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Characterizations of the class of free self decomposable distributions and its subclasses

by

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CHARACTERIZATIONS OF THE CLASS OF FREE SELF DECOMPOSABLE DISTRIBUTIONS AND ITS SUBCLASSES

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ABSTRACT. In this paper, we firstly characterize the class of free self-decomposable distributions as a class of limiting distributions of suitably normalized partial sums of free independent random variables. Secondly we introduce nested classes between the class of free self-decomposable distributions and the class of free stable distributions, characterize them in terms of Lévy measure and show that the limit of the nested classes coincides with the closure of the class of free stable distributions. All results here are the analogue of the results given in classical probability theory.

1. INTRODUCTION

Let X and Y be self-adjoint operators in a W^*-probability space $(\mathcal{A}, \tau)$, where $\mathcal{A}$ is von Neumann algebra and τ is a tracial state on $\mathcal{A}$. Then $X+Y$ is also a self-adjoint operator. If X and Y are freely independent, μ_{X+Y} is uniquely determined by μ_X and μ_Y. We call μ_{X+Y} the free additive convolution of μ_X and μ_Y and denote it by $\mu_X \boxplus \mu_Y$. In classical probability theory, we use characteristic functions to study the sum of independent random variables. Now we introduce a tool called the Cauchy transform to study the sum of free independent random variables. In the following, $\mathcal{P}$ stands for the class of all probability distributions on $\mathbb{R}$.

Definition 1.1. Let $\mu \in \mathcal{P}$. The transformation $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t}\mu(dt) \qquad (z \in \mathbb{C}^+)$$

is called the Cauchy transform.
Let

$$F_\mu(z) = \frac{1}{G_\mu(z)} \qquad (z \in \mathbb{C}^+).$$

$F_\mu(z)$ has right inverse $F_\mu^{-1}(z)$ on the region $\Gamma_{\eta,M}$ for some $M > 0$ and $\eta > 0$, where

$$\Gamma_{\eta,M} := \{z \in \mathbb{C} : |\mathrm{Re}\,(z)| < \eta \mathrm{Im}\,(z), \mathrm{Im}\,(z) > M\}.$$

Definition 1.2. The Voiculescu transform ϕ_μ of μ is defined by

$$\phi_\mu(z) = F_\mu^{-1}(z) - z.$$

The free cummulant $\mathcal{C}_\mu$ of μ is defined by

$$\mathcal{C}_\mu(z) = z\phi_\mu\Big(\frac{1}{z}\Big).$$

As in classical probability theory, the free cummulant of the sum of two free independent self-adjoint operators is the sum of two free cummulant of self-adjoint operators, as seen below.

Proposition 1.3. ([BeVo93]) *Let μ_1, $\mu_2 \in \mathcal{P}$. Then*

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$$

and

$$\mathcal{C}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{C}_{\mu_1}(z) + \mathcal{C}_{\mu_2}(z)$$

for all z in any region $\Gamma_{\eta,M}$, where all appearing functions are defined.

As in classical probability theory, the convergence of probability measures and the convergence of the free cummulant are very connected each other.

Proposition 1.4. ([BeVo93]) *Let $\{\mu_n\} \subset \mathcal{P}$. Then the following statements are equivalent:*
(a) *The sequence $\{\mu_n\}$ converges weakly to a probability measure μ on $\mathbb{R}$.*
(b) *There exist positive numbers η and M, and a function ϕ such that all the functions ϕ, ϕ_{μ_n} are defined on $\Gamma_{\eta,M}$, and such that*
(b1) $\phi_{\mu_n}(z) \to \phi(z)$, $n \to \infty$, *uniformly on compact subsets of* $\Gamma_{\eta,M}$
and
(b2) $\sup_{n\in\mathbb{N}} \left|\frac{\phi_{\mu_n}(z)}{z}\right| \to 0$, *as* $|z| \to \infty$, $z \in \Gamma_{\eta,M}$.

Infinite divisibility with respect to free additive convolution is defined as follows.

Definition 1.5. ([BeVo93]) $\mu \in \mathcal{P}$ is free infinitely divisible (or $\boxplus$-infinitely divisible), if for any $n \in \mathbb{N}$, there exists $\mu_n \in \mathcal{P}$ such that

$$\mu = \underbrace{\mu_n \boxplus \mu_n \boxplus \ldots \boxplus \mu_n}_{\text{n times}}.$$

By $I(\boxplus)$, we denote the class of all free infinitely divisible distributions. In what follows, $I(*)$ denotes the class of all classical infinitely divisible distributions on $\mathbb{R}$.

As in classical probability theory, $\boxplus$-infinitely divisible distributions are characterized by a free Lévy-Khintchine representation:

Proposition 1.6. ([BeVo93], [BaTh06]) *$\mu \in \mathcal{P}$ is $\boxplus$-infinitely divisible if only if there exist $a \geq 0$, $\gamma \in \mathbb{R}$ and a Lévy measure ν such that the free cummulant transform $\mathcal{C}_\mu$ has the representation:*

$$\mathcal{C}_\mu(z) = \gamma z + az^2 + \int_{\mathbb{R}} \left(\frac{1}{1-tz} - 1 - tz 1_{[-1,1]}(t) \right) \nu(dt), \qquad (z \in \mathbb{C}^-). \tag{1.1}$$

Here, the Lévy measure ν is a measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(|x|^2 \wedge 1)\nu(dx) < \infty$. In this case, the triplet (a, ν, γ) is uniquely determined by μ and is called the free characteristic triplet for μ.

We now introduce free Gaussian, stable and self-decomposable distributions, following ([BaTh06]). For $c > 0$, $D_c\mu$ means that the distribution of cX where X is the free random variable whose distribution is μ.

Definition 1.7. $\mu \in \mathcal{P}$ is called the standard semi-circle distribution if

$$\mu(dx) = \frac{1}{2\pi}\sqrt{4-x^2}1_{[-2,2]}(x)dx.$$

By $G(\boxplus)$, we denote the class of all semi-circle distributions, i.e.

$$G(\boxplus) = \{D_c\mu \boxplus \delta_b : c \geq 0, b \in \mathbb{R}, \mu \text{ is the standard semi-circle distribution}\},$$

where δ_b is the delta measure at b.

Since semi-circle distributions appear in the central limit theorem of free probability theory ([BeVo95]), we can regard semi-circle distributions as "the free Gaussian distributions ".

Definition 1.8. $\mu \in \mathcal{P}$ is called free stable ($\boxplus$-stable), if the class

$$\{\psi(\mu) : \psi \text{ is an increasing affine transformation}\}$$

is closed under the operation $\boxplus$. By $S(\boxplus)$, we denote the class of all free stable distributions.

Remark 1.9. ([BeVo93],[BaTh06]) The $\boxplus$-stability of $\mu \in \mathcal{P}$ is equivalent to each of the following.

(1) $\mu \in \mathcal{P}$ such that for any $a, a' > 0$ and for $b, b' \in \mathbb{R}$, there exist $a'' > 0$ and $b'' \in \mathbb{R}$ such that

$$\mathcal{C}_\mu(az) + bz + \mathcal{C}_\mu(a'z) + b'z = \mathcal{C}_\mu(a''z) + b''z.$$

(2) For each $n \in \mathbb{N}$, there exist $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$n\mathcal{C}_\mu(z) = n\mathcal{C}_\mu(a_n z) + b_n. \tag{1.2}$$

a_n in (1.2) can be taken as $a_n = n^{1/\alpha}$ with some $\alpha \in (0, 2]$. The proof of this can be carried out as in the classical case (See, e.g. [Fe71]).

Definition 1.10. $\mu \in \mathcal{P}$ is free self-decomposable ($\boxplus$-self-decomposable) if, for any $b > 0$, there exists $\rho_b \in \mathcal{P}$ such that

$$\mu = D_{b^{-1}}\mu \boxplus \rho_b.$$

By $L(\boxplus)$, we denote the class of all free self-decomposable distributions.

An important connection between the free and classical infinite divisibility was established by Bercovici and Pata ([BePa99]), by using the following bijection $\Lambda : I(*) \to I(\boxplus)$.

Definition 1.11. Suppose that μ is a measure in $I(*)$, and has its classical generating triplet (a, ν, γ) in its Lévy-Khintchine representation. Define $\Lambda(\mu)$ as the probability measure in $I(\boxplus)$ with the same generating triplet (a, ν, γ) as free generating triplet appearing in (1.1). We call this mapping $\Lambda : I(*) \to I(\boxplus)$ the Bercovici-Pata bijection.

Example 1.12. Let μ be the standard Gaussian distribution, i.e.

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)dx.$$

Then $\Lambda(\mu)$ is the standard semi-circle distribution.

Example 1.13. Let μ be the Cauchy distribution, i.e.

$$\mu(dx) = \frac{1}{\pi}\frac{a}{x^2 + a^2}dx.$$

Then $\Lambda(\mu)$ is the Cauchy distribution.

The following can be shown based on the Bercovici-Pata bijection.

Proposition 1.14. ([BaTh06]) *The relationships of the subclasses of free Infinitely divisible distributions is as follow:*

$$G(\boxplus) \subset S(\boxplus) \subset L(\boxplus) \subset I(\boxplus).$$

2. The free analogue of a characterization of the classical self-decomposable distributions

In classical probability theory, the class of self-decomposable distributions is characterized as a class of limiting distributions. We are going to show here that the same is true in the free probability theory. In the following, $\mathcal{L}$ stands for "the law of".

Definition 2.1. Let $\{k_n : n \in \mathbb{N}\}$ be a sequence of positive integers, and let

$$A = \{\mu_{n_j} : n \in \mathbb{N},\, j \in \{1, 2, \cdots, k_n\}\}$$

be an array of probability measures on $\mathbb{R}$. We say then that A is a null array, if the following conditions fulfilled: for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \max_{1\le j\le k_n} \mu_{n_j}(\mathbb{R}\backslash[-\varepsilon, \varepsilon]) = 0.$$

Theorem 2.2. *Let $\{Z_k : k \in \mathbb{N}\}$ be free independent random variables and let $S_n = \sum_{k=1}^{n} Z_k$. Let μ be a non-trivial probability measure on $\mathbb{R}$. Suppose that there are $\{b_n > 0 : n \in \mathbb{N}\}$ and $\{c_n \in \mathbb{R} : n \in \mathbb{N}\}$ such that*

$$\mathcal{L}(b_n S_n + c_n) \to \mu, \tag{2.1}$$

and that

$$\{b_n Z_k : k = 1, ..., n;\, n \in \mathbb{N}\} \text{ is a null array.} \tag{2.2}$$

Then, μ is free self-decomposable. Conversely, for any free self-decomposable distribution μ affiliated with some W^*-probability space $(\mathcal{A}, \tau)$ we can find $\{Z_n\}$ free independent random variables, $b_n > 0$, $\{c_n \in \mathbb{R} : n \in \mathbb{N}\}$ satisfying (2.1), (2.2).

To prove this theorem, we need the following lemma.

Lemma 2.3. *Under the assumptions of Theorem 2.2, we have $b_n \to 0$ and $b_n/b_{n+1} \to 1$ as $n \to \infty$.*

Proof. Let μ_k be the distribution of Z_k. The condition (2.2) says that, for any $\varepsilon > 0$,

$$\max_{1\le k\le n} D_{b_n}\mu_k(\mathbb{R}\backslash[-\varepsilon, \varepsilon]) \to 0 \quad (\text{as } n \to \infty).$$

So,

$$\max_{1\le k\le n} \mu_k \left(\mathbb{R}\backslash \left[-b_n^{-1}\varepsilon, b_n^{-1}\varepsilon\right]\right) \to 0 \quad (\text{as } n \to \infty).$$

Suppose that some subsequence $\{b_{n(l)}\}$ of $\{b_n\}$ tends to a non-zero b. Then it follows that, for any k, $\mu_k(\mathbb{R}\backslash[-b^{-1}\varepsilon, b^{-1}\varepsilon]) = 0$. Therefore μ is trivial, contrary to the assumption. Hence we have $b_n \to 0$. We next show that $b_n/b_{n+1} \to 1$. Write

$$W_n = b_n S_n + c_n.$$

By Proposition 1.4, there exist α and $\beta > 0$ such that

$$\phi_{\mathcal{L}(W_n)}(z) \to \phi_\mu(z) \quad (n \to \infty),$$

uniformly on the compact subsets of the truncated cone $\Gamma_{\alpha,\beta}$. Note

$$W_{n+1} = \frac{b_{n+1}}{b_n} W_n + b_{n+1} Z_{n+1} + \left(c_{n+1} - \frac{c_n b_{n+1}}{b_n} \right)$$

and

$$W_n + b_n Z_{n+1} = \frac{b_n}{b_{n+1}} W_{n+1} + \left(c_n - \frac{c_{n+1} b_n}{b_{n+1}} \right).$$

Since $b_n \to 0$, $c_n \to 0$ and by Proposition 1.4 , there exist α' and $\beta' > 0$ such that

$$\operatorname{Im} \left(\frac{b_{n+1}}{b_n} \phi_{\mathcal{L}(W_n)} \left(\frac{b_n}{b_{n+1}} z \right) \right) \to \operatorname{Im} \phi_\mu(z) \quad (n \to \infty)$$

and

$$\operatorname{Im} \left(\frac{b_n}{b_{n+1}} \phi_{\mathcal{L}(W_{n+1})} \left(\frac{b_{n+1}}{b_n} z \right) \right) \to \operatorname{Im} \phi_\mu(z) \quad (n \to \infty), \tag{2.3}$$

uniformly on the compact subsets of $\Gamma_{\alpha',\beta'} \subset \Gamma_{\alpha,\beta}$. Applying (b2) of Proposition 1.4 once again to $\phi_{\mathcal{L}(W_n)}$, we see that b_n/b_{n+1} is bounded away from zero and infinity. Indeed, suppose that b_n/b_{n+1} is not bounded away, for example, from zero. Then, provided that we pass to a subsequence, we have that, for every fixed z in the truncated cone $\Gamma_{\alpha',\beta'}$, the left-hand side of (2.3) goes to zero. Hence there exists $N > 1$ such that $b_n/b_{n+1} \in [1/N, N]$ for every n.

Let us denote throughout the rest of the proof

$$d_n := \frac{b_n}{b_{n+1}}$$

Selecting $z = iy$, y large enough, we get

$$zd_n \in \left[\frac{iy}{N}, iyN \right] \subset \Gamma_{\alpha',\beta'}.$$

For each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in \mathbb{N}$ such that, for $n \geq n_0(\varepsilon)$,

$$\left| \operatorname{Im} \phi_{\mu_{\mathcal{L}(W_n)}}(d_n z) - \operatorname{Im} \phi_\mu(d_n z) \right| < \frac{\varepsilon}{2N},$$

from which it follows that

$$\left| \operatorname{Im} \frac{1}{d_n} \phi_{\mu_{\mathcal{L}(W_n)}}(d_n z) - \operatorname{Im} \phi_\mu(z) \right| < \frac{\varepsilon}{2N d_n} \leq \frac{\varepsilon}{2},$$

and

$$\left| \operatorname{Im} \frac{1}{d_n} \phi_{\mu_{\mathcal{L}(W_n)}}(d_n z) - \operatorname{Im} \frac{1}{d_n} \phi_\mu(d_n z) \right| < \frac{\varepsilon}{2}.$$

Therefore, combining the last two inequalities, we get that, for $n \geq n_0(\varepsilon)$,

$$\left| \operatorname{Im} \frac{1}{d_n} \phi_\mu(d_n z) - \operatorname{Im} \phi_\mu(z) \right| < \varepsilon.$$

By theorem 1 of [BePa00], μ is $\boxplus$-infinitely divisible and, since we suppose μ non-trivial, by a Proposition 1.6, there exists a positive measure σ and $a \in \mathbb{R}$ such that

$$\phi_\mu(z) = a + \int_{-\infty}^{+\infty} \frac{1+tz}{z-t}\sigma(dt).$$

Therefore

$$\operatorname{Im}\phi_\mu(iy) = \int_{-\infty}^{+\infty} \frac{y(1+t^2)}{t^2+y^2}\sigma(dt),$$

and we get

$$\begin{aligned}
&\left|\operatorname{Im}\frac{1}{d_n}\phi_\mu(d_n z) - \operatorname{Im}\phi_\mu(z)\right| \\
&= y\left|\int_{-\infty}^{+\infty}(1+t^2)\left(\frac{1}{t^2+d_n^2y^2} - \frac{1}{t^2+y^2}\right)\sigma(dt)\right. \\
&= y^3|1-d_n^2|\int_{-\infty}^{+\infty}\frac{1+t^2}{(t^2+c_n^2y^2)(t^2+y^2)}\sigma(dt) \\
&\geq y^3|1-d_n^2|\int_{-\infty}^{+\infty}\frac{1+t^2}{(t^2+N^2y^2)(t^2+y^2)}\sigma(dt).
\end{aligned}$$

Since, once y is fixed,

$$y^3\int_{-\infty}^{+\infty}\frac{1+t^2}{(t^2+N^2y^2)(t^2+y^2)}\sigma(dt) = C \geq 0,$$

it follows that for $n \geq n_0(\varepsilon)$,

$$|1-d_n^2| \leq \frac{\varepsilon}{C}.$$

Thus, letting $\varepsilon \to 0$, we finally get

$$\lim_{n\to\infty} d_n = 1,$$

which yields the result. This completes the proof of lemma 2.3. □

Proof of Theorem 2.2. To show that μ is $\boxplus$-self-decomposable, we show for any $b > 1$ there exist positive numbers α', β' and some $\rho \in \mathcal{P}$ such that

$$\phi_\mu(z) = \phi_{D_{b^{-1}}\mu}(z) + \phi_\rho(z) \quad (z \in \Gamma_{\alpha',\beta'}).$$

Assume that μ is non-trivial. By Lemma 2.3, $b_n \to 0$ and $b_n/b_{n+1} \to 1$ as $n \to \infty$. For any $b > 1$ we can find sequences $\{n(l) : l \in \mathbb{N}\}$ and $\{m(l) : l \in \mathbb{N}\}$ of positive integers going to infinity such that

$$m(l) < n(l) \quad \text{and} \quad b_{m(l)}b_{n(l)}^{-1} \to b,$$

(See, e.g. proof of theorem 15.3 in [Sa99].) Let

$$W_n = b_nS_n + c_n,$$

$$U_l = b_{n(l)} \sum_{k=1}^{m(l)} Z_k + b_{n(l)} b_{m(l)}^{-1} c_{m_l}$$

and

$$V_l = b_{n(l)} \sum_{k=m_l+1}^{n(l)} Z_k + c_{n(l)} - b_{n(l)} b_{m(l)}^{-1} c_{m_l}.$$

Then $W_{n(l)} = U_l + V_l$ and there exist $\alpha(l) > 0$ and $\beta(l) > 0$ such that

$$\phi_{W_{n(l)}}(z) = \phi_{U_l}(z) + \phi_{V_l}(z) \quad (z \in \Gamma_{\alpha(l),\beta(l)})$$

by the free independence. By the hypothesis (2.1) and Proposition 1.4, there exist positive numbers α, β and a function $\phi_\mu(z)$ such that all the functions $\phi_\mu, \phi_{W_{n(l)}}$ are defined on $\Gamma_{\alpha,\beta}$, and such that

$(b1)$ $\phi_{W_{n(l)}}(z) \to \phi_\mu(z)$, uniformly on compact subsets of $\Gamma_{\alpha,\beta}$ as $n \to \infty$.

and

$(b2)$ $\sup_{n\in\mathbb{N}} \left| \frac{\phi_{W_{m(l)}}(z)}{z} \right| \to 0$, as $|z| \to \infty$ in $\Gamma_{\alpha,\beta}$.

Since $U_l = b_{n(l)} b_{m(l)}^{-1} W_{m(l)}$, we have

$$\phi_{U(l)}(z) = \frac{b_{n(l)}}{b_{m(l)}} \phi_{W_{m(l)}}\left(\frac{b_{m(l)}}{b_{n(l)}} z \right).$$

Now we can choose some positive numbers α', β' such that $\Gamma_{\alpha',\beta'} \subset \cap_{l\in\mathbb{N}} \{z \in \mathbb{C}^+ : \frac{b_{m(l)}}{b_{n(l)}} z \in \Gamma_{\alpha,\beta}\}$. On any compact subset $K \subset \Gamma_{\alpha',\beta'}$, for any $\varepsilon > 0$, there exists some large L such that, if $l > L$

$$\begin{aligned} &\sup_{z\in K} \left| \phi_{U_l}(z) - b^{-1}\phi_\mu(bz) \right| \\ &\le \sup_{z\in K} \left| \phi_{U_l}(z) - \frac{b_{n(l)}}{b_{m(l)}} \phi_\mu\left(\frac{b_{m(l)}}{b_{n(l)}} z \right) \right| + \sup_{z\in K} \left| \frac{b_{n(l)}}{b_{m(l)}} \phi_\mu\left(\frac{b_{m(l)}}{b_{n(l)}} z \right) - b^{-1}\phi_\mu(bz) \right| < \varepsilon. \end{aligned}$$

So,

$$\phi_{U_l}(z) \to b^{-1}\phi_\mu(bz), \text{ uniformly on compact subsets of } \Gamma_{\alpha',\beta'} \text{ as } l \to \infty.$$

Also

$$\sup_{l\in\mathbb{N}} \left| \frac{\phi_{U_l}(z)}{z} \right| = \sup_{l\in\mathbb{N}} \left| \frac{\phi_{W_{n(l)}}\left(\frac{b_{m(l)}}{b_{n(l)}} z \right)}{\frac{b_{m(l)}}{b_{n(l)}} z} \right| \to 0, \text{ as } |z| \to \infty, \ z \text{ in } \Gamma_{\alpha,\beta}.$$

So there exist positive numbers α', β' and a function $\phi_{D_{b^{-1}\mu}}(z) = b^{-1}\phi_\mu(bz)$ such that all the functions $\phi_{D_{b\mu}}, \phi_{U_l}$ are defined on $\Gamma_{\alpha',\beta'}$, and such that

(*b*1) $\phi_{U_l}(z) \to \phi_{D_{b^{-1}\mu}}(z)$, uniformly on compact subsets of $\Gamma_{\alpha',\beta'}$ as $l \to \infty$.

and

(*b*2) $\sup_{l\in\mathbb{N}} \left| \frac{\phi_{U_l}(z)}{z} \right| \to 0$, *as* $|z| \to \infty$ in $\Gamma_{\alpha',\beta'}$.

Thus, $\mathcal{L}(U_l) \to D_{b^{-1}}\mu$. Since

$$\phi_{V_l}(z) = -\phi_{U_l}(z) + \phi_{W_{n(l)}}(z) \quad (z \in \Gamma_{\alpha,\beta}),$$

$\mathcal{L}(V_l) \to \rho_b \in \mathcal{P}$, too. So,

$$\phi_\mu(z) = \phi_{D_{b^{-1}\mu}}(z) + \phi_\rho(z) \quad (z \in \Gamma_{\alpha,\beta}).$$

This implies that $\mu = D_{b^{-1}}\mu \boxplus \rho_b$ and thus, μ is $\boxplus$-self-decomposable. The proof of the converse is the same as Theorem 4.25. in [BaTh06]. □

3. Subclasses $L_m(\boxplus)$ of $L(\boxplus)$ and their characterisations

The class of free self-decomposable distributions includes that of free stable distributions. In this section, we introduce nested classes $L_m(\boxplus)$, $m = 1, 2, \cdots$ between the class $L_0(\boxplus)$ of self-decomposable distributions and the class $S(\boxplus)$ of stable distributions, such that

$$L_0(\boxplus) \supset L_1(\boxplus) \supset L_2(\boxplus) \supset ... \supset L_\infty(\boxplus) \supset S(\boxplus).$$

First, basic properties are proved and these classes are characterized as limits of partial sums of free independent random variables whose distributions are in certain classes closed under convolution and convergence.
Secondly, a representation of free cummulants of the classes above is presented, showing, in particular, that $L_\infty(\boxplus)$ contains the class $S(\boxplus)$. The representation of distributions in $L_\infty(\boxplus)$ indicates a clear connection to the representation of free stable distributions. Put $L_0(\boxplus) = L(\boxplus)$. For example, semicircle distributions are $\boxplus$-self-decomposable. It is known that all $\boxplus$-self-decomposable distributions are $\boxplus$-infinitely divisible. We have the following proposition.

Proposition 3.1. ([BaTh06]) *Let $\mu \in L_0(\boxplus)$. Then ρ_b is uniquely determined by μ and b, and both μ and ρ_b are in $I(\boxplus)$.*

Next we define the subclasses of $\boxplus$-self-decomposable distributions.

Definition 3.2. For $m = 1, 2, \cdots$ $\mu \in L_m(\boxplus)$ if, for any $c \in (0, 1)$, there exists $\rho_c \in L_{m-1}(\boxplus)$ such that

$$\mu = D_c\mu \boxplus \rho_c.$$

We also define

$$L_\infty(\boxplus) = \cap_{m=0}^{\infty} L_m(\boxplus).$$

It is immediate, by Proposition 3.1 and Definition 3.2, that $I(\boxplus) \supset L_0(\boxplus) \supset L_m(\boxplus)$ for all $m \geq 1$. Next lemma shows these classes form a nested sequence. Thus, the intersection over all $L_m(\boxplus)$ will give the limiting class.

Lemma 3.3. $I(\boxplus) \supset L_0(\boxplus) \supset L_1(\boxplus) \supset L_2(\boxplus) \supset \cdots.$

Proof. It is immediate. □

Remark 3.4. The class of trivial distributions is contained in $L_\infty(\boxplus)$. Briefly, let δ_{x_0} with $x_0 \in \mathbb{R}$ be the probability measure concentrated at x_0. Then $\delta_{x_0} = \delta_{b^{-1}x_0 + (1-b^{-1})x_0} = \delta_{b^{-1}} \boxplus \delta_{(1-b^{-1})x_0}$ for all $b > 1$. Hence $\delta_{x_0} \in L_m$ for all m.

Classes $L_m(\boxplus)$ are closed under convolution, convergence and type equivalence. These statements are proved in (2), (3) and (4) of following lemma,

Lemma 3.5. *Let $m \in \{0, 1, 2, \ldots, \infty\}$.*
(1) *If μ is in $L_m(*)$, then $\Lambda(\mu)$ is in $L_m(\boxplus)$. The converse is also true, i.e.*

$$\Lambda(L_m(*)) = L_m(\boxplus) \text{ and } \Lambda^{-1}(L_m(\boxplus)) = L_m(*)$$

(2) *If μ_1 and μ_2 are in $L_m(\boxplus)$, then $\mu_1 \boxplus \mu_2 \in L_m(\boxplus)$.*
(3) *If $\mu_n \in L_m(\boxplus)$ and $\mu_n \to \mu$, then $\mu \in L_m(\boxplus)$.*
(4) *If $\mu_1 = \mathcal{L}(X) \in L_m(\boxplus)$ and $\mu_2 = \mathcal{L}(aX + b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $\mu_2 \in L_m(\boxplus)$.*
(5) *If $\mu_1 \in L_m(\boxplus)$, $\mu_2 \in \mathcal{P}$ and $\mathcal{C}_{\mu_2}(z) = a\mathcal{C}_{\mu_1}(z)$ with some $a \geq 0$, then $\mu_2 \in L_m(\boxplus)$.*

Proof. The proof will be given by the induction method. Let μ is in $L_0(*)$ with ρ_b. We map $\mu = D_{b^{-1}}\mu * \rho_b$ by the Bercovici-Pata bijection. Then,

$$\Lambda(\mu) = \Lambda(D_{b^{-1}}\mu * \rho_b) = D_{b^{-1}}\Lambda(\mu) \boxplus \Lambda(\rho_b).$$

Thus, $\Lambda(\mu)$ is in $L_0(\boxplus)$. Assume that the assertion is true for $m - 1$. Let μ be in $L_m(*)$, then, for any $b > 1$, there exists $\rho_b \in L_{m-1}(*)$ such that

$$\mu = D_{b^{-1}}\mu * \rho_b.$$

Then,

$$\Lambda(\mu) = \Lambda(D_{b^{-1}}\mu * \rho_b) = D_{b^{-1}}\Lambda(\mu) \boxplus \Lambda(\rho_b).$$

By the hypothesis, $\Lambda(\rho_b)$ is in $L_{m-1}(\boxplus)$. Therefore, $\Lambda(\mu)$ is in $L_m(\boxplus)$. In the case that $m = \infty$,

$$\mu \in L_\infty(*) \Leftrightarrow \mu \in \cap_{m=0}^{\infty} L_m(*) \Leftrightarrow \Lambda(\mu) \in \cap_{m=0}^{\infty} L_m(\boxplus) \Leftrightarrow \Lambda(\mu) \in L_\infty(\boxplus).$$

The converse is proved in the same way. (2), (3), (4), (5) are proved easily by using the Bercovici-Pata bijection and [RoSa03], Chapter 1, Lemma 8. □

Next we prove that $\boxplus$-stable distributions are contained in the class $L_\infty(\boxplus)$, and that $L_\infty(\boxplus)$ is the smallest class containing the $\boxplus$-stable distributions closed under convolution and convergence once the representation of free cummulants of the class $L_\infty(\boxplus)$ is established.

Lemma 3.6. $L_\infty(\boxplus) \supset S(\boxplus)$.

Proof. Let $\mu \in S(\boxplus)$. Since $\Lambda^{-1}(\mu) \in S(*)$, $\Lambda^{-1}(\mu) \in L_\infty(*)$. By Lemma 3.5 and $S(*) \subseteq L_\infty(*)$ (See [RoSa03] Chapter1), $\mu \in L_\infty(\boxplus)$. □

Lemma 3.5 shows that the class of distributions $L_m(\boxplus)$ is completely closed class, that is, closed under convolution, weak convergence and type equivalence. This property is essential in characterizing this class in terms of limits for sums of independent random variables.

Definition 3.7. Let D be a subclass of $\mathcal{P}$. Define $K(D) \subset \mathcal{P}$ as follows. $\mu \in K(D)$ if there are free independent random variables $\{Z_n : n \in \mathbb{N}\}$, $\{b_n > 0 : n \in \mathbb{N}\}$ and $\{c_n \in \mathbb{R} : n \in \mathbb{N}\}$ satisfying the following conditions.

$$\mathcal{L}(b_n S_n + c_n) \to \mu,$$

$$\{b_n Z_k : k = 1, ..., n;\ n \in \mathbb{N}\} \text{ is a null array.} \tag{3.1}$$

$$\mathcal{L}(Z_k) \in D \quad \text{for each k.}$$

Using this operation K, the class of self-decomposable distributions is comprehended as a class of limit distributions, as well as the class of distributions $L_m(\boxplus)$.

Theorem 3.8. (1) $L_0(\boxplus) = K(\mathcal{P}) = K(I(\boxplus))$.
(2) $L_m(\boxplus) = K(L_{m-1}(\boxplus))$ *for* $m = 1, 2, \cdots$
(3) $L_\infty(\boxplus) = K(L_\infty(\boxplus))$ *and* $L_\infty(\boxplus)$ *is the largest class* D *that satisfies* $D = K(D)$.

Proof. First, We show (1). The first equality can be found in theorem 2.2. Thus $L_0(\boxplus) \supset K(I(\boxplus))$. The same argument as in the proof of (2) below combined with yields $L_0(\boxplus) \subset K(I(\boxplus))$. Second, We show (2). Let $\mu \in L_m(\boxplus)$. For every $b > 1$,

$$\phi_{\rho_b}(z) = \phi_\mu(z) - \phi_{D_{b^{-1}}\mu}(z) \ (z \in \mathbb{C}^+),$$

with $\rho_b \in L_{m-1}(\boxplus)$. First, we compose the free random variables sequence which converge to μ. Let $Z_1, Z_2, \ldots$ be freely independent random variables with

$$\phi_{Z_k}(z) = (k+1)\phi_{\rho_{\frac{k+1}{k}}}\left(\frac{1}{k+1}z\right),$$

and define $S_n = n^{-1}\sum_{k=1}^n Z_k$.

$$\begin{aligned}
\phi_{S_n}(z) &= \sum_{k=1}^n \frac{1}{n}\phi_{Z_k}(nz) \\
&= \sum_{k=1}^n \frac{k+1}{n}\phi_{\rho_{\frac{k+1}{k}}}\left(\frac{n}{k+1}z\right) \\
&= \sum_{k=1}^n \frac{k+1}{n}\left\{\phi_\mu\left(\frac{n}{k+1}z\right) - \phi_{D_{\frac{k}{k+1}}\mu}\left(\frac{n}{k+1}z\right)\right\} \\
&= \sum_{k=1}^n \frac{k+1}{n}\left\{\phi_\mu\left(\frac{n}{k+1}z\right) - \frac{k}{k+1}\phi_\mu\left(\frac{n}{k}z\right)\right\} \\
&= \sum_{k=1}^n \frac{k+1}{n}\phi_\mu\left(\frac{n}{k+1}z\right) - \frac{k}{n}\phi_\mu\left(\frac{n}{k}z\right) \\
&= \frac{n+1}{n}\phi_\mu\left(\frac{n}{n+1}z\right) - \frac{1}{n}\phi_\mu(nz) \\
&\to \phi_\mu(z) \qquad \text{uniformly compact subset of } \mathbb{C}^+
\end{aligned}$$

and

$$\begin{aligned}
\sup_{n\in\mathbb{N}}\left|\frac{\phi_{S_n}(z)}{z}\right| &= \sup_{n\in\mathbb{N}}\left|\frac{\frac{n+1}{n}\phi_\mu\left(\frac{n}{n+1}z\right) - \frac{1}{n}\phi_\mu(nz)}{z}\right| \\
&\le \sup_{n\in\mathbb{N}}\left|\frac{\phi_\mu(\frac{n}{n+1}z)}{\frac{n}{n+1}z}\right| + \sup_{n\in\mathbb{N}}\left|\frac{\phi_\mu(nz)}{nz}\right| \\
&\to 0 \qquad \text{as } |z| \to \infty, \ z \in \mathbb{C}^+.
\end{aligned}$$

So By Proposition 1.4, $\mathcal{L}(S_n) \to \mu$. This sequence satisfies the condition (3.1). The converse can be proved in the same way as Theorem 2.2. Next we show (3). By (2), $L_m(\boxplus) \supset K(L_{m-1}(\boxplus)) \supset K(L_\infty(\boxplus))$ for all $m \in \mathbb{N}$. Thus, $L_\infty(\boxplus) \supset K(L_\infty(\boxplus))$. If $\mu \in L_\infty(\boxplus)$, then $\rho_b \in L_\infty(\boxplus)$. Take Z_k as the proof of (2). So $\mathcal{L}(n^{-1}\sum_{k=0}^n Z_k) \to \mu$ and $\{n^{-1}Z_k\}$ is null array. This means that $\mu \in K(L_\infty(\boxplus))$. So, $K(L_\infty(\boxplus)) = L_\infty(\boxplus)$. Let $D = K(D)$. Then $D = K(D) \subset K(\mathcal{P}) = L_0(\boxplus)$. Since $D \subset L_0(\boxplus)$, we get $D = K(D) \subset K(L_0(\boxplus)) = L_1(\boxplus)$. By a cyclic application of the same argument, $D \subset L_m(\boxplus)$ for every $m \in \mathbb{N}$. Therefore, $D \subset L_\infty(\boxplus)$. □

The following is our final goal in this paper, which is an analogue of the corresponding statement in classical probability theory.

Theorem 3.9. *The class $L_\infty(\boxplus)$ is the smallest class containing $S(\boxplus)$ and closed under $\boxplus$-convolution and weak convergence.*

Proof. It has already been shown that the class $L_\infty(\boxplus)$ contains $S(\boxplus)$ and that it is closed under convolution and convergence. Let $\mathcal{Q}$ be a class of probability measures containing $S(\boxplus)$ and closed under $\boxplus$ and weak convergence. $\mathcal{Q}' := \mathcal{Q} \cap I(\boxplus)$ has the same properties, and hence $\Lambda(\mathcal{Q}')$ contains $S(*)$, is closed under $*$ and weak convergence by Theorem 5.7. and Corollary 5.14. in [BaTh06]. Hence, by the classical analog of Theorem 3.16 (See e.g. theorem 24 in [RoSa03]), $\Lambda(\mathcal{Q}') \supseteq L_\infty(*)$ and therefore $\mathcal{Q} \supseteq \mathcal{Q}' \supseteq \Lambda(L_\infty(*)) = L_\infty(\boxplus)$. □

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