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Percolation in lattices with large holes

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Abstract

Sparse sublattices of $\mathbb{Z}^2$, obtained by recursive removal of increasingly large squares of points, are shown to have critical site percolation probabilities strictly between zero and one. We show that for every such lattice the infinite cluster, if it exists, is unique. The discussion leads to an apparently new type of ergodic theorem of independent interest.

1 Introduction

In this article, we consider independent site percolation on sublattices of $\mathbb{Z}^2$ which are obtained by recursively removing larger and larger squares of points; we call these lattices *lattices with large holes.* The precise definition is given in section 3. To keep the exposition simple, we have chosen a fixed dimension two and a special system for recursive removal; it is easy to replace these with more general choices. Our goals are to show that for such lattices, the critical probability lies strictly between zero and one (section 5) and that for each such lattice, there are at most one infinite cluster with probability one (sections 6 and 7). In section 7, we discuss an apparently new type of ergodic theorem and an interesting question concerning more general strictly ergodic systems.

2 Configuration space

Let

$$\mathbb{Z}(3) := \left\{ z = \sum_{i \geq 0} z_i 3^i : z_i \in \{0, 1, 2\} \text{ for } i \geq 0 \right\}.$$

Under coordinatewise addition modulo 3 with right carry, $\mathbb{Z}(3)$ is the compact Abelian group of 3-adic integers. We identify the usual integers $\mathbb{Z}$ with the subset of $\mathbb{Z}(3)$ whose elements z have either only finitely many nonzero coordinates $z_i\,(z \geq 0)$ or only finitely many coordinates $z_i \neq 2\,(z < 0)$. Thus $\mathbb{Z}$ acts on $\mathbb{Z}(3)$ by translation.

If z^1 and z^2 belong to $\mathbb{Z}(3)$, then we set $z = (z^1, z^2)$ and

$$I(z) := \{i \geq 0 : z_i^1 = z_i^2 = 2\}.$$

The subset

$$\hat{Z}_0 := \{z : I(z) = \emptyset\}$$

of $\mathbb{Z}(3) \times \mathbb{Z}(3)$ has again a product structure, with eight possibilities for each pair (z_i^1, z_i^2) (the possibility $(2, 2)$ is absent). We denote by μ_0 the product probability measure on $\hat{Z}_0$ giving probability $1/8$ to each of these possibilities, independently for $i \geq 0$. If we set

$$Z_0 := \{z \in \hat{Z}_0 : z^1, z^2 \notin \mathbb{Z}\},$$

then it is easy to see that $\mu_0(Z_0) = 1$, since each one-dimensional section of $\hat{Z}_0$ has μ_0-measure 0 and we have removed countably many such sections.

For each nonnegative integer k, we define

$$Z_{0,k} := \{z \in Z_0 : z_i^1 = z_i^2 = 0 \text{ for } 0 \leq i \leq k\}.$$

Then

$$\mu_0(Z_{0,k}) = \left(\frac{1}{8}\right)^{k+1} \qquad (k \geq 0)$$

and

$$Z_0 \supseteq Z_{0,0} \supseteq Z_{0,1} \supseteq \cdots .$$

Further, set

$$I_k := \{(s, t) \in \mathbb{Z} \times \mathbb{Z} : 2 \cdot 3^k \leq s, t < 3^{k+1}\}.$$

That is, the nonnegative integers s and t with $(s, t) \in I_k$ are those having ternary expansions of length $k + 1$ with leading (highest) coordinates equal to 2. Then, the countable collection of sets

$$(*) \qquad Z_{0,k} + (s, t)$$

with $k \geq 0$ and $(s, t) \in I_k$, together with the single set Z_0, are pairwise disjoint and partition the set

$$Z := \left\{z = (z^1, z^2) \in \mathbb{Z}(3) \times \mathbb{Z}(3) : z^1, z^2 \notin \mathbb{Z},\ I(z) \text{ finite}\right\}.$$

We extend the probability measure μ_0 on Z_0 to the σ-finite measure μ on Z by defining μ on each set $(*)$ to be the translation of μ_0 restricted to the corresponding set $Z_{0,k}$. It should be clear from the construction that $\mathbb{Z}^2$ acts on Z by translation, as the action of an element of $\mathbb{Z}$ on an element of $\mathbb{Z}(3) \setminus \mathbb{Z}$ changes at most finitely many coordinates.

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Lemma 2.1. μ is the unique (σ-finite) measure on Z invariant under the action of each element of $\mathbb{Z}^2$ for which $\mu(Z_0) = 1$.

Proof. For each $k \geq 0$, the set $Z_{0,k}$ together with $8^{k+1} - 1$ of its translates disjointly cover Z_0; these partitions generate the σ-algebra restricted to Z_0, and each of them must have measure $1/8^{k+1}$, since $\mu(Z_0) = 1$. Thus μ is uniquely determined on Z_0, and by the above translations on all of Z. $\square$

Corollary 2.2. The action of $\mathbb{Z}^2$ on (Z, μ) is ergodic.

Proof. This follows from the uniqueness of Lemma 2.1. $\square$

We call an element $z \in Z$ a *configuration* and the measure space (Z, μ) *configuration space*.

3 Lattices with large holes

To each $z \in Z$ we assign a subset $\Lambda(z)$ of $\mathbb{Z}^2$ by setting

$$\Lambda(z) := \{(s,t) \in \mathbb{Z}^2 : z + (s,t) \in Z_0\}.$$

The nearest neighbour graph on $\Lambda(z)$ will be called the *lattice with large holes* of z. At each "level" k this $\Lambda(z)$ has a block structure with two types of blocks of size $3^k \times 3^k$. One block is the empty block (no points), whereas the other is obtained by putting together nine blocks of size $3^{k-1} \times 3^{k-1}$, one empty and the other eight the nonempty block at that level. Here are pictures for levels one and two:

$$(1)\quad \begin{array}{ccc} \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{array} \qquad\qquad (2)\quad \begin{array}{ccccccccc} \cdot & \cdot & \circ & \cdot & \cdot & \circ & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \circ & \circ & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \circ & \circ & \circ \\ \cdot & \cdot & \circ & \cdot & \cdot & \circ & \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \circ & \cdot & \cdot & \circ & \cdot & \cdot & \circ \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

where we have circled the absent points. The coordinates of $z = (z^1, z^2)$ govern the placement in $\mathbb{Z}^2$ of these blocks recursively.

4 Site percolation on $\Lambda(z)$

In this and in the following sections, we assume acquaintance with the ideas of site percolation for subgraphs of $\mathbb{Z}^d$, and in particular with the original results of Broadbent and Hammersley [2] concerning critical values and of Aizenman, Kesten, and Newman [1] regarding uniqueness of the infinite cluster when percolation occurs. We refer the reader to the excellent treatise of Grimmett [4], which discusses these ideas (but for bond percolation) and to [3] for the arguments concerning uniqueness.

Our first result, based on ideas in [9], deals with the nontriviality of critical values:

Theorem 4.1. For any $z \in Z$, the critical value for site percolation on $\Lambda(z)$ lies strictly between 0 and 1.

Our second result, regarding uniqueness, is:

Theorem 4.2. For any $z \in Z$ and any value of p at which percolation occurs on $\Lambda(z)$, the infinite cluster is unique with probability 1.

The next three sections provide proofs of these theorems.

5 Proof of Theorem 4.1

If $z \in Z$, then we can translate z by an element of $\mathbb{Z}^2$ to obtain $z \in Z_0$, so that we may assume $z \in Z_0$. (Critical values are the same for any translate.) Since $\Lambda(z) \subseteq \mathbb{Z}^2$, it should be clear that its critical value is strictly positive, since the critical value for site percolation on $\mathbb{Z}^2$ is positive, and cannot decrease if we go to a subgraph. So we only have to show that for some $p < 1$, percolation occurs on $\Lambda(z)$. Actually, we can make this p explicit. Let us consider the polynomial

$$f(q) = q^{64} + 64q^{63}(1-q) - q.$$

Then

$$f(1) = 0$$

and

$$f'(1) = 64 \cdot 63 - 64 \cdot 63 - 1 = -1,$$

so for values of q less than 1 but close to 1, $f(q) \geq 0$. Let p be the largest solution to $f(q) = 0$ which is less than 1 for p. Then $p < 1$ and

$$p^{64} + 64p^{63}(1-p) = p.$$

Now the left hand side is exactly the probability that at least 63 of the 64 points in the picture (2) are open, if each single point is open (independently) with probability p. So if we define the block in (2) to be "open" if at least 63 of the 64 points in the block are open, then the probability of a block (of size 81×81) to be "open" is also equal to p.

Proceeding to the next (even) level, we define a block of size $9^4 \times 9^4$ to be "open" if each of its 64 nonempty subblocks of size $9^2 \times 9^2$ are "open", and we continue this definition recursively for blocks of sizes $9^{2k} \times 9^{2k}$, $k \geq 1$. Then it should be clear that the event that such a block is "open" has probability p for any $k \geq 1$. This shows indeed that percolation occurs for $\Lambda(z)$ at p; if this were not the case, then the probability that the block of size 9×9 containing the origin has a connected open set of size n should tend to 0 as n tends to infinity. However, the above calculation shows that this probability is at least p for any n, because "open" blocks next to each other are obviously connected. (Steps of size two are necessary for this.)

6 Uniqueness for almost every $z \in Z$

Let $0 < p < 1$. We denote by $\mathbb{P}_p$ the probability measure on

$$\Omega = \{0,1\}^{\mathbb{Z}^2}$$

giving probability p to each of the events

$$\omega(s,t) = 1,$$

independently for $(s,t) \in \mathbb{Z} \times \mathbb{Z}$. If now

$$(z, \omega) \in Z \times \Omega,$$

then we say that the site $(s,t) \in \mathbb{Z} \times \mathbb{Z}$ is open if both

$$(s,t) \in \Lambda(z)$$

and

$$\omega(s,t) = 1.$$

Thus for each $z \in Z$ we have a site percolation situation on $\Lambda(z) \subseteq \mathbb{Z} \times \mathbb{Z}$, in which each site in $\Lambda(z)$ is open with probability p and closed with probability $1-p$, independent of the other sites. (We disregard the values of $\omega(s,t)$ for $(s,t) \notin \Lambda(z)$.) In the sequel we assume familiarity with the basic percolation concepts, and with the contents of [3]. On $Z \times \Omega$ we define the simultaneous $\mathbb{Z} \times \mathbb{Z}$ action given by the action on Z above and translation by the same $\mathbb{Z} \times \mathbb{Z}$ element of $\omega \in \Omega$. The product measure $\mu \times \mathbb{P}_p$ is invariant under this action.

Lemma 6.1. The $\mathbb{Z} \times \mathbb{Z}$ action on $Z \times \Omega$ is ergodic with respect to the product measure $\mu \times \mathbb{P}_p$.

Proof. Suppose that $A \subseteq Z \times \Omega$ is $\mathbb{Z} \times \mathbb{Z}$ invariant, and that both

$$\mu \times \mathbb{P}_p(A) > 0$$

and

$$\mu \times \mathbb{P}_p(Z \times \Omega \setminus A) > 0.$$

Set $A^c := Z \times \Omega \setminus A$. Then there exist finite cylinder sets C and D in $Z \times \Omega$ such that both

$$\frac{\mu \times \mathbb{P}_p(C \cap A)}{\mu \times \mathbb{P}_p(C)} > 1/2$$

and

$$\frac{\mu \times \mathbb{P}_p(D \cap A^c)}{\mu \times \mathbb{P}_p(D)} > 1/2;$$

we may assume also that $C, D \subseteq Z_0 \times \Omega$ by translation invariance. However, it is clear from the above that if (s,t) is sufficiently large and has the correct relation modulo 3^n for some n, then

$$\mathbb{P}_p(C \cap ((s,t) + D)) > 0;$$

thus $(s,t) + A^c$ intersects A, which is a contradiction. □

For $(z, \omega) \in Z \times \Omega$, the total number of infinite open clusters is denoted by $N = N(z, \omega)$. Since N is invariant under the spatial shift and $\mu \times \mathbb{P}_p$ is ergodic under this $\mathbb{Z}^2$-action, N is constant $\mu \times \mathbb{P}_p$-almost everywhere, i.e.

$$\mu \times \mathbb{P}_p(\{N = l\}^c) = 0$$

for some $l \in \{0, 1, 2, \cdots, \infty\}$. By Fubini's theorem, this is equivalent to

$$\mu(Z \setminus \{z : \mathbb{P}_p\{\omega : N(z,\omega) = l\} = 1\}) = 0.$$

Proposition 6.2. For μ-almost every $z \in Z$, N is either zero, one, or infinity with $\mathbb{P}_p$-probability one.

Proof. We use the argument in [8]. Let

$$Z_l := \{z : \mathbb{P}_p\{\omega : N(z,\omega) = l\} = 1\},$$

and suppose that $\mu(Z \setminus Z_l) = 0$ for an integer $l \geq 2$. We set

$$\begin{aligned} R_n &:= ([-n,n] \times [-n,n]) \cap \mathbb{Z}^2, \\ \partial R_n &:= \{(s,t) \in \mathbb{Z}^2 : \max\{|s|,|t|\} = n\}, \\ D_n &:= \left\{ \omega : \begin{array}{c} \text{there exist at least two distinct} \\ \text{infinite open clusters starting from } R_n \end{array} \right\}. \end{aligned}$$

For $z \in Z_l$,

$$\mathbb{P}_p \left(\bigcup_{n \geq 1} D_n \right) \geq \mathbb{P}_p\{\omega : N(z,\omega) = l\} = 1.$$

We can choose M so large that $\mathbb{P}_p(D_M) > 1/2$. Then,

$$\begin{aligned} \mathbb{P}_p\{N(z,\omega) < l\} &\geq \mathbb{P}_p(D_M \cap \{\text{all sites in } R_M \text{ are open}\}) \\ &\geq p^{|R_M|}/2 > 0, \end{aligned}$$

which is a contradiction. □

Theorem 6.3. For μ-almost every $z \in Z$,

$$\mathbb{P}_p\{\omega : N(z,\omega) \in \{0,1\}\} = 1.$$

Proof. Suppose that

$$\mu \times \mathbb{P}_p(\{(z,\omega) : N(z,\omega) = \infty\}^c) = 0.$$

We set

$$T_n := \left\{ (z,\omega) : \begin{array}{c} \text{there exist at least three distinct} \\ \text{infinite open clusters starting from } R_n \end{array} \right\}.$$

Then,

$$0 = \mu \times \mathbb{P}_p \left(\left(\bigcup_{n \geq 1} T_n \right)^c \right) = \lim_{n \to \infty} \mu \times \mathbb{P}_p \left(T_n^c \right).$$

We can choose a large M such that

$$\mu \times \mathbb{P}_p \left(T_M^c \right) < \frac{1}{2}.$$

Let

$$\tilde{T}_n := \left\{ \omega : \begin{array}{c} \text{there exist at least three distinct} \\ \text{infinite open clusters starting from } R_n \end{array} \right\}$$

and

$$\tilde{Z}_M := \{z : \mathbb{P}_p(\tilde{T}_M) > 1/2\}.$$

If $\mu(Z \setminus \tilde{Z}_M) > 1$, then

$$\begin{aligned}\mu \times \mathbb{P}_p(T_M^c) &\geq \int_{Z\setminus\tilde{Z}_M} \mu \times \mathbb{P}_p(dz, d\omega) 1_{T_M^c}(z,\omega) \\ &= \int_{Z\setminus\tilde{Z}_M} \mu(dz) \times \mathbb{P}_p(\Omega \setminus \tilde{T}_M) \\ &\geq \mu(Z \setminus \tilde{Z}_M) \cdot \frac{1}{2} > \frac{1}{2},\end{aligned}$$

which is impossible. Thus we have $\mu(Z \setminus \tilde{Z}_M) \leq 1$ and $\mu(\tilde{Z}_M) > 0$.

A point $(s,t) \in \mathbb{Z}^2$ is an *encounter point* for $(z,\omega) \in Z \times \Omega$ if

- (s,t) belongs to an infinite open cluster C of (z,ω), and
- the set $C \setminus \{(s,t)\}$ has no finite components and exactly three infinite components.

For $z \in \tilde{Z}_M$, using the finite energy property (or, simply independence) of $\mathbb{P}_p$ to change ω in R_M, we can see that $(0,0)$ is an encounter point with positive probability.

Let

$$f(z,\omega) := \begin{cases} 1 & \text{if } (0,0) \text{ is an encounter point,} \\ 0 & \text{otherwise,} \end{cases}$$

$$f_0(z,\omega) := \begin{cases} 1 & \text{if } (0,0) \in \Lambda(z), \\ 0 & \text{otherwise.} \end{cases}$$

By the ratio ergodic theorem (see e.g. Krengel [6]),

$$\lim_{R\to\infty} \frac{\sum_{(s,t)\in R} f((s,t)+(z,\omega))}{\sum_{(s,t)\in R} f_0((s,t)+(z,\omega))} = \frac{\int f(z,\omega)\mu \times \mathbb{P}_p(dz,d\omega)}{\int f_0(z,\omega)\mu \times \mathbb{P}_p(dz,d\omega)}$$

$\mathbb{P}_p$-a.s. for μ-a.e. $z \in Z$. Since the integrals of f and g are both positive and finite, the relative density of encounter points in the set of all allowable lattice points is positive. We have

$$\begin{aligned}&\# \text{ of the sites on the boundary of } R \\ &\geq \# \text{ of the distinct infinite open paths starting from the boundary of } R \\ &\geq (\# \text{ of encounter points in } R) + 2 \\ &\geq \varepsilon \cdot \# \text{ of the sites in } R,\end{aligned}$$

which yields a contradiction for large R. □

7 Uniqueness for every $z \in Z$

In the previous section, we showed that μ-almost every $z \in Z$, i.e. for almost every lattice with large holes, there can be at most one infinite cluster with $\mathbb{P}_p$-probability one, for any $0 < p < 1$. Now we sketch the proof that this is true for every $z \in Z$, leaving a number of details to the reader.

Let us begin by consideration of a simpler situation, more general in nature in some aspects. Let

$$\begin{aligned} X &:= \left\{ x = \sum_{i \geq 0} x_i 2^i : x_i \in \{0,1\} \text{ for all } i \geq 0 \right\} \simeq \{0,1\}^{\mathbb{N}}, \\ Sx &:= x + 1 \, (\text{mod 2 coordinatewise with right carry}), \\ \mu &:= (1/2, 1/2)^{\mathbb{N}} \end{aligned}$$

be the well-known binary odometer, and let (Y, T, ν) be a compact separable metric space Y provided with a homeomorphism T and a T-invariant probability measure ν. Consider the compact space $X \times Y$ with the transformation $S \times T$ and the $S \times T$-invariant probability measure $\mu \times \nu$. By the ergodic theorem, for any $f \in \mathbb{L}^1(\mu \times \nu)$, we have that

$$\bar{f}(x,y) := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(S^k x, T^k y)$$

exists for $\mu \times \nu$-almost every $(x, y) \in X \times Y$.

Theorem 7.1. If f is continuous on $X \times Y$, then there exists a subset Y_1 of Y with $\nu(Y_1) = 1$ such that for *every* $x \in X$ and $y \in Y_1$, $\bar{f}(x, y)$ exists.

Proof. Let $g(x) = 1 - x_0$ and let h be a continuous function on Y. Set $f(x, y) = g(x)h(y)$. Then

$$\frac{1}{2n} \sum_{k=0}^{2n-1} f(S^k x, T^k y) = \frac{1}{2n} \sum_{k=0}^{n-1} h(T^{2k} y),$$

so that clearly $\bar{f}(x, y)$ exists for every $x \in X$ and ν-almost every $y \in Y$. It is standard to extend this result to any g on X which depends on finitely many coordinates. Using countability considerations and the fact that linear combinations of such functions are uniformly dense in the continuous functions on $X \times Y$ leads now to the stated result. □

Straightforward but tedious arguments now show that the analogous result holds in our case of interest, namely, the ratio ergodic theorem for the $\mathbb{Z} \times \mathbb{Z}$ action on $Z \times \Omega$ with respect to the infinite invariant measures $\mu \times \mathbb{P}_p$ introduced earlier. We omit the proof.

Finally, we use the encounter point method to complete the proof of Theorem 4.2. Following this method, and returning the notation of the previous section, we say that the origin is an encounter point for the pair (z, ω) if it is contained an infinite cluster and if removal of the origin separates this infinite cluster into exactly three infinite disjoint clusters. Let E denote the set of (z, ω) for which the origin is an encounter point, and denote by E_0 the set of (z, ω) (which does not depend on ω) for which the origin belongs to $\Lambda(z)$. Using finite energy (actually, here we can use independence), it follows that $\mu \times \mathbb{P}_p(E) > 0$, if there are at least three infinite cluster. Also, from our normalization we have that $\mu \times \mathbb{P}_p(E_0) = 1$. The function

$$f_0(z, \omega) = 1_{E_0}(z, \omega)$$

is a continuous function on $Z \times \Omega$, but the function

$$f(z, \omega) = 1_E(z, \omega)$$

is not, and the ratio ergodic theorem discussed above does not apply directly. However

$$\mu \times \mathbb{P}_p(E) = \alpha > 0,$$

and if we define for any n the event E_n to be the set of (z, ω) such that the origin is contained in an open cluster which reaches the boundary of a box R_n with sides of length $2n + 1$ centered at the origin, and such that removal of the origin results in exactly three disjoint open clusters which each reach the boundary. (We call the origin is an *n-encounter point.*) Then it is clear that

$$\bigcap_n E_n = E,$$

so that $\mu \times \mathbb{P}_p(E_n) \geq \alpha$ for every n; moreover, the function

$$f_n(z, \omega) = 1_{E_n}(z, \omega)$$

is continuous for each n, so that we can apply the ratio ergodic theorem to the pair f_n and f_0. This now leads to a contradiction, since by the ergodic theorem, some translate of R_n must have a density of n-encounter points at least $\alpha - \varepsilon$ for every n, which is impossible for large n by following the reasoning in [3].

Remark 7.2. The theorem of this section certainly remains true for some other strictly ergodic systems (X, S, μ). In separate discussions last summer, H. Nakada in Yokohama pointed out that it is valid for Kronecker systems, and B. Weiss in Budapest observed that it remains valid for strictly ergodic systems of zero entropy (because of disjointness - here more hypotheses are apparently needed on Y) and for strictly ergodic K-systems (due to previous articles). We have as yet no counterexample for any strictly ergodic system, although we suspect that such counterexamples exist.

8 Related results

In this article we have established a method which can be used to prove uniqueness of the infinite cluster for finite dimensional lattices with large holes in site percolation, following the encounter point method. We summarize briefly related results. In interesting and remarkable articles, Shinoda [9, 10] considers percolation phase transition for non-oriented and oriented bond percolation on Sierpiński carpet lattices. It is difficult to prove further properties. Using sponge crossing arguments, Kumagai [7] shows the uniqueness of the critical point, the absence of the infinite cluster at criticality, and the uniqueness of the infinite cluster in the supercritical regime for a class of two-dimensional Sierpiński carpet lattices. Introducing a branching process argument, Higuchi and Wu [5] extends the result to the standard Sierpiński case, to which the theorem in [7] cannot be applied. In higher dimensions, the uniqueness of infinite cluster up to now has only been proved for large p (Wu [11, 12]).

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