

Research Report

KSTS/RR-07/009

December 13, 2007

A new formulation of measurement theory

by

Shiro Ishikawa

Shiro Ishikawa
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

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3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

A new formulation of measurement theory

Shiro Ishikawa *

[abstract] As a kind of generalization of dynamical system theory, we recently proposed measurement theory, which has two formulations, i.e., C^* -algebraic formulation and W^* -algebraic formulation. However, we now think that this assertion should be reconsidered, that is, the two formulations should be combined. Here we propose the new compromise formulation, which has both merits of C^* -algebraic formulation and W^* -algebraic formulation.

[key phrases] measurement, mechanical world view, dynamical system theory, statistics, quantum mechanics, axiomatic theory, observable, state, Heisenberg picture, Copernican revolution, thing-in-itself (ding an sich), transcendental idealism, Immanuel Kant, probability

1. Introduction

In [2,3,4], as a kind of generalization of dynamical system theory, we proposed measurement theory, which is formulated in a certain operator algebra (i.e., C^* -algebra and W^* -algebra, *cf.*[7]) as follows.

$$\begin{aligned} \text{“(pure) measurement theory”} &= \underbrace{[\text{measurement}]}_{\text{(Axiom 1)}} + \underbrace{[\text{the relation among systems}]}_{\text{(Axiom 2)}}. \end{aligned} \tag{1}$$

This can be regarded as a kind of mathematical generalization of quantum mechanics:

$$\begin{aligned} \text{“quantum mechanics”} &= \underbrace{[\text{quantum measurement}]}_{\text{(Born's measurement)}} + \underbrace{[\text{unitary time evolution}]}_{\text{(Heisenberg kinetic equation)}} \end{aligned} \tag{2}$$

such as it includes dynamical system theory (=DST) in engineering:

$$\begin{aligned} \text{“DST”} &= \begin{cases} y(t) = g(x(t), u_1(t), t) & \cdots (\text{measurement equation}) \\ \frac{dx(t)}{dt} = f(x(t), u_2(t), t), x(0) = x_0 & \cdots (\text{state equation}) \end{cases} \end{aligned} \tag{3}$$

where u_1 and u_2 are external forces (or noises).

*Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan, E-Mail: ishikawa@math.keio.ac.jp

• The main part of this preprint will be published in Far east journal of dynamical systems.

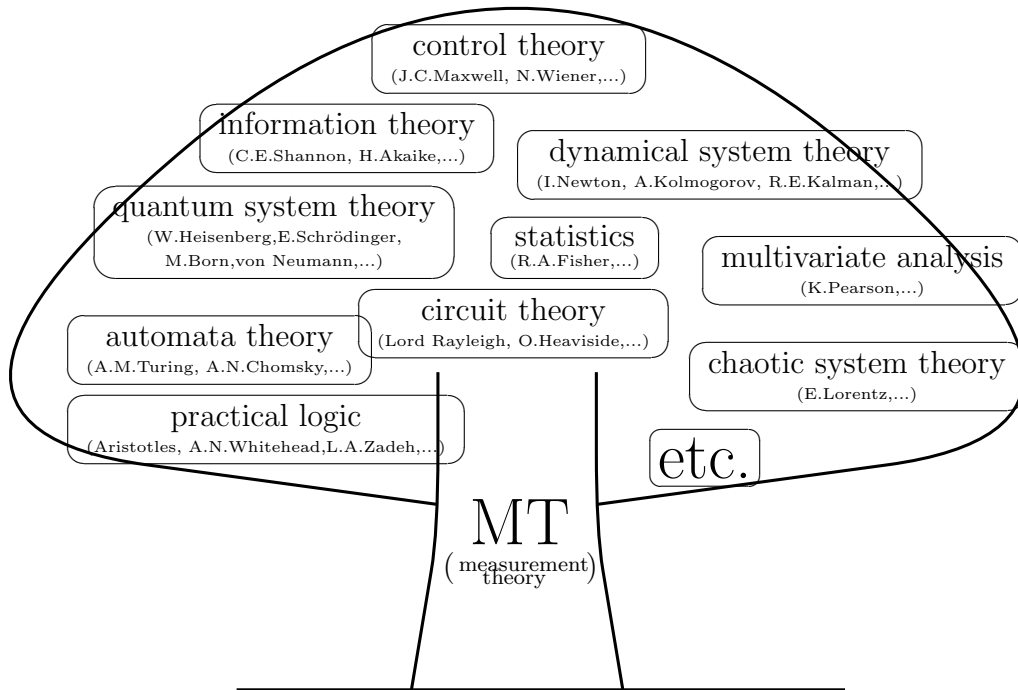
Therefore, we can say $(2)+(3)\subset(1)$. And thus, measurement theory (1) may be also called “*generalized quantum theory*” or “*generalized dynamical system theory (=GDST)*”. Although it is certain that DST(3) is a good mathematical theory to analyze usual phenomena in our usual life (or, economics, psychology, engineering and so on), it should be noted that it is difficult to say that DST(3) is axiomatic. Further, we have the following classification:

$$\text{“(pure) measurement theory (1)”} = \begin{cases} \text{quantum measurement theory} & (4) \\ \text{classical measurement theory,} & (5) \end{cases}$$

where the algebra is either non-commutative or commutative. Thus we consider that $(2)=(4)$, and the (5) is the axiomatic form of DST(3). Therefore, we believe that the (5) is the fundamental theory to describe usual phenomena in our usual life, that is, it is the axiomatic theory of the epistemology called “*the mechanical world view*”.

For completeness,, again note that measurement theory is quite fundamental and wide, as symbolized in the following picture:

The “MT (measurement theory) tree” (extracted from [ref.4: page 323])



As seen in [4] (or, our papers in the references of the book [4]), the above measurement theory (1) has two formulations, i.e., C^* -algebraic formulation and W^* -algebraic formulation. However, we now think that this assertion should be reconsidered, that is, the two formulations should be combined. The purpose of this paper is to propose the new compromise formulation, which has both merits of C^* -algebraic formulation

and W^* -algebraic formulation.

2. Measurement theory (mathematical preparations)

Let V be a complex Hilbert space with the norm $\|\cdot\|_V$. Put

$$B(V) \equiv \{T \mid T \text{ is a bounded linear operator from a Hilbert space } V \text{ into itself}\}.$$

Define $\|T\|_{B(V)} = \sup\{\|Tv\|_V \mid \|v\|_V = 1\}$, and $(T_1T_2)(v) = T_1(T_2v)$ ($\forall v \in V$). And T^* is defined by the adjoint operator of T . Note that it holds that $\|T^*T\|_{B(V)} = \|T\|_{B(V)}^2$ ($\forall T \in B(V)$). And thus, $B(V)$ is a C^* -algebra (cf. [7]).

An element F in $B(V)$ is called *self-adjoint* if it holds that $F = F^*$. A self-adjoint element F in $B(V)$ is called *positive* (and denoted by $F \geq 0$) if there exists an element F_0 in $B(V)$ such that $F = F_0^*F_0$. Also, a positive element F is called a *projection* if $F = F^2$ holds.

A triplet $[\mathcal{A}, \mathcal{N}(\equiv \overline{\mathcal{A}}^{w*}), B(V)]$ is called an *operator algebraic structure*, if $\mathcal{A}(\subseteq B(V))$ is the norm closed sub- $*$ -algebra of $B(V)$, and if $\mathcal{N}(\equiv \overline{\mathcal{A}}^{w*} (\subseteq B(V)))$ is the weak- $*$ -closure of \mathcal{A} in $B(V)$. Thus note that \mathcal{A} and $\overline{\mathcal{A}}^{w*}$ are C^* -algebras. Let \mathcal{A}^* be the dual Banach space of \mathcal{A} . That is, $\mathcal{A}^* \equiv \{\rho \mid \rho : \mathcal{A} \rightarrow \mathbf{C} \text{ is a complex-valued continuous linear function}\}$, and the norm $\|\rho\|_{\mathcal{A}^*}$ is defined by $\sup\{|\rho(F)| \mid \|F\|_{\mathcal{A}} \leq 1\}$. Define the *mixed state space* $\mathfrak{S}^m(\mathcal{A}^*)$ such that:

$$\mathfrak{S}^m(\mathcal{A}^*) \equiv \{\rho \in \mathcal{A}^* \mid \|\rho\|_{\mathcal{A}^*} = 1 \text{ and } \rho(F) \geq 0 \text{ for all } F \geq 0\}.$$

A mixed state $\rho (\in \mathfrak{S}^m(\mathcal{A}^*))$ is called a *pure state* if it satisfies that “ $\rho = \theta\rho_1 + (1-\theta)\rho_2$ for some $\rho_1, \rho_2 \in \mathfrak{S}^m(\mathcal{A}^*)$ and $0 < \theta < 1$ ” implies “ $\rho = \rho_1 = \rho_2$ ”. Define

$$\mathfrak{S}^p(\mathcal{A}^*) \equiv \{\rho^p \in \mathfrak{S}^m(\mathcal{A}^*) \mid \rho^p \text{ is a pure state}\},$$

which is called a *state space* (or *pure state space*, *phase space*). Note that $\mathfrak{S}^p(\mathcal{A}^*)$ is locally compact in the sense of the weak- $*$ topology $\sigma(\mathcal{A}^*; \mathcal{A})$.

It is well known that $\mathcal{N}(\equiv \overline{\mathcal{A}}^{w*})$ is not only a C^* -algebra but also a W^* -algebra, that is, it is a C^* -algebra with the unique pre-dual Banach space \mathcal{N}_* (i.e., $\mathcal{N} = (\mathcal{N}_*)^*$). Also, note that the identity I belongs to $\mathcal{N} (\subseteq B(V))$. Now we can define the *normal state-class* $\mathfrak{S}^n(\mathcal{N}_*)$ such as

$$\mathfrak{S}^n(\mathcal{N}_*) \equiv \{\rho^n \in \mathcal{N}_* \mid \|\rho^n\|_{\mathcal{N}_*} = 1 \text{ and } \rho^n \geq 0 \text{ (i.e., } \rho^n(T^*T) \geq 0 \text{ for all } T \in \mathcal{N})\}.$$

The element ρ^n ($\in \mathfrak{S}^n(\mathcal{N}_*)$) is called a *normal state* (or, *density state*). Although we are not concerned with the normal state in this paper, it plays an important role in statistical measurement theory (9), cf. [4].

Example 1 (i). Let Ω be a locally compact topological space, and let $(\Omega, \mathcal{F}_\Omega, \mu)$ be a measure space such that $0 < \mu(U) \leq \infty$ for any open set $U (\subseteq \Omega)$, and $0 \leq \mu(\{\omega\}) < \infty$ ($\forall \omega \in \Omega$). Define the Banach space $L^r(\Omega, \mu)$, ($r = 1, 2, \infty$), by the set of all complex valued measurable functions on Ω such that the norm $\|f\|_{L^r(\Omega, \mu)}$ is finite, where $\|f\|_{L^r(\Omega, \mu)} = [\int_\Omega |f(\omega)|^r \mu(d\omega)]^{1/r}$ (if $r = 1, 2$), $= \text{ess. sup}_{\omega \in \Omega} |f(\omega)|$ (if $r = \infty$). The operator algebraic structure $[C_0(\Omega), L^\infty(\Omega, \mu), B(L^2(\Omega, \mu))]$ is essential to the classical measurement theory (5). Here, note that

$$\overline{C_0(\Omega)}^{w*} = L^\infty(\Omega, \mu) \subseteq B(L^2(\Omega, \mu))$$

where $C_0(\Omega)$ is the algebra composed of all continuous complex-valued functions *vanishing at infinity* on Ω , and the norm $\|f\|_{C_0(\Omega)}$ is defined by $\|f\|_{C_0(\Omega)} = \max\{|f(\omega)| \mid \omega \in \Omega\}$ ($\forall f \in C_0(\Omega)$). Also, it is well known that

$$\begin{aligned} \mathfrak{S}^p(C_0(\Omega)^*) &= \{\delta_\omega \mid \omega \in \Omega\} = \Omega, \\ \mathfrak{S}^n(L^\infty(\Omega, \mu)_*) &= L^1_{+1}(\Omega, \mu) \equiv \{f \mid f \in L^1(\Omega, \mu), 0 \leq f, \|f\|_{L^1(\Omega, \mu)} = 1\} \end{aligned}$$

where $\delta_\omega (\in C_0(\Omega)^*)$ is the point measure at $\omega (\in \Omega)$.

(ii). Also, the operator algebraic structure $[\mathcal{C}(V), B(V), B(V)]$ is essential to the quantum measurement theory (4). Here, note that

$$\overline{\mathcal{C}(V)}^{w*} = B(V)$$

where $\mathcal{C}(V) \equiv \{T \in B(V) \mid T \text{ is a compact operator}\}$. Also, note that

$$\begin{aligned} \mathfrak{S}^p(\mathcal{C}(V)^*) &= \{|u\rangle\langle u| \mid u \in V\}, \quad (\text{where } |u\rangle\langle u| \text{ is the Dirac notation}) \\ \mathfrak{S}^n(B(V)_*) &= Tr_{+1}(V) \equiv \{T \in Tr(V) \mid T \text{ is a positive such that } \|T\|_{Tr(V)} = 1\} \end{aligned}$$

where $Tr(V)$ is the trace class (with the trace norm $\|\cdot\|_{Tr(V)}$) in $B(V)$.

Definition 2 [Observable]. Let $[\mathcal{A}, \mathcal{N}(\equiv \overline{\mathcal{A}}^{w*}), B(V)]$ be an operator algebraic structure. An observable $\mathbf{O} \equiv (X, \mathcal{F}_X, F)$ in \mathcal{N} is defined such that it satisfies that

- (i) [σ -field]. (X, \mathcal{F}_X) is a measurable space, that is, $\mathcal{F}_X (\subseteq 2^X)$ is a σ -field on X , i.e., it satisfies that

$$\emptyset \in \mathcal{F}_X, \quad \Xi_k \in \mathcal{F}_X (k = 1, 2, \dots) \implies \bigcup_{k=1}^{\infty} \Xi_k \in \mathcal{F}_X, \quad \Xi \in \mathcal{F}_X \implies \Xi^c \in \mathcal{F}_X,$$

- (ii) for every $\Xi \in \mathcal{F}_X$, $F(\Xi)$ is a positive element in \mathcal{N} (i.e., $0 \leq F(\Xi) \in \mathcal{N}$) such that $F(\emptyset) = 0$ and $F(X) = I$,

- (iii) [countably additivity]. For any countable decomposition $\{\Xi_1, \Xi_2, \dots, \Xi_j, \dots\}$ of Ξ , (i.e., $\Xi, \Xi_j \in \mathcal{F}_X, \bigcup_{j=1}^{\infty} \Xi_j = \Xi, \Xi_j \cap \Xi_i = \emptyset$ (if $j \neq i$)), it holds that

$$F(\Xi) = \sum_{j=1}^{\infty} F(\Xi_j)$$

where the series is convergent in the sense of the weak*-topology $\sigma(\mathcal{N}; \mathcal{N}_*)$ in \mathcal{N} .

For each $k = 1, 2, \dots, n$, consider an observable $\mathbf{O}_k \equiv (X_k, \mathcal{F}_{X_k}, F_k)$ in a W^* -algebra $\mathcal{N}(\equiv \overline{\mathcal{A}}^{w*})$. Define the product σ -field $\mathcal{F}_{\times_{k=1}^n X_k}^P (\subseteq 2^{\times_{k=1}^n X_k})$ such as the smallest σ -field (on $\times_{k=1}^n X_k$) that contains $\times_{k=1}^n \Xi_k, \Xi_k \in \mathcal{F}_{X_k}$. An observable $\widehat{\mathbf{O}} \equiv (\times_{k=1}^n X_k, \mathcal{F}_{\times_{k=1}^n X_k}^P, \widehat{F})$ in $\mathcal{N}(\equiv \overline{\mathcal{A}}^{w*})$ is called the *quasi-product observable* of $\{\mathbf{O}_k : k = 1, 2, \dots, n\}$ if it holds that

$$\widehat{F}(X_1 \times \cdots \times X_{k-1} \times \Xi_k \times X_{k+1} \times \cdots \times X_n) = F_k(\Xi_k) \quad (\forall \Xi_k \in \mathcal{F}_{X_k}, \forall k = 1, \dots, n).$$

Here, the $\widehat{\mathbf{O}}$ is denoted by $\mathbf{x}_{k \in \{1, 2, \dots, n\}}^{\text{qp}} \mathbf{O}_k$.

Let $[\mathcal{A}, \mathcal{N}(\equiv \overline{\mathcal{A}}^{w*}), B(V)]$ be an operator algebraic structure. Let W be any element in $\overline{\mathcal{A}}^{w*}$ such that $0 \leq W \leq I$. Define the function $f_W^{(l)} : \mathfrak{S}^p(\mathcal{A}^*) \rightarrow [0, 1]$ [resp. $f_W^{(u)} : \mathfrak{S}^p(\mathcal{A}^*) \rightarrow [0, 1]$] such that, for any $\rho^p \in \mathfrak{S}^p(\mathcal{A}^*)$,

$$f_W^{(l)}(\rho^p) = \sup\{\rho^p(U) \mid U \in \mathcal{A}, 0 \leq U \leq W\}$$

$$[\text{resp. } f_W^{(u)}(\rho^p) = \inf\{1 - \rho^p(U) \mid U \in \mathcal{A}, 0 \leq U \leq I - W\}].$$

Note that $\rho^p(W) = f_W^{(l)}(\rho^p) = f_W^{(u)}(\rho^p)$ ($\forall \rho^p \in \mathfrak{S}^p(\mathcal{A}^*)$) if $W \in \mathcal{A}$. Also, if it holds that $f_W^{(l)}(\rho_0^p) = f_W^{(u)}(\rho_0^p)$ for some $\rho_0^p (\in \mathfrak{S}^p(\mathcal{A}^*))$, then we say that W is *essentially continuous* at ρ_0^p , and define that $\rho_0^p(W) := f_W^{(l)}(\rho_0^p)$ ($= f_W^{(u)}(\rho_0^p)$).

3. Measurement (Axiom 1)

Under the mathematical preparations in the previous sections, now we can propose a new formulation of measurement theory.

With any *system* S , an operator algebraic structure $[\mathcal{A}, \mathcal{N}(\equiv \overline{\mathcal{A}}^{w*}), B(V)]$ can be associated in which measurement theory of that system can be formulated. A *state* of the system S is represented by a pure state ρ_0^p ($\in \mathfrak{S}^p(\mathcal{A}^*)$). Also, an *observable* is represented by an observable $\mathbf{O} \equiv (X, \mathcal{F}_X, F)$ in the W^* -algebra \mathcal{N} . The *measurement of an observable* \mathbf{O} for the system S with (or, in) the state ρ_0^p is represented by $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho_0^p]})$ (or precisely, $\mathbf{M}_{[\mathcal{A}, \overline{\mathcal{A}}^{w*}, B(V)]}(\mathbf{O}, S_{[\rho_0^p]})$). Also, we can take only one measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho_0^p]})$, and obtain a measured value x ($\in X$).

The axiom presented below is analogous to (or, a kind of generalizations of) Born's probabilistic interpretation of quantum mechanics [1,6,8]. We of course assert that the axiom is a principle for all measurements, i.e., classical and quantum measurements.

AXIOM 1 [New measurement axiom]. *Consider a measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O} \equiv (X, \mathcal{F}_X, F), S_{[\rho_0^p]})$ formulated in an operator algebraic structure $[\mathcal{A}, \mathcal{N}(\equiv \overline{\mathcal{A}}^{w*}), B(V)]$. Assume that the measured value x ($\in X$) is obtained by the measurement $\mathbf{M}_{\mathcal{A}}(\mathbf{O}, S_{[\rho_0^p]})$. Then, the probability that the x ($\in X$) belongs to a set Ξ ($\in \mathcal{F}_X$) is given by $\rho_0^p(F(\Xi))$ if $F(\Xi)$ is essentially continuous at ρ_0^p .*

4. The relation among systems (Axiom 2)

Let $[\mathcal{A}_1, \mathcal{N}_1(\equiv \overline{\mathcal{A}}_1^{w*}), B(V_1)]$ and $[\mathcal{A}_2, \mathcal{N}_2(\equiv \overline{\mathcal{A}}_2^{w*}), B(V_2)]$ be operator algebraic structures. Assume that \mathcal{N}_1 and \mathcal{N}_2 have weak*-topologies $\sigma(\mathcal{N}_1, (\mathcal{N}_1)_*)$ and $\sigma(\mathcal{N}_2, (\mathcal{N}_2)_*)$ respectively. A continuous linear operator $\Psi_{1,2} : \mathcal{N}_2 \rightarrow \mathcal{N}_1$ is called a *Markov operator*, if it satisfies that

(i) $\Psi_{1,2}(F_2) \geq 0$ for any positive element F_2 in \mathcal{N}_2 ,

(ii) $\Psi_{1,2}(I_2) = I_1$, where I_k is the identity in \mathcal{N}_k ($k = 1, 2$).

Here note that, for any observable (X, \mathcal{F}_X, F_2) in \mathcal{N}_2 , the $(X, \mathcal{F}_X, \Psi_{1,2}F_2)$ is an observable in \mathcal{N}_1 , which is denoted by $\Psi_{1,2}\mathbf{O}_2$.

Let (T, \leq) be a tree-like partial ordered set, i.e., a partial ordered set such that “ $t_1 \leq t_3$ and $t_2 \leq t_3$ ” implies “ $t_1 \leq t_2$ or $t_2 \leq t_1$ ”. Put $T_{\leq}^2 = \{(t_1, t_2) \in T^2 \mid t_1 \leq t_2\}$.

An element $t_0 \in T$ is called a *root* if $t_0 \leq t$ ($\forall t \in T$) holds. Note that the sub-tree $T_{t_0} \equiv \{t \in T \mid t \geq t_0\}$ has the root t_0 . Thus we always assume that the tree-like ordered set (T, \leq) has the root.

Definition 3 [Sequential observable]. *The family $\{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}$ is called a Markov relation among systems if it satisfies the following conditions (i) and (ii).*

(i) *With each $t \in T$, an operator algebraic structure $[\mathcal{A}_t, \mathcal{N}_t (\equiv \overline{\mathcal{A}_t}^{w*}), B(V_t)]$ is associated.*

(ii) *For every $(t_1, t_2) \in T_{\leq}^2$, Markov operator $\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}$ is defined such that $\Phi_{t_1, t_2} \Phi_{t_2, t_3} = \Phi_{t_1, t_3}$ holds for all $(t_1, t_2), (t_2, t_3) \in T_{\leq}^2$.*

Let an observable $\mathbf{O}_t \equiv (X_t, \mathcal{F}_{X_t}, F_t)$ in a W^* -algebra \mathcal{N}_t be given for each $t \in T$. Then, the pair $[\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}]$ is called a *sequential observable* which is denoted by $[\mathbf{O}_T]$, i.e., $[\mathbf{O}_T] \equiv [\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}]$.

Before we explain Axiom 2, we prepare some notations. For simplicity, assume that T is finite. Let T ($\equiv \{0, 1, \dots, N\}$) be a tree with the root 0. Define the *parent map* $\pi : T \setminus \{0\} \rightarrow T$ such that $\pi(t) = \max\{s \in T \mid s < t\}$. It is clear that the tree $(T \equiv \{0, 1, \dots, N\}, \leq)$ can be identified with the pair $(T \equiv \{0, 1, \dots, N\}, \pi : T \setminus \{0\} \rightarrow T)$.

Let $[\mathbf{O}_T] \equiv [\{\mathbf{O}_t\}_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}]$ be a sequential observable. For each $s \in T$, define iteratively the observable $\tilde{\mathbf{O}}_s \equiv (\times_{t \in T_s} X_t, \mathcal{F}_{\times_{t \in T_s} X_t}^P, \tilde{F}_s)$ in \mathcal{N}_s (where $T_s = \{t \in T \mid s \leq t\}$) such that:

$$\tilde{\mathbf{O}}_s = \begin{cases} \mathbf{O}_s & (\text{if } s \in T \setminus \pi(T)) \\ \mathbf{O}_s \times^{\text{qp}} (\times_{t \in \pi^{-1}(\{s\})} \Phi_{\pi(t), t} \tilde{\mathbf{O}}_t) & (\text{if } s \in \pi(T)). \end{cases} \quad (6)$$

Thus, if the procedure (6) is possible, we can get the observable $\tilde{\mathbf{O}}_0 \equiv (\times_{t \in T} X_t, \mathcal{F}_{\times_{t \in T} X_t}^P, \tilde{F}_0)$ in \mathcal{N}_0 . The $\tilde{\mathbf{O}}_0$ is called *the Heisenberg picture representation of the sequential observable* $[\mathbf{O}_T]$.

Summing up the essential part of the above argument, we can propose the following axiom, which corresponds to “the rule of the relation among systems”. Also, it should be noted that it is essentially the same as Proclaim ^{W^*} 2 in [4].

AXIOM 2 (= Proclaim^{W*} 2 in [4]) [The relation among systems]. For each $t(\in T)$, consider an operator algebraic structure $[\mathcal{A}_t, \mathcal{N}_t(\equiv \overline{\mathcal{A}_t}^{w*}), B(V_t)]$. Then the relation among systems is represented by a Markov relation among systems $\{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}$. Let $[\mathbf{O}_T] \equiv [\{\mathbf{O}_t(\equiv (X_t, \mathcal{F}_{X_t}, F))\}_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}]$ be a sequential observable. If the procedure (6) is possible, the sequential observable $[\mathbf{O}_T]$ can be realized as the observable $\tilde{\mathbf{O}}_0 \equiv (\times_{t \in T} X_t, \mathcal{F}_{\times_{t \in T} X_t}^P, \tilde{F}_0)$ in \mathcal{N}_0 .

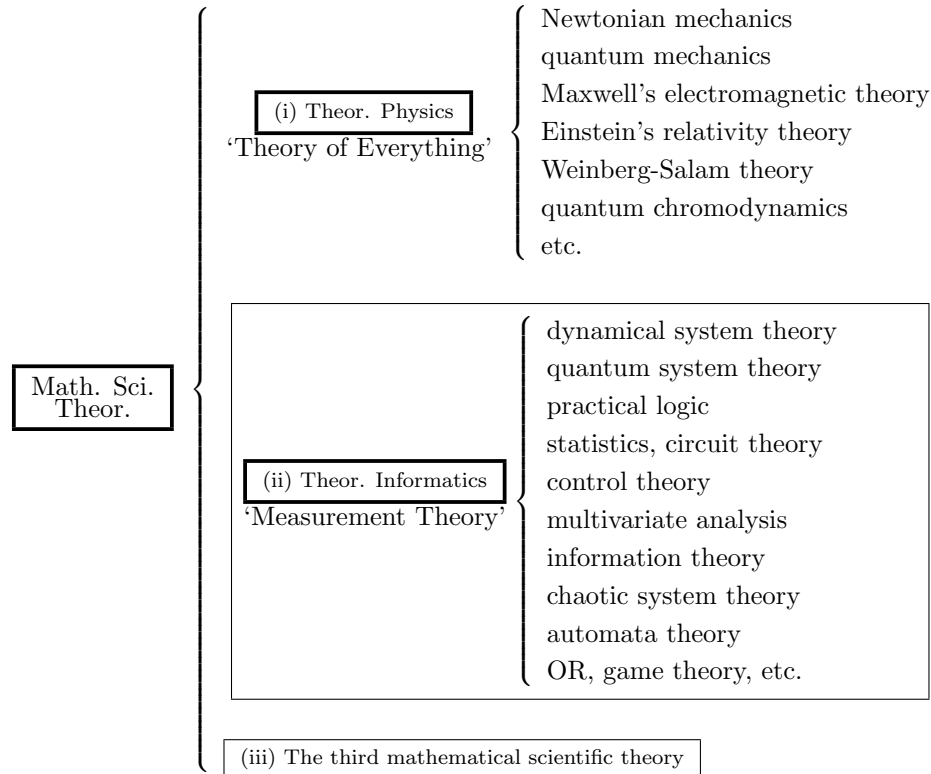
Thus, we can conclude that:

- Let $[\mathbf{O}_T] \equiv [\{\mathbf{O}_t(\equiv (X_t, \mathcal{F}_{X_t}, F))\}_{t \in T}, \{\Phi_{t_1, t_2} : \mathcal{N}_{t_2} \rightarrow \mathcal{N}_{t_1}\}_{(t_1, t_2) \in T_{\leq}^2}]$ be a sequential observable, which is assumed to be realized by $\tilde{\mathbf{O}}_0 \equiv (\times_{t \in T} X_t, \mathcal{F}_{\times_{t \in T} X_t}^P, \tilde{F}_0)$ (due to Axiom 2). Let ρ_0^p ($\in \mathfrak{S}^p(\mathcal{A}_0^*)$) be a pure state. Then, we have the measurement $\mathbf{M}_{\mathcal{A}_0}(\tilde{\mathbf{O}}_0, S_{[\rho_0^p]})$. And thus, Axiom 1 is applicable.

We know many fundamental theories that start from axioms (or, laws, principles), e.g., Newtonian mechanics, electromagnetic theory, the theory of relativity, quantum mechanics, and so on. However, we have no axiomatic theory to describe usual phenomena in our usual life. This seems strange. However, we now have classical measurement theory (5), i.e., the axiomatic theory of “the mechanical world view”.

The following remark will promote the better understanding of the difference between physics and measurement theory.

Remark 4. (a). Although both measurement theory and physical theory (e.g., the theory of relativity, etc.) start from principles (i.e., axioms), measurement theory is not physics but a kind of philosophy, i.e., the axiomatic theory of “the mechanical world view”. In physics, theory must be always tested by experiments. On the other hand, experiments are not necessarily essential for measurement theory, and thus it is somewhat subjective. Thus, there is a reason to consider that we have two mathematical scientific theories (i.e., (i) theoretical physics, (ii) measurement theory) as indicated in the following table (extracted from [ref.4: page 5]):



Here we have no answer to the question: “What is the third mathematical scientific theory (iii) in the above table?”. We believe that it is one of the most important problems in mathematical science. For the further arguments, see Chap. 1 in [4].

(b). In measurement theory, space (i.e., state space Ω) and time (e.g., T in Axiom 2) are not objective. For example, in measurement theory we consider that time is a mathematical handy tool when we want to understand some occurrences in “causal relationship”. In other words, time and space are regarded as merely kinds of parameters. Thus, it should be noted that the measurement theoretical “time and space” is different from that of physics.

(c). We can also see such a subjectivity mentioned in the above (a) and (b) in Kant’s philosophy [5]. Compared with physics (i.e., Newtonian mechanics), he himself called this kind of property of his philosophy “*Copernican revolution*” or “*transcendental idealism*.” The non-measurability of “*thing-in-itself (= ding an sich)*” in his philosophy may corresponds to the spirit (mentioned in the above (a)) of measurement theory such that experiments are not necessarily the highest priority in measurement theory. We can assure that it is interesting to reconsider epistemologies (in the following table)

in light of measurement theory.

measurement theory \Rightarrow	observable	state space	Markov relation	(T) Which is essential? (T)theo.or (E)xperiment
Zeno (BC490-BC430)			Achilles and tortoise What is motion?	(T,E)
Sokrates (BC469-BC399)	What is "brave"? What is "beauty"?			(T)
Plato (BC427-BC347)	idea theory			(T)
Aristotle (BC384-BC322)	form syllogism	material	the efficient cause unmoved mover	(E)
Augustinus (354-430)			What is time?	(T)
Thomas Aquinas (1225-1274)	problem of universals realism			(T)
Ockham (1285-1349)	problem of universals nominalism			(T) Ockham's razor
Rocke (1632-1704)	primary quality secondary quality			(T,E)
Newton (1643-1727)			kinetic equation	(E)
Berkeley (1685 1753)	To be is to be perceived			(T) solipsism
Kant (1724-1804)			subjective time-space	(T) thing in itself ranscendental idealism
Huserl (1859-1938)	intuition of essences			(T) Reduction,epoche
Einestein (1880-1950)			physical time-space	(E)
Schrödinger (1887-1961)			kinetic equation (of state)	(T,E) Schrödinger's cat
Heisenberg (1901-1976)	uncertainty relation		kinetic equation (of observable)	(T,E)
Born (1882-1970)	quantum measurement			(T,E)
Fisher (1890-1962)	maximum likelihood method		regression analysis design of experiments	(T)
Wittgenstein (1889-1951)		The world is everything that is the case		(T) solipsism

5. Conclusions

As shown in [4], we have the following classification:

$$\text{"measurement theory"} = \begin{cases} \text{(pure) measurement theory} \\ \text{statistical measurement theory.} \end{cases} \quad (7)$$

In this paper we devoted ourselves to (pure) measurement theory, and proposed

$$\begin{aligned} \text{“(pure) measurement theory”} &= \underset{\text{(Axiom 1)}}{[\text{measurement}]} + \underset{\text{(Axiom 2 (= Proclaim}^{W^*} \text{ 2 in [4]))}}{[\text{the relation among systems}]}, \end{aligned} \quad (8)$$

which has both merits of C^* -algebraic formulation and W^* -algebraic formulation. Also, note that statistical measurement theory proposed in [4] need not be changed. That is, we have

$$\begin{aligned} &\text{“statistical measurement theory”} \\ &= \underset{\text{(Proclaim}^{W^*} \text{ 1 in [4])}}{[\text{statistical measurement}]} + \underset{\text{(Axiom 2 (= Proclaim}^{W^*} \text{ 2 in [4]))}}{[\text{the relation among systems}]} \end{aligned} \quad (9)$$

Therefore we consider that the measurement theory (7) should be based on (8) and (9). And we believe that the measurement theory (7) is the most useful in all epistemologies (i.e., philosophies).

We hope that our proposal will be examined from various view-points.

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