dagger)^{(0)}S^{(1)} + (g^\dagger)^{(1)}S^{(0)}\}x^\dagger}{|x|^4} + O(|x|^{-4}). \quad (5.24)$$

As a result of this twisting by $(g^\dagger)^{(1)}$, $S^{(1)}$ is not proportional to $S^{(0)}$ in general, and so $tr(S^\dagger S \bar{\sigma}^\mu \sigma^\nu)$ is not proportional to $tr(S^{(0)\dagger} S^{(0)} \bar{\sigma}^\mu \sigma^\nu)$. In fact, taking the trace of (5.23) shows that $Tr\{tr(S^\dagger S \bar{\sigma}^\mu \sigma^\nu)\}$ is deformed by the noncommutative parameter from 0 to $ik\theta^{\mu\nu+}$, where trace Tr is taken with respect to the $k \times k$ matrix indices.

6 Example

In this section, we compute a simple example of a noncommutative instanton that is deformed smoothly from a commutative one. The notation used in this section is given in Appendix B.1.

We start from a $U(2)$ BPST instanton in commutative \mathbb{R}^4 with the instanton number $k = -1$ [24]. Its ADHM data is given by

$$T^\mu = b^\mu, \quad S = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad \rho, b^\mu \in \mathbb{R}. \quad (6.1)$$

The ADHM data (6.1) satisfies

$$[T^\mu, T^\nu]^+ = \frac{1}{2} tr(S^\dagger S \bar{\sigma}^{\mu\nu}).$$

We deform the ADHM equations to the (5.23). For simplicity, we set

$$\theta^{12} = -\theta^{21} = \hbar, \quad \theta^{\mu\nu} = 0 \quad ((\mu, \nu) \neq (1, 2), (2, 1)) \quad (6.2)$$

in this section. Then the ADHM data satisfying (5.23) deforms to

$$T^\mu = b^\mu, \quad S = \begin{pmatrix} \sqrt{\rho^2 + \hbar} & 0 \\ 0 & \sqrt{\rho^2 - \hbar} \end{pmatrix}. \quad (6.3)$$

Note that the data (6.3) connects smoothly to (6.1) in the commutative limit, and the noncommutative deformation of the ADHM data is not unique. By setting $y^\mu = x^\mu - b^\mu$, the solution of $\nabla^\dagger \star \tilde{V} = O$ is given by

$$\tilde{V} = (\tilde{V}_1 \tilde{V}_2) = \begin{pmatrix} \bar{\sigma}_\mu y^\mu \\ M \end{pmatrix}, \quad (6.4)$$

where \tilde{V}_i is a 4-vector and

$$M := -(\bar{\sigma}_\mu y^\mu)_\star^{-1} \star \begin{pmatrix} \sqrt{\rho^2 + \hbar} & 0 \\ 0 & \sqrt{\rho^2 - \hbar} \end{pmatrix} \star (\bar{\sigma}^\nu y_\nu). \quad (6.5)$$

Here $(\bar{\sigma}_\mu y^\mu)_\star^{-1}$ is defined by $(\bar{\sigma}_\mu y^\mu)_\star^{-1} \star (\bar{\sigma}_\mu y^\mu) = I_{2 \times 2}$. Expanding M as $M = \sum_{k=0}^{\infty} M^{(k)} \hbar^k$, we have

$$M^{(0)} = -\rho I_{2 \times 2},$$

$$M^{(1)} = \mathcal{M} + O(|x|^{-2}), \quad \mathcal{M} = -\frac{1}{2\rho|y|^2} y \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} y^\dagger, \quad (6.6)$$

where $y := y^\mu \sigma_\mu$ and $y^\dagger := y^\mu \bar{\sigma}_\mu$. We set the orthonormalization condition $V^\dagger \star V = I_{2 \times 2}$ for the solution of $\nabla^\dagger \star V = O$. This normalization is not elementary because of the \star product, even if we use the Gram-Schmidt process, i.e.

$$\begin{aligned} V_1 &:= \tilde{V}_1 \star |\tilde{V}_1|_\star^{-1}, \\ V_2^\perp &:= \tilde{V}_2 - V_1 \star (V_1^\dagger \star \tilde{V}_2), \\ V_2 &:= V_2^\perp \star |V_2^\perp|_\star^{-1}, \end{aligned} \quad (6.7)$$

where $|\tilde{V}_I|_\star^{-1}$ is defined by $|\tilde{V}_I|_\star |\tilde{V}_I|_\star^{-1} = 1$. The explicit expressions for V_1 and V_2 are given by

$$V_1 = \frac{1}{\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} z_2 \\ -\bar{z}_1 \\ -\rho \\ 0 \end{pmatrix} + \frac{\hbar}{2\rho|y|^2\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} 0 \\ 0 \\ |z_2|^2 - |z_1|^2 \\ 2\bar{z}_1 z_2 \end{pmatrix} + O(\hbar^2) + O(|x|^{-2}), \quad (6.8)$$

$$V_2 = \frac{1}{\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} z_1 \\ \bar{z}_2 \\ 0 \\ -\rho \end{pmatrix} + \frac{\hbar}{2\rho|y|^2\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} 0 \\ 0 \\ 2\bar{z}_2 \bar{z}_1 \\ |z_1|^2 - |z_2|^2 \end{pmatrix} + O(\hbar^2) + O(|x|^{-2}), \quad (6.9)$$

where $z_1 = y_2 + iy_1$ and $z_2 = y_4 + iy_3$. Finally, we obtain

$$V = (V_1 V_2) = \frac{1}{\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} y^\dagger \\ M^{(0)} + \hbar M^{(1)} \end{pmatrix} + O(\hbar^2) + O(|x|^{-2}). \quad (6.10)$$

For this V , the SNCD instanton is given by

$$\begin{aligned} A_\mu &= V^\dagger \star \partial_\mu V \\ &= A_\mu^{(0)} + \frac{\hbar}{\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} -\rho & 0 \\ 0 & \rho \end{pmatrix} \partial_\mu \left\{ \frac{1}{2\rho|y|^2\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} |z_2|^2 - |z_1|^2 & 2\bar{z}_2 z_1 \\ 2\bar{z}_1 z_2 & |z_1|^2 - |z_2|^2 \end{pmatrix} \right\} \\ &\quad + \frac{\hbar}{2\rho|y|^2\sqrt{|y|^2 + \rho^2}} \begin{pmatrix} |z_2|^2 - |z_1|^2 & 2\bar{z}_2 z_1 \\ 2\bar{z}_1 z_2 & |z_1|^2 - |z_2|^2 \end{pmatrix} \begin{pmatrix} -\rho & 0 \\ 0 & \rho \end{pmatrix} \partial_\mu \frac{1}{\sqrt{|y|^2 + \rho^2}} \\ &\quad + O(\hbar^2) + O(|x|^{-4}), \end{aligned} \quad (6.11)$$

where $A_\mu^{(0)}$ is a commutative instanton from [24]:

$$A_\mu^{(0)} = \frac{y_\mu I_{2 \times 2} + \sigma_\mu y^\dagger}{|y|^2 + \rho^2}. \quad (6.12)$$

$A_\mu^{(1)}$, the term in proportional to \hbar in (6.11), is $O(|x|^{-3})$, and this fact is consistent with (2.21) (see also [1]). As we show in Appendix B.2, the zero mode of $\bar{\mathcal{D}}_{A\star}$ is given as $\tilde{\psi} = \frac{1}{\pi}(V^\dagger C) \star f$, where C and f are defined in Appendix B.1. Substituting our ADHM data (6.3) and (6.10), we get

$$\begin{aligned}\tilde{\psi} &= \tilde{\psi}^{(0)} + \hbar\tilde{\psi}^{(1)} + O(\hbar^2) + O(|x|^{-4}) \\ &= \tilde{\psi}^{(0)} + \frac{\hbar}{\pi(|y|^2 + \rho^2)^{\frac{3}{2}}}M^{(1)} + O(\hbar^2) + O(|x|^{-4}) \\ &= \tilde{\psi}^{(0)} + \frac{\hbar}{\pi|y|^3}\mathcal{M} + O(\hbar^2) + O(|x|^{-4}),\end{aligned}\tag{6.13}$$

where $\tilde{\psi}^{(0)}$ is a zero mode of $\bar{\mathcal{D}}_A^{(0)}$:

$$\tilde{\psi}^{(0)} = \frac{-\rho}{(|y|^2 + \rho^2)^{\frac{3}{2}}}I_{2\times 2}.\tag{6.14}$$

By Theorem 2, the $O(|x|^{-3})$ term in the (6.13) should satisfy $D_\mu^{(0)}\tilde{\psi}^{(1)}\sigma^\mu = 0$, as in this example,

$$(\partial_\mu + A_\mu^{(0)})\left(\frac{1}{|y|^3}\mathcal{M}\right)\sigma^\mu = 0.\tag{6.15}$$

This equation is easily verified by direct calculation.

7 Conclusion

Noncommutative deformations of zero modes for the Dirac operator with SNCD instanton backgrounds, the Green's function for SNCD instantons, and the ADHM equations are investigated. From Theorem 1, Theorem 2 and the solutions (3.13), we find that there are no new zero modes of $\mathcal{D}_{A\star}$ and $\bar{\mathcal{D}}_{A\star}$, so the (modified) index of the Dirac operator is unchanged under noncommutative deformation. The asymptotic behavior of the zero mode of $\bar{\mathcal{D}}_{A\star}$ is computed. In particular, the $O(|x|^{-3})$ terms in the zero modes of $\bar{\mathcal{D}}_{A\star}$ are obtained from the zero modes of the Dirac operator in commutative space. This result implies that the term S in the ADHM data is constructed from a linear combination of the corresponding S in the ADHM data of commutative \mathbb{R}^4 . The Green's function with a background SNCD instanton is also constructed recursively. Using these zero modes and the Green's function, we derive the noncommutative ADHM equations and prove a one-to-one correspondence between the ADHM data and SNCD instantons. One simple example is studied as confirmation of our results: we deform $k = -1$ BPST instanton into the

SNCD instanton via the ADHM method. Consistency checks are verified by comparing the term proportional to \hbar in the SNCD instanton and the zero mode of the $\bar{\mathcal{D}}_{A\star}$.

Our method is based on the \hbar expansion, which means that noncommutative instantons whose commutative limits are singular, such as $U(1)$ instantons, are not considered in this article. The relation between the ADHM equations and a noncommutative instanton with a singular commutative limit remains to be investigated in a future work.

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A Derivation of (5.1)

In this section, we derive (5.1). The identity for commutative space is proved by [25,26]. We extend the identity to our formal expansion space. Let us introduce a \hbar -valued spinor propagator $P(x, y) = \sum_{i=0}^{\infty} P^{(i)}(x, y)\hbar^i$ in a k -instanton background by

$$\gamma^\mu D_\mu \star P(x, y) = \delta(x - y) - \sum_{n=1}^k \Psi_n(x) \Psi_n^\dagger(y), \quad (\text{A.1})$$

where we use γ -matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \bar{\sigma}^\mu \\ \sigma^\mu & 0 \end{pmatrix} \quad (\text{A.2})$$

and zero modes $\Psi_n(x)$ of the Dirac operator $\gamma^\mu D_\mu$. We expand (A.1) in \hbar :

$$\gamma^\mu D_\mu^{(0)} P^{(0)}(x, y) = \delta(x - y) - \sum_{i=1}^k \Psi_i^{(0)}(x) \Psi_i^{\dagger(0)}(y), \quad (\text{A.3})$$

$$\begin{aligned} \gamma^\mu D_\mu^{(0)} P^{(1)}(x, y) &= R^{(1)}(x, y), \\ &\vdots \\ \gamma^\mu D_\mu^{(0)} P^{(n)}(x, y) &= R^{(n)}(x, y), \\ &\vdots \end{aligned} \quad (\text{A.4})$$

where

$$\begin{aligned} R^{(n)}(x, y) &:= -\gamma^\mu A_\mu^{(n)} P^{(0)}(x, y) - \sum_{(l; k, m) \in I(n)} \gamma^\mu A_\mu^{(k)} (\overleftarrow{\Delta})^l P^{(m)}(x, y) \\ &\quad - \sum_{i=1}^k \sum_{j=1}^n \Psi_i^{(j)}(x) \Psi_i^{\dagger(n-j)}(y). \end{aligned} \quad (\text{A.5})$$

In the [26], the existence of $P^{(0)}(x, y)$ satisfying (A.3) is shown. We can solve recursively (A.4) for $P^{(n)}(x, y)$ ($n = 1, 2, 3, \dots$) by separating $P^{(n)}(x, y)$ into two parts by chirality and using the similar way of Section 3. Then we find that there exist $P(x, y) = \sum_{i=0}^{\infty} P^{(i)}(x, y) \hbar^i$ such that (A.1).

Next, we derive (5.1). Similar to [25], we take the form

$$P(x, y) = \begin{pmatrix} 0 & s(x, y) \\ \bar{s}(x, y) & 0 \end{pmatrix}, \quad (\text{A.6})$$

then $s(x, y)$ and $\bar{s}(x, y)$ satisfy

$$\bar{\mathcal{D}}_A \star \bar{s}(x, y) = \delta(x - y), \quad (\text{A.7})$$

$$\mathcal{D}_A \star_x s(x, y) = \delta(x - y) - \sum_{i=1}^k \bar{\psi}_i(x) \bar{\psi}_i^\dagger(y). \quad (\text{A.8})$$

Here $\bar{\psi}_i(x)$ is a zero mode of $\bar{\mathcal{D}}_A \star$ given in Section 3. $\bar{s}(x, y)$ is obtained as

$$\bar{s}(x, y) = \mathcal{D}_A \star_x G_A(x, y). \quad (\text{A.9})$$

By multiplying $P^\dagger(z, x)$ from left side of (A.1), we obtain

$$-\bar{s}(z, y) + \int_{\mathbb{R}^4} d^4x \sum_{i=1}^k \bar{\psi}_i(z) \bar{\psi}_i^\dagger(x) \star_x \bar{s}(x, y) = s^\dagger(z, y). \quad (\text{A.10})$$

Because of (A.9),

$$\int_{\mathbb{R}^4} d^4x \sum_{i=1}^k \bar{\psi}_i(z) \bar{\psi}_i^\dagger(x) \star_x \bar{s}(x, y) = - \int_{\mathbb{R}^4} d^4x \sum_{i=1}^k \bar{\psi}_i(z) (\bar{\mathcal{D}}_A \star_x \bar{\psi}_i(x))^\dagger \star_x G_A(x, y) = 0.$$

Then the following relation is obtained:

$$s(x, y) = -\bar{s}^\dagger(x, y). \quad (\text{A.11})$$

From (A.8), (A.9) and (A.11), we obtain (5.1):

$$\star_x \bar{\psi}(x) \bar{\psi}^\dagger(y) \star_y = \star_x \delta(x - y) \star_y - \star_x \mathcal{D}_A \star_x G_A(x, y) \star_y \overleftarrow{\mathcal{D}}_A \star_y.$$

B A One-to-One Correspondence between Instanton and ADHM Data

In this Appendix, we prove a one-to-one correspondence between ADHM data and SNCD instanton solutions. It is shown that instantons can be constructed from ADHM data satisfying (5.23) in [3]. The spinor zero modes of the Dirac operator in a background of noncommutative ADHM instantons are studied, and the index of the Dirac operator is given in [18]. In this paper, we show that the index of the Dirac operator does not depend on noncommutative parameters and the ADHM equations are constructed from SNCD instantons in this article. The proof to show the one-to-one correspondence between ADHM data and SNCD instantons is completed if we show the completeness and the uniqueness. We will prove the completeness and the uniqueness in subsection B.2 and B.3, respectively. In commutative \mathbb{R}^4 , there is the same one-to-one correspondence (see for example [25,27,20,28]). Many parts of the proofs for the completeness and the uniqueness are parallel to the commutative cases.

We use the asymptotic behavior of the SNCD instanton (2.21) and the spinor zero modes (3.17) and other results derived from the decay conditions as needed throughout.

B.1 Notation for The ADHM Construction

In this subsection, we set the notation for the ADHM construction. $(N+2k) \times 2k$ matrices C and ∇ are defined by

$$C := \begin{pmatrix} O_{N \times 2k} \\ I_{2k \times 2k} \end{pmatrix}, \quad \nabla := \begin{pmatrix} S \\ \sigma_\mu(x^\mu - T^\mu) \end{pmatrix}.$$

From this definition, we have

$$\partial_\mu \nabla = \sigma_\mu C. \quad (\text{B.1})$$

If T^μ and S satisfy the ADHM equations (5.23), we have the following:

$$\nabla^\dagger \star \nabla = S^\dagger S + \bar{\sigma}^\mu \sigma^\nu (x^\mu - T^\mu) \star (x^\nu - T^\nu) = \begin{pmatrix} \square & O_{k \times k} \\ O_{k \times k} & \square \end{pmatrix}, \quad (\text{B.2})$$

where

$$\square := \frac{1}{2} \text{tr}(D^\dagger D) + 2T_\mu x^\mu + |x|^2, \quad D = \begin{pmatrix} -S \\ T \end{pmatrix}. \quad (\text{B.3})$$

Here $T = T^\mu \sigma_\mu$.

Let us introduce the $(N+2k) \times N$ matrix V satisfying

$$\nabla^\dagger \star V = O, \quad (\text{B.4})$$

$$V^\dagger \star V = I_{N \times N}, \quad (\text{B.5})$$

$$V \star V^\dagger = I_{(N+2k) \times (N+2k)} - \nabla \star f \star \nabla^\dagger. \quad (\text{B.6})$$

Here

$$f := \square_\star^{-1}, \quad (\text{B.7})$$

and we define g_\star^{-1} , the inverse of g , by $g \star g_\star^{-1} = 1$.

We obtain a noncommutative instanton solution as

$$A_\mu = V^\dagger \star \partial_\mu V. \quad (\text{B.8})$$

B.2 Completeness: ADHM \Rightarrow Instanton \Rightarrow ADHM

In this subsection, we start with ADHM data satisfying the ADHM equations (5.23) is given.

We can obtain an instanton from this ADHM data as in [3]. We show that we can reproduce the ADHM data from the instanton.

In this subsection, ∇ is associated to the given ADHM data.

Let us introduce $\tilde{\psi}$ by

$$\tilde{\psi} = {}^t\bar{\psi}\sigma_2,$$

where the transpose t is with respect to spinor indices. Using $\sigma_2\sigma_\mu\sigma_2 = -{}^t\bar{\sigma}_\mu$, we find that

$$\bar{D}_A \star \bar{\psi} = 0 \Leftrightarrow D_\mu \star \tilde{\psi}\sigma^\mu = 0. \quad (\text{B.9})$$

Therefore, to show $\bar{D}_A \star \bar{\psi} = 0$, it suffices to prove $D_\mu \star \tilde{\psi}\sigma^\mu = 0$.

Lemma 7 Set $\tilde{\psi}$

$$\tilde{\psi} = {}^t\bar{\psi}\sigma_2 = \frac{1}{\pi}V^\dagger \star (Cf), \quad (\text{B.10})$$

where V and f are defined in the previous subsection B.1 with respect to the given ADHM data. Then $\tilde{\psi}$ satisfies

$$D_\mu \star \tilde{\psi}\sigma^\mu = 0. \quad (\text{B.11})$$

PROOF.

$$\begin{aligned} \pi D_\mu \star \tilde{\psi}\sigma^\mu &= D_\mu \star (V^\dagger \star (Cf))\sigma^\mu \\ &= (\partial_\mu V^\dagger + (V^\dagger \star \partial_\mu V) \star V^\dagger) \star (C\sigma^\mu f) + V^\dagger \star C\sigma^\mu \star \partial_\mu f \quad (\text{B.12}) \\ &= \partial_\mu V^\dagger \star (1 - V \star V^\dagger) \star (C\sigma^\mu f) - V^\dagger \star (C\sigma^\mu f) \star \partial_\mu (\nabla^\dagger \star \nabla) \star f, \end{aligned}$$

where we use $I = f \star (\nabla^\dagger \star \nabla)$. Using $1 - V \star V^\dagger = \nabla \star f \star \nabla^\dagger$, (B.12) becomes

$$\partial_\mu V^\dagger \star (\nabla \star f \star \nabla^\dagger) \star (C\sigma^\mu f) - V^\dagger \star (C\sigma^\mu f) \star \partial_\mu (\nabla^\dagger \star \nabla) \star f.$$

Differentiating of $V^\dagger \star \nabla = 0$, we get $(\partial_\mu V^\dagger) \star \nabla = -V^\dagger \partial_\mu \nabla = -V^\dagger \sigma_\mu C$.
Therefore,

$$\begin{aligned} \pi D_\mu \tilde{\psi} \sigma^\mu = & \hspace{15em} (B.13) \\ -V^\dagger \star \{ & (C \sigma_\mu f \star \nabla^\dagger) \star (C \sigma^\mu f) + 4(Cf) \star C^\dagger \nabla \star f - 2(Cf) \star C^\dagger \nabla \star f \}. \end{aligned}$$

Since $\nabla^\dagger C = \bar{\sigma}_\nu(x^\nu - T^\nu)$ and $\sigma_\mu \bar{\sigma}_\nu \sigma^\mu = -2\sigma_\nu$, the first term in (B.13) equals

$$\begin{aligned} -V^\dagger \star (C \sigma_\mu f \star \nabla^\dagger) \star (C \sigma^\mu f) &= -V^\dagger \star C \sigma_\mu f \star \bar{\sigma}_\nu(x^\nu - T^\nu) \sigma^\mu \star f \\ &= 2V^\dagger \star Cf \star C^\dagger \nabla \star f. \end{aligned} \quad (B.14)$$

Then we obtain

$$\pi D_\mu \star \tilde{\psi} \sigma^\mu = 0.$$

□

Next, we show the following lemma.

Lemma 8 *Let $\bar{\psi}$ be the zero mode of $\bar{\mathcal{D}}_{A^\star}$ defined by (B.10). Then,*

$$\bar{\psi}^\dagger \star \bar{\psi} = -\frac{1}{4\pi^2} \partial^2 f. \quad (B.15)$$

PROOF.

$$\begin{aligned} \bar{\psi}^\dagger \star \bar{\psi} &= tr({}^t \bar{\psi}^\dagger \star {}^t \bar{\psi}) \\ &= \frac{1}{\pi^2} tr((fC^\dagger) \star V \star V^\dagger \star (Cf)) \\ &= \frac{1}{\pi^2} tr((fC^\dagger) \star (1_{N+2k} - \nabla \star f \star \nabla^\dagger) \star (Cf)), \end{aligned} \quad (B.16)$$

where tr denote the trace with respect to spinor indices. By definition,

$$(fC^\dagger) \star (\nabla \star f \star \nabla^\dagger) \star (Cf) = f \star ((x^\mu - T^\mu) \sigma_\mu) \star f \star (\bar{\sigma}_\nu(x^\nu - T^\nu)) \star f. \quad (B.17)$$

Differentiating $1 = f \star \square$, where \square is given by (B.3), we get

$$f \star (\bar{\sigma}_\nu(x^\nu - T^\nu)) \star f = -\frac{1}{2} \partial_\nu f \bar{\sigma}^\nu. \quad (B.18)$$

Using (B.18), (B.17) can be rewritten as

$$(fC^\dagger) \star (\nabla \star f \star \nabla^\dagger) \star (Cf) = -\frac{1}{2}f \star ((x^\mu - T^\mu)\sigma_\mu) \star (\partial_\nu f \bar{\sigma}^\nu). \quad (\text{B.19})$$

Since $(fC^\dagger) \star (Cf) = f \star f$, $\text{tr } \sigma_\mu \bar{\sigma}_\nu = \text{tr } \delta_{\mu\nu}$, and by (B.19), (B.16) equals

$$\frac{1}{\pi^2} \text{tr}(f \star f + \frac{1}{2}f \star (x^\mu - T^\mu) \star \partial_\mu f). \quad (\text{B.20})$$

From $\partial^2(\square \star f) = \partial^2 1 = 0$, we obtain the following identity:

$$f \star f + \frac{1}{2}f \star (x^\mu - T^\mu) \star \partial_\mu f = -\frac{1}{8}\partial^2 f. \quad (\text{B.21})$$

Using this identity in (B.20), we obtain

$$\bar{\psi}^\dagger \star \bar{\psi} = \frac{-1}{\pi^2} \text{tr}(\frac{1}{8}\partial^2 f) = -\frac{1}{4\pi^2}\partial^2 f. \quad (\text{B.22})$$

□

For the next step, we show orthonormality.

Lemma 9 *If $\bar{\psi}$ is a $\bar{\mathcal{D}}_{A^\star}$ zero mode given above, we have the orthonormal condition*

$$\int d^4x \bar{\psi}^\dagger \bar{\psi} = 1. \quad (\text{B.23})$$

PROOF. Define $|x|_\star^{-2}$ by $|x|^2 \star |x|_\star^{-2} = 1$. Explicitly, we have

$$\begin{aligned} |x|_\star^{-2} &= \frac{1}{|x|^2} + \hbar^2 \frac{1}{2} \theta_0^{\mu\nu} \theta_{0\mu}{}^\tau \frac{1}{|x|^6} (-\delta_{\nu\tau} + 4 \frac{x_\nu x_\tau}{|x|^2}) + \dots + \hbar^n O(\frac{1}{|x|^{2n+2}}) + \dots \\ &= \frac{1}{|x|^2} + O(|x|^{-6}). \end{aligned} \quad (\text{B.24})$$

Then,

$$f = \square_\star^{-1} = |x|_\star^{-2} \star (1 + \frac{1}{2} \text{tr}(D^\dagger D) |x|_\star^{-2} - 2T_\mu x^\mu \star |x|_\star^{-2})_\star^{-1}. \quad (\text{B.25})$$

By (B.22), (B.24) and (B.25),

$$\bar{\psi}^\dagger \bar{\psi} = \delta^4(x) + \partial^2 O(|x|^{-3}), \quad (\text{B.26})$$

and therefore we obtain $\int d^4x \bar{\psi}^\dagger \bar{\psi} = 1$. □

Now we can show the completeness; that is to say, we can show that the original ADHM data can be reproduced from the noncommutative ADHM instanton by the definition (5.3) and (5.11).

Theorem 10 *Let T and S be ADHM data and let A be a noncommutative instanton constructed from the ADHM data. Let $\bar{\psi}$ be the spinor zero mode of \bar{D}_A given above. Define T' and S' by*

$$\begin{aligned} T'^\mu &= \int d^4x x^\mu \star \bar{\psi}^\dagger \star \bar{\psi}, \\ \tilde{\psi} &= -\frac{g^{-1} S' x^\dagger}{|x|^4} + O(|x|^{-4}). \end{aligned} \quad (\text{B.27})$$

Then

$$T = T' \quad \text{and} \quad S = S'.$$

PROOF.

$$\begin{aligned} T'^\mu &= \int d^4x x^\mu \star \bar{\psi}^\dagger \star \bar{\psi} \\ &= -\frac{1}{4\pi^2} \int d^4x x^\mu \star \partial^2 f \\ &= -\frac{1}{4\pi^2} \int dS^{3\nu} (x^\mu \partial_\nu - \delta_\nu^\mu) \star f \\ &= -\frac{1}{4\pi^2} \int dS^{3\nu} (x^\mu \partial_\nu - \delta_\nu^\mu) \star |x|_\star^{-2} \star \left(1 + \frac{1}{2} \text{tr}(D^\dagger D) |x|_\star^{-2} - 2T_\rho x^\rho \star |x|_\star^{-2}\right)_\star^{-1} \\ &= -\frac{1}{4\pi^2} \int dS^{3\nu} (x^\mu \partial_\nu - \delta_\nu^\mu) \frac{1}{|x|^4} (-2T_\rho x^\rho) \\ &= T^\mu. \end{aligned} \quad (\text{B.28})$$

The proof for $S = S'$ is given by a direct calculation similar to the commutative case. □

B.3 Uniqueness : Instanton \Rightarrow ADHM \Rightarrow Instanton

In this subsection, we start with a some noncommutative instanton A . Let D^μ be the covariant derivative associated with the given noncommutative instanton connection A^μ . We introduce $\tilde{\xi}, \tilde{\chi}$ by

$$D^\mu \star D_\mu \star \tilde{\xi} = 0, \quad D^\mu \star D_\mu \star \tilde{\chi} = -4\pi\tilde{\psi} \quad (\text{B.29})$$

with the boundary conditions as $|x| \rightarrow \infty$:

$$\tilde{\xi} \rightarrow g^\dagger, \quad \tilde{\chi} \rightarrow -\frac{g^\dagger S x^\dagger}{|x|^2}. \quad (\text{B.30})$$

Lemma 11 *Let V be*

$$V = \begin{pmatrix} \tilde{\xi}^\dagger \\ \tilde{\chi}^\dagger \end{pmatrix}. \quad (\text{B.31})$$

Then

$$\nabla^\dagger \star V = 0, \quad V^\dagger \star V = I_{N \times N}. \quad (\text{B.32})$$

PROOF. The identity $D^\mu \star D_\mu \star \tilde{\xi} = 0$ implies that $D_\mu \star \tilde{\xi}$ can be written as a linear combination of $\tilde{\psi}\sigma_\mu$:

$$D_\mu \star \tilde{\xi} = \tilde{\psi}\sigma_\mu L, \quad (\text{B.33})$$

where L is a $2k \times N$ matrix. By orthonormality,

$$\begin{aligned} 4L &= \int d^4x \bar{\sigma}_\mu \tilde{\psi}^\dagger \star D_\mu \star \tilde{\xi} \\ &= \int dS^{3\mu} \bar{\sigma}_\mu \tilde{\psi}^\dagger \star \tilde{\xi} \\ &= \int d\Omega |x|^2 x^\mu \bar{\sigma}_\mu \left(\frac{-x S^\dagger g}{\pi |x|^4} \star g^\dagger \right) = -4\pi S^\dagger, \end{aligned} \quad (\text{B.34})$$

which implies

$$D_\mu \star \tilde{\xi} = -\pi \tilde{\psi} \sigma_\mu S^\dagger. \quad (\text{B.35})$$

A similar computation gives

$$D_\mu \star \tilde{\chi} = \pi \tilde{\psi} \sigma_\mu T^\dagger - \pi \tilde{\psi} \sigma_\mu \star x^\dagger. \quad (\text{B.36})$$

From (B.35) and (B.36), we have

$$D_\mu \star V^\dagger = -\pi \tilde{\psi} \star \sigma_\mu \nabla^\dagger. \quad (\text{B.37})$$

We show that $V^\dagger \star \nabla = 0$. Note that

$$\begin{aligned} D_\mu \star (V^\dagger \star \nabla) &= (D_\mu \star V^\dagger) \star \nabla + V^\dagger \star \partial_\mu \nabla \\ &= -\pi \tilde{\psi} \star (\sigma_\mu \nabla^\dagger) \star \nabla + V^\dagger C \sigma_\mu, \end{aligned} \quad (\text{B.38})$$

where we use (B.37). Then

$$\begin{aligned} D^\mu \star D_\mu (V^\dagger \star \nabla) &= -\pi \tilde{\psi} \star \sigma_\mu ((\partial^\mu \nabla^\dagger) \star \nabla + \nabla^\dagger \star \partial^\mu \nabla) + (D^\mu \star V^\dagger) C \sigma_\mu \\ &= -\pi \tilde{\psi} \star \sigma_\mu (C^\dagger \sigma_\mu \nabla + \nabla^\dagger \sigma_\mu C + \nabla^\dagger C \sigma_\mu) = 0 \end{aligned} \quad (\text{B.39})$$

As we saw in Section 4, the Green's function of $D^\mu \star D_\mu = \Delta_A$ exists. Therefore, we obtain $(V^\dagger \star \nabla) = 0$.

We now verify that $V^\dagger \star V = I_{N \times N}$. $V^\dagger \star V$ is a covariant constant, as

$$\begin{aligned} D_\mu \star (V^\dagger \star V) &= (D_\mu \star V^\dagger) \star V + V^\dagger \star (D_\mu \star V) \\ &= -\pi (\tilde{\psi} \star \sigma_\mu \nabla^\dagger \star V + V^\dagger \star \nabla \bar{\sigma}_\mu \star \tilde{\psi}^\dagger) = 0, \end{aligned} \quad (\text{B.40})$$

By its asymptotic behavior, $V^\dagger \star V \rightarrow g^{-1} \star g = I_{N \times N}$, shows that $V^\dagger \star V = I_{N \times N}$. \square

Finally, we show the uniqueness of the noncommutative ADHM instanton.

Theorem 12 *Let A'_μ be a noncommutative ADHM instanton constructed from V , i.e. $A'_\mu = V^\dagger \star \partial_\mu V$, where V is defined in (B.31). Then, A' is equal to A :*

$$A'_\mu = A_\mu. \quad (\text{B.41})$$

PROOF.

$$\begin{aligned}
A'_\mu &= V^\dagger \star \partial_\mu V \\
&= V^\dagger \star (\partial_\mu V - V \star A_\mu) + V^\dagger \star V \star A_\mu \\
&= V^\dagger \star (D_\mu \star V^\dagger)^\dagger + A_\mu \\
&= -\pi V^\dagger \star \nabla \star \bar{\sigma}_\mu \tilde{\psi}^\dagger + A_\mu = A_\mu.
\end{aligned} \tag{B.42}$$

□

C Gauge Group Elements

In this Appendix, we study the conditions forced by the choice of the $U(N)$ gauge group. If $g \in G$ then $g^\dagger \star g = I_{N \times N}$. By expanding g as $g = \sum_{i=0}^{\infty} g^{(i)} \hbar^i$, each term in the equation $g^\dagger \star g = I_{N \times N}$ is given by

$$\hbar^0 : (g^\dagger)^{(0)} g^{(0)} = I_{N \times N} \tag{C.1}$$

$$\hbar^1 : (g^\dagger)^{(1)} g^{(0)} + (g^\dagger)^{(0)} g^{(1)} + \frac{i}{2} \theta^{\mu\nu} \partial_\mu (g^\dagger)^{(0)} \partial_\nu g^{(0)} = 0 \tag{C.2}$$

⋮

$$\hbar^n : (g^\dagger)^{(n)} g^{(0)} + (g^\dagger)^{(0)} g^{(n)} + \sum_{(p;m,l) \in I(n)} \frac{1}{p!} \left((g^\dagger)^{(m)} (\overleftrightarrow{\Delta})^p g^{(l)} \right) = 0 \tag{C.3}$$

⋮

(C.1) show that $g^{(0)}$ is an element of the $U(N)$ gauge group in commutative space. Let us introduce a $N \times N$ Hermitian matrix $\phi(x)$ by $g^{(0)} = \exp i\epsilon\phi$ with infinitesimal gauge parameter ϵ . By expanding $g^{(1)}$ as

$$g^{(1)} = \sum_{k=0}^{\infty} \epsilon^k g_k^{(1)} = \sum_{k=0}^{\infty} \epsilon^k (H_k^{(1)} + A_k^{(1)}), \tag{C.4}$$

where $H_k^{(1)}$ and $A_k^{(1)}$ are Hermitian part and anti-Hermitian part of $g_k^{(1)}$, respectively, (C.2) becomes

$$\epsilon^0 : H_0^{(1)} = 0 \tag{C.5}$$

$$\epsilon^1 : H_1^{(1)} = \frac{i}{2} \{A_0^{(1)}, \phi\} \tag{C.6}$$

$$\epsilon^2 : H_2^{(1)} = \frac{-1}{2} \left\{ i\{A_0^{(1)}, \phi^2\} + i[H_1^{(1)}, \phi] - i\{A_1^{(1)}, \phi\} + \frac{i}{2} \theta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right\} \tag{C.7}$$

$$\epsilon^3 : H_3^{(1)} = \frac{-1}{2} \left\{ i\{A_0^{(1)}, \phi^3\} - \{H_1^{(1)}, \phi^2\} + [A_1^{(1)}, \phi^2] + i[H_2^{(1)}, \phi] - i\{A_2^{(1)}, \phi\} \right\}$$

\vdots .

These conditions show that we can chose $A_k^{(1)}$ freely, and the choice of $A_k^{(1)}$ determines $H_k^{(1)}$. For example, it is possible to choose $g^{(1)}$ as a non-zero constant matrix in the limit as $|x| \rightarrow \infty$. Therefore we can not ignore the asymptotic effect of $g^{(1)}$ in the estimation of the ADHM data as mentioned in Section 5.

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