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by

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Abstract
We interpret the element $\frac{1}{2i\hbar}(u{*}v{+}v{*}u)$ in the generators u, v of the Weyl algebra W_2 as an indeterminate in $\mathbb{N}{+}\frac{1}{2}$ or $-(\mathbb{N}{+}\frac{1}{2})$, using methods of the transcendental calculus outlined in the announcement [14]. The main purpose of this paper is to give a rigorous proof for the part of [14] which introduces this indeterminate phenomenon. Namely, we discuss how to obtain associativity in the transcendental calculus and show how the Hadamard finite part procedure can be implemented in our context.

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1 Introduction

Deformation quantization, first proposed in [3], is a fruitful approach to developing quantum theory in a purely algebraic framework, and was a prototype for noncommutative calculus on noncommutative spaces. It was first treated as a formal noncommutative calculus, with the Planck constant $\hbar$ regarded as a formal parameter, but has been extended to nonformal cases, as in the studies of noncommutative tori [18] and quantum groups [20]. In fact, the formal and nonformal noncommutative calculus have quite different features.

In [11], we analyzed star exponential functions of quadratic forms in the Weyl algebra and uncovered several mysterious phenomena unanticipated from the formal case. These mysterious phenomena reflect the fact that star exponential functions of quadratic forms (see [6] and [15]) lie outside of the Weyl algebra. These new features suggest a new approach to noncommutative nonformal calculus. In this paper, we show that this new calculus is necessary to treat transcendental elements of the Weyl algebra.

From the papers [11]-[14], we know that the Moyal product, the most typical product on the Weyl algebra, is not sufficient to treat transcendental elements such as star exponential functions. For this reason, we introduced a family of $*_K$-products on the Weyl algebra depending on a complex symmetric matrix K and developed a transcendental nonformal noncommutative calculus specifically formulated to treat star exponential functions of quadratic forms. The transcendental elements of the Weyl algebra have a realization depending on the $*_K$-product, which we called the *K-ordered expression.* Thus, properties of (transcendental) elements of the Weyl algebra depend on the choice of product $*_K$,

We now propose as a principle, called the *Independence of Ordering Principle* (IOP), that the relevant properties of transcendental elements of the Weyl algebra do not depend on the choice of ordered expression, just as properties and objects in differential geometry do not depend on the choice of coordinate expression. Following this principle, in [11] we proposed the notion of a group-like object of star exponential functions of quadratic forms on the Weyl algebra. The IOP seems to be a new outlook on interpreting physical phenomena/mathematical phenomena, especially for treating quantum objects and phenomena from an algebraic point of view.

As a test case, we examine this principle on the nonformal noncommutative calculus for transcendental elements of the Weyl algebra. As part of this approach, we interpret an element as an indeterminate in a discrete set in the case of the Weyl algebra with two generators.

Let W_2 be the Weyl algebra with generators u, v obeying the commutation relation

$$[u, v] = -i\hbar. \tag{1}$$

We consider the element $\frac{1}{i\hbar}u{\circ}v = \frac{1}{2i\hbar}(u{*}v+v{*}u)$ of W_2. We show that $\frac{1}{i\hbar}u{\circ}v$ can be interpreted as an indeterminate in $\mathbb{N}+\frac{1}{2}$ or $-(\mathbb{N}+\frac{1}{2})$, not from a more standard operator theoretic point of view but from a purely algebraic approach, using the IOP that a physical/mathematical object should be independent of its various ordered expressions.

In our approach, we interpret $\frac{1}{i\hbar}u{\circ}v$ in two ways: 1) via the analytic continuation of inverses of $z+\frac{1}{i\hbar}u{\circ}v$ and 2) via the $*$-product of the $*$-sin function and the $*$-gamma function using ordered expressions. These results have been already announced in [14] with outlines of proofs. The main purpose of this paper is to give a rigorous description of method 1) and therefore to realize $\frac{1}{i\hbar}u{\circ}v$ as an indeterminate in the discrete set. The main ingredients of the proof are dealing with associativity in the framework of the transcendental calculus of [14] and applying the Hadamard finite part procedure in this context. For a family of $*_K$-products on the Weyl algebra W_2, we provide rules for the associativity of the extended products $*_K$, and in preparation for the definition of the inverse of $z+\frac{1}{i\hbar}u{\circ}v$, we investigate star exponentials $e_*^{z+\frac{1}{i\hbar}u\circ v}$ and their ordered versions.

We leave the the finite part regularization method for Fréchet algebra valued functions in the subsection 6.1. For a holomorphic function $f(z)$ with a pole at $z{=}z_0$, we define the finite part of $f(z)$ as

$$\mathrm{FP}(f(z)){=}\begin{cases} f(z) & z{\neq}z_0 \\ \mathrm{Res}_{w=0}\frac{1}{w}(f(z_0{+}w)) & z{=}z_0. \end{cases}$$

We first construct the inverses of $z{+}\frac{1}{i\hbar}u{\circ}v$ by using the star exponential function $e_*^{z+\frac{1}{i\hbar}u\circ v}$ and a K-ordered expression. We can construct two inverses of $z{+}\frac{1}{i\hbar}u{\circ}v$ as follows:

$$(z+\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}=\int_{-\infty}^{0}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$$

and

$$(z+\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}=-\int_{0}^{\infty}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$$

(see [7] and [9] for more details). Both inverses have analytic continuations for generic ordered expression. In §6, we mainly study the inverse $(z+\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}$, as the other inverse has similar properties.

In §6, we show the following:

Theorem 1.1 *For generic ordered expressions, the inverses* $(z{+}\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}$, $(z{-}\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}$ *extend to* $\mathcal{E}_{2+}(\mathbb{C}^2)$*-valued holomorphic functions of* z *on* $\mathbb{C}{-}\{-(\mathbb{N}{+}\frac{1}{2})\}$.

Here, we refer the class $\mathcal{E}_{2+}(\mathbb{C}^2)$ in the subsection 2.2.

Employing the Hadamard technique of extracting finite parts of divergent integrals, we now extend the definition of the $*$-product using finite part regularization. We define the new product of $(z{+}\frac{1}{i\hbar}u{\circ}v)_{*\pm}^{-1}$ with either the polynomial $q(u,v)$ or $q(u,v){=}e_*^{s\frac{1}{i\hbar}u\circ v}$ by

$$(z{+}\frac{1}{i\hbar}u{\circ}v)_{*\pm}^{-1}\tilde{*}q(u,v){=}(\mathrm{FP}(z{+}\frac{1}{i\hbar}u{\circ}v)_{*\pm}^{-1}){*}q(u,v). \tag{2}$$

Note that the result may not be continuous in z.

The following is an description of the discrete phenomena for $\frac{1}{i\hbar}u{\circ}v$ via method 1):

Theorem 1.2 *Using definition* (2) *for the* $\tilde{*}$*-product, we have*

$$(z{+}\frac{1}{i\hbar}u{\circ}v)\tilde{*}(z{+}\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}{=}\begin{cases} 1 & z{\notin}{-}(\mathbb{N}{+}\frac{1}{2}) \\ 1{-}\frac{1}{n!}(\frac{1}{i\hbar}u)^n{*}\varpi_{00}{*}v^n & z{=}{-}(n{+}\frac{1}{2}) \end{cases}, \tag{3}$$

$$(z{-}\frac{1}{i\hbar}u{\circ}v)\tilde{*}(z{-}\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}{=}\begin{cases} 1 & z{\notin}{-}(\mathbb{N}{+}\frac{1}{2}) \\ 1{-}\frac{1}{n!}(\frac{1}{i\hbar}v)^n{*}\overline{\varpi}_{00}{*}u^n & z{=}{-}(n{+}\frac{1}{2}) \end{cases}. \tag{4}$$

for generic ordered expressions.

We will interpret this discrete phenomena for $\frac{1}{i\hbar}u{\circ}v$ via method 2) in a forthcoming paper.

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2 General ordered expressions and IOP

We introduce a method to realize the Weyl algebra via a family of ordered expressions. This leads to a transcendental calculus for the Weyl algebra.

2.1 Fundamental product formulas and intertwiners

Let $\mathfrak{S}(n)$ and $\mathfrak{A}(n)$ be the spaces of complex symmetric matrices and skew-symmetric matrices respectively, and set $\mathfrak{M}(n) = \mathfrak{S}(n) \oplus \mathfrak{A}(n)$. We denote by $\boldsymbol{u}$ the set of generators $\boldsymbol{u} = (u_1, \dots, u_{2m})$. For an arbitrary fixed $n \times n$-complex matrix $\Lambda \in \mathfrak{M}(n)$, we define a product $*_{_\Lambda}$ on the space of polynomials $\mathbb{C}[\boldsymbol{u}]$ by the formula

$$(5) \qquad f *_{_\Lambda} g = f e^{\frac{i\hbar}{2}(\sum \overleftarrow{\partial_{u_i}} \Lambda^{ij} \overrightarrow{\partial_{u_j}})} g = \sum_k \frac{(i\hbar)^k}{k! 2^k} \Lambda^{i_1 j_1} \cdots \Lambda^{i_k j_k} \partial_{u_{i_1}} \cdots \partial_{u_{i_k}} f \ \partial_{u_{j_1}} \cdots \partial_{u_{j_k}} g.$$

It is known and not hard to prove that $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ is an associative algebra.

The algebraic structure of $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ is determined by the skew-symmetric part of Λ, if the generators are fixed. In particular, if Λ is a symmetric matrix, $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ is isomorphic to the usual polynomial algebra.

For every symmetric matrix $K \in \mathfrak{S}(n)$, the operator

$$(6) \qquad I_0^{K}(f) = \exp\Big(\frac{i\hbar}{4} \sum_{i,j} K^{ij} \partial_{u_i} \partial_{u_j}\Big) f$$

gives an isomorphism $I_0^{K} : (\mathbb{C}[\boldsymbol{u}], *_{_\Lambda}) \to (\mathbb{C}[\boldsymbol{u}], *_{_{\Lambda+K}})$. Namely, for any $f, g \in \mathbb{C}[\boldsymbol{u}]$:

$$(7) \qquad I_0^{K}(f *_{_\Lambda} g) = I_0^{K}(f) *_{_{\Lambda+K}} I_0^{K}(g).$$

Let $\Lambda = K + J$ be the symmetric/skew symmetric parts of Λ, $K \in \mathfrak{S}(n)$, $J \in \mathfrak{A}(n)$. Changing K while leaving J fixed will be called a *deformation* of the expression of elements, as the algebra remains in the same isomorphism class.

We view these expressions of algebra elements as analogous to the "local coordinate expression" of functions on a manifold. Changing K corresponds to a local coordinate transformation on a manifold. In this context, we call the product formula (5) the K-*ordered expression*, i.e. ignoring the fixed skew part J, and $*_{_K}$ stands sometimes for $*_{_\Lambda}$ with J understood.

The big difference from local coordinate expressions for functions on a manifold is precisely that in our context there is no "underlying topological space".

In the following we set $n = 2m$ and $J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$. $(\mathbb{C}[\boldsymbol{u}], *_{_\Lambda})$ is called the Weyl algebra, with isomorphism class denoted by W_{2m}.

According to the choice of $K = 0, \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}, \begin{bmatrix} 0 & -I_m \\ -I_m & 0 \end{bmatrix}$, the K-ordered expression is called the Weyl ordered, the normal ordered and the anti-normal ordered expressions, respectively. The intertwiner between a K-ordered expression and a K'-ordered expression is given by

$$(8) \qquad I_{K}^{K'}(f) = \exp\Big(\frac{i\hbar}{4} \sum_{i,j} (K'^{ij} - K^{ij}) \partial_{u_i} \partial_{u_j}\Big) f \ \ (= I_0^{K'}(I_0^{K})^{-1}(f)),$$

giving an isomorphism $I_{K}^{K'} : (\mathbb{C}[\boldsymbol{u}]; *_{_{K+J}}) \to (\mathbb{C}[\boldsymbol{u}]; *_{_{K'+J}})$ between algebras.

2.2 Extension of products and intertwiners

In what follows we write $*_K$ for $*_{K+J}$ for simplicity. Let $\mathbb{C}[\boldsymbol{u}][[\hbar]]$ be the space of all formal power series in $\hbar$ with polynomials in $\boldsymbol{u}$ as coefficients. Obviously, the $*_{_K}$-product and the intertwiners extend naturally to $\mathbb{C}[\boldsymbol{u}][[\hbar]]$ by the same formulas. $(\mathbb{C}[\boldsymbol{u}][[\hbar]], *_{_K})$ is an associative algebra and $I_{_K}^{^{K'}}$ is an algebra isomorphism from $(\mathbb{C}[\boldsymbol{u}][[\hbar]], *_{_K})$ to $(\mathbb{C}[\boldsymbol{u}][[\hbar]], *_{_{K'}})$.

Let $Hol(\mathbb{C}^n)$ be the space of all holomorphic functions on $\mathbb{C}^n$ with the topology of uniform convergence topology on compact domains. The following fundamental lemma follows easily from the product formula (5) together with Taylor's formula:

Lemma 2.1 *Let $p(\boldsymbol{u})$ be either a polynomial of $\boldsymbol{u}$ or an exponential function of a linear combination of generators $p(\boldsymbol{u}) = e^{\sum a_i u_i}$. Then the left multiplication $p(\boldsymbol{u})*_{_K}$ (resp. the right multiplication $*_{_K}p(\boldsymbol{u})$) is a continuous linear mapping from $Hol(\mathbb{C}^n)$ to itself. Associativity $(f*_{_K}g)*_{_K}h = f*_{_K}(g*_{_K}h)$ holds if two of f, g, h are polynomials.*

For every positive real number p, we set

$$\mathcal{E}_p(\mathbb{C}^n) = \{f \in Hol(\mathbb{C}^n) \mid \|f\|_{p,s} = \sup |f|\, e^{-s|\xi|^p} < \infty,\ \forall s > 0\} \tag{9}$$

where $|\xi| = (\sum_i |u_i|^2)^{1/2}$. The family of seminorms $\{||\cdot||_{p,s}\}_{s>0}$ induces a topology on $\mathcal{E}_p(\mathbb{C}^n)$ and $(\mathcal{E}_p(\mathbb{C}^n), \cdot)$ is an associative commutative Fréchet algebra, where the dot $\cdot$ is the ordinary product for functions in $\mathcal{E}_p(\mathbb{C}^n)$.

Let H be a polynomial of order p. Then, $e^H \in \mathcal{E}_{p'}(\mathbb{C}^n)$ for every $p' > p$, but $e^H \notin \mathcal{E}_p(\mathbb{C}^n)$. Note also that $exp \sqrt[q]{H} \in \mathcal{E}_{p'/q}$ for every $p' > p$ on a suitable Riemann surface.

It is easily seen that for $0 < p < p'$, there is a continuous embedding

$$\mathcal{E}_p(\mathbb{C}^n) \subset \mathcal{E}_{p'}(\mathbb{C}^n) \tag{10}$$

as commutative Fréchet algebras (cf. [4],[15]), and that $\mathcal{E}_p(\mathbb{C}^n)$ is $Sp(m, \mathbb{C})$-invariant.

It is obvious that every polynomial is contained in $\mathcal{E}_p(\mathbb{C}^n)$ and that $\mathbb{C}[\boldsymbol{u}]$ is dense in $\mathcal{E}_p(\mathbb{C}^n)$ for any $p > 0$ in the Fréchet topology defined by the family of seminorms $\{||\ ||_{p,s}\}_{s>0}$.

Theorem 2.1 *Assume $0 < p \leq 2$. The product formula* (5) *extends in the following way:*
(*a*) *The space $(\mathcal{E}_p(\mathbb{C}^n), *_{_K})$ forms a complete noncommutative topological associative algebra over* $\mathbb{C}$ (cf. [6]).
(*b*) *The intertwiner $I_{_K}^{^{K'}}$ extends to an isomorphism of $(\mathcal{E}_p(\mathbb{C}^n), *_{_K})$ onto itself* (cf.[11]).

See also [15] for the general case with precise proofs and several comments.

It is easily seen that the following identities hold on $\mathcal{E}_p(\mathbb{C}^n)$, $p \leq 2$:

$$I_{_{K'}}^{^K} I_{_K}^{^{K'}} = 1, \quad I_{_{K'}}^{^{K''}} I_{_K}^{^{K'}} = I_{_K}^{^{K''}}. \tag{11}$$

For every $f \in \mathcal{E}_p(\mathbb{C}^n)$ such that $p \leq 2$, $f(K) = I_{_0}^{^K}(f)$ is globally defined on $\mathfrak{S}(n)$.

Thus, we naturally extend our object f to the space of all mutually intertwined sections $\{f(K); K \in \mathfrak{S}(n)\}$ of the trivial bundle $\coprod_{K\in\mathfrak{S}(n)}(\mathcal{E}_p(\mathbb{C}^n), *_{_K})$, $0<p\leq 2$. However, several anomalous phenomena occur in the space $(\mathcal{E}_{2+}(\mathbb{C}^n), *_{_K}) = \bigcap_{p>2}(\mathcal{E}_p(\mathbb{C}^n), *_{_K})$. See [7]-[9], [11]-[13].

Theorem 2.2 *For every pair (p, p') such that $\frac{1}{p} + \frac{1}{p'} \geq 1$ the product* (5) *extends to a continuous bilinear mapping $\mathcal{E}_p(\mathbb{C}^n) \times \mathcal{E}_{p'}(\mathbb{C}^n) \to \mathcal{E}_{p\vee p'}(\mathbb{C}^n)$.*

By Theorems 2.1 and 2.2, associativity $f*(g*h) = (f*g)*h$ holds for $f, g, h \in \mathcal{E}_2(\mathbb{C}^n)$. Moreover if one of f, g, h is in $\mathcal{E}_p(\mathbb{C}^n)$, $p > 2$, then by using the polynomial approximation theorem, we have that associativity holds if the two others are in $\mathcal{E}_{p'}(\mathbb{C}^n)$ such that $\frac{1}{p}+\frac{1}{p'} \geq 1$.

Since $\mathcal{E}_p(\mathbb{C}^n)$ is a Fréchet space, we have

Lemma 2.2 *Let M be a compact domain in $\mathbb{R}^m$, and let $x \mapsto f_x \in \mathcal{E}_p(\mathbb{C}^n)$ be a continuous mapping of M into $\mathcal{E}_p(\mathbb{C}^n)$. Then the integral $\int_M f_x dV_x$ of f_x on M is an element of $\mathcal{E}_p(\mathbb{C}^n)$.*

3 Star exponential functions

In differential geometry, it is widely accepted that geometrical notions should have coordinate free expressions. Obviously, the algebraic structure of $(\mathbb{C}[\boldsymbol{u}], *_\Lambda)$ depends only on the skew part of Λ. This analogy with geometry makes it plausible to introduce the Independence of Ordering Principle (IOP), namely that the algebraic interpretation of physical phenomena should be independent of the choice of ordered expression (cf. [1]).

In fact, this principle for the class $\mathcal{E}_2(\mathbb{C}^n)$ is reflected in Theorem 2.1. However, as will be seen below, we have to think carefully about the true meaning of IOP, since there are many delicate anomalous phenomena in the transcendental calculus of star-exponential functions. In spite of these difficulties, we believe that properties which appear in generic (i.e. almost all/open dense) ordered expressions are fundamental features of this theory. In the end, IOP provides deeper insight into the extended Weyl algebra.

For an element H_* of the Weyl algebra, we define the $*$-exponential function $e_*^{tH_*}$ as the family $\{f_t(K)\}$ of real analytic solutions of the evolution equation

$$\frac{d}{dt} f_t(K) = {:}H_*{:}_K *_K f_t(K), \tag{12}$$

with the initial condition $f_0(K) = 1$. We think of $f_t(K)$ as the K-ordered expression of $e_*^{tH_*}$, and denote it by ${:}e_*^{tH_*}{:}_K = f_t(K)$.

Provided ${:}e_*^{sH_*}{:}_K$ exists for every $s \in \mathbb{C}$, they form a complex one parameter subgroup, for the exponential law holds by the uniqueness of real analytic solutions. If ${:}e_*^{sH_*}{:}_K$ exists for every $s \in \mathbb{R}$, it is a real one parameter subgroup.

If we have the real analytic solution of (12) with initial condition $f_0(K) = g$, then it is natural to denote the solution by ${:}e_*^{tH_*}{:}_K *_K g$. This definition works for $g \in \mathcal{E}_p(\mathbb{C}^{2m})$, $p > 2$.
Warning In general, (12) is a misleading definition, as we can expect neither the existence of a solution of (12), nor any continuity in the initial data. For $H_* = \frac{1}{i\hbar} u{\circ}v$, there are branching singular points in $e_*^{tH_*}$. If H_* is an exponential function such as $e^{au}v$, then (12) is not a partial differential equation, but rather a difference-differential equation (cf.[7]).

If H_* is a quadratic form, ${:}e_*^{sH_*}{:}_K$ is defined with a certain discrete set of singularities, as we shall see in §3.1. In general, there is no reflection symmetry for the domain of existence of the solution of (12).

3.1 General properties of $*$-exponential functions

For a given K, suppose that (12) has real analytic solutions in t on some domain $D(K)$ including 0 for the initial functions 1 and g. We denote the solution of (12) with initial function g by

$${:}e_*^{tH_*}{:}_K *_K g, \quad t \in D(K). \tag{13}$$

Proposition 3.1 *If H_* is a polynomial and ${:}e_*^{tH_*}{:}_K$ is defined on a domain $D(K)$, then ${:}e_*^{tH_*}{:}_K *_K p(\boldsymbol{u})$ is defined for every polynomial $p(\boldsymbol{u})$ on the same domain $D(K)$.*

If $p(\boldsymbol{u}) = \sum A_\alpha(s)\boldsymbol{u}^\alpha$ *is a polynomial whose coefficients depend smoothly on* s*, then the formula*

$$\partial_s^\ell {:}e_*^{tH_*}{:}_K *_K p(\boldsymbol{u}) = {:}e_*^{tH_*}{:}_K *_K \partial_s^\ell p(\boldsymbol{u})$$

holds for every ℓ*.*

Proof Multiplying the defining equation (12) by $*p(\boldsymbol{u})$ and applying the associativity in Lemma 2.1, we have

$$\frac{d}{dt} f_t(K)*p(\boldsymbol{u}) = {:}H_*{:}_K *_K (f_t(K)*p(\boldsymbol{u})), \quad f_0(K) = 1. \tag{14}$$

Since $f_t(K)*p(\boldsymbol{u})$ is a real analytic solution, this is written in our notation as $e_*^{tH_*}*p(\boldsymbol{u})$. Applying ∂_s^ℓ to (14) gives the second assertion by a similar argument. □

Let P_n be the space of polynomials of degree at most n. Then there are natural inclusions $P_n \subset P_{n+1}$. We view $\mathbb{C}[\boldsymbol{u}]$ as the inductive limit $lim_{\rightarrow} P_n$ with the inductive limit topology. The second assertion of Proposition 3.1 then yields continuity with respect to the initial condition in the inductive limit topology. We use this topology in calculations with inverse elements. However, we should remark that $\mathbb{C}[\boldsymbol{u}]$ is *not* a Fréchet space in this topology, as the first axiom of countability fails.

Remark Although ${:}e_*^{tH_*}{:}_K *_K 0 = 0$, since (12) is linear, it does not necessary follow that $\lim_k {:}e_*^{tH_*}{:}_K *_K p_k(\boldsymbol{u}) = 0$, when $\lim_k p_k(\boldsymbol{u}) = 0$ in the uniform convergence topology. Suppose $e_*^{tH_*}$ is singular at $t = t_0$. Since ${:}e_*^{tH_*}{:}_K *_K 0 = 0$ on an open dense domain, the zero function is the real analytic solution of (12), but for a series $c_n \in \mathbb{C}$ such that $\lim_{n\to\infty} c_n = 0$, $\lim_{n\to\infty} {:}e_*^{tH_*}{:}_K *_K c_n$ does not converge to 0 in this topology.

In the following, we often omit the subscript K, and so denote $*_K$, ${:}g_*{:}_K$ simply by $*$, g_* when the context is clear.

Suppose H_* is a polynomial and $G(t;K) = {:}e_*^{tH_*}{:}_K *_K {:}g_*{:}_K$ is defined. Then for every polynomial $p(\boldsymbol{u})$, $G(t;K)$ satisfies

$$\frac{d}{dt} G(t,K) *_K p(\boldsymbol{u}) = ({:}H_*{:}_K *_K G(t,K)) *_K p(\boldsymbol{u}) = {:}H_*{:}_K *_K (G(t,K) *_K p(\boldsymbol{u})),$$
$$G(0,K) *_K p(\boldsymbol{u}) = {:}g_*{:}_K *_K p(\boldsymbol{u}).$$

Since $G(t,K) *_K p(\boldsymbol{u})$ is real analytic in t, we have the following associativity:

Proposition 3.2 *If* $e_*^{tH_*}*g_*$ *is defined for some* K*, then* $e_*^{tH_*}*(g_* * p(\boldsymbol{u}))$ *is defined for* K *and*

$$e_*^{tH_*}*(g_* * p(\boldsymbol{u})) = (e_*^{tH_*}*g_*)*p(\boldsymbol{u}) \quad \textit{for every} \ \ p(\boldsymbol{u}) \in \mathbb{C}[\boldsymbol{u}].$$

Let H_* be a polynomial. Since ${:}e_*^{tH_*}*H_*{:}_K$ and ${:}H_* * e_*^{tH_*}{:}_K$ satisfy the same differential equation with the same initial data, the uniqueness of real analytic solutions gives ${:}e_*^{tH_*}*H_*{:}_K = {:}H_* * e_*^{tH_*}{:}_K$.

Using this, we also have

Proposition 3.3 *If* H_* *is a polynomial such that* $e_*^{tH_*}$ *is defined, then* ${:}e_*^{tH_*}{:}_K$ *is the real analytic solution* $h_t(K)$ *of the equation*

$$\frac{d}{dt} h_t(K) = h_t(K) *_K {:}H_*{:}_K \tag{15}$$

with the initial condition $h_0(K) = 1$*.*

From this fact, we see that $p(\boldsymbol{u}){*}e_*^{tH_*}$ is the solution of (15) with the initial condition $h_0{=}p(\boldsymbol{u})$. Hence the exponential law and the uniqueness of solutions give

$$e_*^{sH_*}{*}(e_*^{tH_*}{*}p(\boldsymbol{u})){=}e_*^{(s+t)H_*}{*}p(\boldsymbol{u}), \quad (p(\boldsymbol{u}){*}e_*^{sH_*}){*}e_*^{tH_*}{=}p(\boldsymbol{u}){*}e_*^{(s+t)H_*}. \tag{16}$$

Let $\mathrm{ad}(H_*)(h){=}[H_*,h]{=}H_*{*}h{-}h{*}H$. If H_* is a quadratic form, then $\mathrm{ad}(H_*)$ defines a linear transformation on the linear hull of the generators. By exponentiation, $\exp s\mathrm{ad}(H_*)$ is a degree preserving linear transformation on the space $\mathbb{C}[\boldsymbol{u}]$ of polynomials such that

$$[(\exp s\mathrm{ad}(H_*))f, (\exp s\mathrm{ad}(H_*))g]{=}(\exp s\mathrm{ad}(H_*))[f,g].$$

Note also that $(\exp t\mathrm{ad}(H_*))(p(\boldsymbol{u}))$ is the solution f_t of

$$\frac{d}{dt}f_t{=}[H_*,f_t], \quad f_0=p(\boldsymbol{u}).$$

Since $p_s(\boldsymbol{u}){=}(\exp s\mathrm{ad}(H_*))(p(\boldsymbol{u}))$ is a polynomial, we see by Proposition 3.1

$$\frac{d}{ds}e_*^{-sH_*}{*}p_s(\boldsymbol{u}){=}e_*^{-sH_*}(-H_*{*}p_s(\boldsymbol{u}){+}[H_*,p_s(\boldsymbol{u})]){=}e_*^{-sH_*}{*}p_s(\boldsymbol{u}){*}(-H_*).$$

Since $e_*^{sH_*}{*}p_s(\boldsymbol{u})|_{s=0}{=}p(\boldsymbol{u})$, we have

$$e^{-sH_*}{*}p_s(\boldsymbol{u}){=}p(\boldsymbol{u}){*}e_*^{-sH_*}. \tag{17}$$

Combining (17) with (16), we get the following associativity:

$$e_*^{sH_*}{*}(p(\boldsymbol{u}){*}e_*^{tH_*}){=}(e_*^{sH_*}{*}p(\boldsymbol{u})){*}e_*^{tH_*}. \tag{18}$$

It also follows that

$$e_*^{sH_*}{*}(p(\boldsymbol{u}){*}e_*^{-sH_*}){=}(e_*^{sH_*}{*}p(\boldsymbol{u})){*}e_*^{-sH_*}{=}(\exp s\mathrm{ad}(H_*))(p(\boldsymbol{u})). \tag{19}$$

3.2 Star-exponentials of quadratic forms in the normal ordered expression

In this section, we set $n{=}2m$ and $u{=}(u_1,\cdots,u_m)$, $v{=}(v_1,\cdots,v_m){=}(u_{m+1},\cdots,u_{2m})$. For every $C=(C_{ij})\in\mathfrak{M}(m)$, we consider $C(u,v)=\sum C_{ij}u_iv_j$. The star exponential function of this special quadratic form is easily obtained in the normal ordered expression, since no anomalous phenomena occur. By setting $\Lambda{=}K_0{+}J$, $K_0{=}\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$ in the product formula (5), a direct calculation gives

$$ge^{\frac{2}{i\hbar}\sum A_{kl}u_kv_l}{*}_{K_0}g'e^{\frac{2}{i\hbar}\sum B_{st}u_sv_t}=gg'e^{\frac{2}{i\hbar}\sum C_{ij}u_iv_j}, \tag{20}$$

where $C=A+B+2AB$. For $(g;A)=ge^{\frac{2}{i\hbar}\sum A_{kl}u_kv_l}$, this product formula becomes

$$(g;A){*}_{K_0}(g';B)=(gg';A+B+2AB), \quad (g;A),(g';B)\in\mathbb{C}{\times}\mathfrak{M}(m). \tag{21}$$

Note that

$$(I+2A)(I+2B)=I+2(A+B+2AB).$$

Under the correspondence $A\leftrightarrow I+2A$, the structure of the usual matrix algebra $\mathfrak{M}(m)$ is carried over to the space $\{e^{\frac{2}{i\hbar}C(u,v)}; C\in\mathfrak{M}(m)\}$. However, note here that 0 corresponds to I. In (21) we see that $(-I){+}(-I){+}2(-I)(-I){=}0$, and $-\frac{1}{2}I{+}C{+}2(-\frac{1}{2}I)C{=}-\frac{1}{2}I$ for every C.

Although these elements are in $\mathcal{E}_{2+}(\mathbb{C}^n)$, associativity still holds for the products

$$(g;A){*_{K_0}}((g';B){*_{K_0}}(g'';C))=((g;A){*_{K_0}}(g';B)){*_{K_0}}(g'';C), \tag{22}$$

and

$$e^{\frac{2}{i\hbar}C(u,v)}{*_{K_0}}e^{\frac{i}{\hbar}I(u,v)}=e^{\frac{i}{\hbar}I(u,v)}{*_{K_0}}e^{\frac{2}{i\hbar}C(u,v)}=e^{\frac{i}{\hbar}I(u,v)}.$$

By (21), we see that

$$e^{\frac{1}{i\hbar}(e^{isC}-I)(u,v)}{*_{K_0}}e^{\frac{1}{i\hbar}(e^{itC}-I)(u,v)}=e^{\frac{1}{i\hbar}(e^{i(s+t)C}-I)(u,v)}. \tag{23}$$

Differentiating the exponential law (23) to obtain the K_0-expression (the normal ordered expression) of the $*$-exponential function, we have

$$:e_*^{\frac{it}{i\hbar}\sum C_{kl}u_k*v_l}:_{K_0}=e^{\frac{1}{i\hbar}\sum(e^{itC}-I)_{kl}u_kv_l}. \tag{24}$$

This is holomorphic in $t{\in}\mathbb{C}$ and the r.h.s of (24) is contained in $\mathcal{E}_{2+}(\mathbb{C}^2)$.

Set

$$a{\circ}b{=}\frac{1}{2}(a{*}b{+}b{*}a).$$

By the exponential law for scalar exponential functions, (24) becomes

$$:e_*^{\frac{it}{i\hbar}\sum C_{kl}u_k\circ v_l}:_{K_0}=e^{\frac{it}{2}\mathrm{Tr}(C)}e^{\frac{1}{i\hbar}\sum(e^{itC}-I)_{kl}u_kv_l}. \tag{25}$$

This is also a holomorphic one parameter group contained in $(\mathcal{E}_{2+}(\mathbb{C}^2);*_{K_0})$. However, this property of $e_*^{\frac{it}{i\hbar}\sum C_{kl}u_k\circ v_l}$ is not generic, as we see in §3.3. Indeed, a generic element has branching singular points periodically distributed in $\mathbb{C}$. On the other hand, for the special case $C{=}I$, we see that $:e_*^{\frac{2\pi i}{i\hbar}\sum u_k\circ v_k}:_{K_0}=(-1)^m$. Intertwiners map scalars to scalars, but may change the sign of the scalar $(-1)^m$ in this equation. The property that $:e_*^{\frac{2\pi i}{i\hbar}\sum u_k\circ v_k}:_K=\pm 1$ is generic.

We note that $\lim_{t\to-\infty}:e_*^{t\frac{1}{i\hbar}\sum u_k\circ v_k}:_{K_0}=0$ but $\lim_{t\to\infty}:e_*^{t\frac{1}{i\hbar}\sum u_k\circ v_k}:_{K_0}=\infty$. It is rather surprising that the finiteness of the integral

$$:\int_{-\infty}^{\infty}e_*^{it\frac{1}{i\hbar}u\circ v}dt:_K\in\mathcal{E}_{2+}(\mathbb{C}^2)$$

is a generic property, as we will see in §3.4.

3.3 Intertwiners for exponential functions of quadratic forms

In this section we extend intertwiners to the space $\mathbb{C}e^{\mathfrak{S}(2m)}$ of exponential functions of quadratic forms $ge^{\langle \boldsymbol{u}Q,\boldsymbol{u}\rangle}$, where $g\in\mathbb{C}, Q\in\mathfrak{S}(2m)$. This will be used to obtain K-ordered expressions of star-exponential functions of quadratic forms.

The exact formula for intertwiners is obtained by solving the evolution equation

$$\frac{d}{dt}g(t)e^{\frac{1}{i\hbar}\langle\boldsymbol{u}Q(t),\boldsymbol{u}\rangle}=\sum_{ij}K^{ij}\partial_{u_i}\partial_{u_j}\big(g(t)e^{\frac{1}{i\hbar}\langle\boldsymbol{u}Q(t),\boldsymbol{u}\rangle}\big),\quad Q(0)=A,\quad g(0)=g$$

by setting

$$e^{t\sum_{ij}K^{ij}\partial_{u_i}\partial_{u_j}}\big(ge^{\frac{1}{i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle}\big)=g(t)e^{\frac{1}{i\hbar}\langle\boldsymbol{u}Q(t),\boldsymbol{u}\rangle}. \tag{26}$$

A direct calculation gives

$$\sum_i K^{ij}\partial_{u_i}\partial_{u_j}\big(g(t)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}Q(t),\boldsymbol{u}\rangle}\big) = g(t)\Big(2\mathrm{Tr}K\frac{1}{i\hbar}Q(t) + 4\frac{1}{(i\hbar)^2}(QKQ)_{ij}u_iu_j\Big)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}Q(t),\boldsymbol{u}\rangle}.$$

To find the intertwiner, we solve the ODE system:

$$\begin{cases} \dfrac{d}{dt}Q(t) = \dfrac{4}{i\hbar}Q(t)KQ(t) \\ \dfrac{d}{dt}g(t) = g(t)(\dfrac{2}{i\hbar}\mathrm{Tr}KQ(t)) \end{cases} \qquad Q(0)=A, \quad g(0)=g.$$

Then $Q(t) = \frac{1}{I-\frac{4t}{i\hbar}AK}A$, $g(t) = g(\det(I-\frac{4t}{i\hbar}AK))^{-1/2}$ is the solution of the ODE system by the uniqueness of real analytic solutions.

Here the inverse matrix of X is denoted by $\frac{1}{X}$. Note also that $\frac{1}{X}\frac{1}{Y} = \frac{1}{YX}$. It is easy to check that $\frac{1}{I-AK}A$ is a symmetric matrix by the identity:

$$\frac{1}{I-AK}A = A\frac{1}{I-KA}. \tag{27}$$

Setting $t=\frac{\hbar i}{4}$, we can build the intertwiner $I_0^{{}_K}$ from

$$Q(\frac{\hbar i}{4}) = \frac{1}{I-AK}A, \quad g(\frac{\hbar i}{4}) = g(\det(I-AK))^{-\frac{1}{2}}. \tag{28}$$

as follows. For $ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle} = (g;A)$ as before, we call g and A the *amplitude* and *phase* part of $(g;A)$, respectively. In this notation, we see that

$$I_0^{{}_K}(g;A) = \Big(g\det(I-AK)^{-\frac{1}{2}}; T_K(A)\Big),$$

where $T_K : \mathfrak{S}(2m)\to\mathfrak{S}(2m)$, $\quad T_K(A) = \frac{1}{I-AK}A$ is the phase part of the intertwiner $I_0^{{}_K}$.

Computing the inverse $I_{{}_K}^0 = (I_0^{{}_K})^{-1}$ and taking the composition $I_0^{{}_{K'}}I_{{}_K}^0$, we easily obtain

$$I_{{}_K}^{{}_{K'}}(g;A) = \Big(g\det(I-A(K'-K))^{-\frac{1}{2}}; \frac{1}{I-A(K'-K)}A\Big). \tag{29}$$

The mapping (29) is singular at those A where either $\det(I-A(K'-K))$=0 or the sign ambiguity in the square root cannot be removed. We denote the phase part of the intertwiner $I_{{}_K}^{{}_{K'}}$ by $T_{{}_K}^{{}_{K'}}(A)=\frac{1}{I-A(K'-K)}A$.

Note that the identities

$$T_{{}_K}^{{}_{K'}}\sim T_{K'}(T_K)^{-1}, \quad I_{{}_K}^{{}_{K'}}\sim I_0^{{}_{K'}}I_{{}_K}^0$$

hold in the same sense as the algebraic identities x/x=1, $\sqrt{1+x}/\sqrt{1+x}$=1, i.e. whenever the denominator is nonzero. Here we use the notation $\sim$ to distinguish such an algebraic calculation. Singularities are moving by this algebraic trick.

By setting $B=\frac{1}{I-A(K'-K)}A$, the r.h.s of (29) is $\Big(g\det(I+B(K'-K))^{\frac{1}{2}}; B\Big)$. Moving branching singularities are a remarkable feature of this calculus.

For every A, $ge^{\frac{1}{i\hbar}\langle \boldsymbol{u}A,\boldsymbol{u}\rangle}$ is an element of $\mathcal{E}_{2+}(\mathbb{C}^n)$. If $(g(\cdot),A(\cdot))$ is a a continuous mapping from a compact manifold M into $\mathbb{C}\times\mathfrak{S}(n)$, then Lemma 2.2 shows that

$$\int_M g(x)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}A(x),\boldsymbol{u}\rangle}dV_x \in \mathcal{E}_{2+}(\mathbb{C}^n).$$

Suppose further that M is simply connected. Since the intertwiner $I_{{}_K}^{{}_{K'}}$ is given in a concrete form, we see the following:

Lemma 3.1 *For every $K, K'\in\mathfrak{S}(n)$,*

$$I_{_K}^{^{K'}}\Big(\int_M g(x)e^{\frac{1}{i\hbar}\langle \boldsymbol{u}A(x),\boldsymbol{u}\rangle}dV_x\Big) = \int_M g(x)I_{_K}^{^{K'}}\Big(e^{\frac{1}{i\hbar}\langle \boldsymbol{u}A(x),\boldsymbol{u}\rangle}\Big)dV_x$$

is also an element of $\mathcal{E}_{2+}(\mathbb{C}^n)$ whenever $\det(I{-}A(x)(K'{-}K))$ is nowhere zero on M.

3.4 The general ordered expression of $e_*^{t(z+\frac{1}{i\hbar}u\circ v)}$

From here on, we set $n{=}2m{=}2$, and $(u_1, u_2){=}(u, v)$. We are mainly concerned with functions of $u{\circ}v = \frac{1}{2}(u{*}v{+}v{*}u)$ alone. The general ordered expression $:e_*^{t(z+\frac{1}{i\hbar}u\circ v)}:_{_K}$ will be given by applying intertwiners to the normal ordered expression.

For this purpose, we set $2u{\circ}v{=}\langle \boldsymbol{u}A, \boldsymbol{u}\rangle$, where $\boldsymbol{u}{=}(u, v)$, $A{=}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$. The intertwiner $I_{_K}^{^{K'}}$ is given by (29).

We determine the formula of a general ordering expression $:e_*^{t\frac{1}{i\hbar}2u\circ v}:_{_K}$, $K{=}\begin{bmatrix}\delta' & \lambda\\ \lambda & \delta\end{bmatrix}$, $\lambda, \delta, \delta'\in\mathbb{C}$.

Setting $B{=}\begin{bmatrix}0 & \beta\\ \beta & 0\end{bmatrix}$, we note that

$$(I{-}B(K-K_0))^{-1}B{=}\frac{1}{(1{-}\beta(\lambda{-}1))^2{-}\beta^2\delta\delta'}\begin{bmatrix}\beta^2\delta & (1{-}\beta(\lambda{-}1))\beta\\ (1{-}\beta(\lambda{-}1)\beta & \beta^2\delta'\end{bmatrix}$$

Recalling the formulas (24) and (29), we have

$$\begin{aligned}:e_*^{t\frac{1}{i\hbar}2u\circ v}:_{_K}{=}&\frac{2}{\sqrt{\Delta^2{-}(e^t{-}e^{-t})^2\delta\delta'}}\\ &\times\exp\frac{1}{i\hbar}\frac{e^t{-}e^{-t}}{\Delta^2{-}(e^t{-}e^{-t})^2\delta\delta'}\Big((e^t{-}e^{-t})\delta u^2{+}\Delta 2uv{+}(e^t{-}e^{-t})\delta' v^2\Big)\end{aligned}$$

where $\Delta{=}(e^t{+}e^{-t}){-}\lambda(e^t{-}e^{-t})$. Here we note that the sign ambiguity of the square root is removed by choosing a path from the $t = 0$ to t on which no singular point appears, and by choosing the initial condition $e_*^{0\frac{1}{i\hbar}2u\circ v}{=}1$ at $t{=}0$.

Replacing t by it, we see that

$$:e_*^{\frac{it}{i\hbar}2u\circ v}:_{_K}{=}\frac{1}{\sqrt{\Delta_{(K)}(t)}}\exp\frac{1}{\hbar}\frac{\sin t}{\Delta_{(K)}(t)}\langle \boldsymbol{u}\begin{bmatrix}i\delta\sin t & \cos t{-}i\lambda\sin t\\ \cos t{-}i\lambda\sin t & i\delta'\sin t\end{bmatrix},\ \boldsymbol{u}\rangle, \tag{30}$$

where

$$\Delta_{(K)}(t){=}\big(\cos t{-}i(\lambda{+}\sqrt{\delta\delta'})\sin t\big)\big(\cos t{-}i(\lambda{-}\sqrt{\delta\delta'})\sin t\big) \tag{31}$$

Note that $\lambda{+}\sqrt{\delta\delta'}$ and $\lambda{-}\sqrt{\delta\delta'}$ can be arbitrary complex numbers. Both (30) and (31) are π-periodic. Here we note that the sign of $\sqrt{\Delta_{(K)}(t)}$ depends on the ordered expression parameter K. It follows that $:e_*^{\pi i\frac{1}{i\hbar}2u\circ v}:_{_K}{=}\frac{2}{\sqrt{(-2)^2}}$, which is ± 1 depending on K and the path from 0 to πi.

In the remainder of this section, we comment on the appearance of these singular points. The sign ambiguity of $\sqrt{\ }$ cannot be removed on the whole complex plane. Thus these $*$-exponentials are double valued functions of $t\in\mathbb{C}$ in general (cf. [11], [12]). The sign ambiguity is removed only when $\delta\delta'{=}0$ by choosing the initial condition $e_*^{0\frac{1}{i\hbar}2u\circ v}{=}1$ at $t{=}0$. In this case, cusp singular points appear π (and not 2π)-periodically along a line parallel to the real axis. However, singular points are not stable under general intertwiners, as intertwiners are double valued in general (cf. [11]).

From these observations we see that in generic ordered expressions the singular points of $:e_*^{it\frac{1}{i\hbar}2u\circ v}:_K$ appear π-periodically on two lines parallel to the real axis and the ordered expression has $e^{-|t|}$-decay on any line parallel to the imaginary axis. Moreover, the generic ordered expression does not have singular points, and the existence of $\int_{\mathbb{R}} e_*^{t\frac{1}{i\hbar}u\circ v}dt$ is a generic property. However, we see there are several categories for the behavior of expression parameters.

To fix the notation, we denote by $\mathfrak{D}$ the open dense domain of expression parameters K such that $:e_*^{\frac{it}{i\hbar}2u\circ v}:_K$ has no singular point on either the real or imaginary axis. Generic patterns of the properties for $:e_*^{\frac{it}{i\hbar}2u\circ v}:_K$ are as follows:

(1) On a domain $\mathfrak{D}_+$ (resp. $\mathfrak{D}_-$) for the parameter K, the singular set of $:e_*^{\frac{it}{i\hbar}2u\circ v}:_K$ appears only in the open lower (resp. upper) half plane, and the $*$-exponential functions form a complex semi-group over the upper (resp. lower) half plane without sign ambiguity by demanding the value 1 at t=0. $:e_*^{\pm\frac{it}{i\hbar}2u\circ v}:_K$, is alternating π-periodic on the real axis (we call $f(z)$ *alternating* π *periodic* if $f(z+n\pi)=(-1)^nf(z)$ for any integer n).

(2) On a domain $\mathfrak{D}_0$ for the parameter K, the singular set occurs in both the upper and lower half-planes, but not on the real axis. In this domain, $:e_*^{\pm\frac{it}{i\hbar}2u\circ v}:_K$, is π-periodic on the real axis by demanding the value 1 at t=0.

Note Some delicate arguments about the winding number are required to determine the periodicity of $:e_*^{\pm\frac{it}{i\hbar}2u\circ v}:_K$, as will be discussed in a forthcoming paper.

3.5 Star exponential functions of general quadratic forms

In this section we give without proof formulas for K-ordered expressions of star exponential functions of general quadratic form, with details in [12].

As in [17], star exponential functions $e_*^{\frac{1}{i\hbar}\langle\boldsymbol{\xi},\boldsymbol{u}\rangle}$ for a linear form $\langle\boldsymbol{\xi},\boldsymbol{u}\rangle$ are well defined as the family $\{e^{\frac{1}{4i\hbar}\langle\boldsymbol{\xi}K,\boldsymbol{\xi}\rangle}e^{\frac{1}{i\hbar}\langle\boldsymbol{\xi},\boldsymbol{u}\rangle}, K{\in}\mathfrak{S}(n)\}$. However, for a quadratic form $\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*=\sum A_{kl}u_k{\circ}u_l$, the star exponential function $e_*^{\frac{t}{i\hbar}\langle\boldsymbol{u}A,\boldsymbol{u}\rangle_*}$ will be defined only on a dense domain of K-ordered expressions, and is in general a double valued function of $t{\in}\mathbb{C}$ (cf. [12]).

For every $\alpha{\in}sp(m,\mathbb{C})$, we first consider the one parameter subgroup $e^{-2t\alpha}$ of $Sp(m,\mathbb{C})$, and consider the inverse image of the twisted Cayley transform $C_\kappa^{-1}(e^{-2t\alpha})$: For $\kappa{\in}sp(m,\mathbb{C})$, we set

$$C_\kappa^{-1}(e^{-2t\alpha})=\frac{1}{(I{-}\kappa){+}e^{-2t\alpha}(I{+}\kappa)}(I{-}e^{-2t\alpha})=\frac{1}{\cosh t\alpha{-}(\sinh t\alpha)\kappa}\sinh t\alpha. \tag{32}$$

The exponential function must lie in a certain submanifold $\widetilde{\mathcal{D}}_\kappa$ through $(1;0)$, and points of this manifold are determined by their phases. Setting $\kappa=JK$, we have

$$:e_*^{s\frac{1}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K=\big(\det(I{+}C_\kappa^{-1}(e^{-2s\alpha})(I{+}\kappa))\big)^{\frac{1}{2}}e^{\frac{1}{i\hbar}\langle\boldsymbol{u}(C_\kappa^{-1}(e^{-2s\alpha})J),\boldsymbol{u}\rangle}. \tag{33}$$

More precisely, for every $\alpha{\in}sp(m,\mathbb{C})$, the K-ordered expression of the $*$-exponential function is given as follows (see [11]-[14] for special cases):

$$:e_*^{\frac{t}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K=\frac{2^m}{\sqrt{\det(I{-}\kappa{+}e^{-2t\alpha}(I{+}\kappa))}}e^{\frac{t}{i\hbar}\langle\boldsymbol{u}\frac{1}{I-\kappa+e^{-2t\alpha}(I+\kappa)}(I-e^{-2t\alpha})J,\boldsymbol{u}\rangle}. \tag{34}$$

It is not hard to see that (34) is the real analytic solution of (12). Note that $\det e^{t\alpha}I{=}1$ for every $\alpha{\in}sp(m,\mathbb{C})$. Thus (34) can be rewritten as

$$:e_*^{\frac{t}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K=\frac{2^m}{\sqrt{\det(e^{t\alpha}(I{-}\kappa){+}e^{-t\alpha}(I{+}\kappa))}}e^{\frac{t}{i\hbar}\langle\boldsymbol{u}\frac{1}{e^{t\alpha}(I-\kappa)+e^{-t\alpha}(I+\kappa)}(e^{t\alpha}-e^{-t\alpha})J,\boldsymbol{u}\rangle} \tag{35}$$

In spite of the sign ambiguity of the square root, the exponential law

$$:e_*^{s\frac{1}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K*_K:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K{=}:e_*^{(s+t)\frac{1}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K \tag{36}$$

holds using $\sqrt{a}\sqrt{b}{=}\sqrt{ab}$ without regard to sign ambiguities, as the exponential law and associativity hold on the group $Sp(m,\mathbb{C})$. Note however that we allow $\sqrt{1}{=}\pm 1$.

To treat these formulas without sign ambiguity, we always have to specify a path with no singular points from $t=0$ to the considered point.

From (34) we derive the following:

Proposition 3.4 *If $e^{2\pi\alpha}{=}I$ (e.g. $\alpha{=}J$), then $:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K=\sqrt{1}$ independent of K.*

The sign of $\sqrt{1}$ depends on the K-ordered expression and also on a path from 0 to π as above.

Hence, even though $(\sqrt{1})^2{=}1$ is trivial, the strict exponential law may fail, that is, $:e_*^{2\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K=1$ or $:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K*_K:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K=1$ may not hold automatically. If $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K$ has a singular point on the interval $[0,2\pi]$, then it may happen that

$$(e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*})^2\neq e_*^{2\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*},$$

although equality holds up to sign. In spite of this, we have

Proposition 3.5 *If $e^{2\pi\alpha}{=}I$, then $(e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*})^2{=}1$ for every K-ordered expression such that $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K$ has no singular point on the interval $[0,\pi]$.*

Proof Note first that this is by no means trivial, because $:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K$ may have a singular point on the interval $[\pi,2\pi]$. Since $:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K=\pm 1$, one can define

$$:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K*_K:1:_K,\quad\text{or}\quad :e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K*_K:(-1):_K$$

by the solution of the evolution equation (12). By Proposition 3.2,

$$:e_*^{t\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K*_K:e_*^{\pi\frac{1}{i\hbar}\langle\boldsymbol{u}\alpha J,\boldsymbol{u}\rangle_*}:_K$$

is the solution of (12). This gives the result. □

By (34), we also see that $:e_*^{s\frac{1}{i\hbar}\langle\boldsymbol{u}(\alpha J),\boldsymbol{u}\rangle_*}:_K$ has in general discrete branching singularities in $\mathbb{C}$ with some periodicity depending on the parameter $\kappa{=}JK$.

4 Criteria for associativity

In this section, we give several criteria which imply associativity for the extended product $*_K$. However, we note that there is no generally applicable lemma guaranteeing associativity. For simplifying notation, we often omit the subscript K of the product $*_K$ and the expression $:\,:_K$ if it contains no confusion.

4.1 Remarks on star exponential functions

We first note how to define rigorously the product of star exponential functions and a general function $e_*^{z\frac{1}{i\hbar}u\circ v}*f(u,v)$. There are essentially two approaches. The first is, as mentioned in §3.1, to use the real analytic solution f_t of

$$\frac{d}{dt}f_t{=}\frac{1}{i\hbar}u{\circ}v{*}f_t,$$

with the initial condition $f_0{=}f(u,v)$ provided such a solution exists. The second approach is to define

$$e_*^{z\frac{1}{i\hbar}u\circ v}*f(u,v){=}\lim_{n\to\infty}e_*^{z\frac{1}{i\hbar}u\circ v}*f_n(u,v),\quad \text{if}\ \ f(u,v){=}\lim_{n\to\infty}f_n(u,v),$$

where f_n are polynomials. These two definitions do not coincide in general, since multiplication by $e_*^{z\frac{1}{i\hbar}u\circ v}*$ is not a continuous linear mapping of $Hol(\mathbb{C}^2)$ to itself (cf. (42), (43)). Note that $e_*^{z\frac{1}{i\hbar}u\circ v}{\in}\mathcal{E}_{2+}(\mathbb{C}^2)$. If $f(u,v){\in}\mathcal{E}_{2-}(\mathbb{C}^2){=}\bigcup_{p<2}\mathcal{E}_p(\mathbb{C}^2)$ with the inductive limit topology, then the two definitions coincide.

Since star exponential functions of quadratic forms are elements of $\mathcal{E}_{2+}(\mathbb{C}^n)$, their product may not be defined, and even if the product is defined associativity may not hold.

We show that $\int_{-\infty}^{\infty}e_*^{t\frac{1}{i\hbar}u\circ v}dt{\in}\mathcal{E}_{2+}(\mathbb{C}^2)$ in the Weyl ordered expression. In the Weyl ordered expression, we have $:e_*^{t\frac{1}{i\hbar}u\circ v}:_0{=}\frac{1}{\cosh\frac{t}{2}}e^{(\tanh\frac{t}{2})\frac{1}{i\hbar}2uv}$. Thus,

$$:\int_{\mathbb{R}}e_*^{t\frac{1}{i\hbar}u\circ v}dt{:}_0=\int_{-\infty}^{\infty}\frac{1}{\cosh\frac{t}{2}}e^{(\tanh\frac{t}{2})\frac{1}{i\hbar}2uv}dt.$$

For $\cos s{=}\tanh\frac{t}{2}$, $-2\sin s ds{=}\sin^2 s dt$, the integral on the r.h.s. of the last equation becomes

$$2\int_{-\pi}^{0}e^{(\cos s)\frac{1}{i\hbar}2uv}ds{=}\int_{-\pi}^{\pi}e^{(\cos s)\frac{1}{i\hbar}2uv}ds.$$

Since $g(s)=e^{(\cos s)\frac{1}{i\hbar}uv}$ is a continuous curve in $\mathcal{E}_{2+}(\mathbb{C}^2)$, Lemma 2.2 implies that the last integral belongs to $\mathcal{E}_{2+}(\mathbb{C}^2)$. Hence, by Lemma 3.1 this property is generic.

Using the intertwiner I_0^K, we see that $:\int_{\mathbb{R}}e_*^{t\frac{1}{i\hbar}u\circ v}dt{:}_K=\int_{-\pi}^{\pi}{:}e^{(\cos s)\frac{1}{i\hbar}2u\circ v}{:}_K ds\in\mathcal{E}_{2+}(\mathbb{C}^2)$. Then we have

Proposition 4.1 *In generic ordered expressions, the integral* $:\int_{-\infty}^{\infty}e_*^{t\frac{1}{i\hbar}u\circ v}dt{:}_K$ *is in* $\mathcal{E}_{2+}(\mathbb{C}^2)$. *Furthermore, integration by parts gives* $\frac{d}{d\theta}\int_{-\infty}^{\infty}e_*^{e^{i\theta}t\frac{1}{i\hbar}u\circ v}e^{i\theta}dt{=}0$ *whenever the integral is defined.*

We have seen that in generic ordered expressions, $\frac{1}{i\hbar}u{\circ}v$ has two different inverses,

$$(\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}{=}\int_{-\infty}^{0}e_*^{t\frac{1}{i\hbar}u\circ v}dt,\quad (\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}{=}{-}\int_{0}^{\infty}e_*^{t\frac{1}{i\hbar}u\circ v}dt$$

in the space $\mathcal{E}_{2+}(\mathbb{C}^n)$, which implies the failure of associativity in general:

$$\Big((\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}{*}(\frac{1}{i\hbar}u{\circ}v)\Big){*}(\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}\neq(\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}{*}\Big((\frac{1}{i\hbar}u{\circ}v){*}(\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}\Big).\tag{37}$$

Indeed $(\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}{*}(\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}$ diverges in any ordered expression. This gives an example where $(f{*}g){*}h{=}f{*}(g{*}h)$ does not hold even if g is a polynomial.

4.2 Basic criteria for associativity for the extended product

Suppose $f, g \in Hol(\mathbb{C}^n)$ are given by $f = \lim f_k, g = \lim g_\ell$ in the topology of $Hol(\mathbb{C}^n)$ for sequences $\{f_k\}, \{g_\ell\} \subset Hol(\mathbb{C}^n)$. Even if $f{*}g$ and $\lim_\ell f{*}g_\ell$ exist, $f{*}g$ may not equal $\lim_\ell f{*}g_\ell$, since $f{*}$ is not continuous in general. Moreover, it may happen that even though $\lim g_\ell$ diverges, $\lim_\ell f{*}g_\ell$ exists.

If $f = \lim f_k, g = \lim g_\ell$, we have

$$\lim_k f_k{*}p(\boldsymbol{u}) = f{*}p(\boldsymbol{u}), \quad \lim_\ell p(\boldsymbol{u}){*}g_\ell = p(\boldsymbol{u}){*}g$$

for every polynomial. However, as we saw in (37), we may have

$$\lim_k(\lim_\ell f_k{*}(p(\boldsymbol{u}){*}g_\ell) \neq \lim_\ell(\lim_k f_k{*}p(\boldsymbol{u})){*}g_\ell),$$

even if both sides exist. In this case, $\lim_{(k,\ell)} f_k{*}p(\boldsymbol{u}){*}g_\ell$ does not converge.

Suppose $f_k{*}g_\ell$ converges to an element h in $Hol(\mathbb{C}^n)$. Then we define $f{*}g = h$, i.e.

$$f{*}g = \lim_{(k,\ell)\to\infty} f_k{*}g_\ell = h. \tag{38}$$

where in the limit $k^2 + \ell^2 \to \infty$. The same definition is also employed for the product

$$\int_{-\infty}^0 f(s)e_*^{s(z+\frac{1}{i\hbar}u\circ v)}ds{*}\int_{-\infty}^0 g(t)e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt = \lim_{(S,T)} \iint_{-(S,T)}^{(0,0)} f(s)g(t)e_*^{(s+t)(z+\frac{1}{i\hbar}u\circ v)}dsdt \tag{39}$$

although these integrals are not in $\mathcal{E}_2(\mathbb{C}^n)$.

Suppose $f, g \in Hol(\mathbb{C}^n)$ are given as $f = \lim f_k, g = \lim g_\ell$ in the topology of $Hol(\mathbb{C}^n)$ as above. For polynomials $p(\boldsymbol{u})$, $q(\boldsymbol{u})$, Lemma 2.1 gives that $\lim_k p(\boldsymbol{u}){*}f_k = p(\boldsymbol{u}){*}f$, $\lim_k q(\boldsymbol{u}){*}g_k = q(\boldsymbol{u}){*}g$.

Lemma 4.1 *Suppose that associativity holds for the approximating series:*

$$(p(\boldsymbol{u}){*}f_k){*}(q(\boldsymbol{u}){*}g_\ell) = ((p(\boldsymbol{u}){*}f_k){*}q(\boldsymbol{u})){*}g_\ell),$$

and $\lim_{(k,\ell)}(p(\boldsymbol{u}){*}f_k){*}(q(\boldsymbol{u}){*}g_\ell)$ *converges to an element* h *in* $Hol(\mathbb{C}^n)$. *Then* $(p(\boldsymbol{u}){*}f){*}(q(\boldsymbol{u}){*}g)$ *equals* h*, and the following associativity holds:*

$$(p(\boldsymbol{u}){*}f){*}(q(\boldsymbol{u}){*}g) = (p(\boldsymbol{u}){*}f{*}q(\boldsymbol{u})){*}g.$$

Proof By definition, we have $(p(\boldsymbol{u}){*}f){*}(q(\boldsymbol{u}){*}g) = \lim_{(k,\ell)}(p(\boldsymbol{u}){*}f_k){*}(q(\boldsymbol{u}){*}g_\ell)$. Using the associativity of the inside of the r.h.s. of the last equation in Lemma 4.1, we have

$$\lim_{(k,\ell)}(p(\boldsymbol{u}){*}f_k){*}(q(\boldsymbol{u}){*}g_\ell) = \lim_{(k,\ell)}(p(\boldsymbol{u}){*}f_k{*}q(\boldsymbol{u})){*}g_\ell.$$

From Lemma 2.1, we see that $\lim_k p(\boldsymbol{u}){*}(f_k{*}q(\boldsymbol{u})) = p(\boldsymbol{u}){*}f{*}q(\boldsymbol{u})$. It follows that

$$(p(\boldsymbol{u}){*}f){*}(q(\boldsymbol{u}){*}g) = \lim_{(k,\ell)}(p(\boldsymbol{u}){*}(f_k{*}q(\boldsymbol{u}))){*}g_\ell = (p(\boldsymbol{u}){*}f{*}q(\boldsymbol{u})){*}g.$$

□

Note that if the approximating series are in $\mathcal{E}_2(\mathbb{C}^n)$, then associativity holds before the limiting procedure.

Lemma 4.2 *Suppose* $f, g, f{*}g$ *are given as in* (38)*. Then, for any polynomials* $p(\boldsymbol{u}), q(\boldsymbol{u})$*, the product* $(p(\boldsymbol{u}){*}f){*}(g{*}q(\boldsymbol{u}))$ *is defined and associativity holds:*

$$(p(\boldsymbol{u}){*}f){*}(g{*}q(\boldsymbol{u})) = p(\boldsymbol{u}){*}(f{*}g){*}q(\boldsymbol{u}).$$

Proof By Lemma 2.1, we see that $p(\boldsymbol{u}){*}f{=}\lim_k p(\boldsymbol{u}){*}f_k,\ g{*}q(\boldsymbol{u}){=}\lim_\ell(g_\ell{*}q(\boldsymbol{u}))$, and the product is defined by

$$(p(\boldsymbol{u}){*}f){*}(g{*}q(\boldsymbol{u}))=\lim_{(k,\ell)}(p(\boldsymbol{u}){*}f_k){*}(g_\ell{*}q(\boldsymbol{u}))=\lim_{(k,\ell)}p(\boldsymbol{u}){*}(f_k{*}g_\ell){*}q(\boldsymbol{u}).$$

Hence Lemma 2.1 gives $(p(\boldsymbol{u}){*}f){*}(g{*}q(\boldsymbol{u}))=p(\boldsymbol{u}){*}(f{*}g){*}q(\boldsymbol{u})$. □

It does not seem that the existence of $\lim_{(k,\ell)} f_k{*}g_\ell$ yields that of $\lim_{(k,\ell)} f_k{*}(p(\boldsymbol{u}){*}g_\ell)$ or $\lim_{(k,\ell)}(f_k{*}p(\boldsymbol{u})){*}g_\ell$ for every polynomial $p(\boldsymbol{u})$. In spite of this, we have the following for the special element $u{\circ}v$:

Lemma 4.3 *If* (39) *is defined, then*

$$\int_{-\infty}^{0} f(s)e_*^{s(z+\frac{1}{i\hbar}u\circ v)}ds{*}p(\boldsymbol{u}){*}\int_{-\infty}^{0} g(t)e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$$

is defined for every polynomial $p(\boldsymbol{u})$.

Proof Using the "bumping identity" :

$$v{*}f(u{*}v)=f(v{*}u){*}v$$

several times, we find a polynomial $\tilde{p}(\boldsymbol{u})$ such that

$$p(\boldsymbol{u}){*}\int_{-\infty}^{0} g(t)e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt=\int_{-\infty}^{0} g(t)p(\boldsymbol{u}){*}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt=\int_{-\infty}^{0} g(t)e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt{*}\tilde{p}(\boldsymbol{u}).$$

Hence Lemma 4.2 gives the result. □

In the general setting, suppose the limits $f{*}g{=}\lim_{(k,\ell)} f_k{*}g_\ell$ in (38) and $\lim_{(k,\ell)} \partial^\alpha f_k{*}\partial^\beta g_\ell$ exist for every α,β. Then it is not hard to show the existence of $\lim_{(k,\ell)} f_k{*}(p(\boldsymbol{u}){*}g_\ell)$ and $\lim_{(k,\ell)}(f_k{*}p(\boldsymbol{u})){*}g_\ell$ for every polynomial $p(\boldsymbol{u})$.

The following is useful in concrete computations. Note that for $(\mathbb{C}[\boldsymbol{u}][[\hbar]],*_{_K})$, the space of formal power series in $\hbar$, the $*_{_K}$-product is always defined by the product formula (5) and associativity holds. The elements of $\mathcal{E}_{2+}(\mathbb{C}^n)$ are often given as a real analytic function of $\hbar$ defined on a certain interval containing $\hbar=0$.

The following is easy to see:

Lemma 4.4 *Suppose* $f(\hbar,\boldsymbol{u})$, $g(\hbar,\boldsymbol{u})$ *and* $h(\hbar,\boldsymbol{u})$ *are given as real analytic functions of* $\hbar$ *in some interval* $[0,H]$.

If $f(\hbar,\boldsymbol{u}){*}g(\hbar,\boldsymbol{u})$, $(f(\hbar,\boldsymbol{u}){*}g(\hbar,\boldsymbol{u})){*}h(\hbar,\boldsymbol{u})$, $g(\hbar,\boldsymbol{u}){*}h(\hbar,\boldsymbol{u})$ *and* $f(\hbar,\boldsymbol{u}){*}(g(\hbar,\boldsymbol{u}){*}h(\hbar,\boldsymbol{u}))$ *are defined as real analytic functions on* $[0,H]$*, then the following associativity holds:*

$$\big(f(\hbar,\boldsymbol{u}){*}g(\hbar,\boldsymbol{u})\big){*}h(\hbar,\boldsymbol{u}) = f(\hbar,\boldsymbol{u}){*}\big(g(\hbar,\boldsymbol{u}){*}h(\hbar,\boldsymbol{u})\big)$$

Remark In the following, elements are often given in the form $f(\frac{1}{i\hbar}\varphi(t),\boldsymbol{u})$ for a real analytic function $f(t,\boldsymbol{u})$ in $t{\in}[0,T]$, where $\varphi(t)$ is a real analytic function such that $\varphi(0){=}0$ (cf. (24)). In such a case, replacing t by $s\hbar$ gives a real analytic function of $\hbar$, and such an element lies in $(\mathbb{C}[\boldsymbol{u}][[\hbar]],*_{_K})$. Thus, we can apply Lemma 4.4.

However, there are many elements in $\mathcal{E}_{2+}(\mathbb{C}^n)$ of the form $f(\frac{1}{i\hbar}\varphi(t),\boldsymbol{u})$ such that $\varphi(0){\neq}0$. For these elements we have to use Lemmas 4.1 and 4.2 carefully.

As mentioned before, we know that $:e_*^{t\frac{1}{i\hbar}u\circ v}:_{_K}{\in}Hol(\mathbb{C}^2)$ for every fixed t whenever defined. We also see that $:e_*^{t\frac{1}{i\hbar}u\circ v}:_{_K}$ is rapidly decreasing with respect to t in a generic ordered expression.

5 Vacuums and their matrix element expressions

In this section, we give properties of vacuums which we can compare to similar properties in operator theory.

Noting that $v{*}u{=}u{\circ}v{+}\frac{1}{2}i\hbar$, we begin with the following:

Proposition 5.1 *In generic ordered expressions with no singular points on the real axis, we have*

$$\lim_{t\to-\infty} e_*^{t\frac{1}{i\hbar}2v*u}{=}0, \quad \lim_{t\to\infty} e_*^{t\frac{1}{i\hbar}2u*v}{=}0,$$

and the following limits exist:

$$\lim_{t\to\infty} e_*^{t\frac{1}{i\hbar}2v*u}{=}\varpi_{00}, \quad \lim_{t\to-\infty} e_*^{t\frac{1}{i\hbar}2u*v}{=}\overline{\varpi}_{00}.$$

We call ϖ_{00} and $\overline{\varpi}_{00}$ the **vacuum** and **bar-vacuum** respectively. Strictly speaking, such vacuums should be defined together with the one parameter semigroups $e_*^{t\frac{1}{i\hbar}2v*u}$, $e_*^{-t\frac{1}{i\hbar}2v*u}$, $t{\geq}0$, for they depend on the K-ordered expression and may change sign if there are singular points on $t{\geq}0$. When the ordered expressions $K(s)$, $s{\in}I$, move along a curve, we require that $:e_*^{t\frac{1}{i\hbar}2u*v}:_{K(s)}$ has no singular point on $[0,\infty){\times}I$. Since the $*$-exponential function $e_*^{t\frac{1}{i\hbar}2u*v}$ can be defined as a single valued element by requiring it equal 1 at $t{=}0$, the sign ambiguity does not occur in the K-ordered expression. Thus, we have

$$\begin{aligned}
&\lim_{t\to\infty} :e_*^{t\frac{1}{i\hbar}2v*u}:_K{=}\frac{2}{\sqrt{(1{-}\lambda)^2{+}\delta\delta'}}e^{\frac{1}{i\hbar}\frac{1}{(1-\lambda)^2-\delta\delta'}(\delta u^2+(1-\lambda)2uv+\delta' v^2)},\\
&\lim_{t\to-\infty} :e_*^{t\frac{1}{i\hbar}2u*v}:_K{=}\frac{2}{\sqrt{(1{+}\lambda)^2{+}\delta\delta'}}e^{-\frac{1}{i\hbar}\frac{1}{(1+\lambda)^2-\delta\delta'}(\delta u^2+(1+\lambda)2uv+\delta' v^2)},\\
&\lim_{t\to-\infty} :e_*^{t\frac{1}{i\hbar}2v*u}:_K{=}0, \quad \lim_{t\to\infty} :e_*^{t\frac{1}{i\hbar}2u*v}:_K{=}0.
\end{aligned} \tag{40}$$

The exponential law gives

$$\varpi_{00}{*_0}\varpi_{00}{=}\varpi_{00}, \quad \overline{\varpi}_{00}{*_0}\overline{\varpi}_{00}{=}\overline{\varpi}_{00}.$$

However, we easily see

Theorem 5.1 *The product $\varpi_{00}{*_0}\overline{\varpi}_{00}$ diverges in any ordered expression.*

The existence of the limits (40) also gives

$$u{*}v{*}\varpi_{00} = 0 = \varpi_{00}{*}u{*}v,$$

but the bumping identity $v{*}f(u{*}v){=}f(v{*}u){*}v$ gives the following:

Lemma 5.1 *$v{*}\varpi_{00}{=}0{=}\varpi_{00}{*}u$ in generic ordered expressions.*

Proof Using the continuity of $v*$, we see that $v{*}\lim_{t\to-\infty} e_*^{t\frac{1}{i\hbar}2u*v}{=}\lim_{t\to-\infty} v{*}e_*^{t\frac{1}{i\hbar}2u*v}$. Hence, the bumping identity proved by the uniqueness of the real analytic solution for linear ODE and (40) give $\lim_{t\to-\infty} e_*^{t\frac{1}{i\hbar}2v*u}{*}v{=}0$. □.

The following identities ensure associativity:

Lemma 5.2 $\varpi_{00}{*}(u^p{*}\varpi_{00}){=}0,$ *and* $(\varpi_{00}{*}v^p){*}\varpi_{00}{=}0.$

Proof By the formal power series expansion in $i\hbar$ for e_*^{su*v}, associativity for the equations in Lemma 5.2 holds, and the following computation is justified by the bumping identity:

$$e_*^{su*v}*(u^p*e_*^{tu*v})=(e_*^{su*v}*u^p)*e_*^{tu*v}=u^p*e_*^{(s+t)u*v+i\hbar ps}.$$

The r.h.s of this equation is continuous in s,t. In particular,

$$\lim_{t\to a} e_*^{su*v}*(u^p*e_*^{tu*v})=e_*^{su*v}*\lim_{t\to a}(u^p*e_*^{tu*v}).$$

Using the bumping identity, we have

$$e_*^{su*v}*(u^p*\lim_{t\to-\infty} e_*^{tu*v})=e_*^{su*v}*\lim_{t\to-\infty} u^p*e_*^{tu*v}=\lim_{t\to-\infty} u^p*e_*^{(s+t)u*v+i\hbar ps}$$
$$=u^p*\lim_{t\to-\infty} e_*^{(s+t)u*v+i\hbar ps}=u^p e^{i\hbar ps}*\varpi_{00}.$$

It follows that

$$\varpi_{00}*(u^p*\varpi_{00})=\lim_{s\to-\infty} e_*^{s\frac{1}{i\hbar}u*v}*(\lim_{t\to-\infty} u^p*e_*^{t\frac{1}{i\hbar}u*v})=\lim_{s\to-\infty} u^p e^{ps}*\varpi_{00}=0.$$

Similarly, we also have $(\varpi_{00}*v^p)*\varpi_{00}=0$. □

Lemma 5.3 *For every polynomial* $f(u,v)=\sum a_{pq}u^p*v^q$,

$$\varpi_{00}*(f(u,v)*\varpi_{00})=f(0,0)\varpi_{00}=(\varpi_{00}*f(u,v))*\varpi_{00}.$$

Consequently, associativity holds for $\varpi_{00}*f(u,v)*\varpi_{00}$ *for all polynomials* $f(u,v)$.

Reasoning as above, we see that

$$(e_*^{su*v}*v^q)*(u^p*e_*^{tu*v})=e_*^{su*v}*(v^q*u^p*e_*^{tu*v})=e^{(q-p)ti\hbar}e_*^{(s+t)u*v}*v^q*u^p \quad \text{for } q\geq p,$$
$$(e_*^{su*v}*v^q)*(u^p*e_*^{tu*v})=e_*^{su*v}*(v^q*u^p*e_*^{tu*v})=v^q*u^p*e_*^{(s+t)u*v}*e^{(p-q)si\hbar} \quad \text{for } q\leq p.$$

Replacing s,t by $\frac{1}{i\hbar}s$, $\frac{1}{i\hbar}t$ and taking the limits $t\to-\infty$ and $s\to\infty$ for the case $p\geq q$ and $q\geq p$ respectively, we have

$$(41)\qquad (\varpi_{00}*v^q)*(u^p*\varpi_{00})=\delta_{p,q}p!(i\hbar)^p=\varpi_{00}*(v^q*u^p*\varpi_{00})=(\varpi_{00}*v^q*u^p)*\varpi_{00}.$$

Since $\varpi_{00}*v^q*u^p*\varpi_{00}=\delta_{p,q}p!(i\hbar)^p\varpi_{00}$, we have the following:

Proposition 5.2 $\frac{1}{\sqrt{p!q!(i\hbar)^{p+q}}}u^p*\varpi_{00}*v^q$ *is the* (p,q)*-matrix element.*

As mentioned in the Remark in §3.4, we have two definitions of $e_*^{z\frac{1}{i\hbar}u\circ v}*f(u,v)$. However, both definitions satisfy

$$(42)\qquad e_*^{z\frac{1}{i\hbar}u\circ v}*\varpi_{00}=e^{-\frac{1}{2}z}*\varpi_{00}.$$

Remark In contrast, since $\frac{1}{i\hbar}u\circ v*\delta_*(\frac{1}{\hbar}u\circ v)=0$, where $\delta_*(\frac{1}{\hbar}u\circ v)=\int_\infty^\infty e^{s\frac{1}{i\hbar}u\circ v}$, we must set $e_*^{t\frac{1}{i\hbar}u\circ v}*\delta_*(\frac{1}{\hbar}u\circ v)=\delta_*(\frac{1}{\hbar}u\circ v)$ as the real analytic solution of $\frac{d}{dt}f_t=\frac{1}{i\hbar}u\circ v*f_t$. However, computing

$$\lim_{N\to\infty} e_*^{t\frac{1}{i\hbar}u\circ v}*\int_{-N}^N e_*^{s\frac{1}{i\hbar}u\circ v}ds=\lim_{N\to\infty}\int_{-N}^N e_*^{(t+s)\frac{1}{i\hbar}u\circ v}ds$$

gives the following:

$$(43)\qquad e_*^{(x+iy)\frac{1}{i\hbar}u\circ v}*\delta_*(\frac{1}{i\hbar}u\circ v)=e_*^{iy\frac{1}{i\hbar}u\circ v}*\delta_*(\frac{1}{\hbar}u\circ v).$$

Note that $e_*^{i\pi\frac{1}{i\hbar}u\circ v}=-1$ in the Weyl ordered expression. Thus, (42) is holomorphic with respect to z, while (43) is only continuous and not real analytic with respect to $z=x+iy$.

6 Inverses and their analytic continuation

6.1 The Hadamard finite part procedure

We first recall the Hadamard finite part procedure, a well known technique in distribution theory to extract a finite quantity from a divergent expression. (cf. [19]). We reformulate this procedure on abstract Fréchet algebra in order to extract information on the eigenspaces of a given matrix via its inverse. We conclude that the element $\frac{1}{i\hbar}u{\circ}v$ is an indeterminate lying in a discrete set. Let $(\mathcal{A};*)$ be a complex, complete, topological associative Fréchet algebra with 1 and $\tilde{\mathcal{A}}$ a Fréchet space with a $(\mathcal{A};*)$-bimodule structure (i.e. a continuous bilinear product $*$ is defined for $\mathcal{A}{\times}\tilde{\mathcal{A}}, \tilde{\mathcal{A}}\times\mathcal{A}$ into $\tilde{\mathcal{A}}$ with the natural associativity). We call $\lambda{\in}\mathbb{C}$ a resolvent of $X{\in}\mathcal{A}$ if $\lambda{-}X$ has inverse $(\lambda{-}X)^{-1}$ in $\tilde{\mathcal{A}}$.

Suppose the resolvent set $\rho(X)$ of $X{\in}\mathcal{A}$ is open and dense in $\mathbb{C}$, and $(\zeta{-}X)^{-1}$ is holomorphic in $\zeta{\in}\rho(X)$. Since $(\zeta{-}X){*}(\zeta{-}X)^{-1}{=}1$ on the open dense domain $\rho(X)$, the singularities of this equation are all removable in the usual complex analysis sense.

An isolated singular point z_0 of $(\zeta{-}X)^{-1}$ is a pole, if $(\zeta{-}X)^{-1}$ can be expressed in the form

$$(\zeta{-}X)^{-1}{=}\frac{A_{-d}}{(\zeta{-}z_0)^d}{+}\cdots{+}\frac{A_{-1}}{\zeta{-}z_0}{+}A_0{+}\cdots$$

on a neighborhood of z_0. We call A_0 the finite part of $(\zeta{-}X)^{-1}$ and denote the finite part by $\mathrm{FP}((\zeta{-}X)^{-1})$.

In general, for an $\tilde{\mathcal{A}}$-valued holomorphic function $f(z)$ with a pole at $z{=}z_0$ the finite part $\mathrm{FP}(f(z))$ is defined as follows:

$$\mathrm{FP}(f(z)){=}\begin{cases} f(z) & z{\neq}z_0 \\ \mathrm{Res}_{w=0}\frac{1}{w}(f(z_0{+}w)) & z{=}z_0. \end{cases}$$

This definition is valid for z in a neighborhood of z_0 containing no other pole. Although $(\zeta{-}X){*}(\zeta{-}X)^{-1}{=}1$ for $\zeta{\neq}z_0$, we have

$$(\zeta{-}X){*}\mathrm{FP}(\zeta{-}X)^{-1}{=}\begin{cases} 1 & \zeta{\neq}z_0 \\ 1{-}A_{-1} & \zeta{=}z_0 \end{cases}$$

where we use $(z_0{-}X)A_0{+}A_{-1}{=}1$, which follows easily from the identity $(\zeta{-}X){*}(\zeta{-}X)^{-1}{=}1$. We will employ this trick to analyze singularities of $(\zeta{-}X)^{-1}$ in calculations in extensions of star algebras. In particular, we use this procedure to define a new product by

$$(\zeta{-}X)\tilde{*}(\zeta{-}X)^{-1}{=}(\zeta{-}X){*}\mathrm{FP}(\zeta{-}X)^{-1}.$$

Note that this trick applied to the usual matrix algebra naturally relates to generalized eigenspaces. For a matrix X of finite rank with the eigenvalues $\lambda_1, \dots, \lambda_n$, we have

$$(zI{-}X)\tilde{*}(zI{-}X)^{-1}{=}\begin{cases} I & z{\neq}\lambda_1, \dots, \lambda_n \\ I{-}P_i & z{=}\lambda_i \end{cases}$$

where P_i is the projection to the generalized eigenspace corresponding to the eigenvalue λ_i.

Since the inverse $(\zeta{-}X)^{-1}$ is given very often via the Laplace transform $\int_{-\infty}^0 e_*^{t(\zeta-X)}dt$, we have the following theorem:

Theorem 6.1 *Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be as above. Suppose $X{\in}\mathcal{A}$ is an element such that the equation*

$$\frac{d}{dz}f(z){=}X{*}f(z), \quad f(0){=}1 \tag{44}$$

has a complex analytic solution in the Fréchet space $\tilde{\mathcal{A}}$ defined on a connected open domain D. If $\lambda{+}X$ has an inverse in the Fréchet algebra $\mathcal{A}$ for some $\lambda{\in}\mathbb{C}$, then D is simply connected and $f(z){=}\sum\frac{z^n}{n!}X_*^n$.

Proof The proof is elementary. Denote the solution by e_*^{zX}. Let $\Sigma(X)$ be the set of singular points of e_*^{zX} in $\mathbb{C}$. If $\mathbb{C}\setminus\Sigma(X)$ is not simply connected, there is a closed curve C in D surrounding a singular point z_0. □

By the uniqueness of real analytic solutions, the exponential law $e_*^{zX}{*}e_*^{wX}{=}e_*^{(z+w)X}$ holds, provided all three terms exist. Suppose there is a $\lambda{\in}\mathbb{C}$ such that $(\lambda{+}X)^{-1}{\in}\mathcal{A}$. Since $e^{z\lambda}e_*^{zX}$ is the solution of the equation $\frac{d}{dz}f_z{=}(\lambda{+}X){*}f_z$, we derive a second exponential law $e^{z\lambda}e_*^{zX}{=}e_*^{z(\lambda+X)}$. It follows that $\Sigma(\lambda{+}X){=}\Sigma(X)$.

Obviously, for every integer $k\geq 0$ the contour integral $\int_C(z{-}z_0)^k e_*^{z(\lambda+X)}dz$ gives an element of $\tilde{\mathcal{A}}$. It follows that

$$(\lambda{+}X)^{k+1}{*}\int_C(z{-}z_0)^k e_*^{z(\lambda+X)}dz{=}\int_C(z{-}z_0)^k(\lambda{+}X)^{k+1}{*}e_*^{z(\lambda+X)}dz$$
$$={\int_C}(z{-}z_0)^k\frac{d^{k+1}}{dz^{k+1}}e_*^{z(\lambda+X)}dz{=}(-1)^k\int_C\frac{d}{dz}e_*^{z(\lambda+X)}dz{=}0.$$

The existence of $(\lambda{+}X)^{-1}$ gives $\int_C(z{-}z_0)^k e_*^{z(\lambda+X)}dz{=}0$ for every integer k, which implies that z_0 is not a singular point. Thus, D is an open simply connected neighborhood of the origin. Standard Taylor series methods yield $f(z){=}e_*^{zX}{=}\sum\frac{1}{n!}(zX)^n$. □

This theorem suggests that we have to go beyond the category of Fréchet algebra valued meromorphic functions to treat the inverse of $z{+}\frac{1}{i\hbar}u{\circ}v$, as $e_*^{t\frac{1}{i\hbar}u\circ v}$ has discrete singular points in general ordered expressions. The regularized product $(\zeta{-}X)\tilde{*}(\zeta{-}X)^{-1}$ seems to be a good method to treat singularities.

6.2 Basic properties of the inverse of $z{+}\frac{1}{i\hbar}u{\circ}v$

We first study basic properties of the inverse of $z{+}\frac{1}{i\hbar}u{\circ}v$.

By the results of §4, the integrals

$$:\int_{-\infty}^{0}e^{tz}e_*^{t\frac{1}{i\hbar}u\circ v}dt{:}_0=\int_{-\infty}^{0}\frac{e^{tz}}{\cosh\frac{1}{2}t}e^{\frac{1}{i\hbar}2uv\tanh\frac{1}{2}t}dt,\quad \mathrm{Re}\,z>-\frac{1}{2}, \tag{45}$$

$$:-\int_{0}^{\infty}e^{tz}e_*^{t\frac{1}{i\hbar}u\circ v}dt{:}_0=-\int_{0}^{\infty}\frac{e^{tz}}{\cosh\frac{1}{2}t}e^{\frac{1}{i\hbar}2uv\tanh\frac{1}{2}t}dt,\quad \mathrm{Re}\,z<\frac{1}{2} \tag{46}$$

converge in the Weyl ordered expression.

One can analyze the r.h.s. of (45) and (46) more closely via a change of variables as in Proposition 4.1. For $-\frac{1}{2}<\mathrm{Re}\,z\leq 0$, the change of variables $\tanh\frac{1}{2}t{=}\cos s$ transforms the r.h.s of (45) into

$$2\int_{-\pi}^{0}(\frac{1{+}\cos s}{1{-}\cos s})^z e^{(\cos s)\frac{1}{i\hbar}2uv}ds.$$

For $0\leq\mathrm{Re}\,z{<}\frac{1}{2}$ and for $-\cos s{=}\tanh\frac{t}{2}$, $2\sin s ds{=}\sin^2 s dt$, the r.h.s. of (46) transforms into

$$2\int_{0}^{\pi}(\frac{1{+}\cos s}{1{-}\cos s})^{-z}e^{(\cos s)\frac{1}{i\hbar}2uv}ds.$$

Hence, Lemmas 2.2, 3.1 give that $\int_{-\infty}^{\infty}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt$ is an element of $Hol(\mathbb{C}^2)$ in generic ordered expressions. Thus, both (45) and (46) give inverses of $z{+}\frac{1}{i\hbar}u{\circ}v$ for generic ordered expressions, which will be denoted by $(z{+}\frac{1}{i\hbar}u{\circ}v)_{*+}^{-1}$, $(z{+}\frac{1}{i\hbar}u{\circ}v)_{*-}^{-1}$, respectively.

The following may be viewed as a Sato hyperfunction:

Proposition 6.1 *If* $-\frac{1}{2} < \mathrm{Re}\, z < \frac{1}{2}$*, then the difference of the two inverses is given by*

$$(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}-(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}=\int_{-\infty}^{\infty} e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt. \tag{47}$$

The difference is holomorphic in this strip for generic ordered expressions.

An elementary change of variables gives

$$((-z)+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}=-\int_0^{\infty} e_*^{-t(z-\frac{1}{i\hbar}u\circ v)}dt=-\int_{-\infty}^{0} e_*^{(z-\frac{1}{i\hbar}u\circ v)}dt.$$

Thus, for generic ordered expressions, we see that

$$(z-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}=-((-z)+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}. \tag{48}$$

This is holomorphic on the domain $\mathrm{Re}\, z> -\frac{1}{2}$, on which $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$ is also holomorphic. All of these results are easily proved for the Weyl ordered expression. However, for generic K-ordered expression, $:e_*^{t\frac{1}{i\hbar}u\circ v}:_K$ is rapidly decreasing in t, and the same computation gives the following:

Proposition 6.2 *For generic ordered expressions,* $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$ *and* $(z-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}$ *are defined for* $\mathrm{Re}\, z> -\frac{1}{2}$*.*

The product $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}{*}(w+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$ is naturally defined for $z, w{\not\in}-(\mathbb{N}+\frac{1}{2})$ by the usual resolvent identity. $\{(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}; z{\not\in}-(\mathbb{N}+\frac{1}{2})\}$ forms an associative algebra. in $\mathcal{E}_{2+}(\mathbb{C}^{2m})$.

Note that $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}{*}(-z-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}$ diverges for any ordered expression. However, the standard resolvent formula gives the following:

Proposition 6.3 *If* $z{+}w{\neq}0$*, then*

$$\frac{1}{z+w}\Big((z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}+(w-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}\Big)$$

is an inverse of $(z+\frac{1}{i\hbar}u{\circ}v){*}(w-\frac{1}{i\hbar}u{\circ}v)$*. In particular, for every positive integer* n*, and for every complex number* z *such that* $\mathrm{Re}\, z> -(n+\frac{1}{2})$*,*

$$\frac{1}{2n}\big((1+\frac{1}{n}(z+\frac{1}{i\hbar}u{\circ}v))^{-1}_{*+}+(1-\frac{1}{n}(z+\frac{1}{i\hbar}u{\circ}v))^{-1}_{*-}\big)$$

is an inverse of $1-\frac{1}{n^2}(z+\frac{1}{i\hbar}u{\circ}v)^2_*$ *for generic ordered expressions.*

6.3 Analytic continuation of inverses

Recall that $(z{\pm}\frac{1}{i\hbar}u{\circ}v)^{-1}_{\pm*}$ is holomorphic on the domain $\mathrm{Re}\, z > -\frac{1}{2}$ for generic ordered expressions. It is natural to expect that $(z{\pm}\frac{1}{i\hbar}u{\circ}v)^{-1}_{\pm*}{=}C(C(z{\pm}\frac{1}{i\hbar}u{\circ}v))^{-1}_{\pm*}$ for any non-zero constant C. To confirm this, we set $C{=}e^{i\theta}$ and consider the θ-derivative of

$$e^{i\theta}\int_{-\infty}^{0} e_*^{e^{i\theta}t(z\pm\frac{1}{i\hbar}u\circ v)}dt.$$

In generic K-ordered expressions, the phase part of the integrand is bounded in t and the amplitude is given by

$$\frac{2e^{i\theta}tz}{(1-\kappa)e^{e^{i\theta}t/2}+(1+\kappa)e^{-e^{i\theta}t/2}}, \quad \kappa\neq 1.$$

The integral converges whenever $\mathrm{Re}\, e^{i\theta}(z\pm\frac{1}{2}) > 0$, and by integration by parts this convergence is independent of θ. It follows that $(z\pm\frac{1}{i\hbar}u{\circ}v)^{-1}_{\pm *}$ is holomorphic on the domain $\mathbb{C}-\{z; -\infty<z<-\frac{1}{2}\}$.

Next, it is natural to expect that the bumping identity $(u{\circ}v){*}v=v{*}(u{\circ}v-i\hbar)$ gives the following "sliding identities"

$$v^{-1}_{*+}{*}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}{*}v=(z-1+\frac{1}{i\hbar}u{\circ}v)^{-1}_{+*}, \quad v^{-1}_{*+}{*}(z-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}{*}v=(z+1-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}$$

whenever the inverse of v exists in a particular ordered expression. In this section, analytic continuation will be produced via these sliding identities.

However, the existence of v^{-1}_{*+} is not a generic property. As a result, instead of using v^{-1}_{*+} we will apply the sliding identity to the left inverse v° of v given below.

Remark There is a K-ordered expression such that $:\int_{-\infty}^{0} e_*^{tv}dt:_K$ converges to give an inverse of $:v^{-1}_{*+}:_K$ of v (cf. [17]), but it is easy to see that $:v^{-1}_{*+}{*}\varpi_{00}:_K$ diverges.

First, we remark that the formula in Proposition 6.1 gives

$$(u{*}v)^{-1}_{*-}=-\frac{1}{i\hbar}\int_0^{\infty} e_*^{t\frac{1}{i\hbar}u*v}dt, \quad (v{*}u)^{-1}_{*+}=\frac{1}{i\hbar}\int_{-\infty}^{0} e_*^{t\frac{1}{i\hbar}v*u}dt$$

for generic ordered expressions. Then

$$v^{\circ}=u{*}(v{*}u)^{-1}_{*+}, \quad u^{\bullet}=v{*}(u{*}v)^{-1}_{*-},$$

are left and right inverses of v and u respectively. That is,

$$v{*}v^{\circ}=1, \quad v^{\circ}{*}v=1-\varpi_{00}, \quad u{*}u^{\bullet}=1, \quad u^{\bullet}{*}u=1-\varpi_{00}.$$

The bumping identity gives

$$v{*}(z+\frac{1}{i\hbar}u{\circ}v){*}v^{\circ}=z+1+\frac{1}{i\hbar}u{\circ}v, \quad v^{\circ}{*}(z+\frac{1}{i\hbar}u{\circ}v){*}v=(1-\varpi_{00}){*}(z-1+\frac{1}{i\hbar}u{\circ}v).$$

Successive applications of the bumping identity give the following useful formula:

$$(u{*}(v{*}u)^{-1}_{*+})^n{*}\varpi_{00}=\frac{1}{n!}(\frac{1}{i\hbar}u)^n{*}\varpi_{00}. \tag{49}$$

Using v° instead of v^{-1}_{*+}, we can produce the analytic continuation of inverses. However, we have to be careful about the continuity of the $*$-product. We compute

$$\begin{aligned} v^{\circ}{*}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}&=\Big(u*\int_{-\infty}^{0} e_*^{t(\frac{1}{i\hbar}u\circ v+\frac{1}{2})}dt\Big)*\int_{-\infty}^{0} e_*^{s(z+\frac{1}{i\hbar}u\circ v)}ds\\ &=u*\int_{-\infty}^{0}\int_{-\infty}^{0} e_*^{t(\frac{1}{i\hbar}u\circ v+\frac{1}{2})}{*}e_*^{s(z+\frac{1}{i\hbar}u\circ v)}dtds \quad (\text{cf. } (39))\\ &=\int_{-\infty}^{0}\int_{-\infty}^{0} e^{t\frac{1}{2}+sz}u{*}e_*^{(t+s)\frac{1}{i\hbar}u\circ v}dtds \quad (\text{cf. Lemma 2.1})\\ &=\int_{-\infty}^{0}\int_{-\infty}^{0} e^{t\frac{1}{2}+sz-(t+s)}e_*^{(t+s)\frac{1}{i\hbar}u\circ v}{*}udtds. \end{aligned}$$

Hence, whenever both sides are defined, we obtain

$$\begin{aligned}(v^{\circ}*(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+})*v=&\int_{-\infty}^{0}\int_{-\infty}^{0}e^{-t\frac{1}{2}+s(z-1)}e_*^{(t+s)\frac{1}{i\hbar}u\circ v}*(u*v)dtds\\ =&\int_{-\infty}^{0}(u*v)*e_*^{t\frac{1}{i\hbar}u*v}dt*\int_{-\infty}^{0}e_*^{s(z-1+\frac{1}{i\hbar}u\circ v)}ds\\ =&(1-\varpi_{00})*(z-1+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}.\end{aligned}$$

Noting that

$$\varpi_{00}*(z-1+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}=(z-1+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}*\varpi_{00}=(z-\frac{1}{2})^{-1}\varpi_{00},$$

whenever $(z-1+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$ is defined, we have

$$\big(v^{\circ}*(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}\big)*v+(z-\frac{1}{2})^{-1}\varpi_{00}=\big(z-1+\frac{1}{i\hbar}u{\circ}v\big)^{-1}_{*+}. \tag{50}$$

Since $(z-\frac{1}{2})^{-1}\varpi_{00}$ is always defined, we see that the functional equation (50) gives an analytic continuation for $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$. Namely, we have the following (see [7] and [9] for more details):

Theorem 6.2 *For generic ordered expressions, the inverses* $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$, $(z-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}$ *extend to* $\mathcal{E}_{2+}(\mathbb{C}^2)$*-valued holomorphic functions of* z *on* $\mathbb{C}-\{-(\mathbb{N}+\frac{1}{2})\}$.

In particular, $(z^2-(\frac{1}{i\hbar}u{\circ}v)^2)^{-1}_{\pm *}$ *extends to a holomorphic function of* z *on this domain.*

The residue at a singular point z_0 is defined as usual by $\frac{1}{2\pi i}\int_{C_{z_0}}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*\pm}dz$. The analytic continuation formula gives the following:

Theorem 6.3 $\mathrm{Res}((z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+},-(n+\frac{1}{2}))=\frac{1}{(i\hbar)^n n!}u^n*\varpi_{00}*v^n$ *for generic ordered expressions.*

For the proof, we remark that $(z+n+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*\pm}$ is holomorphic for sufficiently large n, and the contour integral is an integral on a compact set. Note that $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$ is singular at $z=n+\frac{1}{2}$, but $(z+\frac{1}{i\hbar}u{\circ}v)*(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}=1$ for $z\not\in-(\mathbb{N}+\frac{1}{2})$ for generic ordered expressions.

Note also that if we exchange $(z+\frac{1}{i\hbar}u{\circ}v)*$ and the integration, then

$$\int_{-\infty}^{0}(z+\frac{1}{i\hbar}u{\circ}v)*e_*^{t(z+\frac{1}{i\hbar}u\circ v)}dt=\begin{cases}1 & \mathrm{Re}\,z>-\frac{1}{2}\\ 1-\varpi_{00} & z=-\frac{1}{2}\end{cases},$$

$$\int_{-\infty}^{0}(z-\frac{1}{i\hbar}u{\circ}v)*e_*^{t(z-\frac{1}{i\hbar}u\circ v)}dt=\begin{cases}1 & \mathrm{Re}\,z>-\frac{1}{2}\\ 1-\overline{\varpi}_{00} & z=-\frac{1}{2}\end{cases}.$$

As suggested by these formulas and Hadamard's technique of extracting finite parts of divergent integrals, we now extend the definition of the $*$-product using the finite part regularization mentioned in the introduction.

We consider the inductive limit topology on the space $\mathbb{C}[\boldsymbol{u}]$. We define the new product of $(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*\pm}$ with either polynomials $q(u,v)$ or $q(u,v)=e_*^{s\frac{1}{i\hbar}u\circ v}$ by

$$(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*\pm}\tilde{*}q(u,v)=(\mathrm{FP}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*\pm})*q(u,v), \tag{51}$$

where $\mathrm{FP}f(z)$ denotes its finite part of f at z. The result may not be continuous in z.

For $\mathrm{Re}\,z > -\frac{1}{2}$ we easily see that

$$(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}\tilde{*}q(u,v)=\lim_{N\to\infty}\int_{-N}^{0}e_*^{t(z+\frac{1}{i\hbar}u\circ v)}*q(u,v)dt$$

Hence we have the formula

$$(z+\frac{1}{i\hbar}u{\circ}v)\tilde{*}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}=\begin{cases}1 & \mathrm{Re}\,z>-\frac{1}{2}\\ 1-\varpi_{00} & z=-\frac{1}{2}\end{cases}. \tag{52}$$

Using $(v^{\circ})^n*(z+\frac{1}{i\hbar}u{\circ}v)*(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}*v^n=(v^{\circ})^n*(z+\frac{1}{i\hbar}u{\circ}v)*v^n*(v^{\circ})^n*(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}*v^n$ and (50), we have the following:

Theorem 6.4 *Using definition* (51) *for the* $\tilde{*}$*-product, we have*

$$(z+\frac{1}{i\hbar}u{\circ}v)\tilde{*}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}=\begin{cases}1 & z\not\in-(\mathbb{N}+\frac{1}{2})\\ 1-\frac{1}{n!}(\frac{1}{i\hbar}u)^n*\varpi_{00}*v^n & z=-(n+\frac{1}{2})\end{cases}, \tag{53}$$

$$(z-\frac{1}{i\hbar}u{\circ}v)\tilde{*}(z-\frac{1}{i\hbar}u{\circ}v)^{-1}_{*-}=\begin{cases}1 & z\not\in-(\mathbb{N}+\frac{1}{2})\\ 1-\frac{1}{n!}(\frac{1}{i\hbar}v)^n*\overline{\varpi}_{00}*u^n & z=-(n+\frac{1}{2})\end{cases}. \tag{54}$$

for generic ordered expressions.

Although $z{=}-(n{+}\frac{1}{2})$, $n=0,1,2,\cdots$ are all removable singularities for (53) and (54) as a function of z, it is better to retain these singular points.

In these computations, elements are often given via a limiting procedure. As usual, $*$-products of such elements depend delicately on the limiting procedure. There is no general rule guaranteeing associativity.

Via the identity $(1{+}\frac{1}{m}(z{+}\frac{1}{i\hbar}u{\circ}v))^{-1}_{*+}{=}m(m{+}z{+}\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}$, we have, in particular

$$(1+\frac{1}{m}(z+\frac{1}{i\hbar}u{\circ}v))\tilde{*}(1+\frac{1}{m}(z+\frac{1}{i\hbar}u{\circ}v))^{-1}_{*+}=\begin{cases}1 & z\not\in-(\mathbb{N}+m+\frac{1}{2})\\ 1-\frac{1}{k!}(\frac{1}{i\hbar}u)^k*\varpi_{00}*v^k & z=-(k+m+\frac{1}{2})\end{cases} \tag{55}$$

for every fixed positive integer m and for arbitrary $k\in\mathbb{N}$ for generic ordered expressions.

By the associativity stated in Lemma 2.1, we see the following:

Theorem 6.5

$$(-n-\frac{1}{2}+\frac{1}{i\hbar}u{\circ}v)*u^n*\varpi_{00}=u^n*(\frac{1}{i\hbar}u{*}v)\varpi_{00}=0.$$

Thus, we have

$$(1-\frac{1}{\ell}(z+\frac{1}{i\hbar}u{\circ}v))*\Big((1+\frac{1}{m}(z+\frac{1}{i\hbar}u{\circ}v))\tilde{*}((1+\frac{1}{m}(z+\frac{1}{i\hbar}u{\circ}v))^{-1}_{*+}\Big)$$
$$=\begin{cases}1-\frac{1}{\ell}(z+\frac{1}{i\hbar}u{\circ}v) & z\not\in-(\mathbb{N}+m+\frac{1}{2})\\ 1-\frac{1}{\ell}(z+\frac{1}{i\hbar}u{\circ}v) & z=-(\ell+\frac{1}{2})\\ (1-\frac{1}{\ell}(z+\frac{1}{i\hbar}u{\circ}v))*(1-\frac{1}{k!}(\frac{1}{i\hbar}u)^k*\varpi_{00}*v^k) & z=-(k+\frac{1}{2}),\ z\neq-(\ell+\frac{1}{2}).\end{cases},$$

for generic ordered expressions.

We note here that singularities such as $\frac{1}{\ell!}(\frac{1}{i\hbar}u)^{\ell}*\varpi_{00}*v^{\ell}$ disappear from the r.h.s. of the above equality because of the term $1{-}\frac{1}{\ell}(z{+}\frac{1}{i\hbar}u{\circ}v)$.

We also define a $\tilde{*}$-product for a certain class of elements by

$$f(\boldsymbol{u})\tilde{*}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}=f(\boldsymbol{u})*\big(\mathrm{FP}(z+\frac{1}{i\hbar}u{\circ}v)^{-1}_{*+}\big).$$

These formulas will be applied to the computation of

$$\sin_*(z+\frac{1}{i\hbar}u{\circ}v)\tilde{*}(1+\frac{1}{m}(z+\frac{1}{i\hbar}u{\circ}v))^{-1}_{*+}$$

along with an infinite product formula for $\sin_*(z{+}\frac{1}{i\hbar}u{\circ}v)$ in a forthcoming paper.

References

[1] G. S. Agawal and E. Wolf, *Calculus for functions of noncommuting operators and general phase-space method of functions*, Physical Review D, **2** (1970), 2161-2186.

[2] G. Andrews, R. Askey and R. Roy, Special Functions, Encyclopedia Math, Appl. **71**, Cambridge, 2000.

[3] Bayen, M. Flato, Frosdental, A. Lichnerowicz, D. Sternheimer, *Deformation quantizations*, Ann. Physics, **111** (1978), 61-110 .

[4] I. M. Gel'fand and G. E. Shilov, Generalized Functions, 2 Academic Press, 1968.

[5] V. Guillemin and S. Sternberg, Geometric Asymptotics. A.M.S. Mathematical Surveys, **14**, 1977.

[6] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Deformation quantization of Fréchet-Poisson algebras, –Convergence of the Moyal product–*, Math. Phys. Stud. **22**, (2000), 233-246.

[7] H. Omori, Physics in Mathematics, (in Japanese) Tokyo Univ. Publ., 2004.

[8] H.Omori, *Toward geometric quantum theory*, Progr. Math. **252** (2006), 213-251.

[9] H. Omori and Y. Maeda, Quantum Theoretic Calculus, (in Japanese) Springer-Verlag, Tokyo, 2004.

[10] H. Omori and T. Kobayashi, *Singular star-exponential functions*, SUT J. Math. **37** (2001), 137-152.

[11] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Strange phenomena related to ordering problems in quantizations*, J. Lie Theory, **13**, (2003), 481-510.

[12] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Star exponential functions as two-valued elements*, Progr. Math. **232** (2004), 483-492..

[13] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Geometric objects in an approach to quantum geometry*, Progr. Math. **252** (2006), 303-324.

[14] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka,: *Expressions of elements of algebras and transcendental noncommutative calculus,* Noncommutative Geometry and Physics 2005 (2007), 3-30.

[15] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Non-formal deformation quantization of Fréchet-Poisson algebras: The Heisenberg Lie algebra case,* Contemp. Math. **434**, (2007), AMS, 99-123.

[16] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Noncommutative Minkowski space and transcendental calculus*, Progress of Theoretical Physics Suppl. 171, (2007), 184-195.

[17] H. Omori, Y. Maeda, N. Miyazaki and A. Yoshioka, *Orderings and nonformal deformation quantizations,* Lett. Math. Phys. **82** (2007), 153-175.

[18] M. Rieffel, *Noncommutative tori—a case study of noncommutative differentiable manifolds*, Contemp. Math., **105** (1990), 191-211

[19] L. Schwarz, Theory of Distributions, Academic Press 1966.

[20] S.L. Woronowicz, *Differential calculus on compact matrix pseudogroup(quantum group)*, Comm. Math. Phys. **122** (1989), 25-70.

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