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Asymptotic behaviour at a tip of a rigid line inclusion in linearized elasticity

by

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Abstract

We consider an asymptotic behaviour of a solution near a tip of a rigid line inclusion in two dimensional homogeneous isotropic linearized elasticity. By means of Goursat-Kolosov-Muskhelishvili stress functions we derive convergent expansions of the solution around there. Furthermore, we give expressions of the invariant integral and the Irwin's formula.

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KEY WORDS: linearized elasticity, singularities at a tip of rigid line inclusion, delamination, invariant integral, Irwin's formula

1 Introduction

Analysis of the stress fields in elastic bodies induced by inhomogeneities such as cracks, inclusions, voids plays a major role in fracture mechanics. A great deal of progress has been made to the studies of stress concentrations around these defects of various shapes, in particular, for crack analysis many research results have been reported (e.g. [3], [8]–[19], [21], [22], [24]–[26]).

Meanwhile, a rigid line inclusion, also called stiffener or anticrack, is a mathematical model used in solid mechanics to describe a fiber, embedded in a matrix material. In cases where a material has both cracks and anticracks, it is very important to express the interaction of the thin inclusions with cracks because inclusions cause delamination from the matrix material and forming a new crack. The mathematical model was considered in [16, 17]. And in [4, 1], the stress concentration near a rigid line inclusion was discussed under some specific situations.

From the mathematical point of view, the theory of boundary value problems in domains with non-smooth boundaries has been established in [7], [20] and [2] by employing weighted Sobolev spaces with weights with respect to the distance to the singular points. The weights correspond to singularity of the solution of the boundary value problems and depend on various components such as the governing equation, material properties, boundary conditions and so on. In [8, 9] the explicit convergent expansions of the solution around the tip of a linear crack in the linearized elastic plate are derived under various boundary conditions on the crack by means of Goursat-Kolosov-Muskhelishvili stress functions.

In this paper we treat with an equilibrium problem for two dimensional homogeneous isotropic linearized elasticity with a rigid line inclusion. Both cases of not delaminated and delaminated inclusions are considered. Then we explicitly derive the convergent expansions of the solutions in the vicinity of the tip of the rigid line inclusion without and with delamination in a linear situation, respectively. Here the rigid line inclusion without delamination means to have no elastic deformations and only have rigid body motions, that is, three degrees of freedom in two dimensions, and with delamination in a linear situation means that one side of which is completely bonded to the elastic medium and have rigid body motions, while the traction of the another side is free which implies to allow delamination. In the first case, derivation of the convergent expansion is based on the use of Goursat-Kolosov-Muskhelishvili stress functions and their Riemann-Hilbert problem, which are the same method constructed in [5, 23, 24].

However, in the second case due to mixed boundary condition on the rigid line inclusion it needs to solve a system of Riemann-Hilbert problems, which is slightly different from the other cases. Moreover, in both cases, we verify that the derivative of the energy functional with respect to the inclusion length is represented as the invariant integral and derive the Irwin's formula [11] which originally implies that the energy release rate is expressed only by the coefficients of singular terms of elastic fields, called Stress Intensity Factors in fracture mechanics.

2 Formulation of the problem

Let Ω be a bounded domain of $\mathbb{R}^2$ with Lipschitz boundary, which represents an isotropic homogeneous linearized elasticity. We denote Lipschitz domains $\Omega^{(1)}$ and $\Omega^{(2)}$ by $\Omega^{(1)} = \Omega \cap \{x_2 > 0\}$ and $\Omega^{(2)} = \Omega \cap \{x_2 < 0\}$, respectively. We define an interface of $\Omega^{(k)}$ $(k = 1, 2)$ by Γ'. Let Γ be a rigid line inclusion on Γ' and have the two tips are located at the origin $\boldsymbol{O} \notin \partial\Omega$ of the coordinates system $\boldsymbol{x} = (x_1, x_2)$ and a point $\boldsymbol{P}(-\ell, 0) \notin \partial\Omega$, $\ell > 0$. And let Γ_N be an arbitrary nonempty open subset of $\partial\Omega^{(1)} \setminus \Gamma'$ such that $\Gamma_N \cap \overline{\Gamma'} = \emptyset$ and $\Gamma_D = \partial\Omega \setminus \Gamma_N$, see figure 1 for an illustration of the geometry.

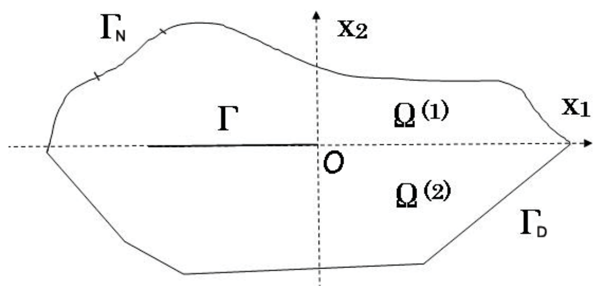

Figure 1: domain

By $\boldsymbol{u} = (u_i)_{i=1,2}$ and $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1,2}$ we denote the displacement vector and the stress tensor, respectively. In addition, throughout this paper, the frequently used constant c is a generic positive constant whose value may be different under different context.

We introduce a notation meaning the jump, for example, the jump of $\boldsymbol{u}$ at Γ is denoted by the formula

$$[\boldsymbol{u}]_\Gamma := \boldsymbol{u}^+ - \boldsymbol{u}^- \quad \text{on} \quad \Gamma,$$

where $\boldsymbol{u}^\pm$ fit to the positive and negative faces of Γ with respect to the normal vector $\boldsymbol{n} = (n_1, n_2)$.

In $\Omega \setminus \overline{\Gamma}$ we suppose the stationary equilibrium conditions without any body forces hold, which are described as

$$\frac{\partial}{\partial x_j}\sigma_{ij} = 0, \quad i = 1, 2. \tag{2.1}$$

Then, the linearized elasticity equations for $\boldsymbol{u}$ are given by

$$A\boldsymbol{u} := \mu\triangle\boldsymbol{u} + (\tilde{\lambda} + \mu)\nabla(\nabla\cdot\boldsymbol{u}) = \boldsymbol{0} \quad \text{in} \quad \Omega\setminus\overline{\Gamma}.$$

Here and in what follows we use the summation convention,

$$\tilde{\lambda} = \begin{cases} \lambda \quad \text{(plane strain)}, \\ \dfrac{2\lambda\mu}{\lambda + 2\mu} \quad \text{(plane stress)}, \end{cases}$$

λ and μ are *Lamé* constants of the elastic medium. Since both the shear modulus and the bulk modulus are required to be positive, we suppose $\mu > 0$ and $3\lambda + 2\mu > 0$, in which case it is easy to see that the operator A is elliptic. And we define $\tilde{\kappa} = \frac{\tilde{\lambda}+3\mu}{\tilde{\lambda}+\mu}$. Moreover we introduce the boundary stress operator T and the stress vector $T\boldsymbol{u}$ expressed by $T\boldsymbol{u} := \boldsymbol{\sigma n}$, where $\boldsymbol{n} = (n_1, n_2)$ the unit outward normal vector field on $\partial\Omega$ and

$$\boldsymbol{\sigma} = \tilde{\lambda}(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \mu\{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\}, \tag{2.2}$$

where $\boldsymbol{I}$ is the second order identity tensor.

In the two dimensional case a rigid displacement can be written in the form

$$\boldsymbol{\rho}(\boldsymbol{x}) = (c_1 + c_0 x_2, c_2 - c_0 x_1)$$

with a constant vector $\boldsymbol{c} = (c_1, c_2, c_0)$. We denote the set of all rigid displacements on D by $\mathcal{R}(D)$.

Now we consider the following boundary value problems, see [16, 17];

Problem 1 (without delamination). *For given* $\boldsymbol{g} \in L^2(\Gamma_N)$*, find* $\boldsymbol{u} \in H^1(\Omega)$ *and* $\boldsymbol{\rho}_0 \in \mathcal{R}(\Gamma)$ *satisfying*

$$(*)\begin{cases} A\boldsymbol{u} = \boldsymbol{0} & \text{in} \quad \Omega \setminus \overline{\Gamma}, \\ T\boldsymbol{u} = \boldsymbol{g} & \text{on} \quad \Gamma_N, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \Gamma_D, \\ \boldsymbol{u} = \boldsymbol{\rho}_0 & \text{on} \quad \Gamma, \\ \int_\Gamma [\boldsymbol{\sigma n}]_\Gamma \cdot \boldsymbol{\rho}\, \mathrm{d}S_{\boldsymbol{x}} = 0 & \text{for} \quad \forall \boldsymbol{\rho} \in \mathcal{R}(\Gamma). \end{cases}$$

Problem 2 (with delamination in a linear situation). *For given* $\boldsymbol{g} \in L^2(\Gamma_N)$*, find* $\boldsymbol{u} \in H^1(\Omega \setminus \overline{\Gamma})$ *and* $\boldsymbol{\rho}_0 \in \mathcal{R}(\Gamma)$ *satisfying*

$$(\dagger)\begin{cases} A\boldsymbol{u} = \boldsymbol{0} & \text{in} \quad \Omega \setminus \overline{\Gamma}, \\ T\boldsymbol{u} = \boldsymbol{g} & \text{on} \quad \Gamma_N, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on} \quad \Gamma_D, \\ T\boldsymbol{u}^+ = \boldsymbol{0}, \quad \boldsymbol{u}^- = \boldsymbol{\rho}_0 & \text{on} \quad \Gamma, \\ \int_\Gamma [\boldsymbol{\sigma n}]_\Gamma \cdot \boldsymbol{\rho}\, \mathrm{d}S_{\boldsymbol{x}} = 0 & \text{for} \quad \forall \boldsymbol{\rho} \in \mathcal{R}(\Gamma). \end{cases}$$

Here we take $\boldsymbol{n} = (0, 1)$ in the fourth equation of $(*)$ and $(\dagger)$. Note that the condition $\boldsymbol{u} = \boldsymbol{\rho}_0$ on Γ, $\boldsymbol{\rho}_0 \in \mathcal{R}(\Gamma)$ does not mean, in general, that $\boldsymbol{\sigma} = \boldsymbol{0}$ on Γ.

3 Problem 1 for the rigid line inclusion without delamination

3.1 The weak solution and the regularity

Now we define the functional of potential energy of $\boldsymbol{u} \in H^1_{\Gamma_D}(\Omega) := \{\boldsymbol{u} \in H^1(\Omega) \mid \boldsymbol{u}|_{\Gamma_D} = \boldsymbol{0}\}$ for the solid,

$$\Pi(\boldsymbol{u}) := \frac{1}{2}\mathcal{E}_\Omega(\boldsymbol{u}, \boldsymbol{u}) - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{u}\, \mathrm{d}S_{\boldsymbol{x}},$$

where a bilinear form

$$\mathcal{E}_\Omega(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \sigma_{ij}\frac{\partial}{\partial x_j} v_i \,\mathrm{d}\boldsymbol{x}.$$

Here the stress tensor in $\mathcal{E}$ is given by substituting the first element $\boldsymbol{u}$ of $\mathcal{E}_\Omega(\boldsymbol{u},\boldsymbol{v})$ into the displacement vector in (2.2). Moreover, note that $\Pi(\boldsymbol{u})$ is a positive, convex, continuous and differentiable functional on $H^1(\Omega)$.

Considering the boundary condition on Γ, we define the convex set of admissible displacements

$$\mathcal{K} := \{\boldsymbol{v} \in H^1_{\Gamma_D}(\Omega) \mid \boldsymbol{v}|_\Gamma \in \mathcal{R}(\Gamma)\}.$$

Then the boundary value problem $(*)$ can be reduced to the following minimization problem:

$$\Pi(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\mathcal{K}} \Pi(\boldsymbol{v}). \tag{3.1}$$

One can see that existence of the minimization problem (3.1) guarantees the solvability of the Euler equation

$$\Pi'_{\boldsymbol{u}}(\boldsymbol{w}) = 0 \quad \forall \boldsymbol{w} \in \mathcal{K}, \tag{3.2}$$

that is, there exists a solution $\boldsymbol{u} \in \mathcal{K}$ such that for any $\boldsymbol{w} \in \mathcal{K}$

$$\mathcal{E}_\Omega(\boldsymbol{u},\boldsymbol{w}) - \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{w}\,\mathrm{d}S_{\boldsymbol{x}} = 0. \tag{3.3}$$

In order to provide the boundary stress with an exact meaning we employ Green formulae written in the Lipschitz domains $\Omega^{(1)}$ and $\Omega^{(2)}$ as

$$0 = -\int_{\Omega^{(k)}} A\boldsymbol{u}\cdot\boldsymbol{v}\,\mathrm{d}\boldsymbol{x} = \mathcal{E}_{\Omega^{(k)}}(\boldsymbol{u},\boldsymbol{v}) - \langle \boldsymbol{\sigma n}, \boldsymbol{v}\rangle_{\partial\Omega^{(k)}} \tag{3.4}$$

for all $\boldsymbol{v} \in H^1(\Omega^{(k)})$, $k = 1, 2$. Due to (2.1) and (2.2) we have $A\boldsymbol{u} \in L^2(\Omega\setminus\overline{\Gamma})$, then the stress vectors $\boldsymbol{\sigma n}$ are determined in $H^{-1/2}(\partial\Omega^{(k)})$. At $\partial\Omega^{(k)}\cap\partial\Omega$ we suppose that $\boldsymbol{\sigma n} = T\boldsymbol{u}$ are the L^2-functions. At $\partial\Omega^{(k)}\cap\Gamma'$, the stress vectors $\boldsymbol{\sigma n} = (-1)^k(\sigma_{12},\sigma_{22})$ are the bounded measures over $C_0(\Gamma')$. Since $C_0(\Gamma')$ are dense in $H^{1/2}_0(\Gamma') = H^{1/2}(\Gamma')$, this defines well the duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\Gamma'}$ between the boundary traces $v_i \in H^{1/2}(\Gamma')$ and the $H^{-1/2}$-distributions σ_{i2}, $i = 1, 2$.

Now we introduce the space the Lions–Magenes space $H^{1/2}_{00}(\Gamma')$ endowed with the norm

$$\|\boldsymbol{w}\|^2_{H^{1/2}_{00}(\Gamma')} := \|\boldsymbol{w}\|^2_{H^{1/2}(\Gamma')} + \int_{\Gamma'} \frac{|\boldsymbol{w}|^2}{\mathrm{dist}(\boldsymbol{x},\partial\Gamma')}\,\mathrm{d}S_{\boldsymbol{x}}$$

and its dual space denoted by $H^{-1/2}_{00}(\Gamma')$. Let $\overline{\boldsymbol{w}}$ be an extension of $\boldsymbol{w}$ defined in Γ' by zero outside to $\partial\Omega^{(k)}$, that is,

$$\overline{\boldsymbol{w}} = \begin{cases} \boldsymbol{w} & \text{on} \quad \Gamma', \\ \boldsymbol{0} & \text{on} \quad \partial\Omega^{(k)}\setminus\Gamma'. \end{cases}$$

Then $\overline{\boldsymbol{w}} \in H^{1/2}(\partial\Omega^{(k)})$ if and only if $\boldsymbol{w} \in H^{1/2}_{00}(\Gamma')$, for the detail see [12]. Hence, together (3.3) with (3.4) gives

$$\langle [\boldsymbol{\sigma n}]_{\Gamma'}, \boldsymbol{v}\rangle^{00}_{\Gamma'} = 0, \quad \forall \boldsymbol{v} \in \mathcal{K} \tag{3.5}$$

with the notation $\langle\cdot,\cdot\rangle^{00}_{\Gamma'}$ for a duality pairing between $H^{-1/2}_{00}(\Gamma')$ and $H^{1/2}_{00}(\Gamma')$. Thus, since $[\sigma_{i2}]_{\Gamma'\setminus\Gamma} = 0$ $(i = 1, 2)$ and $\boldsymbol{v} = \boldsymbol{\rho}$ on Γ, the condition

$$\int_\Gamma [\boldsymbol{\sigma n}]_\Gamma\cdot\boldsymbol{\rho}\,\mathrm{d}S_{\boldsymbol{x}} = 0 \quad \forall \boldsymbol{\rho} \in \mathcal{R}(\Gamma) \tag{3.6}$$

make sense as (3.5).

Next, we state the solvability of (3.1).

Let us start with some preliminaries. We suppose that the bilinear form in (3.3) possesses the first Korn inequality: there exist $0 < \underline{C}_0 \leq \overline{C}_0 < \infty$ such that

$$\underline{C}_0 \|\boldsymbol{u}\|^2_{H^1(\Omega)} \leq \mathcal{E}_\Omega(\boldsymbol{u}, \boldsymbol{u}) \leq \overline{C}_0 \|\boldsymbol{u}\|^2_{H^1(\Omega)} \quad \forall \boldsymbol{u} \in H^1_{\Gamma_D}(\Omega). \tag{3.7}$$

This allows us to introduce the equivalent norm in $H^1_{\Gamma_D}(\Omega)$ as

$$\|\boldsymbol{u}\|^2_{1,\Omega} := \mathcal{E}_\Omega(\boldsymbol{u}, \boldsymbol{u}).$$

The constants $\underline{C}_0$, $\overline{C}_0$ in (3.7) depend on the material parameters λ, μ and on the geometry of Ω.

Then, since it follows from (3.7) and the continuity property of the trace operator at $\partial\Omega$ that

$$\Pi(\boldsymbol{u}) \geq \underline{C}_0 \|\boldsymbol{u}\|^2_{H^1(\Omega)} - c\|g\|_{L^2(\Gamma_N)} \|\boldsymbol{u}\|_{H^1(\Omega)},$$

one sees $\Pi(\boldsymbol{u})$ is coercive on $H^1_{\Gamma_D}(\Omega)$. Moreover, from convexity of $\Pi(\boldsymbol{u})$ one knows $\Pi(\boldsymbol{u})$ is weakly lower semicontinuous. Consequently, noting that $\mathcal{K}$ is a closed convex subset of a reflexive Banach space, we have the following existence theorem:

Theorem 3.1. *There exists a unique solution $\boldsymbol{u} \in \mathcal{K}$ of the minimization problem* (3.1).

For the need of further asymptotic analysis we formulate the following lemma on the local smoothness of the solution.

Lemma 3.1. *The solution $\boldsymbol{u} \in \mathcal{K}$ of* (3.1) *obeys the interior C^∞-regularity in $\Omega^{(1)}$ and $\Omega^{(2)}$. The boundary stress components σ_{i2}, $i = 1, 2$ are pointwise functions inside Γ.*

Indeed, the interior C^∞-regularity of $\boldsymbol{u}$ is ensured by the equilibrium equation $A\boldsymbol{u} = \boldsymbol{0}$ in the standard way (e.g., [6]). The interior regularity at Γ follows from the uniform estimate of the stress inside Γ.

3.2 The convergent expansion of the solution near $\boldsymbol{O}$

In this section we derive convergent expansions of the solution constructed in Theorem 3.1. Now we introduce a polar coordinate system $(x_1, x_2) = (r\cos\theta, r\sin\theta)$ with respect to the origin $\boldsymbol{O}$. And we fix some notations

$$B_a := B_a(\boldsymbol{O}), \quad B_a^{(1)} := B_a \cap \Omega^{(1)}, \quad B_a^{(2)} := B_a \cap \Omega^{(2)}$$

with a sufficiently small $a > 0$ such that $a < \ell$,

$$\Gamma_1 := B_a \cap \Gamma, \quad \Gamma'' := B_a \cap \Gamma', \quad \Gamma_2 := \Gamma'' \setminus \overline{\Gamma_1}.$$

Then, we construct Goursat-Kolosov-Muskhelishvili stress functions, see [23], in each $B_a^{(k)}$. The interior and boundary regularity results of Lemma 3.1 ensure that σ_{ij} is in $C^\infty(B_a^{(k)})$ and satisfies the condition on Γ_1 in the pointwise sense. From this fact and Poincaré lemma we obtain two holomorphic functions $\phi^{(k)}(z)$, $\omega^{(k)}(z)$ in $B_a^{(k)}$ $(k = 1, 2)$, of the complex variable $z = x_1 + ix_2$. Moreover, it follows from generalized Poincaré lemma (e.g., [8]) that $\phi^{(k)}(z)$, $\omega^{(k)}(z) \in H^1(B_a^{(k)})$. Then for each $k = 1, 2$ displacement $\boldsymbol{u}$ and stress fields $\boldsymbol{\sigma}$ in the plane isotropic elasticity $B_a^{(k)}$ can be represented as

$$2\mu(u_1 + iu_2) = \tilde{\kappa}\phi^{(k)}(z) - \overline{\omega^{(k)}(z)} + (\overline{z} - z)\overline{\phi^{(k)\prime}(z)}, \tag{3.8}$$

$$\sigma_{11} + \sigma_{22} = 2(\phi^{(k)\prime}(z) + \overline{\phi^{(k)\prime}(z)}), \tag{3.9}$$

$$\sigma_{22} - i\sigma_{12} = \phi^{(k)\prime}(z) + \overline{\omega^{(k)\prime}(z)} + (z - \overline{z})\overline{\phi^{(k)\prime\prime}(z)}, \tag{3.10}$$

where $\phi^{(k)'}(z) = \mathrm{d}\phi^{(k)}/\mathrm{d}z$ and overbar of functions denotes the complex conjugate.

Next, taking into account the boundary conditions on Γ'', we derive the Riemann–Hilbert problem for the stress functions (e.g. [24] and [5]). Firstly, since $[\boldsymbol{\rho}_0]_\Gamma = \mathbf{0}$ and the displacement vector is continuous on Γ_2, it follows that $[\boldsymbol{u}]_{\Gamma''} = \mathbf{0}$ and from (3.8) we obtain

$$\frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\phi^{(1)}(x_1) - \frac{1}{\mu^{(1)}}\overline{\omega^{(1)}(x_1)} = \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\phi^{(2)}(x_1) - \frac{1}{\mu^{(2)}}\overline{\omega^{(2)}(x_1)} \quad \text{on} \quad \Gamma''.$$

Differentiating both sides of this relation with respect to x_1 yields

$$\frac{\tilde{\kappa}^{(1)}}{\mu^{(1)}}\phi^{(1)'}(x_1) - \frac{1}{\mu^{(1)}}\overline{\omega^{(1)'}(x_1)} = \frac{\tilde{\kappa}^{(2)}}{\mu^{(2)}}\phi^{(2)'}(x_1) - \frac{1}{\mu^{(2)}}\overline{\omega^{(2)'}(x_1)} \quad \text{on} \quad \Gamma''.$$

Here it is easy to see that $\overline{\omega^{(1)'}(\overline{z})}$ and $\overline{\omega^{(2)'}(\overline{z})}$ are holomorphic in $B_a^{(2)}$ and $B_a^{(1)}$, respectively. Then we can define a holomorphic function $\Phi(z)$ in B_a as

$$\Phi(z) = \begin{cases} \frac{\tilde{\kappa}}{\mu}\phi^{(1)'}(z) + \frac{1}{\mu}\overline{\omega^{(2)'}(\overline{z})} & \text{in} \quad B_a^{(1)}, \\ \frac{\tilde{\kappa}}{\mu}\phi^{(2)'}(z) + \frac{1}{\mu}\overline{\omega^{(1)'}(\overline{z})} & \text{in} \quad B_a^{(2)}. \end{cases}$$

Secondly, it follows from continuity of $\boldsymbol{\sigma}$ on Γ_2 that

$$[\sigma_{22} - i\sigma_{12}]_{\Gamma_2} = 0.$$

Since $z = \overline{z}$ on Γ'', from (3.10) we have

$$\phi^{(1)'}(x_1) + \overline{\omega^{(1)'}(x_1)} = \phi^{(2)'}(x_1) + \overline{\omega^{(2)'}(x_1)} \quad \text{on} \quad \Gamma_2.$$

From this we can define a sectionally holomorphic function in B_ρ cut along Γ_1, i.e. holomorphic in $B_a \setminus \overline{\Gamma_1}$, sectionally continuous in the neighbourhood of Γ_1, weakly singular at the end points ($z = 0$, $z = -a$),

$$\Psi(z) = \begin{cases} \phi^{(1)'}(z) - \overline{\omega^{(2)'}(\overline{z})} & \text{in} \quad B_a^{(1)}, \\ \phi^{(2)'}(z) - \overline{\omega^{(1)'}(\overline{z})} & \text{in} \quad B_a^{(2)}. \end{cases}$$

Next, by using functions $\Phi(z)$, $\Psi(z)$ we express the functions $\phi^{(k)}(z)$, $\omega^{(k)}(z)$ ($k = 1, 2$) as follows,

$$\phi^{(k)'}(z) = \frac{\mu}{\tilde{\kappa}+1}\left(\frac{1}{\mu}\Psi(z) + \Phi(z)\right) \quad \text{in} \quad B_a^{(k)}, \tag{3.11}$$

$$\omega^{(k)'}(z) = \frac{\mu}{\tilde{\kappa}+1}\left(-\frac{\tilde{\kappa}}{\mu}\overline{\Psi(\overline{z})} + \overline{\Phi(\overline{z})}\right) \quad \text{in} \quad B_a^{(k)}. \tag{3.12}$$

Finally, now we take into account the condition on Γ_1, that is, $\boldsymbol{u} = \boldsymbol{\rho}_0 := (\beta_1 + \alpha x_2, \beta_2 - \alpha x_1)$ on Γ_1. Then it follows from (3.8), (3.11) and (3.12) that

$$\lim_{x_2 \to 0+}\left\{\frac{\tilde{\kappa}}{2\mu}\cdot\frac{\mu}{\tilde{\kappa}+1}\left(\frac{1}{\mu}\Psi(z) + \Phi(z)\right) - \frac{1}{2\mu}\cdot\frac{\mu}{\tilde{\kappa}+1}\left(-\frac{\tilde{\kappa}}{\mu}\Psi(\overline{z}) + \Phi(\overline{z})\right)\right\} = -\alpha i \quad \text{on} \quad \Gamma_1.$$

Since $\Phi(z)$ is continuous on Γ_1 and from (3.11) it yields

$$\Psi(z) = (\tilde{\kappa}+1)\phi^{(1)'}(z) - \mu\Phi(z) \quad \text{in} \quad B_a^{(1)},$$

we obtain the Riemann–Hilbert problem

$$\phi^{(1)'}(z) + \phi^{(1)'}(\overline{z}) = \frac{\mu}{\tilde{\kappa}}\Phi(z) - \frac{2\mu}{\tilde{\kappa}}\alpha i \quad \text{on} \quad \Gamma_1. \tag{3.13}$$

According to [5, 9, 23], the general solution of (3.13) can be given by

$$\phi^{(1)'}(z) = \frac{X(z)}{2\pi i}\int_{\Gamma_1}\frac{1}{X^+(t)(t-z)}\left(\frac{\mu}{\tilde{\kappa}}\Phi(t) - \frac{2\mu}{\tilde{\kappa}}\alpha i\right)\,\mathrm{d}t + X(z)\chi(z), \tag{3.14}$$

where $\chi(z)$ is holomorphic in B_a and

$$X(z) := z^{-\frac{1}{2}}(z+a)^{-\frac{1}{2}}.$$

Note here that $X(z)$ is defined in the whole plane and has branch points at $z=0$, $z=-a$. In order to define $X(z)$ uniquely we define $\arg z$ and $\arg(z+a)$ as $-\pi < \arg z,\ \arg(z+a) < \pi$. Then, it is easy to see that $X(z)$ is sectionally holomorphic in the plane cut along Γ_1. Moreover, the integral in (3.14) can be calculated by the Cauchy's integral theorem and thus

$$\phi^{(1)'}(z) = \frac{\mu}{2\tilde{\kappa}}\Phi(z) - \frac{\mu}{\tilde{\kappa}}\alpha i + X(z)\chi(z).$$

Indeed, it is obvious that $\frac{\mu}{2\tilde{\kappa}}\Phi(z) - \frac{\mu}{\tilde{\kappa}}\alpha i$ is a special solution of (3.13). Resetting $\chi(z)$ defined in $B_{a'}$ with $a' < a$ gives

$$\phi^{(k)'}(z) = z^{-\frac{1}{2}}\chi(z) + \frac{\mu}{2\tilde{\kappa}}\Phi(z) - \frac{\mu}{\tilde{\kappa}}\alpha i. \tag{3.15}$$

Analogously, from (3.12) we find the expressions of the other functions

$$\omega^{(k)'}(z) = -\tilde{\kappa}z^{-\frac{1}{2}}\overline{\chi(\overline{z})} + \frac{\mu}{2}\overline{\Phi(\overline{z})} - \mu\alpha i. \tag{3.16}$$

Since $\chi(z)$ and $\Phi(z)$ are holomorphic in $B_{a'}$, they can be written as local Taylor series expansions

$$\chi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \Phi(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{3.17}$$

which are generalized uniform convergent in $B_{a'}$. Moreover, since the coefficients a_n, b_n can be given by

$$a_n = \frac{1}{2\pi i}\int_{|w|=r}\frac{\chi(w)}{w^{n+1}}\,\mathrm{d}w \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_{|w|=r}\frac{\Phi(w)}{w^{n+1}}\,\mathrm{d}w$$

for $0 < r < a'$, by virtue of Cauchy–Schwarz inequality it is easy to verify the following estimates

$$\begin{aligned} |a_n| &\leq c\sqrt{2n+1}\,(a')^{-\left(n+\frac{1}{2}\right)}\|\chi\|_{L^2(B_{a'})}, \\ |b_n| &\leq c\sqrt{n+1}\,(a')^{-(n+1)}\|\Phi\|_{L^2(B_{a'})}. \end{aligned}$$

Now we set $a_n := A_n + iB_n$, $b_n := C_n + iD_n$ and

$$P_m = \begin{cases} \frac{A_n}{\frac{1}{2}+n} & (m = 2n+1), \\ \frac{\mu C_n}{2\tilde{\kappa}(n+1)} & (m = 2(n+1)), \end{cases} \qquad Q_m = \begin{cases} \frac{B_n}{\frac{1}{2}+n} & (m = 2n+1), \\ \frac{\mu D_n}{2\tilde{\kappa}(n+1)} & (m = 2(n+1)). \end{cases}$$

By substituting (3.15) and (3.16) into (3.8) and using (3.17) we obtain the convergent expansion of $\boldsymbol{u}$ near $\boldsymbol{O}$.

Proposition 3.1. *There exist real numbers P_m, Q_m and constants α, α_1, β_1, β_2 such that*

$$\boldsymbol{u}(r,\theta) = \sum_{m=1}^{\infty}\frac{1}{2\mu}P_m r^{\frac{m}{2}}\boldsymbol{R}_{1,m}(\theta) - \frac{1}{2\mu}Q_m r^{\frac{m}{2}}\boldsymbol{S}_{1,m}(\theta) + \boldsymbol{F}_1,$$

where

$$\boldsymbol{R}_{1,m}(\theta) = \begin{pmatrix} \tilde{\kappa}(1-(-1)^m)\cos\frac{m}{2}\theta + \frac{m}{2}\cos\frac{m}{2}\theta - \frac{m}{2}\cos\left(\frac{m}{2}-2\right)\theta \\ \tilde{\kappa}(1+(-1)^m)\sin\frac{m}{2}\theta - \frac{m}{2}\sin\frac{m}{2}\theta + \frac{m}{2}\sin\left(\frac{m}{2}-2\right)\theta \end{pmatrix},$$

$$\boldsymbol{S}_{1,m}(\theta) = \begin{pmatrix} \tilde{\kappa}(1+(-1)^m)\sin\frac{m}{2}\theta + \frac{m}{2}\sin\frac{m}{2}\theta - \frac{m}{2}\sin\left(\frac{m}{2}-2\right)\theta \\ -\tilde{\kappa}(1-(-1)^m)\cos\frac{m}{2}\theta + \frac{m}{2}\cos\frac{m}{2}\theta - \frac{m}{2}\cos\left(\frac{m}{2}-2\right)\theta \end{pmatrix},$$

$$\boldsymbol{F}_{\mathbf{1}} = \begin{pmatrix} \alpha_1 r\sin\theta + \beta_1 \\ -\alpha r\cos\theta + \beta_2 \end{pmatrix}.$$

The series are convergent, absolutely in $H^1(B_{a'})$ and generalized uniform in $B_{a'}$. For $m \geq 1$, P_m and Q_m satisfy

$$|P_m| + |Q_m| \leq c\frac{1}{\sqrt{m}}\,(a')^{-\frac{m}{2}}\|\nabla\boldsymbol{u}\|_{L^2(B_{a'})}.$$

Here the estimates of coefficients can be obtained from (3.9)–(3.10).

Note that it follows form Proposition 3.1 that

$$\begin{aligned} \sigma_{12}|_{\theta=\pi} - \sigma_{12}|_{\theta=-\pi} &= \sum_{m=1}^{\infty}(\tilde{\kappa}+1)mP_m r^{\frac{m}{2}-1}\sin\frac{m}{2}\pi, \\ \sigma_{22}|_{\theta=\pi} - \sigma_{22}|_{\theta=-\pi} &= \sum_{m=1}^{\infty}(\tilde{\kappa}-1)mQ_m r^{\frac{m}{2}-1}\sin\frac{m}{2}\pi. \end{aligned}$$

3.3 The invariant integral and the Irwin's formula

To derive the invariant integral and the Irwin's formula [11] we use the smooth perturbations method [18, 21, 25]. Choose the cut-off function $\eta \in C_0^\infty(\Omega)$ such that $\eta = 1$ in a domain D in Ω with the boundary ∂D. Moreover, we assume that

$$\boldsymbol{P} \notin \operatorname{supp}\eta, \quad \boldsymbol{O} \in D. \tag{3.18}$$

For all sufficiently small $\delta > 0$ we construct the one-to-one coordinate transformation $\boldsymbol{y} = \Phi_\delta(\boldsymbol{x})$, where

$$y_1 = x_1 + \delta\eta(x_1, x_2), \quad y_2 = x_2, \ (x_1, x_2) \in \Omega, \tag{3.19}$$

with the Jacobian $J_\delta = 1 + \delta\,\eta_{,1}$. Then, we consider the perturbed rigid inclusion $\Gamma_\delta = \Phi_\delta(\Gamma)$. By definition of the transformation (3.19), we have $\Omega = \Phi_\delta(\Omega)$, $\Omega^{(i)} = \Phi_\delta(\Omega^{(i)})$, $i = 1, 2$; $\Gamma' = \Phi_\delta(\Gamma')$, $\Gamma_\delta \subset \Gamma'$.

Now we define the convex set of admissible displacements

$$\mathcal{K}^\delta := \{\boldsymbol{v} \in H^1_{\Gamma_D}(\Omega) \mid \boldsymbol{v}|_{\Gamma_\delta} \in \mathcal{R}(\Gamma_\delta)\}$$

and consider the following minimization problem:

$$\Pi(\boldsymbol{u}^{\boldsymbol{\delta}}) = \inf_{\boldsymbol{v}\in\mathcal{K}^\delta}\Pi(\boldsymbol{v}), \tag{3.20}$$

which is equivalent to the variational equality

$$\mathcal{E}_\Omega(\boldsymbol{u}, \boldsymbol{w}) - \int_{\Gamma_N}\boldsymbol{g}\cdot\boldsymbol{w}\,\mathrm{d}S_{\boldsymbol{x}} = 0 \quad \forall w \in \mathcal{K}^\delta. \tag{3.21}$$

Let $\bar{\boldsymbol{v}}(\boldsymbol{x})$ be the transformed function $\boldsymbol{v}(\boldsymbol{y})$, that is,

$$\boldsymbol{v}(y_1, y_2) = \boldsymbol{v}(x_1 + \delta\, \eta(x_1, x_2), x_2) = \bar{\boldsymbol{v}}(x_1, x_2).$$

We apply the inverse transformation of (3.19) to functions and integrals in (3.21). Since $\eta = 0$ on $\partial\Omega$, it holds the following variational equality

$$\mathcal{E}_\Omega(\bar{\boldsymbol{u}}^\delta, \bar{\boldsymbol{w}}) + \delta \int\limits_\Omega A_1(\eta; \bar{\boldsymbol{u}}^\delta, \bar{\boldsymbol{w}}) \mathrm{d}\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \bar{\boldsymbol{w}} \, \mathrm{d}S_{\boldsymbol{x}} + o(\delta) R_1(\delta; \boldsymbol{u}_\delta, \bar{\boldsymbol{w}}) = 0 \ \forall \bar{\boldsymbol{w}} \in \mathcal{K}_\delta, \quad (3.22)$$

where

$$\begin{aligned}
&A_1(\eta; \boldsymbol{u}, \boldsymbol{v}) = \eta_{,1} \sigma_{ij}(\boldsymbol{u}) \frac{\partial}{\partial x_j} v_i - \sigma_{ij}(\boldsymbol{u}) E_{ij}(\eta; \boldsymbol{u}) - \sigma_{ij}(\boldsymbol{v}) E_{ij}(\eta; \boldsymbol{v}), \\
&E_{11}(\eta; \boldsymbol{u}) = \eta_{,1} u_{1,1}, \ \ E_{22}(\eta; \boldsymbol{u}) = \eta_{,2} u_{2,1}, \\
&E_{12}(\eta; \boldsymbol{u}) = E_{21}(\eta; \boldsymbol{u}) = \frac{1}{2}(\eta_{,1} u_{2,1} + \eta_{,2} u_{1,1}), \\
&|R_1(\delta; \boldsymbol{u}, \boldsymbol{w})| \le c \|\boldsymbol{u}\|_{H^1_{\Gamma_D}(\Omega)} \cdot \|\boldsymbol{w}\|_{H^1_{\Gamma_D}(\Omega)}, \\
&\mathcal{K}_\delta := \{\boldsymbol{v} \in H^1_{\Gamma_D}(\Omega) \mid \boldsymbol{v}|_\Gamma = (c_1 + c_0 x_2, c_2 - c_0 x_1 - \delta c_0 \eta(x_1, x_2))\}.
\end{aligned} \quad (3.23)$$

It is important to note that the variational equality (3.22) has the unique solution $\bar{\boldsymbol{u}}^\delta \in \mathcal{K}_\delta$. Furthermore, taking into account of the following chain of equalities

$$\inf_{\boldsymbol{v} \in \mathcal{K}^\delta} \Pi(\boldsymbol{v}) = \Pi(\boldsymbol{u}^{\boldsymbol{\delta}}) = \Pi_\delta(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}) = \inf_{\boldsymbol{v} \in \mathcal{K}_\delta} \Pi_\delta(\boldsymbol{v}), \quad (3.24)$$

the function $\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}$ minimizes the functional

$$\Pi_\delta(\boldsymbol{v}) = \frac{1}{2} \mathcal{E}_\Omega(\boldsymbol{v}, \boldsymbol{v}) + \frac{1}{2}\, \delta \int\limits_\Omega A_1(\eta; \boldsymbol{v}, \boldsymbol{v}) \mathrm{d}\boldsymbol{x} - \int_{\Gamma_N} \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}S_{\boldsymbol{x}} + o(\delta) R_1(\delta; \boldsymbol{v}, \boldsymbol{v}).$$

Lemma 3.2. *The solution $\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}$ of the variational equality* (3.22) *converges strongly to the solution of the variational equality* (3.3) *in $H^1_{\Gamma_D}(\Omega)$ as $\delta \to 0$. Moreover, it holds the estimate*

$$\|\bar{\boldsymbol{u}}^{\boldsymbol{\delta}} - \boldsymbol{u}\|_{H^1_{\Gamma_D}(\Omega)} \le c\sqrt{\delta}.$$

Proof. By substituting $\bar{\boldsymbol{w}} = \bar{\boldsymbol{u}}^{\boldsymbol{\delta}}$ into the variational equality (3.22) and using (3.23) we obtain

$$\|\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}\|_{H^1_{\Gamma_D}(\Omega)} \le c. \quad (3.25)$$

Then, we consider two auxiliary functions

$$\begin{aligned}
\boldsymbol{v}^{\boldsymbol{\delta 1}} &= (0, \alpha\eta), \\
\boldsymbol{v}^{\boldsymbol{\delta 2}} &= (0, \alpha^\delta \eta),
\end{aligned}$$

where $\boldsymbol{u}^{\boldsymbol{\delta}}|_{\Gamma_\delta} = \boldsymbol{\rho}^{\boldsymbol{\delta}} := (\beta_1^\delta + \alpha^\delta x_2, \beta_2^\delta - \alpha^\delta x_1)$. From (3.25) and the continuity of the trace operator we get

$$|\alpha^\delta| \le c, \ \ |\beta_i^\delta| \le c, \ \ i = 1, 2. \quad (3.26)$$

Since $\eta = 0$ on the boundary $\partial\Omega$, we have $\boldsymbol{v}^{\boldsymbol{\delta i}} \in H^1_{\Gamma_D}(\Omega)$ for $i = 1, 2$. Moreover, we have

$$\begin{aligned}
(\boldsymbol{u} - \delta \boldsymbol{v}^{\boldsymbol{\delta 1}})|_\Gamma &= (\beta_1 + \alpha x_2, \beta_2 - \alpha x_1 - \delta\alpha\eta), \\
(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}} + \delta \boldsymbol{v}^{\boldsymbol{\delta 2}})|_\Gamma &= (\beta_1^\delta + \alpha^\delta x_2, \beta_2^\delta - \alpha^\delta x_1).
\end{aligned}$$

It means that the following inclusions hold;

$$\boldsymbol{u} - \delta \boldsymbol{v}^{\boldsymbol{\delta 1}} \in \mathcal{K}_\delta, \quad \bar{\boldsymbol{u}}^{\boldsymbol{\delta}} + \delta \boldsymbol{v}^{\boldsymbol{\delta 2}} \in \mathcal{K}. \tag{3.27}$$

We substitute $\boldsymbol{u} - \bar{\boldsymbol{u}}^{\boldsymbol{\delta}} - \delta \boldsymbol{v}^{\boldsymbol{\delta 2}} \in \mathcal{K}$ and $\bar{\boldsymbol{u}}^{\boldsymbol{\delta}} - \boldsymbol{u} + \delta \boldsymbol{v}^{\boldsymbol{\delta 1}} \in \mathcal{K}_\delta$ in variational equalities (3.3) and (3.22), respectively, and combine the received identities. By (3.23), (3.25), (3.26) we obtain

$$\mathcal{E}_\Omega(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}} - \boldsymbol{u}, \bar{\boldsymbol{u}}^{\boldsymbol{\delta}} - \boldsymbol{u}) = O(\delta).$$

By virtue of Korn's inequality the statement in lemma is followed.

□

It follows from Lemma 3.2 that

$$\alpha^\delta \to \alpha, \ \beta_i^\delta \to \beta_i, i = 1, 2,$$

as $\delta \to 0$. Therefore,

$$\boldsymbol{v}^{\boldsymbol{\delta i}} \to \boldsymbol{p} := (0, \alpha\eta) \quad \text{strongly in} \quad H^1_{\Gamma_D}(\Omega), \ i = 1, 2. \tag{3.28}$$

Let us derive the derivative of the energy functional with respect to δ, that is,

$$\Pi'(\boldsymbol{u}) = \lim_{\delta \to 0+} \frac{\Pi(\boldsymbol{u}^\delta) - \Pi(\boldsymbol{u})}{\delta}.$$

Proposition 3.2. *The following formula is valid;*

$$\begin{aligned} \Pi'(\boldsymbol{u}) \quad = \quad & \frac{1}{2} \int_\Omega \eta_{,1} \sigma_{ij}(\boldsymbol{u}) \frac{\partial}{\partial x_j} v_i \mathrm{d}\boldsymbol{x} - \int_\Omega (\eta_{,1}\sigma_{i1}(\boldsymbol{u}) + \eta_{,2}\sigma_{i2}(\boldsymbol{u})) u_{i,1} \, \mathrm{d}\boldsymbol{x} \\ & - \alpha \int_\Omega (\sigma_{21}(\boldsymbol{u})\eta_{,1} + \sigma_{22}(\boldsymbol{u})\eta_{,2}) \, \mathrm{d}\boldsymbol{x}. \end{aligned} \tag{3.29}$$

Proof. Firstly, by (3.24) we have

$$\frac{\Pi(\boldsymbol{u}^{\boldsymbol{\delta}}) - \Pi(\boldsymbol{u})}{\delta} = \frac{\Pi_\delta(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}) - \Pi(\boldsymbol{u})}{\delta}. \tag{3.30}$$

On the other hand, by (3.24) and (3.27) we have the following chain of inequalities:

$$\frac{\Pi_\delta(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}) - \Pi(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}} + \delta \boldsymbol{v}^{\boldsymbol{\delta 2}})}{\delta} \le \frac{\Pi_\delta(\bar{\boldsymbol{u}}^{\boldsymbol{\delta}}) - \Pi(\boldsymbol{u})}{\delta} \le \frac{\Pi_\delta(\boldsymbol{u} - \delta \boldsymbol{v}^{\boldsymbol{\delta 1}}) - \Pi(\boldsymbol{u})}{\delta}. \tag{3.31}$$

Using Lemma 3.2 it is possible to pass to the limit as $\delta \to 0$ in the left and right sides in (3.31). Both limits are equal to the right side of the formulae (3.29). Therefore, the derivative $\Pi'(\boldsymbol{u})$ exists and is given by (3.29). The proof is completed.

□

Note that from the inclusion $\mathcal{K}^\delta \subset \mathcal{K}$ it follows that $\Pi'(\boldsymbol{u}) \ge 0$. Moreover, using standard arguments [21, 26] it can be shown that the derivative (3.29) does not depend on the function η. It implies that for any cut-off function η which is equal to 1 in D and satisfies (3.18), the value of the derivative (3.29) is the same. On the other hand, it means that the derivative (3.29) is independent of the domain D. Due to this fact it yields the invariant integral around the tip of the rigid inclusion.

Since $\eta = 1$ in D, we have $\eta_{,i} = 0$ in D, $i = 1, 2$. And since $\eta = 0$ outside of $\operatorname{supp} \eta$, we can consider the integrals in (3.29) over $\operatorname{supp} \eta \setminus (\overline{D} \cup \overline{\Gamma})$. By the interior and boundary regularity result of Lemma 3.1 we can apply the Green formula. Taking into account of the equilibrium equations we get

$$\Pi'(\boldsymbol{u}) = - \int_{\partial(\operatorname{supp} \eta \setminus (\overline{D} \cup \overline{\Gamma}))} \eta \left(\frac{1}{2} \sigma_{ij}(\boldsymbol{u}) n_1 \frac{\partial}{\partial x_j} u_i - \sigma_{ij}(\boldsymbol{u}) n_j u_{i,1} - \alpha \sigma_{2j}(\boldsymbol{u}) n_j \right) \mathrm{d}S_{\boldsymbol{x}},$$

where $\boldsymbol{n} = (n_1, n_2)$ is the unit internal normal vector to $\operatorname{supp}\eta \setminus (\overline{D} \cup \overline{\Gamma})$. Let divide $\partial(\operatorname{supp}\eta \setminus (\overline{D} \cup \overline{\Gamma}))$ into four parts: $\gamma_1 = \partial(\operatorname{supp}\eta)$, $\gamma_2 = \partial D$, $\gamma_3^{\pm} = (\operatorname{supp}\eta \setminus \overline{D}) \cap \Gamma^{\pm}$ and note that $\eta = 0$ on γ_1, $\eta = 1$ on γ_2, the vector $\boldsymbol{n} = (0, \pm 1)$ on $\gamma_3^{\pm}$. It follows from these conditions that

$$\begin{aligned} \Pi'(\boldsymbol{u}) &= -\int\limits_{\partial D} \left(\frac{1}{2}\sigma_{ij}(\boldsymbol{u}) n_1 \frac{\partial}{\partial x_j} u_i - \sigma_{ij}(\boldsymbol{u}) n_j u_{i,1} - \alpha\sigma_{2j}(\boldsymbol{u}) n_j \right) \mathrm{d}S_{\boldsymbol{x}} \\ &\quad + \int\limits_{\gamma_3^{\pm}} \eta(\pm\sigma_{2j}(\boldsymbol{u}) u_{j,1} \pm \alpha\sigma_{22}(\boldsymbol{u}))\, \mathrm{d}S_{\boldsymbol{x}}. \end{aligned} \tag{3.32}$$

Let us consider the last terms in (3.32). We have $\boldsymbol{u} = \boldsymbol{\rho}_0 := (\beta_1 + \alpha x_2, \beta_2 - \alpha x_1)$ on $\gamma_3^{\pm}$, then $u_{1,1} = 0$, $u_{2,1} = -\alpha$ on $\gamma_3^{\pm}$. Therefore, the last integral in (3.32) vanishes. Finally, since the derivative $\Pi'(\boldsymbol{u})$ does not depend on D, we get the invariant integral

$$J = -\int\limits_{S} \left(\frac{1}{2}\sigma_{ij}(\boldsymbol{u}) n_1 \frac{\partial}{\partial x_j} u_i - \sigma_{ij}(\boldsymbol{u}) n_j u_{i,1} - \alpha\sigma_{2j}(\boldsymbol{u}) n_j \right) \mathrm{d}S_{\boldsymbol{x}}, \tag{3.33}$$

where S is any closed curve around the tip $\boldsymbol{O}$ of the rigid inclusion which does not contains the second tip $\boldsymbol{P}$ inside of S.

Next we are in position to derive the Irwin's formula. By using the local polar coordinates we can rewrite the derivatives as

$$\begin{aligned} u_{,1}(x_1, x_2) &= \cos\theta u_{,r}(r,\theta) - \frac{\sin\theta}{r} u_{,\theta}(r,\theta), \\ u_{,2}(x_1, x_2) &= \sin\theta u_{,r}(r,\theta) + \frac{\cos\theta}{r} u_{,\theta}(r,\theta). \end{aligned}$$

Then the components of the stress tensor are given by

$$\begin{aligned} \sigma_{11}(\boldsymbol{u}) &= (2\mu + \tilde{\lambda}) \left(\cos\theta u_{1,r} - \frac{\sin\theta}{r} u_{1,\theta} \right) + \tilde{\lambda} \left(\sin\theta u_{2,r} + \frac{\cos\theta}{r} u_{2,\theta} \right), \\ \sigma_{12}(\boldsymbol{u}) = \sigma_{21}(\boldsymbol{u}) &= \mu \left(\sin\theta u_{1,r} + \frac{\cos\theta}{r} u_{1,\theta} + \cos\theta u_{2,r} - \frac{\sin\theta}{r} u_{2,\theta} \right), \\ \sigma_{22}(\boldsymbol{u}) &= (2\mu + \tilde{\lambda}) \left(\sin\theta u_{2,r} + \frac{\cos\theta}{r} u_{2,\theta} \right) + \tilde{\lambda} \left(\cos\theta u_{1,r} - \frac{\sin\theta}{r} u_{1,\theta} \right). \end{aligned}$$

Let a circumference of the radius $r < a'$ be the curve of integrating in (3.33). In this case, the invariant integral (3.33) takes the following form:

$$J = -\int\limits_{-\pi}^{\pi} r f(r,\theta)\, \mathrm{d}\theta, \tag{3.34}$$

where

$$
\begin{aligned}
&f(r,\theta) = \\
&\quad -\frac{1}{2}\Big((2\mu+\tilde{\lambda})\big(\cos\theta u_{1,r} - \frac{\sin\theta}{r}u_{1,\theta}\big) + \tilde{\lambda}\big(\sin\theta u_{2,r} + \frac{\cos\theta}{r}u_{2,\theta}\big)\Big)\Big(\cos\theta u_{1,r} - \frac{\sin\theta}{r}u_{1,\theta}\Big)\cos\theta \\
&\quad +\frac{1}{2}\Big((2\mu+\tilde{\lambda})\big(\sin\theta u_{2,r} + \frac{\cos\theta}{r}u_{2,\theta}\big) + \tilde{\lambda}\big(\cos\theta u_{1,r} - \frac{\sin\theta}{r}u_{1,\theta}\big)\Big)\Big(\sin\theta u_{2,r} + \frac{\cos\theta}{r}u_{2,\theta}\Big)\cos\theta \\
&\quad +\frac{\mu}{2}\Big(\sin\theta u_{1,r} + \frac{\cos\theta}{r}u_{1,\theta} + \cos\theta u_{2,r} - \frac{\sin\theta}{r}u_{2,\theta}\Big)\times \\
&\qquad \times\Big(\sin\theta u_{1,r} + \frac{\cos\theta}{r}u_{1,\theta} - \cos\theta u_{2,r} + \frac{\sin\theta}{r}u_{2,\theta}\Big)\cos\theta \\
&\quad -\mu\Big(\sin\theta u_{1,r} + \frac{\cos\theta}{r}u_{1,\theta} + \cos\theta u_{2,r} - \frac{\sin\theta}{r}u_{2,\theta}\Big)\Big(\cos\theta u_{1,r} - \frac{\sin\theta}{r}u_{1,\theta}\Big)\sin\theta \\
&\quad -\Big((2\mu+\tilde{\lambda})\big(\sin\theta u_{2,r} + \frac{\cos\theta}{r}u_{2,\theta}\big) + \tilde{\lambda}\big(\cos\theta u_{1,r} - \frac{\sin\theta}{r}u_{1,\theta}\big)\Big)\Big(\cos\theta u_{2,r} - \frac{\sin\theta}{r}u_{2,\theta}\Big)\sin\theta \\
&\quad -\alpha\mu\Big(\sin\theta u_{1,r} + \frac{\cos\theta}{r}u_{1,\theta} + \cos\theta u_{2,r} - \frac{\sin\theta}{r}u_{2,\theta}\Big)\cos\theta \\
&\quad -\alpha\Big((2\mu+\tilde{\lambda})\big(\sin\theta u_{2,r} + \frac{\cos\theta}{r}u_{2,\theta}\big) + \tilde{\lambda}\big(\cos\theta u_{1,r} - \frac{\sin\theta}{r}u_{1,\theta}\big)\Big)\sin\theta.
\end{aligned}
$$

The series in Proposition 3.1 converges uniformly in $B_{a'}$, then, substituting the convergent expansion in (3.34), after simple calculations, we get the Irwin's formula

$$J = \frac{\pi\tilde{\kappa}(\tilde{\lambda}+2\mu)}{2\mu(\tilde{\lambda}+\mu)}(P_1^2 + Q_1^2).$$

We see again that $J \geq 0$.

4 Problem 2 for the rigid line inclusion with delamination

In this section we consider Problem 2 for a model of rigid inclusion with delamination in a linear situation.

4.1 The weak solution and the regularity

For the boundary value problem (†), we introduce the set of admissible displacements

$$\mathcal{K}^- := \{\boldsymbol{v} \in H^1_{\Gamma_D}(\Omega \setminus \overline{\Gamma}) \mid \boldsymbol{v}^-|_\Gamma \in \mathcal{R}(\Gamma)\}.$$

Then, since $\mathcal{K}^-$ is weakly closed, in a similar way to Problem 1 one sees that there exists a solution $\boldsymbol{u} \in \mathcal{K}^-$ satisfying the minimization problem

$$\Pi(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in\mathcal{K}^-} \Pi(\boldsymbol{v}) \tag{4.1}$$

which implies that there exists a solution $\boldsymbol{u} \in \mathcal{K}^-$ such that for any $\boldsymbol{w} \in \mathcal{K}^-$

$$\mathcal{E}_\Omega(\boldsymbol{u},\boldsymbol{w}) - \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{w}\,\mathrm{d}S_{\boldsymbol{x}} = 0. \tag{4.2}$$

In the same manner as Problem 1 the solution $\boldsymbol{u} \in \mathcal{K}^-$ satisfies (3.5). Therefore, since $[\sigma_{i2}]_{\Gamma'\setminus\Gamma} = 0$ $(i = 1,2)$ and $T\boldsymbol{u}^+ = \boldsymbol{0}$, $\boldsymbol{v}^- = \boldsymbol{\rho}$ on Γ, the condition (3.6) holds in the sense (3.5).

Summing up the above, we have the existence theorem and the regularity result:

Theorem 4.1. *There exists a unique solution $\boldsymbol{u} \in \mathcal{K}^-$ of the minimization problem* (4.1)*.*

Lemma 4.1. *The solution $\boldsymbol{u} \in \mathcal{K}^-$ of* (4.1) *obeys the interior C^∞-regularity in $\Omega^{(1)}$ and $\Omega^{(2)}$. The boundary stress components σ_{i2}, $i = 1,2$ are pointwise functions inside Γ.*

4.2 The convergent expansion of the solution near O

In this section we derive convergent expansions of the solution constructed in Theorem 4.1. We use the same notations as in section 3.2. Goursat-Kolosov-Muskhelishvili stress functions are also defined in each $B_a^{(k)}$. Namely, from Lemma 3.1 and Poincaré lemma we obtain two holomorphic functions $\phi^{(k)}(z)$, $\omega^{(k)}(z)$ in $B_a^{(k)}$ $(k = 1, 2)$, $\phi^{(k)}(z), \omega^{(k)}(z) \in H^1(B_a^{(k)})$.

Next, taking into account the boundary conditions on Γ'', we derive the Riemann–Hilbert problem for the stress functions, similar to section 3.2.

First, since the displacement vector and the stress tensor are continuous on Γ_2, we can define sectionally holomorphic functions $\Phi(z)$, $\Psi(z)$ in B_a in the same forms as in section 3.2. However, note here that $\Phi(z)$ is not a holomorphic function in contrast to the one in section 3.2.

Second, we take into account the condition on Γ_1. From (3.10) the condition $T\boldsymbol{u}^+ = \mathbf{0}$ implies

$$\lim_{x_2 \to 0+} \left\{ \phi^{(1)'}(z) + \overline{\omega^{(1)'}(z)} \right\} = 0 \quad \text{on} \quad \Gamma_1.$$

Then it follows from (3.11) and (3.12) that

$$\lim_{x_2 \to 0+} \left\{ \Psi(z) - \tilde{\kappa}\Psi(\overline{z}) + \mu \left(\Phi(z) + \Phi(\overline{z}) \right) \right\} = 0 \quad \text{on} \quad \Gamma_1. \tag{4.3}$$

On the other hand, the condition $\boldsymbol{u}^- = \boldsymbol{\rho}_0 := (\beta_1 + \alpha x_2, \beta_2 - \alpha x_1)$ on Γ_1 leads to

$$\lim_{x_2 \to 0-} \left\{ \frac{\tilde{\kappa}}{2\mu} \cdot \frac{\mu}{\tilde{\kappa}+1} \left(\frac{1}{\mu}\Psi(z) + \Phi(z) \right) - \frac{1}{2\mu} \cdot \frac{\mu}{\tilde{\kappa}+1} \left(-\frac{\tilde{\kappa}}{\mu}\Psi(\overline{z}) + \Phi(\overline{z}) \right) \right\} = -\alpha i \quad \text{on} \quad \Gamma_1,$$

that is,

$$\lim_{x_2 \to 0-} \left\{ \tilde{\kappa} \left(\Psi(z) + \Psi(\overline{z}) \right) + \mu \left(-\Phi(z) + \tilde{\kappa}\Phi(\overline{z}) \right) \right\} = -2\mu(\tilde{\kappa}+1)\alpha i \quad \text{on} \quad \Gamma_1. \tag{4.4}$$

Conditions (4.3) and (4.4) yield that on Γ_1

$$\Psi(z) + \mu\Phi(\overline{z}) = -2\mu\alpha i, \tag{4.5}$$

$$\tilde{\kappa}\Psi(\overline{z}) - \mu\Phi(z) = -2\mu\alpha i. \tag{4.6}$$

Therefore, we need to solve a system of the Riemann– Hilbert problems (4.5) and (4.6) in this case.

Then, adding (4.5) multiplied by $\sqrt{\tilde{\kappa}} i$ to (4.6) multiplied by -1, we obtain the Riemann–Hilbert problem

$$\left\{ \sqrt{\tilde{\kappa}} i \Psi(z) + \mu\Phi(z) \right\} + \sqrt{\tilde{\kappa}} i \left\{ \sqrt{\tilde{\kappa}} i \Psi(\overline{z}) + \mu\Phi(\overline{z}) \right\} = -2\mu(\sqrt{\tilde{\kappa}} i - 1)\alpha i \quad \text{on} \quad \Gamma_1. \tag{4.7}$$

By using the same procedure as in section 3.2, the general solution of (4.7) can be given by

$$\sqrt{\tilde{\kappa}} i \Psi(z) + \mu\Phi(z) = z^{-\frac{1}{4} - \epsilon i}(z+a)^{-\frac{3}{4} + \epsilon i}\chi(z) - \frac{2\mu(\sqrt{\tilde{\kappa}} i - 1)}{1 + \sqrt{\tilde{\kappa}} i}\alpha i, \tag{4.8}$$

where $\chi(z)$ is holomorphic whole B_a and

$$\epsilon := \frac{\log \tilde{\kappa}}{4\pi}.$$

Next, adding (4.5) multiplied by $\sqrt{\tilde{\kappa}} i$ to (4.6), we obtain the Riemann–Hilbert problem

$$\left\{ \sqrt{\tilde{\kappa}} i \Psi(z) - \mu\Phi(z) \right\} - \sqrt{\tilde{\kappa}} i \left\{ \sqrt{\tilde{\kappa}} i \Psi(\overline{z}) - \mu\Phi(\overline{z}) \right\} = -2\mu(\sqrt{\tilde{\kappa}} i + 1)\alpha i \quad \text{on} \quad \Gamma_1. \tag{4.9}$$

The general solution of (4.9) can be given by

$$\sqrt{\tilde{\kappa}}i\Psi(z) - \mu\Phi(z) = z^{-\frac{3}{4}-\epsilon i}(z+a)^{-\frac{1}{4}+\epsilon i}\tilde{\chi}(z) - \frac{2\mu(\sqrt{\tilde{\kappa}}i+1)}{1-\sqrt{\tilde{\kappa}}i}\alpha i, \tag{4.10}$$

where $\tilde{\chi}(z)$ is holomorphic whole B_a.

Lastly, adding (4.8) to (4.10), and subtracting (4.10) from (4.8), we obtain

$$\Psi(z) = \frac{1}{2\sqrt{\tilde{\kappa}}i}z^{-\frac{1}{4}-\epsilon i}(z+a)^{-\frac{3}{4}+\epsilon i}\chi(z) + \frac{1}{2\sqrt{\tilde{\kappa}}i}z^{-\frac{3}{4}-\epsilon i}(z+a)^{-\frac{1}{4}+\epsilon i}\tilde{\chi}(z) - \frac{4\mu\alpha i}{\tilde{\kappa}+1}, \tag{4.11}$$

$$\Phi(z) = \frac{1}{2\mu}z^{-\frac{1}{4}-\epsilon i}(z+a)^{-\frac{3}{4}+\epsilon i}\chi(z) - \frac{1}{2\mu}z^{-\frac{3}{4}-\epsilon i}(z+a)^{-\frac{1}{4}+\epsilon i}\tilde{\chi}(z) - \frac{2(\tilde{\kappa}-1)\alpha i}{\tilde{\kappa}+1}. \tag{4.12}$$

Resetting $\chi(z)$ and $\tilde{\chi}(z)$ defined in $B_{a'}$ with $a' < a$ and substituting into (3.11) and (3.12) give

$$\phi^{(k)\prime}(z) = z^{-\frac{1}{4}-\epsilon i}\chi(z) + z^{-\frac{3}{4}-\epsilon i}\tilde{\chi}(z) - \frac{2\mu\alpha i}{\tilde{\kappa}+1}, \tag{4.13}$$

$$\omega^{(k)\prime}(z) = -\sqrt{\tilde{\kappa}}iz^{-\frac{1}{4}+\epsilon i}\overline{\chi(\overline{z})} + \sqrt{\tilde{\kappa}}iz^{-\frac{3}{4}+\epsilon i}\overline{\tilde{\chi}(\overline{z})} - \frac{2\mu\alpha i}{\tilde{\kappa}+1}. \tag{4.14}$$

Since $\chi(z)$ and $\tilde{\chi}(z)$ are holomorphic whole $B_{a'}$, they can be written as local Taylor series expansions

$$\chi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \tilde{\chi}(z) = \sum_{n=0}^{\infty} b_n z^n, \tag{4.15}$$

which are generalized uniform convergent in $B_{a'}$. Moreover, since the coefficients a_n, b_n can be given by

$$a_n = \frac{1}{2\pi i}\int_{|w|=r}\frac{\chi(w)}{w^{n+1}}\,\mathrm{d}w \quad \text{and} \quad b_n = \frac{1}{2\pi i}\int_{|w|=r}\frac{\tilde{\chi}(w)}{w^{n+1}}\,\mathrm{d}w$$

for $0 < r < a'$, by virtue of Cauchy–Schwarz inequality it is easy to verify the following estimates

$$|a_n| \le c\sqrt{4n+3}\,(a')^{-(n+\frac{3}{4})}\|\chi\|_{L^2(B_{a'})},$$
$$|b_n| \le c\sqrt{4n+1}\,(a')^{-(n+\frac{1}{4})}\|\tilde{\chi}\|_{L^2(B_{a'})}.$$

Now we set

$$c_m = \begin{cases} \frac{b_n}{n+\frac{1}{4}-\epsilon i} & (m = 2n+1), \\ \frac{a_n}{n+\frac{3}{4}-\epsilon i} & (m = 2(n+1)), \end{cases}$$

By substituting (4.13) and (4.14) into (3.8) and using (4.15) we obtain the convergent expansion of $\boldsymbol{u}$ near $\boldsymbol{O}$.

Proposition 4.1. *There exist a complex numbers c_m and real constants α, β_1, β_2 such that*

$$\boldsymbol{u}(r,\theta) = \sum_{m=1}^{\infty}\frac{1}{2\mu}\mathbf{Re}[c_m r^{-\epsilon i}]r^{\frac{2m-1}{4}}e^{\epsilon\theta}\boldsymbol{R}_{2,m}(\theta) - \frac{1}{2\mu}\mathbf{Im}[c_m r^{-\epsilon i}]r^{\frac{2m-1}{4}}e^{\epsilon\theta}\boldsymbol{S}_{2,m}(\theta) + \boldsymbol{F_2},$$

where

$$\boldsymbol{R}_{2,m}(\theta) =$$
$$\begin{pmatrix} \left(\tilde{\kappa} + \frac{2m-1}{4}\right)\cos\frac{2m-1}{4}\theta - \left((-1)^m\sqrt{\tilde{\kappa}}e^{-2\epsilon\theta} - \epsilon\right)\sin\frac{2m-1}{4}\theta - \frac{2m-1}{4}\cos\frac{2m-9}{4}\theta - \epsilon\sin\frac{2m-9}{4}\theta \\ \left(\tilde{\kappa} - \frac{2m-1}{4}\right)\sin\frac{2m-1}{4}\theta - \left((-1)^m\sqrt{\tilde{\kappa}}e^{-2\epsilon\theta} - \epsilon\right)\cos\frac{2m-1}{4}\theta + \frac{2m-1}{4}\sin\frac{2m-9}{4}\theta - \epsilon\cos\frac{2m-9}{4}\theta \end{pmatrix},$$

$$\boldsymbol{S}_{2,m}(\theta) =$$
$$\begin{pmatrix} \left(\tilde{\kappa}+\frac{2m-1}{4}\right)\sin\frac{2m-1}{4}\theta - \left((-1)^m\sqrt{\tilde{\kappa}}e^{-2\epsilon\theta}+\epsilon\right)\cos\frac{2m-1}{4}\theta - \frac{2m-1}{4}\sin\frac{2m-9}{4}\theta + \epsilon\cos\frac{2m-9}{4}\theta \\ -\left(\tilde{\kappa}-\frac{2m-1}{4}\right)\cos\frac{2m-1}{4}\theta + \left((-1)^m\sqrt{\tilde{\kappa}}e^{-2\epsilon\theta}+\epsilon\right)\sin\frac{2m-1}{4}\theta - \frac{2m-1}{4}\cos\frac{2m-9}{4}\theta - \epsilon\sin\frac{2m-9}{4}\theta \end{pmatrix},$$

$$\boldsymbol{F_2} = \begin{pmatrix} \alpha r\sin\theta + \beta_1 \\ -\alpha r\cos\theta + \beta_2 \end{pmatrix}.$$

The series are convergent, absolutely in $H^1(B_{a'})$ and generalized uniform in $B_{a'}$. For $m \geq 1$, c_m satisfies

$$|c_m| \leq c\frac{1}{\sqrt{2m-1}}\,(a')^{-\frac{m}{2}+\frac{1}{4}}\|\nabla\boldsymbol{u}\|_{L^2(B_{a'})}.$$

Note that it follows form Proposition 4.1 that

$$\begin{aligned}
&\sigma_{12}|_{\theta=\pi} - \sigma_{12}|_{\theta=-\pi} = 0 - \sigma_{12}|_{\theta=-\pi} \\
&= \sum_{m=1}^{\infty}\tilde{\kappa}^{-\frac{1}{4}}(\tilde{\kappa}+1)r^{\frac{2m-5}{4}}\Big\{\mathbf{Re}[c_m r^{-\epsilon i}]\left(\epsilon\cos\frac{2m-1}{4}\pi + \frac{2m-1}{4}\sin\frac{2m-1}{4}\pi\right) \\
&\qquad + \mathbf{Im}[c_m r^{-\epsilon i}]\left(\epsilon\sin\frac{2m-1}{4}\pi - \frac{2m-1}{4}\cos\frac{2m-1}{4}\pi\right)\Big\} \\
&= \sum_{m=1}^{\infty}\tilde{\kappa}^{-\frac{1}{4}}(\tilde{\kappa}+1)r^{\frac{2m-5}{4}} \times \\
&\qquad \times\Big\{\mathbf{Re}[c_m]\left(\epsilon\cos\left(\epsilon\log r + \frac{2m-1}{4}\pi\right) + \frac{2m-1}{4}\sin\left(\epsilon\log r + \frac{2m-1}{4}\pi\right)\right) \\
&\qquad + \mathbf{Im}[c_m]\left(\epsilon\sin\left(\epsilon\log r + \frac{2m-1}{4}\pi\right) - \frac{2m-1}{4}\cos\left(\epsilon\log r + \frac{2m-1}{4}\pi\right)\right)\Big\},
\end{aligned}$$

$$\begin{aligned}
&\sigma_{22}|_{\theta=\pi} - \sigma_{22}|_{\theta=-\pi} = 0 - \sigma_{22}|_{\theta=-\pi} \\
&= \sum_{m=1}^{\infty}\tilde{\kappa}^{-\frac{1}{4}}(\tilde{\kappa}+1)r^{\frac{2m-5}{4}}\Big\{\mathbf{Re}[c_m r^{-\epsilon i}]\left(-\epsilon\sin\frac{2m-1}{4}\pi + \frac{2m-1}{4}\cos\frac{2m-1}{4}\pi\right) \\
&\qquad + \mathbf{Im}[c_m r^{-\epsilon i}]\left(\epsilon\cos\frac{2m-1}{4}\pi + \frac{2m-1}{4}\sin\frac{2m-1}{4}\pi\right)\Big\}, \\
&= \sum_{m=1}^{\infty}\tilde{\kappa}^{-\frac{1}{4}}(\tilde{\kappa}+1)r^{\frac{2m-5}{4}} \times \\
&\qquad \times\Big\{\mathbf{Re}[c_m]\left(-\epsilon\sin\left(\epsilon\log r + \frac{2m-1}{4}\pi\right) + \frac{2m-1}{4}\cos\left(\epsilon\log r + \frac{2m-1}{4}\pi\right)\right) \\
&\qquad + \mathbf{Im}[c_m]\left(\epsilon\cos\left(\epsilon\log r + \frac{2m-1}{4}\pi\right) + \frac{2m-1}{4}\sin\left(\epsilon\log r + \frac{2m-1}{4}\pi\right)\right)\Big\}.
\end{aligned}$$

4.3 The Irwin's formula

Analogous to section 3.3, we derive the Irwin's formula in the delamination case. It is easy to see that we get the invariant integral which has the same expression as (3.33) also in this case. Then, substituting the series expansion in Proposition 4.1 into (3.34), we have the Irwin's formula

$$J = -\frac{\pi(\tilde{\lambda}+2\mu)\sqrt{\tilde{\kappa}}}{4\mu(\tilde{\lambda}+\mu)}\left\{(3+16\epsilon^2)\mathbf{Im}[c_1\overline{c_2}] - 8\epsilon\mathbf{Re}[c_1\overline{c_2}]\right\}.$$

5 Conclusion

We derived the complete asymptotic expansions of the displacement near the tip of the rigid line inclusion without and with delamination, written in Proposition 3.1 and 4.1, respectively. It gets the exact forms with respect to the distance to the crack tip as well as the explicit expression of the angular functions around there. Moreover, the convergence proof of expansions is obtained for an arbitrary solution. Simultaneously, it enables us to have an a priori regularity of the solution, that is, it follows from Proposition 3.1 and 4.1 that generally, $\boldsymbol{u} \notin H^{3/2}(B_{a'})$ in both cases. Proposition 3.1 and 4.1 also showed that regularities of the solutions are the same as ones in the case of Laplace equation under assumptions of Dirichlet conditions and mixed conditions on the line Γ, respectively. However, we saw that the difference is appearance of the oscillating singularity in Proposition 4.1.

And using Proposition 3.1 and 4.1 we also show that the invariant integral (3.33) is expressed only by coefficients of singular terms in the expansions, respectively.

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