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ON THE TOPOLOGICAL DERIVATIVE DUE TO KINK OF A CRACK WITH NON-PENETRATION

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ABSTRACT. We define a topological derivative caused by kinking of a crack, thus, representing the topology change. Using variational methods, the objective function of the potential energy is expanded with respect to an incipient crack branch. For the sensitivity analysis we provide a Saint-Venant principle, and we decompose the solution of a model problem in the Fourier series.

INTRODUCTION

The problem of kinking and determining of the direction in which a crack will propagate is the subject for discussion in the literature on fracture mechanics, see [2], [4], [8], [24], [26]. There is no explicit formulas of the kink angle even in the simple, linear setting of crack problems. We investigate a non-linear model problem for the crack subject to non-penetration conditions [13]. To emphasize the main difficulties arising here, in the paper we rely on a model scalar-valued problem and on a piecewise-linear path of the crack. We are aimed to derive an expansion of the objective function of the potential energy with respect to an incipient crack branch.

Our task lies within the general framework of structure optimization. To account the common approaches adopted in shape and topology optimization, we refer to [1], [7], [11], [19], [25]. In spite of the known techniques, singular character of solutions dealing with cracked geometries requires always separate investigation. In the crack context, variational methods of the shape sensitivity analysis were developed in

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[13], [16], [17], [20], [21], [22], and other works by the authors. For the appropriate numerical methods, see [12], [28].

Utilizing the optimization approach due to [6], evolution of a crack with kink and non-penetration is described recently in [14]. The evolution process implies that it is global in time. Nevertheless, local characteristics at the time, when a kink occurs, are of especial interest. Indeed, the kinking phenomenon implies arrest of the tangential movement along the pre-kinked crack and appearance of a different branch at the point of kink. In this sense, it is close to the phenomena of crack splitting into few branches as well as appearance of a crack-like defect in a continuum. From a geometric viewpoint, these features present the change of topology.

Treating changes of topology is the key difficulty in the structure analysis and optimization. The generic change of topology due to creating infinitesimal holes in a continuum was introduced successfully in the works [9], [27]. The mathematical formalism exploits a respective topological derivative of the objective function when the hole diminishes. The topological sensitivity based approach was tested numerically for the problem of crack identifiability in [3]. However, the disadvantage concerns the fact that the topological derivative was defined for a-priori smooth geometries only. Pre-described cracks are not the case. By these reasons, in the paper we adapt the notation of the topological derivative specifically for the phenomenon of kink.

Denoting by r and ϕ the length of a branch and its angle with the pre-kinked path of a crack, respectively, we will consider the objective function (of the potential energy) $r \mapsto \Pi(r, \phi) : (0, R) \mapsto \mathbb{R}$ for arbitrary fixed $\phi \in (-\pi, \pi)$. Under suitable regularity assumptions, it can be decomposed as

$$(A) \quad \Pi(r, \phi) = \Pi(0) + \int_0^r \Pi'(t, \phi) dt,$$

where $\Pi(0) := \Pi(0, \phi)$ does not depend on ϕ , and the shape derivatives

$$(B) \quad \Pi'(r, \phi) := \lim_{s \rightarrow 0} \left\{ \frac{1}{s} (\Pi(r+s, \phi) - \Pi(r, \phi)) \right\}$$

are well defined for $r > 0$ by the smooth velocity arguments of [14]. Restating a Saint-Venant principle (see [5], [18]) for the constrained crack problem, we obtain the uniform estimate

$$(C) \quad \int_0^r \Pi'(t, \phi) dt = O(r).$$

In the general case, (C) admits bounded oscillations when $r \rightarrow 0$. For particular cases it can be specified in more details. If an expansion holds

$$(D) \quad \int_0^r \Pi'(t, \phi) dt = r\Pi'(0, \phi) + o(r),$$

then we can define a *topological derivative* (for $\phi \neq 0$) as the first asymptotic term in (D). Obviously, from (A) and (D) it follows that

$$(E) \quad \Pi'(0, \phi) = \lim_{r \rightarrow 0} \left\{ \frac{1}{r} (\Pi(r, \phi) - \Pi(0, \phi)) \right\}.$$

Note that according to (B) and (E), generally speaking,

$$\Pi'(0, \phi) \neq \lim_{r \rightarrow 0} \Pi'(r, \phi)$$

in view of interchanging the limits. Using a Fourier series arguments of [10], [15], [23], for the linearized crack problem we will specify these expressions in the terms of stress intensity factors, which are of the first importance for engineers.

1. FORMULATION OF THE MODEL PROBLEM WITH CRACK

Let Ω be a bounded domain in \mathbb{R}^2 with Lipschitz boundary $\partial\Omega$ and normal vector q at $\partial\Omega$. We assume that the origin O of a Cartesian coordinate system $x = (x_1, x_2) \in \mathbb{R}^2$ is located strictly inside Ω . Denoting with B_δ a ball of radius $\delta > 0$ centered at O , this assumes that $R > 0$ exists such that $B_R \subset \Omega$.

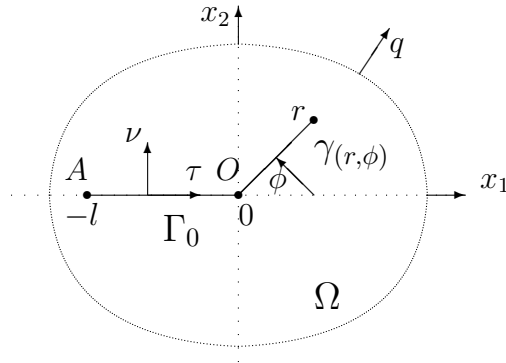


FIGURE 1. Example configuration of the kinked crack $\Gamma_{(r,\phi)}$.

We consider the reference crack Γ_0 as a segment AO of length $l > 0$ posed in Ω along the x_1 -axis. Its right end-point lying at the origin O

will be associated with the point of kink. We specify shape parameters $r \in [0, R]$ and $\phi \in (-\pi, \pi)$ such that kinked cracks $\Gamma_{(r,\phi)}$ will be formed by two parts: fixed one Γ_0 and varying branches $\gamma_{(r,\phi)}$. The branch $\gamma_{(r,\phi)}$ is assumed to be a rectilinear segment of the length r starting from O with the kink of angle ϕ counter-clockwise from the x_1 -axis. An example configuration is illustrated in Figure 1. The tangential vector τ and the normal vector ν at $\Gamma_{(r,\phi)}$ are:

$$\begin{cases} \tau(0) = (1, 0), & \nu(0) = (0, 1) & \text{on } \Gamma_0, \\ \tau(\phi) = (\cos \phi, \sin \phi), & \nu(\phi) = (-\sin \phi, \cos \phi) & \text{on } \gamma_{(r,\phi)}. \end{cases}$$

With $\Omega_{(r,\phi)}$ we denote $\Omega \setminus \Gamma_{(r,\phi)}$. For the further use we fix the radius $0 < R < l$ of the ball B_R inscribed in Ω , thus the left end-point A is located outside of B_R .

Starting modeling we fix r and ϕ . To model a solid which occupies the domain with crack $\Omega_{(r,\phi)}$, we rely on the scalar-valued setting of the problem.

Let $\partial\Omega$ consist of two parts Γ_N and Γ_D such that $\text{meas}(\Gamma_D) > 0$. The volume force $f \in C^1(\bar{\Omega})$ and the boundary traction $g \in L^2(\Gamma_N)$ are given. For $x \in \bar{\Omega}_{(r,\phi)}$, unknown displacements $u(x)$ are assumed to be zero at Γ_D . Along the crack, they are restricted by non-penetration conditions due to possible contact between the opposite crack faces:

$$(1) \quad \llbracket u \rrbracket := u|_{\Gamma_{(r,\phi)}^+} - u|_{\Gamma_{(r,\phi)}^-} \geq 0 \quad \text{on } \Gamma_{(r,\phi)}.$$

The positive $\Gamma_{(r,\phi)}^+$ and the negative $\Gamma_{(r,\phi)}^-$ faces can be distinguished geometrically as the limit of points x going to $\Gamma_{(r,\phi)}$ "from above" and "from below", respectively.

The potential energy of a solid is represented by the domain-dependent functional

$$(2) \quad \Pi(u; \Omega_{(r,\phi)}) = \frac{1}{2} \int_{\Omega_{(r,\phi)}} |\nabla u|^2 dx - \int_{\Omega_{(r,\phi)}} f u dx - \int_{\Gamma_N} g u dx$$

defined over the Sobolev space

$$(3) \quad H(\Omega_{(r,\phi)}) = \{u \in H^1(\Omega_{(r,\phi)}) : u = 0 \quad \text{on } \Gamma_D\}.$$

It is equipped with the norm

$$(4) \quad \|u\|_{H(\Omega_{(r,\phi)})}^2 = \int_{\Omega_{(r,\phi)}} |\nabla u|^2 dx,$$

which is equivalent to the standard H^1 -norm due to the Dirichlet boundary condition at Γ_D . The non-penetration condition (1) accounts

for the set of admissible displacements

$$(5) \quad K(\Omega_{(r,\phi)}) = \{u \in H(\Omega_{(r,\phi)}) : \llbracket u \rrbracket \geq 0 \text{ on } \Gamma_{(r,\phi)}\},$$

which is a convex cone in $H(\Omega_{(r,\phi)})$.

The equilibrium of a solid with crack is described by the following constrained minimization problem: Find $u^{(r,\phi)} \in K(\Omega_{(r,\phi)})$ such that

$$(6) \quad \Pi(u^{(r,\phi)}; \Omega_{(r,\phi)}) \leq \Pi(v; \Omega_{(r,\phi)}) \quad \text{for all } v \in K(\Omega_{(r,\phi)}).$$

Optimality conditions for (6) are expressed by the variational inequality

$$(7) \quad \begin{aligned} & \int_{\Omega_{(r,\phi)}} \nabla u^{(r,\phi)} \cdot \nabla (v - u^{(r,\phi)}) dx \geq \int_{\Omega_{(r,\phi)}} f(v - u^{(r,\phi)}) dx \\ & + \int_{\Gamma_N} g(v - u^{(r,\phi)}) dx \quad \text{for all } v \in K(\Omega_{(r,\phi)}). \end{aligned}$$

By the Lax–Milgram theorem, there exists the unique solution to problem (6), equivalently, (7). Variational inequality (7) describes a weak solution to the boundary-value problem:

$$(8a) \quad -\Delta u^{(r,\phi)} = f \quad \text{in } \Omega_{(r,\phi)},$$

$$(8b) \quad u^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial u^{(r,\phi)}}{\partial \nu} = g \quad \text{on } \Gamma_N,$$

$$(8c) \quad \begin{aligned} & \left[\frac{\partial u^{(r,\phi)}}{\partial \nu} \right] = 0, \quad \frac{\partial u^{(r,\phi)}}{\partial \nu} \leq 0, \\ & \llbracket u^{(r,\phi)} \rrbracket \geq 0, \quad \frac{\partial u^{(r,\phi)}}{\partial \nu} \llbracket u^{(r,\phi)} \rrbracket = 0 \quad \text{on } \Gamma_{(r,\phi)}. \end{aligned}$$

To give an exact sense to the boundary terms in (8c), we introduce a Lions–Magenes space $H_{00}^{1/2}(\Gamma_{(r,\phi)})$ which is equipped with the norm:

$$(9) \quad \begin{aligned} \|u\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)})}^2 &= \|u\|_{H^{1/2}(\Gamma_{(r,\phi)})}^2 + \int_{\Gamma_{(r,\phi)}} \frac{u^2(x(s))}{\text{dist}(x(s), \partial\Gamma_{(r,\phi)})} dx(s), \\ &= \int_{\Gamma_{(r,\phi)}} u^2(x(s)) dx(s) + \int_{\Gamma_{(r,\phi)}} \int_{\Gamma_{(r,\phi)}} \frac{|u(x(t)) - u(x(s))|^2}{|x(t) - x(s)|^2} dx(s) dx(t) \end{aligned}$$

for parameters $s, t \in (-l, r)$ of the length along the crack $\Gamma_{(r,\phi)}$. The function $\text{dist}(x(s), \partial\Gamma_{(r,\phi)})$ of distance from $x(s)$ to the crack end-points $\partial\Gamma_{(r,\phi)}$ has the order $l + s$ as $s \rightarrow -l$, and $r - s$ as $s \rightarrow r$. With

$H_{00}^{1/2}(\Gamma_{(r,\phi)})^*$ we denote the formally dual space to $H_{00}^{1/2}(\Gamma_{(r,\phi)})$. Then the following proposition holds, see [13] for its detailed proof.

Proposition 1. *The solution of (7) possesses the properties:*

$$(10) \quad \begin{aligned} u^{(r,\phi)} &\in K(\Omega_{(r,\phi)}), \quad \Delta u^{(r,\phi)} \in L^2(\Omega_{(r,\phi)}), \\ \llbracket u^{(r,\phi)} \rrbracket &\in H_{00}^{1/2}(\Gamma_{(r,\phi)}), \quad \frac{\partial u^{(r,\phi)}}{\partial \nu} \in H_{00}^{1/2}(\Gamma_{(r,\phi)})^*, \end{aligned}$$

and the solution is H^2 -smooth up to $\Gamma_{(r,\phi)}^\pm$ apart from the kink and end points of the crack.

In what follows we keep the kink angle ϕ fixed and pass the branch length r to zero. For $r = 0$, data of the crack problem do not depend on ϕ , so we exclude ϕ from notation for simplicity:

$$\Gamma_{(0,\phi)} = \Gamma_0, \quad \Omega_{(0,\phi)} = \Omega_0, \quad u^{(0,\phi)} = u^0.$$

The reference solution $u^0 \in K(\Omega_0)$ minimizes

$$(11) \quad \Pi(u^0; \Omega_0) \leq \Pi(v; \Omega_0) \quad \text{for all } v \in K(\Omega_0),$$

or, equivalently, satisfies the variational inequality

$$(12) \quad \begin{aligned} \int_{\Omega_0} \nabla u^0 \cdot \nabla (v - u^0) dx &\geq \int_{\Omega_0} f(v - u^0) dx \\ + \int_{\Gamma_N} g(v - u^0) dx &\quad \text{for all } v \in K(\Omega_0). \end{aligned}$$

It describes a weak solution to the respective boundary-value problem:

$$(13a) \quad -\Delta u^0 = f \quad \text{in } \Omega_0,$$

$$(13b) \quad u^0 = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial u^0}{\partial q} = g \quad \text{on } \Gamma_N,$$

$$(13c) \quad \left[\left[\frac{\partial u^0}{\partial \nu} \right] \right] = 0, \quad \frac{\partial u^0}{\partial \nu} \leq 0, \quad \llbracket u^0 \rrbracket \geq 0, \quad \frac{\partial u^0}{\partial \nu} \llbracket u^0 \rrbracket = 0 \quad \text{on } \Gamma_0.$$

Firstly, we find the shape derivative at finite $r > 0$ defined as

$$(14) \quad \Pi'(r, \phi) = \lim_{s \rightarrow 0} \frac{\Pi(u^{(r+s,\phi)}; \Omega_{(r+s,\phi)}) - \Pi(u^{(r,\phi)}; \Omega_{(r,\phi)})}{s}.$$

With the help of (14), second we restate the *topological derivative* as the following limit at $r = 0$ (if it exists)

$$(15) \quad \Pi'(0, \phi) = \lim_{r \rightarrow 0} \left(\frac{1}{r} \int_0^r \Pi'(t, \phi) dt \right).$$

It expresses the derivative of energy at the kink point in the direction of $\tau(\phi)$.

2. DERIVATIVE OF THE ENERGY FUNCTIONAL AT $r > 0$

For fixed $r \in (0, R)$ we can apply the regular perturbation arguments. For this reason, we construct a time-independent velocity

$$(16) \quad V(x) = x\eta(x) \in W^{1,\infty}(\mathbb{R}^2)^2, \quad V = 0 \quad \text{on } \partial\Omega,$$

where $x \mapsto \eta : \mathbb{R}^2 \rightarrow [0, 1]$ is a suitable cut-off function supported in Ω such that $\eta \equiv 1$ in a ball B_δ of radius $\delta \in (r, R)$ and $\eta \equiv 0$ outside B_R . The usual solvability arguments provide existence of a unique solution

$$(17) \quad \Phi(t, x) \in C^1([-T, T]; W^{1,\infty}(\Omega))^2, \quad T > 0,$$

of the Cauchy problem for a nonlinear ODE

$$(18) \quad \frac{d}{dt}\Phi(t, \cdot) = V(\Phi(t, \cdot)) \text{ for } t \neq 0, \quad \Phi(0, x) = x.$$

Since (18) is an autonomous system, we obtain the identities

$$(19) \quad \Phi(-t, \Phi(t, x)) = \Phi(t, \Phi(-t, x)) = x$$

implying that $\Phi(-t, x)$ is an inverse function to $\Phi(t, x)$. In the ball B_δ where $\eta \equiv 1$, the solution to (18) can be calculated analytically as

$$(20) \quad \Phi(t, x) = x e^t \quad \text{when } x e^t \in B_\delta.$$

Relations (16)–(20) argue the following proposition, see [14] for details.

Proposition 2. *There exists $T > 0$ such that, for $t \in [-T, T]$, the coordinate extention $y = \Phi(t, x)$ yields a bijective mapping between the domains $\Omega_{(r,\phi)}$ and $\Omega_{(re^t,\phi)}$, and between sets (5) in the following sense:*

$$(21) \quad \begin{aligned} &\text{if } u \in K(\Omega_{(r,\phi)}), \text{ then } u \circ \Phi(-t) \in K(\Omega_{(re^t,\phi)}); \\ &\text{if } u \in K(\Omega_{(re^t,\phi)}), \text{ then } u \circ \Phi(t) \in K(\Omega_{(r,\phi)}). \end{aligned}$$

Next, we rewrite the potential energy functional (2) in the perturbed domain $\Omega_{(re^t,\phi)}$

$$(22) \quad \Pi(u; \Omega_{(re^t,\phi)}) = \frac{1}{2} \int_{\Omega_{(re^t,\phi)}} |\nabla u|^2 dx - \int_{\Omega_{(re^t,\phi)}} fu dx - \int_{\Gamma_N} gu dx$$

for $u \in H(\Omega_{(re^t,\phi)})$

and expand it in small $t \rightarrow 0$. In fact, transformation $y = \Phi(t, x)$ applied to (22) yields

$$(23) \quad \Pi(u; \Omega_{(re^t,\phi)}) = \Pi \circ \Phi(t)(u \circ \Phi(t); \Omega_{(r,\phi)}) \quad \text{for } u \in H(\Omega_{(re^t,\phi)})$$

with the perturbed functional

$$\begin{aligned}
& \Pi \circ \Phi(t)(u; \Omega_{(r,\phi)}) \\
&= \frac{1}{2} \int_{\Omega_{(r,\phi)}} (\nabla u)^\top \frac{\partial \Phi^{-1}}{\partial x}(t) \left(\frac{\partial \Phi^{-1}}{\partial x}(t) \right)^\top \nabla u \left| \det \left(\frac{\partial \Phi}{\partial x}(t) \right) \right| dx \\
(24) \quad & - \int_{\Omega_{(r,\phi)}} f \circ \Phi(t) u \left| \det \left(\frac{\partial \Phi}{\partial x}(t) \right) \right| dx - \int_{\Gamma_N} g u dx \\
& \text{for } u \in H(\Omega_{(r,\phi)}).
\end{aligned}$$

Using the asymptotic formula due to (18)

$$(25) \quad \Phi(t, x) = x + tV(x) + \text{Res}_t, \quad \|\text{Res}_t\|_{W^{1,\infty}(\Omega)^2} = o(t),$$

differentiating (25) with respect to x , and substituting the result into (24) we derive the asymptotic expansion

$$\begin{aligned}
(26) \quad & \Pi \circ \Phi(t)(u; \Omega_{(r,\phi)}) = \Pi(u; \Omega_{(r,\phi)}) + t \Pi_V^1(u, u, f; \Omega_{(r,\phi)}) + \text{Res}_t(u), \\
& |\text{Res}_t(u)| \leq c(t) (\|u\|_{H(\Omega_{(r,\phi)})}^2 + \text{const}), \quad 0 \leq c(t) = o(t).
\end{aligned}$$

With Res we denote respective residuals. The first asymptotic term is associated to a quadratic form:

$$\begin{aligned}
(27) \quad & \Pi_V^1(u, v, f; \Omega_{(r,\phi)}) = \frac{1}{2} \int_{\Omega_{(r,\phi)}} \nabla u \cdot \left(\text{div}(V) I - \frac{\partial V}{\partial x} - \frac{\partial V^\top}{\partial x} \right) \nabla v dx \\
& - \int_{\Omega_{(r,\phi)}} \text{div}(Vf)v dx \quad \text{for } u, v \in H(\Omega_{(r,\phi)}).
\end{aligned}$$

Note, that (27) is not symmetric with respect to u and v in the latter, linear term. For the detailed derivation of (26), see, for instance, [20].

Proposition 3. *Since (21) and (26) hold true for problem (6), the directional derivative (14) exists, and it can be expressed by formula*

$$(28) \quad 0 \geq \Pi'(r, \phi) = \frac{1}{r} \Pi_V^1(u^{(r,\phi)}, u^{(r,\phi)}, f; \Omega_{(r,\phi)}).$$

Proof. The complete proof is given in [14], we sketch it briefly.

We consider the perturbed problem (6) for $u^{(re^t, \phi)} \in K(\Omega_{(re^t, \phi)})$ such that minimizes

$$(29) \quad \Pi(u^{(re^t, \phi)}; \Omega_{(re^t, \phi)}) \leq \Pi(v; \Omega_{(re^t, \phi)}) \quad \text{for all } v \in K(\Omega_{(re^t, \phi)}).$$

With the help of (21) and (22), it follows from (29) that $u^{(re^t, \phi)} \circ \Phi(t) \in K(\Omega_{(r, \phi)})$ is the unique solution minimizing

$$(30) \quad \begin{aligned} \Pi \circ \Phi(t)(u^{(re^t, \phi)} \circ \Phi(t); \Omega_{(r, \phi)}) &\leq \Pi \circ \Phi(t)(v; \Omega_{(r, \phi)}) \\ &\text{for all } v \in K(\Omega_{(r, \phi)}). \end{aligned}$$

Substituting u^0 as a test function into (30) results in the uniform estimate due to (26):

$$\|u^{(re^t, \phi)} \circ \Phi(t)\|_{H(\Omega_{(r, \phi)})}^2 \leq c_0 + c_1 \|u^0\|_{H(\Omega_{(r, \phi)})}^2 + O(t).$$

Hence, a subsequence of $u^{(re^t, \phi)} \circ \Phi(t)$ exists which converges weakly to $u^{(r, \phi)}$. By the usual arguments of monotone operators we arrive at

$$(31) \quad u^{(re^t, \phi)} \circ \Phi(t) \rightarrow u^{(r, \phi)} \quad \text{strongly in } H(\Omega_{(r, \phi)}) \text{ as } t \rightarrow 0.$$

Due to (23), (26), and (30) we evaluate the increment of energy from above:

$$(32) \quad \begin{aligned} &\Pi(u^{(re^t, \phi)}; \Omega_{(re^t, \phi)}) - \Pi(u^{(r, \phi)}; \Omega_{(r, \phi)}) \\ &= \Pi \circ \Phi(t)(u^{(re^t, \phi)} \circ \Phi(t); \Omega_{(r, \phi)}) - \Pi(u^{(r, \phi)}; \Omega_{(r, \phi)}) \\ &\leq \Pi \circ \Phi(t)(u^{(r, \phi)}; \Omega_{(r, \phi)}) - \Pi(u^{(r, \phi)}; \Omega_{(r, \phi)}) \\ &= t \Pi_V^1(u^{(r, \phi)}, u^{(r, \phi)}, f; \Omega_{(r, \phi)}) + \text{Res}_t(u^{(r, \phi)}), \quad \text{Res}_t(u^{(r, \phi)}) = o(t). \end{aligned}$$

On the other hand, (6) implies similar estimation from below:

$$(33) \quad \begin{aligned} &\Pi(u^{(re^t, \phi)}; \Omega_{(re^t, \phi)}) - \Pi(u^{(r, \phi)}; \Omega_{(r, \phi)}) \\ &\geq \Pi \circ \Phi(t)(u^{(re^t, \phi)} \circ \Phi(t); \Omega_{(r, \phi)}) - \Pi(u^{(re^t, \phi)} \circ \Phi(t); \Omega_{(r, \phi)}) \\ &= t \Pi_V^1(u^{(re^t, \phi)} \circ \Phi(t), u^{(re^t, \phi)} \circ \Phi(t), f; \Omega_{(r, \phi)}) \\ &\quad + \text{Res}_t(u^{(re^t, \phi)} \circ \Phi(t)), \quad \text{Res}_t(u^{(re^t, \phi)} \circ \Phi(t)) = o(t). \end{aligned}$$

For $re^t = r + s$, we have $t = \ln(1 + s/r) = s/r + o(s/r)$. Dividing (32) and (33) with s , passing $s \rightarrow 0$ due to (31) we infer (28).

The sign of the derivative follows from the general fact that $r \mapsto \Pi(r, \phi)$ is a nonincreasing function of the crack length. \square

Corollary 1. *If $f = 0$ in B_{δ_f} , $\delta_f > 0$, the derivative in (28) can be expressed equivalently by the domain integral over $\Omega_0 \setminus B_\delta$ only*

$$(34) \quad \Pi'(r, \phi) = \frac{1}{r} \Pi_V^1(u^{(r, \phi)}, u^{(r, \phi)}, f; \Omega_0 \setminus B_\delta),$$

or by the contour integral over ∂B_δ

$$(35) \quad \Pi'(r, \phi) = \frac{\delta}{r} \int_{\partial B_\delta} \left\{ \frac{1}{2} |\nabla u^{(r, \phi)}|^2 - \left(\frac{\partial u^{(r, \phi)}}{\partial n} \right)^2 \right\} dx$$

for arbitrary $\delta \in (0, \delta_f)$. The notation uses the normal vector

$$(36) \quad n := \frac{x}{|x|} \quad \text{on } \partial B_\delta.$$

In the general case we have

$$(37) \quad u^{(r,\phi)} \in H^2(B_\delta \setminus \Gamma_{(r,\phi)}) \Rightarrow \Pi'(r, \phi) = 0.$$

Proof. Let us rewrite (28) with the help of (27) explicitly as

$$(38) \quad \Pi'(r, \phi) = \frac{1}{r} \int_{\Omega_{(r,\phi)}} \left\{ \frac{1}{2} \nabla u^{(r,\phi)} \cdot \left(\operatorname{div}(V) I - \frac{\partial V}{\partial x} - \frac{\partial V^\top}{\partial x} \right) \nabla u^{(r,\phi)} - \operatorname{div}(Vf)u^{(r,\phi)} \right\} dx.$$

In B_δ we can take $\eta \equiv 1$, thus $V(x) = x$, and the density of the integral in (38) is $-\operatorname{div}(xf)u^{(r,\phi)}$ due to identity

$$\operatorname{div}(x) I - \frac{\partial x}{\partial x} - \frac{\partial x^\top}{\partial x} = 0.$$

Henceforth, $f = 0$ in B_δ implies (34).

In $B_R \setminus B_\delta$ the solution $u^{(r,\phi)}$ is H^2 -smooth according to Proposition 1. Therefore, we can differentiate the domain integral by parts and obtain due to $V = 0$ in $\Omega_0 \setminus B_R$:

$$\begin{aligned} & \int_{\Omega_0 \setminus B_\delta} \left\{ \frac{1}{2} \nabla u^{(r,\phi)} \cdot \left(\operatorname{div}(V) I - \frac{\partial V}{\partial x} - \frac{\partial V^\top}{\partial x} \right) \nabla u^{(r,\phi)} - \operatorname{div}(Vf)u^{(r,\phi)} \right\} dx \\ &= \int_{B_R \setminus B_\delta} (\Delta u^{(r,\phi)} + f)(V \cdot \nabla u^{(r,\phi)}) dx \\ &+ \int_{\partial B_\delta} \left\{ \frac{1}{2} (n \cdot V) |\nabla u^{(r,\phi)}|^2 - \frac{\partial u^{(r,\phi)}}{\partial n} (V \cdot \nabla u^{(r,\phi)}) \right\} dx \\ &- \int_{\Gamma_0 \cap (B_R \setminus B_\delta)} \left[\frac{1}{2} (\nu \cdot V) |\nabla u^{(r,\phi)}|^2 - \frac{\partial u^{(r,\phi)}}{\partial \nu} (V \cdot \nabla u^{(r,\phi)}) \right] dx. \end{aligned}$$

Using relations (8), the integrals over $B_R \setminus B_\delta$ as well as $\Gamma_0 \cap (B_R \setminus B_\delta)$ are zero. At ∂B_δ it holds $V = x$. As the result, from (34) we arrive at formula (35). The detailed derivation is given in [20].

If $u^{(r,\phi)} \in H^2(B_\delta \setminus \Gamma_{(r,\phi)})$, then the integration by parts over the whole ball B_R results in $\Pi'(r, \phi) = 0$. \square

Corollary 2. *Due to the property that $y = \Phi(t, x)$ maps Ω_0 and $K(\Omega_0)$ into themselves, it holds*

$$(39) \quad \Pi_V^1(u^0, u^0, f; \Omega_0) = 0.$$

Indeed, (39) is argued by the proof of Proposition 3 implying that

$$\begin{aligned} 0 &= \frac{d}{dt} \Pi(u^0; \Omega_0) = \lim_{t \rightarrow 0} \frac{\Pi \circ \Phi(t)(u^0 \circ \Phi(t); \Omega_0) - \Pi(u^0; \Omega_0)}{t} \\ &= \Pi_V^1(u^0, u^0, f; \Omega_0). \end{aligned}$$

3. CONVERGENCE OF THE SOLUTIONS AS $r \rightarrow 0$

In this section, we are aimed to evaluate with respect to $r \rightarrow 0$ the increment of solutions, which we will denote by

$$(40) \quad w^{(r,\phi)} := u^{(r,\phi)} - u^0 \in H(\Omega_{(r,\phi)}).$$

For this aim, we will employ the normal derivatives at the crack.

Indeed, following Proposition 1, by the surjectivity of the trace operator at a boundary, distribution $\frac{\partial u^{(r,\phi)}}{\partial \nu} \in H_{00}^{1/2}(\Gamma_{(r,\phi)})^*$ is defined from the Green formula by the identity

$$(41) \quad \left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} = \int_{\Omega_{(r,\phi)}} (-\nabla u^{(r,\phi)} \cdot \nabla v + f v) dx + \int_{\Gamma_N} g v dx$$

for $v \in H(\Omega_{(r,\phi)})$,

where $\langle \cdot, \cdot \rangle_{\Gamma_{(r,\phi)}}$ means the duality pairing between $H_{00}^{1/2}(\Gamma_{(r,\phi)})$ and its dual space $H_{00}^{1/2}(\Gamma_{(r,\phi)})^*$. Variational inequality (7) together with (41) imply that

$$(42) \quad \left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu}, \llbracket v - u^{(r,\phi)} \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \leq 0 \quad \text{for all } v \in K(\Omega_{(r,\phi)}).$$

Apart from the kink and end points of the crack, where $u^{(r,\phi)}$ is smooth, from (42) we can derive the boundary conditions (8c) pointwisely.

Similarly, we can extend the normal derivative of the solution $u^0 \in H(\Omega_0) \subset H(\Omega_{(r,\phi)})$ of problem (11) from Γ_0 to the crack $\Gamma_{(r,\phi)}$ as the following distribution

$$(43) \quad \left\langle \frac{\partial u^0}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} = \int_{\Omega_{(r,\phi)}} (-\nabla u^0 \cdot \nabla v + f v) dx + \int_{\Gamma_N} g v dx$$

for $v \in H(\Omega_{(r,\phi)})$.

It fulfills the natural boundary conditions in the following sense

$$(44) \quad \left\langle \frac{\partial u^0}{\partial \nu}, \llbracket v - u^0 \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \leq 0 \quad \text{for } v \in K(\Omega_{(r,\phi)}) : \llbracket v \rrbracket = 0 \text{ on } \gamma_{(r,\phi)},$$

where the test functions form $K(\Omega_0)$. We recall that $\gamma_{(r,\phi)} = \Gamma_{(r,\phi)} \setminus \Gamma_0$. Apart from the end points of crack Γ_0 , smooth u^0 satisfies relations (13c) pointwisely.

Subtracting (43) from (41) gets a variational formulation for the increment $w^{(r,\phi)} \in H(\Omega_{(r,\phi)})$:

$$(45) \quad \int_{\Omega_{(r,\phi)}} \nabla w^{(r,\phi)} \cdot \nabla v \, dx = - \left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu} - \frac{\partial u^0}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)}}$$

for all $v \in H(\Omega_{(r,\phi)})$.

It satisfies weakly the following boundary-value problem:

$$(46a) \quad -\Delta w^{(r,\phi)} = 0 \quad \text{in } \Omega_{(r,\phi)},$$

$$(46b) \quad w^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w^{(r,\phi)}}{\partial q} = 0 \quad \text{on } \Gamma_N,$$

$$(46c) \quad \frac{\partial w^{(r,\phi)}}{\partial \nu} = \frac{\partial u^{(r,\phi)}}{\partial \nu} - \frac{\partial u^0}{\partial \nu} \quad \text{on } \Gamma_{(r,\phi)}.$$

In what follows we evaluate the solution of (45) with respect to $r \rightarrow 0$.

To estimate the norm of $w^{(r,\phi)}$ from (45), we observe the following facts. Substituting $u^0 \in K(\Omega_0) \subset K(\Omega_{(r,\phi)})$ into (42) as a test function provides the inequality

$$(47) \quad - \left\langle \frac{\partial u^{(r,\phi)}}{\partial \nu}, \llbracket w^{(r,\phi)} \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \leq 0.$$

In contrast, $u^{(r,\phi)}$ can not be substituted into (44). By this reason, we partition $\Gamma_{(r,\phi)}$ with the help of a suitable cut-off function $x \mapsto \chi_r(x) : \Omega \rightarrow [0, 1]$ such that satisfies the following relations:

$$(48) \quad \chi_r(x) = 1 \quad \text{as } x \in \gamma_{(r,\phi)}, \quad \chi_r(x) = 0 \quad \text{as } x \in \Gamma_0 \setminus B_r.$$

Consequently, $(1 - \chi_r)\llbracket u^{(r,\phi)} \rrbracket = 0$ at $\gamma_{(r,\phi)}$, and $v = (1 - \chi_r)u^{(r,\phi)} + \chi_r u^0$ can be taken in (44), thus providing the inequality

$$(49) \quad \left\langle \frac{\partial u^0}{\partial \nu}, \llbracket w^{(r,\phi)} \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \leq \left\langle \frac{\partial u^0}{\partial \nu}, \chi_r \llbracket w^{(r,\phi)} \rrbracket \right\rangle_{\Gamma_{(r,\phi)}}.$$

Substituting $w^{(r,\phi)}$ into (45), from (47) and (49) we end with the estimate

$$(50) \quad \int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \leq \left\langle \frac{\partial u^0}{\partial \nu}, \chi_r \llbracket w^{(r,\phi)} \rrbracket \right\rangle_{\Gamma_{(r,\phi)} \cap B_r},$$

where $\langle \cdot, \cdot \rangle_{\Gamma_{(r,\phi)} \cap B_r}$ means the duality pairing between $H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)$ and its dual space $H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^*$.

Note that the right-hand side of (50) employs the cut-off function χ_r supported locally in a neighborhood of the branch $\gamma_{(r,\phi)}$ only. To use this feature in the further estimation we need to restate the usual result on traces in a local sense.

Lemma 1. *For $u \in H(\Omega_{(r,\phi)})$, continuity of the trace operator yields the following estimates holding at the crack locally in B_δ with $\delta \in (r, R)$:*

$$(51) \quad \begin{aligned} \|\llbracket u \rrbracket\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}^2 &\leq \frac{c}{\delta^2} \int_{B_\delta \setminus \Gamma_{(r,\phi)}} |u|^2 dx + c \int_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 dx, \\ \text{if } \llbracket u \rrbracket &\in H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r), \end{aligned}$$

for the jump, and for the normal derivative

$$(52) \quad \begin{aligned} \left\| \frac{\partial u}{\partial \nu} \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^*}^2 &\leq c \int_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 dx, \\ \text{if } \left[\left[\frac{\partial u}{\partial \nu} \right] \right] &= 0 \text{ and } \Delta u = 0 \text{ in } B_\delta \setminus \Gamma_{(r,\phi)}. \end{aligned}$$

Moreover, a Poincaré inequality implies that

$$(53) \quad \frac{1}{\delta^2} \int_{B_\delta \setminus \Gamma_{(r,\phi)}} |u|^2 dx \leq c \int_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 dx, \quad \text{if } \int_{B_\delta \setminus \Gamma_{(r,\phi)}} u dx = 0.$$

All constants c are independent of δ .

Proof. We start deriving an equivalent formulation of the $H_{00}^{1/2}$ -norm from (9). In fact, for $\delta > r$ we can take a closed extension $\tilde{\Gamma}_\delta$ of $\Gamma_{(r,\phi)} \cap B_r$ in B_δ and extend $\llbracket u \rrbracket$ with zero along $\tilde{\Gamma}_\delta$. Accounting identities

$$\frac{1}{r - |s|} = \int_r^\infty \frac{1}{|t - s|^2} dt = \int_{-\infty}^{-r} \frac{1}{|t - s|^2} dt$$

and taking the distance function $\text{dist}(x(s), \partial(\Gamma_{(r,\phi)} \cap B_r)) = (r - |s|)/2$, it follows from (9) that

$$(54) \quad \begin{aligned} & \|\llbracket u \rrbracket\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}^2 = \int_{\tilde{\Gamma}_\delta} \llbracket u(x(s)) \rrbracket^2 dx(s) \\ & + \int_{\tilde{\Gamma}_\delta} \int_{\tilde{\Gamma}_\delta} \frac{\|\llbracket u(x(t)) - u(x(s)) \rrbracket\|^2}{|x(t) - x(s)|^2} dx(s) dx(t) = \|\llbracket u \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_\delta)}^2. \end{aligned}$$

To derive (51)–(53), we employ homogeneity arguments. With the uniform extension of coordinates $x = \delta y$, integrals in the right-hand side of (54) are transformed onto the image $\tilde{\Gamma}_1$ of $\tilde{\Gamma}_\delta$ in B_1 as

$$\begin{aligned} \|\llbracket u \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_\delta)}^2 &= \int_{\tilde{\Gamma}_1} \int_{\tilde{\Gamma}_1} \frac{\|\llbracket u(\delta y(t)) - u(\delta y(s)) \rrbracket\|^2}{|y(t) - y(s)|^2} dy(s) dy(t) \\ &+ \delta \int_{\tilde{\Gamma}_1} \llbracket u(\delta y(s)) \rrbracket^2 dy(s) \leq \max(1, R) \|\llbracket u(\delta y) \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_1)}^2. \end{aligned}$$

In B_1 , the usual trace theorem provides standard estimation

$$\begin{aligned} \|\llbracket u(\delta y) \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_1)}^2 &\leq c_1 \|u(\delta y)\|_{H^1(B_1 \setminus \tilde{\Gamma}_1)}^2 \\ &= c_1 \left\{ \int_{B_1 \setminus \tilde{\Gamma}_1} |u(\delta y)|^2 dy + \int_{B_1 \setminus \tilde{\Gamma}_1} |\nabla u(\delta y)|^2 dy \right\}, \end{aligned}$$

with constant c_1 , which is independent of δ . Thus, applying the inverse transformation $y = x/\delta$ in $B_1 \setminus \tilde{\Gamma}_1$ we obtain

$$\|\llbracket u \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_\delta)}^2 \leq \frac{c_2}{\delta^2} \int_{B_\delta \setminus \tilde{\Gamma}_\delta} |u(x)|^2 dx + c_2 \int_{B_\delta \setminus \tilde{\Gamma}_\delta} |\nabla u(x)|^2 dx,$$

and together with (54) we infer (51).

Conversely, an extension operator exists such that

$$\|\tilde{u}(\delta y)\|_{H^1(B_1 \setminus \tilde{\Gamma}_1)}^2 \leq c_3 \|\llbracket u(\delta y) \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_1)}^2,$$

with constant c_3 independent of δ . This implies the inequality

$$\begin{aligned} & \int_{B_1 \setminus \tilde{\Gamma}_1} |\nabla u(\delta y)|^2 dy(s) \leq c_3 \left\{ \int_{\tilde{\Gamma}_1} \llbracket u(\delta y(s)) \rrbracket^2 dy(s) \right. \\ & \left. + \int_{\tilde{\Gamma}_1} \int_{\tilde{\Gamma}_1} \frac{\|\llbracket u(\delta y(t)) - u(\delta y(s)) \rrbracket\|^2}{|y(t) - y(s)|^2} dy(s) dy(t) \right\}. \end{aligned}$$

After transformation $y = x/\delta$ it reads

$$\begin{aligned} \int_{B_\delta \setminus \tilde{\Gamma}_\delta} |\nabla u(x)|^2 dx &\leq c_3 \left\{ \frac{1}{\delta} \int_{\tilde{\Gamma}_\delta} \llbracket u(x(s)) \rrbracket^2 dx(s) \right. \\ &\quad \left. + \int_{\tilde{\Gamma}_\delta} \int_{\tilde{\Gamma}_\delta} \frac{\llbracket u(x(t)) - u(x(s)) \rrbracket^2}{|x(t) - x(s)|^2} dx(s) dx(t) \right\}. \end{aligned}$$

Since $\llbracket u(x(s)) \rrbracket = 0$ for $|s| > r$, we can evaluate

$$\begin{aligned} \frac{1}{\delta} \int_{\tilde{\Gamma}_\delta} \llbracket u(x(s)) \rrbracket^2 dx(s) &= \int_{\tilde{\Gamma}_\delta} \frac{r - |s|}{\delta} \frac{\llbracket u(x(s)) \rrbracket^2}{r - |s|} dx(s) \\ &\leq \int_{\tilde{\Gamma}_\delta} \frac{\llbracket u(x(s)) \rrbracket^2}{r - |s|} dx(s) = \int_r^\infty \int_{\tilde{\Gamma}_\delta} \frac{|0 - \llbracket u(x(s)) \rrbracket|^2}{|t - s|^2} dx(s) dt \\ &\leq \int_{\tilde{\Gamma}_\delta} \int_{\tilde{\Gamma}_\delta} \frac{\llbracket u(x(t)) - u(x(s)) \rrbracket^2}{|x(t) - x(s)|^2} dx(s) dx(t). \end{aligned}$$

As a consequence, the following uniform estimate holds with $c_4 = 2c_3$:

$$(55) \quad \int_{B_\delta \setminus \tilde{\Gamma}_\delta} |\nabla u|^2 dx \leq c_4 \|\llbracket u \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_\delta)}^2 \quad \text{for } u \in H^1(B_\delta \setminus \Gamma_{(r,\phi)}).$$

The distribution $\partial u / \partial \nu \in H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^*$ is defined well from a generic Green formula by the relation

$$\left\langle \frac{\partial u}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)} \cap B_r} = \int_{B_\delta \setminus \Gamma_{(r,\phi)}} (-\nabla u \cdot \nabla v - v \Delta u) dx$$

for $v \in H^1(B_\delta \setminus (\Gamma_{(r,\phi)} \cap B_r))$, $u = 0$ on ∂B_δ ,

provided that $\Delta u \in L^2(B_\delta \setminus \Gamma_{(r,\phi)})$ and $\llbracket \partial u / \partial \nu \rrbracket = 0$ at $\tilde{\Gamma}_\delta$. If $\Delta u = 0$, we evaluate the norm as

$$\begin{aligned} \left\| \frac{\partial u}{\partial \nu} \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^*} &:= \sup_{\|\llbracket v \rrbracket\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)} = 1} \left| \left\langle \frac{\partial u}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)} \cap B_r} \right| \\ &= \sup_{\|\llbracket v \rrbracket\|_{H^{1/2}(\tilde{\Gamma}_\delta)} = 1} \left| \int_{B_\delta \setminus \Gamma_{(r,\phi)}} \nabla u \cdot \nabla v dx \right| \leq c_4 \left(\int_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u|^2 dx \right)^{1/2} \end{aligned}$$

due to (55), thus providing (52).

If the integral of u over $B_\delta \setminus \Gamma_{(r,\phi)}$ is zero, a Poincaré inequality in B_1 reads:

$$\int_{B_1 \setminus \tilde{\Gamma}_1} |u(\delta y)|^2 dy \leq c \int_{B_1 \setminus \tilde{\Gamma}_1} |\nabla u(\delta y)|^2 dy \quad \text{for} \quad \int_{B_1 \setminus \tilde{\Gamma}_1} u(\delta y) dy = 0.$$

Consequently, with the help of $y = x/\delta$ we obtain (53). \square

For the following use we decompose in $B_R \setminus \Gamma_{(r,\phi)}$:

$$(56) \quad w^{(r,\phi)} = \bar{w}^{(r,\phi)} + W^{(r,\phi)}, \quad \bar{w}^{(r,\phi)} := \frac{1}{2\pi} \left(\int_{-\pi}^{\phi} + \int_{\phi}^{\pi} \right) w^{(r,\phi)} d\theta,$$

using polar coordinates $x = \rho(\cos \theta, \sin \theta)$. The polar angle $\theta \in (-\pi, \phi) \cup (\phi, \pi)$ starts from the x_1 -axis counter-clockwise around the origin O . The polar radius $\rho = |x| = \sqrt{x_1^2 + x_2^2}$. Firstly, integrating (56) with respect to θ gets

$$(57) \quad \left(\int_{-\pi}^{\phi} + \int_{\phi}^{\pi} \right) W^{(r,\phi)} d\theta = 0.$$

Second, we evaluate $\bar{w}^{(r,\phi)}$.

Lemma 2. *The function $x \mapsto \bar{w}^{(r,\phi)}$ is constant for $x \in B_R \setminus \Gamma_{(r,\phi)}$.*

Proof. We take a smooth cut-off function $\xi(\rho)$ supported in B_R and substitute $v = \xi$ into (45). In view of $[[\xi]] = 0$ at the crack, the right-hand side turns to be zero. Due to $\Delta w^{(r,\phi)} \in L^2(B_R \setminus \Gamma_{(r,\phi)})$ we can integrate by parts in $B_R \setminus \Gamma_{(r,\phi)}$. Thus, accounting $\xi = 0$ at ∂B_R , $[[\xi]] = 0$ and $[\partial w^{(r,\phi)}/\partial \nu] = 0$ at $\Gamma_{(r,\phi)}$ we obtain that

$$\begin{aligned} 0 &= \int_{\Omega_{(r,\phi)}} \nabla w^{(r,\phi)} \cdot \nabla \xi dx = - \int_{B_R \setminus \Gamma_{(r,\phi)}} \Delta w^{(r,\phi)} \xi dx \\ &= - \left(\int_{-\pi}^{\phi} + \int_{\phi}^{\pi} \right) \int_0^R \left\{ \frac{\partial}{\partial \rho} \left(\rho \frac{\partial w^{(r,\phi)}}{\partial \rho} \right) + \frac{1}{\rho} \frac{\partial^2 w^{(r,\phi)}}{\partial \theta^2} \right\} \xi(\rho) d\rho d\theta \\ &= -2\pi \int_0^R \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \bar{w}^{(r,\phi)}}{\partial \rho} \right) \xi d\rho - \int_0^R \left[\left[\frac{1}{\rho} \frac{\partial w^{(r,\phi)}}{\partial \theta} \right] \right]_{\theta=\pm\pi, \theta=\phi} \xi d\rho \\ &= -2\pi \int_0^R \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \bar{w}^{(r,\phi)}}{\partial \rho} \right) \xi d\rho. \end{aligned}$$

Since ξ is arbitrary we conclude with the identity

$$\frac{\partial}{\partial \rho} \left(\rho \frac{\partial \bar{w}^{(r,\phi)}}{\partial \rho} \right) = 0 \quad \text{for all } \rho \in (0, R).$$

A general solution to this differential equation gets $\bar{w}^{(r,\phi)} = c_1 + c_2 \ln \rho$. But the logarithmic term would contradict to the inclusion $w^{(r,\phi)} \in H^1(B_R \setminus \Gamma_{(r,\phi)})$. Indeed, if $c_2 \neq 0$, we can derive from (56) that

$$\begin{aligned} \int_{B_R \setminus \Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx &\geq \int_{-\pi}^{\pi} \int_0^R \left(\frac{\partial w^{(r,\phi)}}{\partial \rho} \right)^2 \rho d\rho d\theta \\ &\geq 2\pi c_2^2 \int_0^R \frac{1}{\rho} d\rho + 2c_2 \int_0^R \frac{\partial}{\partial \rho} \left(\int_{-\pi}^{\pi} W^{(r,\phi)} d\theta \right) d\rho = +\infty \end{aligned}$$

due to (57). Henceforth, $c_2 = 0$ implies the assertion of lemma. \square

Corollary 3. *If $f = 0$ in B_{δ_f} with $\delta_f \in (0, R)$ and*

$$u^0 = \bar{u}^0 + U^0 \quad \text{in } B_{\delta_f} \setminus \Gamma_0, \quad \bar{u}^0 := \frac{1}{2\pi} \int_{-\pi}^{\pi} u^0 d\theta,$$

then U^0 satisfies $\int_{-\pi}^{\pi} U^0 d\theta = 0$, and $\bar{u}^0(x)$ is constant for $x \in B_{\delta_f} \setminus \Gamma_0$.

Proof. We take a suitable cut-off function $\xi(\rho)$ supported in B_{δ_f} and substitute $v = u^0 \pm \xi$ into (12). In view of $f = 0$ in B_{δ_f} and $\xi = 0$ on Γ_N we obtain the identity

$$\int_{B_{\delta_f} \setminus \Gamma_0} \nabla u^0 \cdot \nabla \xi dx = 0.$$

Henceforth, repeating the arguments used in the proof of Lemma 2 implies that \bar{u}^0 is constant in $B_{\delta_f} \setminus \Gamma_0$. \square

The next two results restate a Saint–Venant principle for the crack problem.

Lemma 3. *Apart from the kink point, the following estimate holds for $r < \delta_0 \leq \delta < R$:*

$$(58) \quad \int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 dx \leq \frac{\delta_0}{\delta} \int_{\Omega_0 \setminus B_{\delta_0}} |\nabla w^{(r,\phi)}|^2 dx.$$

Proof. We have $w^{(r,\phi)} \in H^1(\Omega_0 \setminus B_\delta)$, and $\Delta w^{(r,\phi)} = 0$ due to (46a). Therefore, with the help of a regular extension $\tilde{\Gamma}_0$ of Γ_0 from its left end-point A up to the external boundary $\partial\Omega$, a generic Green formula can be written in $D := (\Omega \setminus \tilde{\Gamma}_0) \setminus B_\delta$. This gets

$$(59) \quad \int_D \nabla w^{(r,\phi)} \cdot \nabla v \, dx = \left\langle \frac{\partial w^{(r,\phi)}}{\partial q}, v \right\rangle_{\partial D} \quad \text{for } v \in H^1(D),$$

where the normal derivative $\partial w^{(r,\phi)}/\partial q$ is defined in a distributional sense at ∂D . The boundary ∂D is a closed curve consisted of $\partial\Omega$, ∂B_δ , and two faces of $\tilde{\Gamma}_0 \setminus B_\delta$.

For $R < l$, the crack end-point A is not contained in $B_R \setminus B_\delta$. Then $w^{(r,\phi)}$ is smooth in $(B_R \setminus B_\delta) \setminus \Gamma_0$ up to the boundary ∂B_δ and $\Gamma_0 \cap (B_R \setminus B_\delta)$ due to Proposition 1. Let $\xi : \Omega \rightarrow [0, 1]$ be a suitable cut-off function supported in B_R such that $\xi(x) = 1$ for $x \in \bar{B}_\delta$. With the partition $v = \xi v + (1 - \xi)v$, using the local smoothness of $w^{(r,\phi)}$ and accounting the boundary conditions in (46), from (59) we infer that

$$(60) \quad \begin{aligned} & \int_{\Omega_0 \setminus B_\delta} \nabla w^{(r,\phi)} \cdot \nabla v \, dx = - \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} v \, dx \\ & - \int_{\Gamma_0 \cap (B_R \setminus B_\delta)} \frac{\partial w^{(r,\phi)}}{\partial \nu} \xi[v] \, dx - \left\langle \frac{\partial w^{(r,\phi)}}{\partial \nu}, (1 - \xi)[v] \right\rangle_{\Gamma_0 \setminus B_\delta} \\ & \text{for } v \in H(\Omega_0 \setminus B_\delta). \end{aligned}$$

Substituting $v = \xi u^{(r,\phi)} + (1 - \xi)u^0$ into (42) and $v = (1 - \xi)u^{(r,\phi)} + \xi u^0$ into (44) gets

$$- \left\langle \frac{\partial w^{(r,\phi)}}{\partial \nu}, (1 - \xi)[w^{(r,\phi)}] \right\rangle_{\Gamma^{(r,\phi)}} \leq 0.$$

The next pointwise relations are derived from conditions (8c) and (13c)

$$(61) \quad \frac{\partial w^{(r,\phi)}}{\partial \nu} [w^{(r,\phi)}] \geq 0 \quad \text{on } \Gamma_0 \cap (B_R \setminus B_r).$$

After substitution of $w^{(r,\phi)}$ into (60), the lines between (60) and (61) result in local estimation of the norm as

$$(62) \quad \int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 \, dx \leq - \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} w^{(r,\phi)} \, dx.$$

To evaluate the right-hand side of (62) we observe that

$$\begin{aligned} \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} w^{(r,\phi)} dx &= \int_{-\pi}^{\pi} \frac{\partial(\bar{w}^{(r,\phi)} + W^{(r,\phi)})}{\partial \rho} (\bar{w}^{(r,\phi)} + W^{(r,\phi)}) \delta d\theta \\ &= \int_{-\pi}^{\pi} \frac{\partial W^{(r,\phi)}}{\partial \rho} W^{(r,\phi)} \delta d\theta + \left(\frac{\partial}{\partial \rho} \int_{-\pi}^{\pi} W^{(r,\phi)} \delta d\theta \right) \bar{w}^{(r,\phi)} = \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} W^{(r,\phi)} dx \end{aligned}$$

due to Lemma 2 and (57). Therefore, we can apply to $W^{(r,\phi)}$ the Poincaré inequality along the circle

$$(63) \quad \int_{-\pi}^{\pi} u^2 d\theta \leq 4 \int_{-\pi}^{\pi} \left(\frac{\partial u}{\partial \theta} \right)^2 d\theta \quad \text{for } u \text{ such that } \int_{-\pi}^{\pi} u d\theta = 0$$

and estimate

$$\begin{aligned} \left| \int_{\partial B_\delta} \frac{\partial w^{(r,\phi)}}{\partial n} w^{(r,\phi)} dx \right| &\leq \int_{-\pi}^{\pi} \left| \frac{\partial w^{(r,\phi)}}{\partial \rho} W^{(r,\phi)} \right| \delta d\theta \\ (64) \quad &\leq \delta \int_{-\pi}^{\pi} \left\{ \delta \left(\frac{\partial w^{(r,\phi)}}{\partial \rho} \right)^2 + \frac{1}{4\delta} (W^{(r,\phi)})^2 \right\} d\theta \\ &\leq \delta \int_{-\pi}^{\pi} \left\{ \delta \left(\frac{\partial w^{(r,\phi)}}{\partial \rho} \right)^2 + \frac{1}{\delta} \left(\frac{\partial W^{(r,\phi)}}{\partial \theta} \right)^2 \right\} d\theta = \delta \int_{\partial B_\delta} |\nabla w^{(r,\phi)}|^2 dx. \end{aligned}$$

On the other hand, differentiating with respect to δ we find that

$$\begin{aligned} \frac{d}{d\delta} \int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 dx &= - \lim_{\varepsilon \rightarrow 0} \int_{(B_{\delta+\varepsilon} \setminus B_\delta) \cap \Omega_0} \frac{1}{\varepsilon} |\nabla w^{(r,\phi)}|^2 dx \\ (65) \quad &= - \int_{\partial B_\delta} |\nabla w^{(r,\phi)}|^2 dx. \end{aligned}$$

Combining estimates (62)–(65) implies the differential inequality

$$\int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 dx \leq -\delta \frac{d}{d\delta} \int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 dx.$$

Applying a Grönwall inequality we arrive at the desired relation (58). \square

Lemma 4. *Around the kink point, if $f = 0$ in B_{δ_f} , the following estimate holds for $0 < \delta_0 \leq \delta < \min(R, \delta_f)$:*

$$(66) \quad \int_{B_{\delta_0} \setminus \Gamma_0} |\nabla u^0|^2 dx \leq \frac{\delta_0}{\delta} \int_{B_\delta \setminus \Gamma_0} |\nabla u^0|^2 dx.$$

Proof. The statement for a general inhomogeneous load is proven in [15]. As $f = 0$, we simplify the proof with the arguments used in Lemma 3.

The extended crack $\Gamma_{(R,\phi)}$ splits a ball B_δ into two subdomains which we denote by B_δ^+ and B_δ^- . Since $u^0 \in H^1(B_\delta^\pm)$ and $-\Delta u^0 = f \in L^2(B_\delta^\pm)$, the normal derivative $\partial u^0 / \partial q$ is well defined at the boundaries ∂B_δ^\pm in a distributional sense. From Green formulas holding in B_δ^\pm , due to $f = 0$ in B_δ we have

$$(67) \quad \int_{B_\delta^\pm} \nabla u^0 \cdot \nabla v dx = \left\langle \frac{\partial u^0}{\partial q}, v \right\rangle_{\partial B_\delta^\pm} \quad \text{for } v \in H^1(B_\delta^\pm).$$

For $v \in H^1(B_\delta \setminus \Gamma_0)$, we apply to $[[v]]$ the partition $1 = \chi_r + (1 - \chi_r)$ at $\partial B_\delta^+ \cap \partial B_\delta^-$ with a non-negative cut-off function χ_r suitably supported in B_δ and satisfying (48). Accounting the local smoothness of u^0 in $B_R \setminus B_r$, using $[[\partial u^0 / \partial \nu]] = 0$ at $\Gamma_{(R,\phi)}$ and $[[v]] = 0$ at $\Gamma_{(R,\phi)} \setminus \Gamma_0$ result (67) in the identity

$$(68) \quad \begin{aligned} \int_{B_\delta \setminus \Gamma_0} \nabla u^0 \cdot \nabla v dx &= \int_{\partial B_\delta} \frac{\partial u^0}{\partial n} v dx \\ &- \int_{\Gamma_0 \cap B_\delta} \frac{\partial u^0}{\partial \nu} (1 - \chi_r) [[v]] dx - \left\langle \frac{\partial u^0}{\partial \nu}, \chi_r [[v]] \right\rangle_{\Gamma_0 \cap B_r}. \end{aligned}$$

Substituting $v = (1 \pm \chi_r)u^0$ into (44) gets

$$\left\langle \frac{\partial u^0}{\partial \nu}, \chi_r [[u^0]] \right\rangle_{\Gamma_0 \cap B_r} = 0.$$

With the help of boundary conditions (13c) holding pointwisely at the crack apart from its end-points, we obtain

$$(1 - \chi_r) \frac{\partial u^0}{\partial \nu} [[u^0]] = 0 \quad \text{on } \Gamma_0 \cap B_\delta.$$

Thus, from (68) with $v = u^0$ we infer equality (compare with (62)):

$$(69) \quad \int_{B_\delta \setminus \Gamma_0} |\nabla u^0|^2 dx = \int_{\partial B_\delta} \frac{\partial u^0}{\partial n} u^0 dx.$$

Differentiating the left-hand side of (69) with respect to δ similarly to (65) implies the relation

$$(70) \quad \frac{d}{d\delta} \int_{B_\delta \setminus \Gamma_0} |\nabla u^0|^2 dx = \int_{\partial B_\delta} |\nabla u^0|^2 dx.$$

In view of Corollary 3 we can repeat for u^0 the arguments used in (64) and evaluate the right-hand side of (69) as

$$(71) \quad \left| \int_{\partial B_\delta} \frac{\partial u^0}{\partial n} u^0 dx \right| \leq \delta \int_{\partial B_\delta} |\nabla u^0|^2 dx.$$

From (69)–(71) we conclude with the differential inequality

$$\int_{B_\delta \setminus \Gamma_0} |\nabla u^0|^2 dx \leq \delta \frac{d}{d\delta} \int_{B_\delta \setminus \Gamma_0} |\nabla u^0|^2 dx.$$

Applying the Grönwall inequality results in (66). □

Finally, we state the main result of this section.

Theorem 1. *If $f = 0$ in B_{δ_f} with $\delta_f \in (0, R)$, the increment of solutions $w^{(r,\phi)} = u^{(r,\phi)} - u^0$ converges to zero as $r \rightarrow 0$ strongly in $H(\Omega_{(\delta,\phi)})$ for arbitrary fixed $\delta \in (0, \delta_f)$ with the following uniform estimates:*

$$(72) \quad \|w^{(r,\phi)}\|_{H(\Omega_{(r,\phi)})} \leq c \sqrt{r},$$

$$(73) \quad \|w^{(r,\phi)}\|_{H(\Omega_0 \setminus B_\delta)} \leq cr.$$

Proof. The proof is based on evaluation of (50) due to Lemma 1–Lemma 4. Indeed, the right-hand side of (50) admits the Cauchy inequality

$$\begin{aligned} \|w^{(r,\phi)}\|_{H(\Omega_{(r,\phi)})}^2 &= \int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \leq \left| \left\langle \frac{\partial u^0}{\partial \nu}, \chi_r \llbracket w^{(r,\phi)} \rrbracket \right\rangle_{\Gamma_{(r,\phi)} \cap B_r} \right| \\ &\leq \left\| \frac{\partial u^0}{\partial \nu} \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^*} \left\| \chi_r \llbracket w^{(r,\phi)} \rrbracket \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}. \end{aligned}$$

Applying estimate (52) with $\delta = 2r$ to u^0 , which obeys $-\Delta u^0 = f = 0$ in B_{2r} , gets further

$$(74) \quad \|w^{(r,\phi)}\|_{H(\Omega_{(r,\phi)})}^2 \leq c_1 \|u^0\|_{H(\Omega_0 \setminus B_{2r})} \left\| \chi_r \llbracket w^{(r,\phi)} \rrbracket \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}$$

with constant c_1 which does not depend on r .

We assume that the cut-off function χ_r in (74), which satisfies relations (48) in B_r , can be extended appropriately to B_{2r} such that

$$(75) \quad 0 \leq \chi_r(x) \leq 1, \quad |\nabla \chi_r(x)| \leq \frac{c}{r} \quad \text{for } x \in B_{2r}.$$

In view of Lemma 2, constant $\bar{w}^{(r,\phi)}$ can be avoided from the right-hand side of (74) since $[\bar{w}^{(r,\phi)}] = 0$. Henceforth, applying estimate (51) as $\delta = 2r$ to $\chi_r W^{(r,\phi)}$ we infer that

$$\begin{aligned} \|\chi_r [w^{(r,\phi)}]\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}^2 &= \|\chi_r [W^{(r,\phi)}]\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}^2 \\ &\leq c_2 \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla(\chi_r W^{(r,\phi)})|^2 dx + \frac{c_2}{r^2} \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\chi_r W^{(r,\phi)}|^2 dx. \end{aligned}$$

Using (75) and Poincaré inequality (53) as $\delta = 2r$ due to (57) proceeds

$$\begin{aligned} \|\chi_r [w^{(r,\phi)}]\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}^2 &\leq c_2 \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla W^{(r,\phi)}|^2 dx \\ &+ \frac{c_3}{r^2} \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |W^{(r,\phi)}|^2 dx \leq c_4 \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla W^{(r,\phi)}|^2 dx. \end{aligned}$$

Adding $\nabla \bar{w}^{(r,\phi)} = 0$ to the right-hand side of the above inequality yields

$$(76) \quad \|\chi_r [w^{(r,\phi)}]\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}^2 \leq c_4 \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx.$$

From (74) and (76) we have the estimate

$$\int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \leq c_5 \left(\int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla u^0|^2 dx \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \right)^{1/2}.$$

Now, with the help of Lemma 4 as $\delta_0 = 2r$, for fixed $\delta \in (2r, \delta_f)$ we infer that

$$\begin{aligned} \int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx &\leq c_5 \left(\frac{2r}{\delta} \int_{B_\delta \setminus \Gamma_{(r,\phi)}} |\nabla u^0|^2 dx \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \right)^{1/2} \\ &\leq c_6 \sqrt{r} \left(\int_{\Omega_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \right)^{1/2}, \end{aligned}$$

thus (72). It implies also the strong convergence $w^{(r,\phi)} \rightarrow 0$ in the $H(\Omega_{(\delta,\phi)})$ -norm as $r \rightarrow 0$.

To derive (73), we represent (72) as

$$(77) \quad \int_{\Omega_0 \setminus B_{2r}} |\nabla w^{(r,\phi)}|^2 dx + \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \leq c_7 r$$

and apply Lemma 3 for $\delta_0 = 2r$:

$$(78) \quad \frac{1}{r} \int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 dx \leq c_8 \int_{\Omega_0 \setminus B_{2r}} |\nabla w^{(r,\phi)}|^2 dx.$$

Evidently, (77) and (78) imply the estimate

$$\frac{1}{r} \int_{\Omega_0 \setminus B_\delta} |\nabla w^{(r,\phi)}|^2 dx + \int_{B_{2r} \setminus \Gamma_{(r,\phi)}} |\nabla w^{(r,\phi)}|^2 dx \leq c r,$$

thus (73). □

Corollary 4. *A weak limit $\dot{w}^\phi \in H(\Omega_0 \setminus B_\delta)$ exists such that*

$$(79) \quad \frac{1}{r_n} w^{(r_n,\phi)} \rightarrow \dot{w}^\phi \quad \text{weakly in } H(\Omega_0 \setminus B_\delta) \quad \text{as } r_n \rightarrow 0.$$

Indeed, this is a direct consequence of the uniform estimate (73).

4. EXPANSION OF THE ENERGY FUNCTIONAL AT $r = 0$

The uniform estimation of solutions from Theorem 1 provides statements for the energy functional formulated below. During the rest of the paper we assume always that $f = 0$ in B_{δ_f} with fixed $\delta_f \in (0, R)$.

Proposition 4. *The sequence of derivatives $\Pi'(r, \phi)$ is bounded uniformly with respect to $r \rightarrow 0$.*

Proof. Due to $f = 0$ in B_{δ_f} we can apply Corollary 1 and rewrite $\Pi'(r, \phi)$ with the help of decomposition $u^{(r,\phi)} = u^0 + w^{(r,\phi)}$ as

$$(80) \quad \begin{aligned} \Pi'(r, \phi) &= \frac{1}{r} \Pi_V^1(u^0, u^0, f; \Omega_0 \setminus B_\delta) \\ &+ \Pi_V^1\left(2u^0, \frac{w^{(r,\phi)}}{r}, f; \Omega_0 \setminus B_\delta\right) + \Pi_V^1\left(w^{(r,\phi)}, \frac{w^{(r,\phi)}}{r}, 0; \Omega_0 \setminus B_\delta\right). \end{aligned}$$

Using Corollary 2 and estimate (73) yields the assertion. □

For fixed $s > 0$, definition (14) and Proposition 3 imply expansion of $\Pi(s + r, \cdot)$ with respect to $r > 0$ in the form

$$\Pi(u^{(s+r,\phi)}; \Omega_{(s+r,\phi)}) = \Pi(u^{(s,\phi)}; \Omega_{(s,\phi)}) + \int_0^r \Pi'(s + t, \phi) dt.$$

Passing here to the limit as $s \rightarrow 0$ due to the strong convergence (31) and Proposition 4, the Lebesgue dominated convergence theorem gets

$$(81) \quad \Pi(u^{(r,\phi)}; \Omega_{(r,\phi)}) = \Pi(u^0; \Omega_0) + \int_0^r \Pi'(t, \phi) dt.$$

Details arguing passage to the limit can be found in [14]. Moreover, we employ decomposition (80) to $\Pi'(t, \phi)$ in (81) for $t \in (0, r)$ and conclude with the following result.

Theorem 2. *The expansion of the potential energy holds as $r \rightarrow 0$:*

$$(82) \quad \Pi(r, \phi) = \Pi(0) + \int_0^r \Pi_V^1\left(2u^0, \frac{w^{(t,\phi)}}{t}, f; \Omega_0 \setminus B_\delta\right) dt + O(r^2),$$

and

$$(83) \quad 0 \geq \int_0^r \Pi_V^1\left(2u^0, \frac{w^{(t,\phi)}}{t}, f; \Omega_0 \setminus B_\delta\right) dt = O(r),$$

for arbitrary fixed $\delta \in (0, \delta_f)$.

Corollary 5. *If the limit function $\dot{w}^\phi \in H(\Omega_0 \setminus B_\delta)$ from Corollary 4 is unique and it satisfies in $\Omega_0 \setminus B_\delta$ the relation*

$$w^{(r,\phi)} = r\dot{w}^\phi + \text{Res}_r, \quad \|\text{Res}_r\|_{H^1(\Omega_0 \setminus B_\delta)} = o(r),$$

then the topological derivative in (15) exists uniquely given by

$$\Pi'(0, \phi) = \Pi_V^1(2u^0, \dot{w}^\phi, f; \Omega_0 \setminus B_\delta).$$

In the general case, Theorem 2 guarantees existence only of the limit superior and the limit inferior of the quotient in (15). For particular cases, formulas in Theorem 2 can be specified in more details. For this reason, we rely on a linearized setting of the crack problem (8) in the next section.

5. SPECIFICATION OF EXPANSIONS FOR A LINEAR PROBLEM

Avoiding the non-penetration constraint (1) will result in linear formulation of the crack problem. For this particular case, we express the integral Π_V^1 in expansion (82) in the terms of stress intensity factors.

We restate problems (6)–(8): Find $u^{(r,\phi)} \in H(\Omega_{(r,\phi)})$ such that

$$(84) \quad \Pi(u^{(r,\phi)}; \Omega_{(r,\phi)}) \leq \Pi(v; \Omega_{(r,\phi)}) \quad \text{for all } v \in H(\Omega_{(r,\phi)}),$$

which is equivalent to the variational equation

$$(85) \quad \int_{\Omega(r,\phi)} \nabla u^{(r,\phi)} \cdot \nabla v \, dx = \int_{\Omega(r,\phi)} f v \, dx + \int_{\Gamma_N} g v \, dx \quad \text{for } v \in H(\Omega(r,\phi)),$$

and it describes a weak solution to the linear boundary-value problem:

$$(86a) \quad -\Delta u^{(r,\phi)} = f \quad \text{in } \Omega(r,\phi),$$

$$(86b) \quad u^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial u^{(r,\phi)}}{\partial \nu} = g \quad \text{on } \Gamma_N,$$

$$(86c) \quad \frac{\partial u^{(r,\phi)}}{\partial \nu} = 0 \quad \text{on } \Gamma(r,\phi).$$

At $r = 0$, the reference solution $u^0 \in H(\Omega_0)$ satisfies (compare with (11)–(13)):

$$(87) \quad \Pi(u^0; \Omega_0) \leq \Pi(v; \Omega_0) \quad \text{for all } v \in H(\Omega_0),$$

variational equation

$$(88) \quad \int_{\Omega_0} \nabla u^0 \cdot \nabla v \, dx = \int_{\Omega_0} f v \, dx + \int_{\Gamma_N} g v \, dx \quad \text{for all } v \in H(\Omega_0),$$

and the respective boundary-value problem:

$$(89a) \quad -\Delta u^0 = f \quad \text{in } \Omega_0,$$

$$(89b) \quad u^0 = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial u^0}{\partial \nu} = g \quad \text{on } \Gamma_N,$$

$$(89c) \quad \frac{\partial u^0}{\partial \nu} = 0 \quad \text{on } \Gamma_0.$$

All the previous results remain true for the linear problems (84)–(89). In what follows we will refine expansions of the solutions (56) with the first-order asymptotic terms in a Fourier series with respect to the polar angle θ at the point of kink.

Proposition 5. *Around the kink point, the following expansion holds*

$$(90) \quad u^0 = \bar{u}^0 + K \sqrt{\rho} \sin \frac{\theta}{2} + U_1^0 \quad \text{in } B_{\delta_f} \setminus \Gamma_0$$

with the constant stress intensity factor $K \in \mathbb{R}$ given by

$$(91) \quad K = \frac{1}{\pi \sqrt{\rho}} \int_{-\pi}^{\pi} u^0(\rho, \theta) \sin \frac{\theta}{2} \, d\theta \quad \text{for arbitrary } \rho \in (0, \delta_f),$$

and the reminder term $U_1^0 \in H^1(B_{\delta_f} \setminus \Gamma_0)$ satisfying

$$(92a) \quad \int_{-\pi}^{\pi} U_1^0(\rho, \theta) d\theta = \int_{-\pi}^{\pi} U_1^0(\rho, \theta) \sin \frac{\theta}{2} d\theta = 0 \quad \text{for any } \rho \in (0, \delta_f),$$

$$(92b) \quad \int_{B_{\delta_0} \setminus \Gamma_0} |\nabla U_1^0|^2 dx \leq \left(\frac{\delta_0}{\delta}\right)^2 \int_{B_{\delta} \setminus \Gamma_0} |\nabla U_1^0|^2 dx, \quad 0 < \delta_0 \leq \delta < \delta_f.$$

Proof. Indeed, consider the zero-order decomposition from Corollary 3:

$$(93) \quad \begin{aligned} u^0 &= \bar{u}^0 + U^0 \quad \text{in } B_{\delta_f} \setminus \Gamma_0, \\ \bar{u}^0 &:= \frac{1}{2\pi} \int_{-\pi}^{\pi} u^0 d\theta = \text{const}, \quad \int_{-\pi}^{\pi} U^0 d\theta = 0. \end{aligned}$$

Let us define in $B_{\delta_f} \setminus \Gamma_0$

$$(94) \quad a(\rho) := \frac{1}{\pi} \int_{-\pi}^{\pi} U^0 \sin \frac{\theta}{2} d\theta, \quad U_1^0 := U^0 - a(\rho) \sin \frac{\theta}{2}.$$

Property (92a) follows immediately from (93) and (94).

Repeating the arguments of Lemma 2 we take a smooth cut-off function $\xi(\rho)$ supported in B_{δ_f} and substitute $v = \xi \sin \theta/2$ into equation (88). Accounting $f = 0$ in B_{δ_f} , decomposition (94) and (92a) get

$$\begin{aligned} 0 &= \int_{B_{\delta_f} \setminus \Gamma_0} \nabla u^0 \cdot \nabla \left(\xi \sin \frac{\theta}{2}\right) dx = \int_{B_{\delta_f} \setminus \Gamma_0} \nabla \left(a \sin \frac{\theta}{2}\right) \cdot \nabla \left(\xi \sin \frac{\theta}{2}\right) dx \\ &= \int_{-\pi}^{\pi} \int_0^{\delta_f} \left\{ \rho \frac{\partial(a \sin \frac{\theta}{2})}{\partial \rho} \frac{\partial(\xi \sin \frac{\theta}{2})}{\partial \rho} + \frac{1}{\rho} \frac{\partial(a \sin \frac{\theta}{2})}{\partial \theta} \frac{\partial(\xi \sin \frac{\theta}{2})}{\partial \theta} \right\} d\rho d\theta \\ &= \pi \int_0^{\delta_f} \left\{ -\frac{\partial}{\partial \rho} \left(\rho \frac{\partial a}{\partial \rho} \right) + \frac{a}{4\rho} \right\} \xi d\rho + \pi \left(\rho \frac{\partial a}{\partial \rho} \xi \right)_{\rho=0}^{\rho=\delta_f}. \end{aligned}$$

Since ξ is arbitrary we derive the ordinary differential equation

$$-\frac{\partial}{\partial \rho} \left(\rho \frac{\partial a}{\partial \rho} \right) + \frac{a}{4\rho} = 0, \quad \rho \in (0, \delta_f).$$

Its general solution implies

$$(95) \quad a(\rho) = K\sqrt{\rho} + \frac{c}{\sqrt{\rho}}, \quad K, c \in \mathbb{R}.$$

Using (95) we can evaluate from below the norm

$$\begin{aligned} \int_{B_{\delta_f} \setminus \Gamma_0} |\nabla u^0|^2 dx &= \pi \int_0^{\delta_f} \left\{ \rho \left(\frac{\partial a}{\partial \rho} \right)^2 + \frac{a^2}{4\rho} \right\} d\rho + \int_{B_{\delta_f} \setminus \Gamma_0} |\nabla U_1^0|^2 dx \\ &\geq \frac{\pi}{2} \int_0^{\delta_f} \left(K^2 + \frac{c^2}{\rho^2} \right) d\rho = +\infty. \end{aligned}$$

This fact contradicts to $u^0 \in H^1(B_{\delta_f} \setminus \Gamma_0)$ and concludes necessarily that $c = 0$. From (95) with $c = 0$, (93) and (94) we infer (90) and (91).

To derive (92b) we repeat the arguments of Lemma 4 modified for U_1^0 . Substituting expansion of the solution (90) into equality (69) and using (92a) gets for $\delta \in (0, \delta_f)$:

$$\begin{aligned} \int_{B_\delta \setminus \Gamma_0} |\nabla u^0|^2 dx &= \frac{\pi}{2} K^2 \delta + \int_{B_\delta \setminus \Gamma_0} |\nabla U_1^0|^2 dx \\ &= \int_{\partial B_\delta} \frac{\partial u^0}{\partial n} u^0 dx = \frac{\pi}{2} K^2 \delta + \int_{\partial B_\delta} \frac{\partial U_1^0}{\partial n} U_1^0 dx, \end{aligned}$$

which implies the equality

$$(96) \quad \int_{B_\delta \setminus \Gamma_0} |\nabla U_1^0|^2 dx = \int_{\partial B_\delta} \frac{\partial U_1^0}{\partial n} U_1^0 dx.$$

Moreover, (92a) guarantees for U_1^0 the following Poincaré inequality

$$\int_{-\pi}^{\pi} (U_1^0)^2 d\theta \leq \int_{-\pi}^{\pi} \left(\frac{\partial U_1^0}{\partial \theta} \right)^2 d\theta$$

in comparison with (63). Henceforth, from (96) we can estimate

$$\begin{aligned} \int_{B_\delta \setminus \Gamma_0} |\nabla U_1^0|^2 dx &\leq \frac{\delta}{2} \int_{-\pi}^{\pi} \left\{ \delta \left(\frac{\partial U_1^0}{\partial \rho} \right)^2 + \frac{1}{\delta} (U_1^0)^2 \right\} (\delta) d\theta \\ &\leq \frac{\delta}{2} \int_{\partial B_\delta} |\nabla U_1^0|^2 dx = \frac{\delta}{2} \frac{d}{d\delta} \int_{B_\delta \setminus \Gamma_0} |\nabla U_1^0|^2 dx. \end{aligned}$$

Applying the Grönwall inequality ends with the desired property (92b).

Note that U_1^0 obeys the H^2 -smoothness property, see [10]. A generalization of this result for the nonlinear crack problem (12) is given in [15]. \square

Next we consider the increment of solutions $w^{(r,\phi)} := u^{(r,\phi)} - u^0$, which describes the boundary-value problem (compare with (46)):

$$(97a) \quad -\Delta w^{(r,\phi)} = 0 \quad \text{in } \Omega_{(r,\phi)},$$

$$(97b) \quad w^{(r,\phi)} = 0 \quad \text{on } \Gamma_D, \quad \frac{\partial w^{(r,\phi)}}{\partial q} = 0 \quad \text{on } \Gamma_N,$$

$$(97c) \quad \frac{\partial w^{(r,\phi)}}{\partial \nu} = 0 \quad \text{on } \Gamma_0, \quad \frac{\partial w^{(r,\phi)}}{\partial \nu} = -\frac{\partial u^0}{\partial \nu} \quad \text{on } \gamma_{(r,\phi)}.$$

Its weak solution $w^{(r,\phi)} \in H(\Omega_{(r,\phi)})$ satisfies the variational equation

$$(98) \quad \int_{\Omega_{(r,\phi)}} \nabla w^{(r,\phi)} \cdot \nabla v \, dx = \left\langle \frac{\partial u^0}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \quad \text{for all } v \in H(\Omega_{(r,\phi)}).$$

Accounting representation (90), we state the following auxiliary result.

Lemma 5. *The solution of equation (98) admits the decomposition*

$$(99) \quad w^{(r,\phi)} = \frac{K}{2} \cos \frac{\phi}{2} h^{(r,\phi)} + Q_1, \\ \|Q_1\|_{H(\Omega_{(r,\phi)})} = O(r), \quad \|Q_1\|_{H(\Omega_0 \setminus B_\delta)} = O(r^{3/2}) \quad \text{for } \delta \in (0, \delta_f),$$

where $h^{(r,\phi)} \in H(\Omega_{(r,\phi)})$ solves the problem

$$(100) \quad \int_{\Omega_{(r,\phi)}} \nabla h^{(r,\phi)} \cdot \nabla v \, dx = - \left\langle \frac{1}{\sqrt{\rho}} \mathcal{H}_{\gamma_{(r,\phi)}}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \quad \text{for } v \in H(\Omega_{(r,\phi)}), \\ \mathcal{H}_{\gamma_{(r,\phi)}} = 1 \quad \text{on } \gamma_{(r,\phi)}, \quad \mathcal{H}_{\gamma_{(r,\phi)}} = 0 \quad \text{on } \Gamma_0.$$

Proof. From (90), (98) and (100) we derive that $Q_1 \in H(\Omega_{(r,\phi)})$ fulfills the equation

$$(101) \quad \int_{\Omega_{(r,\phi)}} \nabla Q_1 \cdot \nabla v \, dx = \left\langle \frac{\partial U_1^0}{\partial \nu}, \llbracket v \rrbracket \right\rangle_{\Gamma_{(r,\phi)}} \quad \text{for all } v \in H(\Omega_{(r,\phi)}).$$

Substituting $v = \chi_r Q_1 + (1 - \chi_r) Q_1$ into (100) with the cut-off function χ_r from (48) gets estimation due to $\partial U_1^0 / \partial \nu = \partial u^0 / \partial \nu = 0$ at Γ_0

$$\int_{\Omega_{(r,\phi)}} |\nabla Q_1|^2 \, dx \leq \left\| \frac{\partial U_1^0}{\partial \nu} \right\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)^*} \|\chi_r \llbracket Q_1 \rrbracket\|_{H_{00}^{1/2}(\Gamma_{(r,\phi)} \cap B_r)}.$$

Since $\Delta(\sqrt{\rho} \sin \theta/2) = 0$, then $\Delta U_1^0 = 0$ in B_{δ_f} , hence we can apply estimate (52) in Lemma 1 to U_1^0 . Repeating arguments used in the

proof of Theorem 1 we continue the estimation

$$\begin{aligned} \int_{\Omega(r,\phi)} |\nabla Q_1|^2 dx &\leq c_1 \left(\int_{B_{2r} \setminus \Gamma(r,\phi)} |\nabla U_1^0|^2 dx + \int_{B_{2r} \setminus \Gamma(r,\phi)} |\nabla Q_1|^2 dx \right)^{1/2} \\ &\leq c_2 r \left(\int_{B_\delta \setminus \Gamma(r,\phi)} |\nabla U_1^0|^2 dx + \int_{B_{2r} \setminus \Gamma(r,\phi)} |\nabla Q_1|^2 dx \right)^{1/2} \leq c_3 r \left(\int_{\Omega(r,\phi)} |\nabla Q_1|^2 dx \right)^{1/2} \end{aligned}$$

due to (92b), thus $\|Q_1\|_{H(\Omega(r,\phi))} = O(r)$.

In $\Omega_0 \setminus B_\delta$ the Green formula provides the following identity for Q_1 (similarly to (62))

$$\int_{\Omega_0 \setminus B_\delta} |\nabla Q_1|^2 dx = - \int_{\partial B_\delta} \frac{\partial Q_1}{\partial n} Q_1 dx.$$

Henceforth, we can apply Lemma 3 to Q_1 and obtain

$$\int_{\Omega_0 \setminus B_\delta} |\nabla Q_1|^2 dx \leq \frac{\delta_0}{\delta} \int_{\Omega_0 \setminus B_{\delta_0}} |\nabla Q_1|^2 dx, \quad r < \delta_0 \leq \delta < \delta_f.$$

Taking $\delta_0 = 2r$, the estimation finishes with $\|Q_1\|_{H(\Omega_0 \setminus B_\delta)} = O(r^{3/2})$. \square

From (100) it is interesting to observe that $h^{(r,\phi)}$ in decomposition (99) is independent of the particular choice of the forces f and g , but it depends on the geometry of $\Omega(r,\phi)$ only. We proceed with an expansion of $h^{(r,\phi)}$ in the Fourier series.

Lemma 6. *The following expansion holds*

$$(102) \quad h^{(r,\phi)}(\rho, \theta) = \left(\bar{h}^{(r,\phi)} + b^{(r,\phi)}(\rho) \sin \frac{\theta}{2} \right) \chi(\rho) + Q_2(\rho, \theta) \quad \text{in } \Omega(r,\phi),$$

where $\bar{h}^{(r,\phi)} = \text{const}$, χ is a cut-off function supported in B_R ,

$$(103) \quad b^{(r,\phi)}(\rho) = C_{1/2}^{(r,\phi)} \sqrt{\rho} - C_{-1/2}^{(r,\phi)} \frac{1}{\sqrt{\rho}} \quad \text{for } \rho > r,$$

stress intensity factors $C_{\pm 1/2}^{(r,\phi)} \in \mathbb{R}$ are given by:

$$(104) \quad \begin{aligned} C_{1/2}^{(r,\phi)} &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\partial}{\partial \rho} (\sqrt{\rho} h^{(r,\phi)}) \sin \frac{\theta}{2} d\theta, \\ C_{-1/2}^{(r,\phi)} &= \frac{1}{\pi} \int_{-\pi}^{\pi} \rho^2 \frac{\partial}{\partial \rho} \left(\frac{h^{(r,\phi)}}{\sqrt{\rho}} \right) \sin \frac{\theta}{2} d\theta \quad \text{for } \rho \in (r, R), \end{aligned}$$

and the residual term yields

$$(105) \quad \|Q_2\|_{H(\Omega_0 \setminus B_\delta)} = O(r^{3/2}) \quad \text{for fixed } \delta \in (r, R).$$

Proof. Since (100) is a particular case of the variational equation (98), all the results of $w^{(r,\phi)}$ remain valid for $h^{(r,\phi)}$, too. By this reason, we can apply Lemma 3 and conclude with

$$h^{(r,\phi)} = \bar{h}^{(r,\phi)} + B^{(r,\phi)} \quad \text{in } B_R \setminus \Gamma_{(r,\phi)},$$

$$\bar{h}^{(r,\phi)} := \frac{1}{2\pi} \left(\int_{-\pi}^{\phi} + \int_{\phi}^{\pi} \right) h^{(r,\phi)} d\theta = \text{const}, \quad \left(\int_{-\pi}^{\phi} + \int_{\phi}^{\pi} \right) B^{(r,\phi)} d\theta = 0.$$

We expand further in $B_R \setminus \Gamma_{(r,\phi)}$:

$$(106) \quad b^{(r,\phi)} := \frac{1}{\pi} \left(\int_{-\pi}^{\phi} + \int_{\phi}^{\pi} \right) h^{(r,\phi)} \sin \frac{\theta}{2} d\theta,$$

$$B_1^{(r,\phi)} := B^{(r,\phi)} - b^{(r,\phi)} \sin \frac{\theta}{2},$$

which implies that

$$(107) \quad \int_{-\pi}^{\pi} B_1^{(r,\phi)} d\theta = \int_{-\pi}^{\pi} B_1^{(r,\phi)} \sin \frac{\theta}{2} d\theta = 0 \quad \text{for } \rho \in (0, R).$$

For arbitrary cut-off function $\xi(\rho)$ supported in $[r, R]$, the substitution of $v = \xi \sin \theta/2$ as a test function into (100) gets

$$0 = \int_{B_R \setminus B_r} \nabla h^{(r,\phi)} \cdot \nabla \left(\xi \sin \frac{\theta}{2} \right) dx = \pi \int_r^R \left\{ -\frac{\partial}{\partial \rho} \left(\rho \frac{\partial b^{(r,\phi)}}{\partial \rho} \right) + \frac{b^{(r,\phi)}}{4\rho} \right\} \xi d\rho.$$

This proves the representation of $b^{(r,\phi)}$ in the form of (103), similarly to (95). Then (103) and (106) result in the equality

$$C_{1/2}^{(r,\phi)} \sqrt{\rho} - C_{-1/2}^{(r,\phi)} \frac{1}{\sqrt{\rho}} = \frac{1}{\pi} \int_{-\pi}^{\pi} h^{(r,\phi)} \sin \frac{\theta}{2} d\theta \quad \text{for } \rho \in (r, R).$$

Differentiating it with respect to ρ yields (104).

With the help of a cut-off function χ supported in B_R and such that $\chi \equiv 1$ in B_δ , $\delta \in (r, R)$, we can define a function $Q_2 \in H(\Omega_{(r,\phi)})$ by

$$Q_2 := h^{(r,\phi)} - \left(\bar{h}^{(r,\phi)} + b^{(r,\phi)} \sin \frac{\theta}{2} \right) \chi, \quad Q_2 = B_1^{(r,\phi)} \text{ in } B_\delta \setminus \Gamma_{(r,\phi)}.$$

The Green formula written in $\Omega_0 \setminus B_\delta$ gets for $v \in H(\Omega_0 \setminus B_\delta)$:

$$\begin{aligned} \int_{\Omega_0 \setminus B_\delta} \nabla h^{(r,\phi)} \cdot \nabla v \, dx &= \int_{\Omega_0 \setminus B_\delta} \nabla \left(Q_2 + b^{(r,\phi)} \chi \sin \frac{\theta}{2} \right) \cdot \nabla v \, dx \\ &= - \int_{\partial B_\delta} \frac{\partial h^{(r,\phi)}}{\partial n} v \, dx = - \int_{\partial B_\delta} \left\{ \frac{\partial}{\partial n} \nabla \left(B_1^{(r,\phi)} + b^{(r,\phi)} \sin \frac{\theta}{2} \right) \right\} v \, dx. \end{aligned}$$

Substituting $v = Q_2$ as a test function here we obtain the equality, which is similar to (96), and estimate it in the same way

$$\begin{aligned} \int_{\Omega_0 \setminus B_\delta} |\nabla Q_2|^2 \, dx &= - \int_{\partial B_\delta} \frac{\partial B_1^{(r,\phi)}}{\partial n} B_1^{(r,\phi)} \, dx \\ &\leq \frac{\delta}{2} \int_{\partial B_\delta} |\nabla B_1^{(r,\phi)}|^2 \, dx = - \frac{\delta}{2} \frac{d}{d\delta} \int_{\Omega_0 \setminus B_\delta} |\nabla Q_2|^2 \, dx \end{aligned}$$

due to property (107). The Grönwall inequality provides

$$(108) \quad \int_{\Omega_0 \setminus B_\delta} |\nabla Q_2|^2 \, dx \leq \left(\frac{\delta_0}{\delta} \right)^2 \int_{\Omega_0 \setminus B_{\delta_0}} |\nabla Q_2|^2 \, dx, \quad 0 < \delta_0 \leq \delta < R.$$

Taking $\delta_0 = 2r$ we proceed (108):

$$\begin{aligned} \int_{\Omega_0 \setminus B_\delta} |\nabla Q_2|^2 \, dx &\leq c_1 r^2 \int_{\Omega_0 \setminus B_{2r}} |\nabla Q_2|^2 \, dx \leq c_2 r^2 \int_{\Omega(r,\phi)} |\nabla h^{(r,\phi)}|^2 \, dx \\ &\leq c_3 r^2 \int_{\Omega(r,\phi)} |\nabla w^{(r,\phi)}|^2 \, dx + O(r^4) = O(r^3) \end{aligned}$$

in view of Lemma 5 and Theorem 1. The latter estimate implies (105), and this ends the proof. \square

From Lemma 5 and Lemma 6 we derive the following.

Proposition 6. *Apart from the kink point, the representation holds*

$$(109) \quad \begin{aligned} w^{(r,\phi)}(\rho, \theta) &= \bar{w}^{(r,\phi)} + \frac{K}{2} \cos \frac{\phi}{2} \left(C_{1/2}^{(r,\phi)} \sqrt{\rho} - C_{-1/2}^{(r,\phi)} \frac{1}{\sqrt{\rho}} \right) \sin \frac{\theta}{2} \\ &+ Q(\rho, \theta), \quad \|Q\|_{H(B_R \setminus B_\delta)} = O(r^{3/2}) \quad \text{for fixed } \delta \in (r, \delta_f). \end{aligned}$$

Finally, we state the main result of this section.

Theorem 3. *For the linear crack problems (84)–(89), expansion of the potential energy at the point of kink as $r \rightarrow 0$ reads*

$$(110) \quad \Pi(r, \phi) = \Pi(0) - \frac{\pi}{4} K^2 \cos \frac{\phi}{2} \int_0^r \frac{1}{t} C_{-1/2}^{(t, \phi)} dt + O(r^{3/2}),$$

with the stress intensity factors K and $C_{-1/2}^{(t, \phi)}$ defined in (91) and (104),

$$(111) \quad 0 \leq \int_0^r \frac{1}{t} C_{-1/2}^{(t, \phi)} dt = O(r).$$

Proof. Let us calculate the integral $\Pi_V^1(2u^0, w^{(r, \phi)}, f; \Omega_0 \setminus B_\delta)$ from Theorem 2 explicitly by substituting here the representation of solutions (90) and (109). In this way we will specify the decomposition of energy (82)–(83) for the linear problem in the form of (110)–(111).

We choose the cut-off function $\eta(\rho)$ for the velocity V in (16) and the cut-off function $\chi(\rho)$ in representation (102) such that:

$$\begin{aligned} \eta &\equiv 1 \quad \text{in } B_\delta, \quad \text{supp}(\eta) \subset B_{\delta_f}, \\ \chi &\equiv 1 \quad \text{in } B_{\delta_f}, \quad \text{supp}(\chi) \subset B_R \quad \text{for } \delta < \delta_f < R. \end{aligned}$$

Using (99) and (102) due to Corollary 1 and Proposition 6 gets

$$\begin{aligned} &\frac{1}{r} \Pi_V^1(2u^0, w^{(r, \phi)}, f; \Omega_0 \setminus B_\delta) \\ &= \frac{K}{2r} \cos \frac{\phi}{2} \Pi_V^1\left(2u^0, \chi\left(\bar{h}^{(r, \phi)} + b^{(r, \phi)} \sin \frac{\theta}{2}\right), f; \Omega_0 \setminus B_\delta\right) + O(\sqrt{r}), \end{aligned}$$

with $b^{(r, \phi)}(\rho)$ given in (103). Integrating the following integral by parts in $B_{\delta_f} \setminus B_\delta$, where the solutions are smooth and $\chi \equiv 1$,

$$\begin{aligned} &\Pi_V^1\left(2u^0, \chi\left(\bar{h}^{(r, \phi)} + b^{(r, \phi)} \sin \frac{\theta}{2}\right), f; \Omega_0 \setminus B_\delta\right) \\ &= \int_{B_{\delta_f} \setminus B_\delta} \left\{ \nabla u^0 \cdot \left(\text{div}(V) I - 2 \frac{\partial V}{\partial x} \right) \nabla \left(b^{(r, \phi)} \sin \frac{\theta}{2} \right) \right. \\ &\quad \left. - \text{div}(V f) \left(\bar{h}^{(r, \phi)} + b^{(r, \phi)} \sin \frac{\theta}{2} \right) \right\} dx, \end{aligned}$$

similarly to the proof of Corollary 1 we obtain

$$\begin{aligned}
& \Pi_V^1 \left(2u^0, \chi \left(\bar{h}^{(r,\phi)} + b^{(r,\phi)} \sin \frac{\theta}{2} \right), f; \Omega_0 \setminus B_\delta \right) \\
&= \int_{B_{\delta_f} \setminus B_\delta} \left\{ (\Delta u^0 + f) \left(V \cdot \nabla \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) \right) + \Delta \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) (V \cdot \nabla u^0) \right\} dx \\
&+ \delta \int_{\partial B_\delta} \left\{ \nabla u^0 \cdot \nabla \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) - 2 \frac{\partial u^0}{\partial n} \frac{\partial}{\partial n} \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) \right\} dx \\
&+ \int_{\Gamma_0 \cap (B_{\delta_f} \setminus B_\delta)} \left\{ \frac{\partial u^0}{\partial \nu} \left[\left(b^{(r,\phi)} \sin \frac{\theta}{2} \right)_{,1} \right] + \frac{\partial}{\partial \nu} \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) \left[u_{,1}^0 \right] \right\} dx \\
&= \delta \int_{\partial B_\delta} \left\{ \nabla u^0 \cdot \nabla \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) - 2 \frac{\partial u^0}{\partial n} \frac{\partial}{\partial n} \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) \right\} dx
\end{aligned}$$

in view of (89a), (89c), and relations

$$\Delta \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) = 0 \text{ in } \Omega_0, \quad \frac{\partial}{\partial \nu} \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) = 0 \text{ on } \Gamma_0$$

due to (103). Applying decomposition (90), from Proposition 5 together with the orthogonality conditions (92a) it proceeds further

$$\begin{aligned}
& \delta \int_{\partial B_\delta} \left\{ \nabla u^0 \cdot \nabla \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) - 2 \frac{\partial u^0}{\partial n} \frac{\partial}{\partial n} \left(b^{(r,\phi)} \sin \frac{\theta}{2} \right) \right\} dx \\
&= K \int_{-\pi}^{\pi} \left\{ \frac{\partial(\sqrt{\rho} \sin \frac{\theta}{2})}{\partial \theta} \frac{\partial(b^{(r,\phi)} \sin \frac{\theta}{2})}{\partial \theta} - \delta^2 \frac{\partial(\sqrt{\rho} \sin \frac{\theta}{2})}{\partial \rho} \frac{\partial(b^{(r,\phi)} \sin \frac{\theta}{2})}{\partial \rho} \right\}_{\rho=\delta} d\theta \\
&= \pi K \left\{ \frac{\sqrt{\delta}}{4} \left(\sqrt{\delta} C_{1/2}^{(r,\phi)} - \frac{C_{-1/2}^{(r,\phi)}}{\sqrt{\delta}} \right) - \frac{\delta^2}{2\sqrt{\delta}} \left(\frac{C_{1/2}^{(r,\phi)}}{2\sqrt{\delta}} + \frac{C_{-1/2}^{(r,\phi)}}{2\delta\sqrt{\delta}} \right) \right\} = -\frac{\pi}{2} K C_{-1/2}^{(r,\phi)}.
\end{aligned}$$

Thus, we arrive at the equality for the linear crack problem

$$\begin{aligned}
(112) \quad \Pi'(r, \phi) &= \frac{1}{r} \Pi_V^1(2u^0, w^{(r,\phi)}, f; \Omega_0 \setminus B_\delta) \\
&= -\frac{\pi}{4r} K^2 \cos \frac{\phi}{2} C_{-1/2}^{(r,\phi)} + O(\sqrt{r}).
\end{aligned}$$

Henceforth, Theorem 2 argues (110) and (111). \square

We finish with few remarks on Theorem 3.

Corollary 6. *If $K = 0$ in (91), i.e., the solution u^0 is H^2 -smooth around the kink point, then (110) implies that*

$$K = 0 \Rightarrow \quad \Pi'(0, \phi) = 0, \quad \Pi(r, \phi) = \Pi(0) + O(r^{3/2}).$$

Corollary 7. *If $\phi = 0$, i.e., there is no kink, then it is known that the derivative exists and $\Pi'(0, 0) = -\pi/4 K^2$, for example, see [20]. Therefore, from (110) and (111) we infer that*

$$C_{-1/2}^{(r,0)} = r + o(r).$$

CONCLUSION

The asymptotic representations resulting formal analysis are also of practical meaning for engineers. In fact, the expansion of potential energy (110) is expressed via stress intensity factors K and $C_{-1/2}^{(r,\phi)}$. They can be calculated as a path-independent integrals by explicit formulas (91) and (104). The former constant K depends on the specific choice of data of the reference crack problem before kinking, while the latter $(r, \phi) \mapsto C_{-1/2}^{(r,\phi)}$ are universal functions depending on the geometry of the kinked domain $\Omega_{(r,\phi)}$ only. These implicit quantities are to be determined from a generic problem of the crack kinking (100).

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