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RIEMANNIAN GEOMETRY ON LOOP SPACES

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ABSTRACT. A Riemannian metric on a manifold M induces a family of Riemannian metrics on the loop space LM depending on a Sobolev space parameter. In Part I, we compute the Levi-Civita connection for these metrics. The connection and curvature forms take values in pseudodifferential operators (ΨDOs), and we compute the top symbols of these forms. In Part II, we develop a theory of Chern-Simons classes $CS^W_{2k-1} \in H^{2k-1}(LM, \mathbb{R})$, using the Wodzicki residue on ΨDOs. By results in Part I, for stably parallelizable manifolds these "Wodzicki-Chern-Simons" classes are defined for all metrics and are smooth invariants of M for $k > 2$. We produce examples of nontrivial three dimensional Wodzicki-Chern-Simons classes.

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1. **Introduction**

The loop space LM of a manifold M appears frequently in mathematics and mathematical physics. In this paper, using an infinite dimensional version of Chern-Simons theory, we develop a nontrivial, computable theory of secondary characteristic classes on certain infinite rank bundles including the tangent bundles to loop spaces.

The theory of primary characteristic classes on infinite rank bundles was treated via Chern-Weil theory in [20]. While these classes can be nonzero, the Pontrjagin classes vanish for loop spaces (Corollary 6.15).

In fact, the Pontrjagin forms vanish on loop spaces, which is the precondition for defining Chern-Simons classes. For technical reasons, we can only define Chern-Simons classes for stably parallelizable manifolds, (i.e. manifolds whose tangent bundle is stably trivial), but these "Wodzicki-Chern-Simons" classes $CS_{2k-1}^W \in H^{2k-1}(LM, \mathbb{R})$ are stronger than their finite dimensional counterparts in several ways:

- they are defined for any metric on M for $k \geq 2$ (Theorem 6.11);
- they are frame independent, and hence give real (as opposed to $\mathbb{R}/\mathbb{Z}$) classes (Proposition 6.13) ;
- they are metric independent (i.e. smooth invariants) for $k \geq 3$ (Theorem 7.2);
- they are potentially nontrivial in all odd dimensions (Remark 6.4).

We produce computer examples of metrics on $M = SO(3) \times S^1$ with $CS_3^W \neq 0$, so the theory is nontrivial at least in this dimension.

Since Chern-Weil and Chern-Simons theory are geometric, it is necessary to understand connections and curvature on loop spaces. A Riemannian metric g on M induces a family of metrics g^s on LM parametrized by a Sobolev space parameter $s \geq 0$, where $s = 0$ gives the usual L^2 metric. This metric is too weak for analysis on LM, so we think of s as a necessary but annoying regularizing parameter.

In Part I, we compute the connection and curvature forms for the Levi-Civita connection for g^s. These forms take values in pseudodifferential operators (ΨDOs) acting on a trivial bundle over S^1, as first shown by Freed for loop groups [12]. We are able to write down the Levi-Civita connection one-form fairly explicitly for integer Sobolev parameter; the noninteger case is more technical and is quarantined to a separate section. The symbol calculus for ΨDOs effectively computes symbols of the connection and curvature forms, so this theory is as computable as finite dimensional curvature calculations. In particular, it is easy to spot the parameter-independent portion of the calculations.

In Part II, we develop a theory of Chern-Simons classes on loop spaces, The structure group of the tangent bundle of (LM, g^s) is a group of ΨDOs, so we need to find invariant polynomials on the corresponding Lie algebra. We could use the standard polynomials $\mathrm{Tr}(\Omega^k)$ on the curvature $\Omega = \Omega^s$, where Tr is the operator trace, but this trace is impossible to compute in general. Instead, as in [20] we use the locally computable Wodzicki residue, the only trace on the full algebra of ΨDOs. Following Chern-Simons [6] as much as possible, we build a theory of Wodzicki-Chern-Simons (WCS) classes. The main difference from the finite dimensional theory is the absence of a Narasimhan-Ramanan universal connection theorem. As a result, we can only define WCS classes for stably parallelizable manifolds. We define parameter-free or regularized WCS classes by setting $s = 1$ is the final formulas (Definition 6.4); this "continuation to $s = 1$" is not the same as computing WCS classes in the $s = 1$ metric on LM.

The paper is organized as follows. Part I treats the family of metrics g^s on LM associated to (M, g). §2 discusses the relatively easy case $s \in \mathbb{Z}^+$. After some preliminary material, we compute the Levi-Civita connection one-form for g^s (Theorem 2.1) and show that the one-form takes values in ΨDOs of order zero (Proposition 2.2).

§3 discusses the case $s \notin \mathbb{Z}^+$. In this case, the Levi-Civita one-form takes values in bounded operators $GL(\mathcal{H})$ on a specific Hilbert space $\mathcal{H}$; these operators are in some sense a limit of ΨDOs of increasing order. As a result, we have to extend the structure group for the frame bundle FLM to $GL(\mathcal{H})$. The corresponding extended frame bundle is trivial, so we lose the topological information in the original frame bundle. We can define an extension of the Wodzicki residue to the appropriate operators, except possibly when s is a half integer.

In §4, we compute some symbols of the Levi-Civita connection one-form and the curvature two-form for the g^s metric. The key results are Theorems 4.3, 4.4, which state that the curvature takes values in ΨDOs of order at most -1, for s not a half integer, and compute the top order symbol. This unexpected negative order underlies the main results bulleted above. We finish Part I in §5 with a comparison of our results with Freed's on loop groups [12].

Part II covers Wodzicki-Chern-Simons classes. In §6 we review finite dimensional Chern-Weil and Chern-Simons theory for $O(n)$-bundles. As mentioned above, we replace the ordinary matrix trace by the Wodzicki residue to define characteristic and secondary classes on LM. (An alternative trace given by the leading order symbol is also discussed in §6.3) We show that the Wodzicki-Pontrjagin classes vanish on LM and more generally on $Maps(N, M)$, the space of maps from one Riemannian manifold to another. For M stably parallelizable, WCS classes and regularized WCS classes are defined in §6.4 and are shown to be independent of the trivializing frame.

In §7, we show that the WCS classes $CS^W_{2k-1}(LM, \mathbb{R})$ are independent of the metric on stably parallelizable M for $k \geq 3$, and CS^W_3 is an invariant of conformal families of Einstein metrics. In §8, we use Mathematica calculations to produce examples (up to high precision) of metrics on $SO(3) \times S^1$ with nonvanishing CS^W_3.

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Part I. The Levi-Civita Connection on the Loop Space LM

In this part of the paper, we compute the Levi-Civita connection on LM associated to a Riemannian metric on M and a Sobolev parameter s. The main result is Theorem 2.1, which computes the Levi-Civita connection explicitly except for one term denoted A_XY. This term is analyzed more concretely in Proposition 2.2 for $s \in \mathbb{Z}^+$.

Part I is organized as follows. In §2, we review background material on LM and pseudodifferential operators on manifolds, and prove the main theorem for $s \in \mathbb{Z}^+$. In §3, we cover the more technical case of $s \notin \mathbb{Z}^+$. In §4, we compute symbols of the Levi-Civita connection one-form and the curvature two-form. In §5, we compare our results with earlier work of Freed [12] on loop groups.

2. **The Levi-Civita Connection for Integer Sobolev Parameters**

This section covers background material and computes the Levi-Civita connection on LM for integer Sobolev parameter. In §2.1, we review material on LM, and in §2.2 we review pseudodifferential operators and the Wodzicki residue. In §2.3, we give the main computation of the connection one-form for the Levi-Civita connection on LM. In §2.4, we give a more complete calculation of the Levi-Civita connection for integer Sobolev parameter. In §2.5, we prove a technical lemma allowing us to reduce local coordinate computations on LM to local computations on M.

2.1. **Preliminaries on LM.** Let $(M, \langle\ ,\ \rangle)$ be a closed oriented Riemannian n-manifold with loop space $LM = C^\infty(S^1, M)$ of smooth loops. LM is a smooth infinite dimensional manifold, but it is technically simpler to work with the smooth manifold of loops in some Sobolev class $s \gg 0$, as we now recall. For $\gamma \in LM$, the formal tangent space $T_\gamma LM$ is $\Gamma(\gamma^*TM)$, the space of smooth sections of the pullback bundle $\gamma^*TM \longrightarrow S^1$. For $s > 1/2$, we complete $\Gamma(\gamma^*TM \otimes \mathbb{C})$ with respect to the Sobolev inner product

$$\langle X, Y\rangle_s = \frac{1}{2\pi}\int \langle (1+\Delta)^s X(\alpha), Y(\alpha)\rangle_{\gamma(\alpha)} d\alpha,\ \ X, Y \in \Gamma(\gamma^*TM).$$

Here $\Delta = D^*D$, with $D = D/d\gamma$ the covariant derivative along γ. We need the complexified pullback bundle, denoted from now on just as γ^*TM, in order to apply the ΨDO $(1+\Delta)^s$. The construction of $(1+\Delta)^s$ is reviewed in §2.2. We denote this completion by $H^s(\gamma^*TM)$.

A small real neighborhood U_γ of the zero section in $H^s(\gamma^*TM)$ is a coordinate chart near γ in the space of H^s loops via the pointwise exponential map

$$\exp_\gamma : U_\gamma \longrightarrow LM,\ X \mapsto \left(\alpha \mapsto \exp_{\gamma(\alpha)} X(\alpha)\right). \tag{2.1}$$

The differentiability of the transition functions $\exp_{\gamma_1}^{-1} \cdot \exp_{\gamma_2}$ is proved in [8] and [12, Appendix A]. Here γ_1, γ_2 are close loops in the sense that a geodesically convex neighborhood of $\gamma_1(\theta)$ contains $\gamma_2(\theta)$ and vice versa for all θ. Since γ^*TM is (non-canonically) isomorphic to the trivial bundle $\mathcal{R} = S^1 \times \mathbb{R}^n \longrightarrow S^1$, the model space for LM is the set of H^s sections of this trivial bundle.

The tangent bundle TLM has transition functions $d(\exp_{\gamma_1}^{-1} \circ \exp_{\gamma_2})$. Under the isomorphisms $T_{\gamma_1}LM \simeq \mathcal{R} \simeq T_{\gamma_2}LM$, the transition functions lie in the gauge group $\mathcal{G}(\mathcal{R})$, so this is the structure group of TLM.

2.2. **Review of ΨDO Calculus.** We recall the construction of pseudodifferential operators (ΨDOs) on a manifold M from [13, 22], assuming knowledge of ΨDOs on $\mathbb{R}^n$. We emphasize how to calculate global symbols in local coordinates, since subprincipal terms are coordinate dependent (e.g. (2.2)).

A linear operator $P : C^\infty(M) \longrightarrow C^\infty(M)$ is a ΨDO of order d if for every open chart $U \subset M$ and functions $\phi, \psi \in C_c^\infty(U)$, $\phi P \psi$ is a ΨDO of order d on $\mathbb{R}^n$, where we do not distinguish between U and its diffeomorphic image in $\mathbb{R}^n$. Let $\{U_i\}$ be a cover of M with subordinate partition of unity $\{\phi_i\}$. Let $\psi_i \in C_c^\infty(U_i)$ have $\psi_i \equiv 1$ on $\mathrm{supp}(\phi_i)$ and set $P_i = \psi_i P \phi_i$. Then $\sum_i \phi_i P_i \psi_i$ is a ΨDO of M, and P differs from $\sum_i \phi_i P_i \psi_i$ by a smoothing operator, denoted $P \sim \sum_i \phi_i P_i \psi_i$. In particular, this sum is independent of the choices up to smoothing operators.

An example is the ΨDO $(1+\Delta-\lambda)^{-1}$ for Δ a positive order nonnegative elliptic ΨDO and λ outside the spectrum of $1+\Delta$. In each U_i, we construct a parametrix P_i for $A_i = \psi_i(1+\Delta-\lambda)\phi_i$ by formally inverting $\sigma(A_i)$ and then constructing a ΨDO with the inverted symbol. By [1, App. A], $B = \sum_i \phi_i P_i \psi_i$ is a parametrix for $(1+\Delta-\lambda)^{-1}$. Since $B \sim (1+\Delta-\lambda)^{-1}$, $(1+\Delta-\lambda)^{-1}$ is itself a ΨDO. For $x \in U_i$, by definition

$$\sigma((1+\Delta-\lambda)^{-1})(x,\xi) = \sigma(P)(x,\xi) = \sigma(\phi P \phi)(x,\xi),$$

where ϕ is a bump function with $\phi(x) = 1$ [13, p. 29]; the symbol depends on the choice of (U_i, ϕ_i).

The operator $(1+\Delta)^s$ for $\mathrm{Re}(s) < 0$, which exists as a bounded operator on $L^2(M)$ by the functional calculus, is also a ΨDO. To see this, we construct the putative symbol σ_i of $\psi_i(1+\Delta)^s\phi_i$ in each U_i by a contour integral $\int_\Gamma \lambda^s \sigma[(1+\Delta-\lambda)^{-1}]d\lambda$ around the spectrum of $1+\Delta$. We then construct a ΨDO Q_i on U_i with $\sigma(Q_i) = \sigma_i$, and set $Q = \sum_i \phi_i Q_i \psi_i$. By arguments in [22], $(1+\Delta)^s \sim Q$, so $(1+\Delta)^s$ is a ΨDO.

For $\alpha = (\alpha_1, \dots, \alpha_n)$, let $\partial_x^\alpha = (\partial^{\alpha_1}/\partial x_1^{\alpha_1}) \dots (\partial^{\alpha_n}/\partial x_n^{\alpha_n})$ in some local coordinates. For any ΨDO $P \sim \sum_i \phi_i P_i \psi_i$ and fixed $x \in U_{i_0}$, the symbol of P in U_{i_0} coordinates is

$$\begin{aligned}
\sigma(P)(x,\xi) &= \sigma(\phi(\sum_i \phi_i P_i \psi_i)\phi) = \sum_i \phi(x)\phi_i(x)\sigma(P\psi_i\phi) \\
&= \sum_i \phi(x)\phi_i(x) \sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_\xi^\alpha \sigma(P_i) \partial_x^\alpha(\psi_i\phi) \\
&= \sum_i \phi(x)\phi_i(x)\sigma(P_i)(x,\xi)\psi_i(x)\phi(x) \\
&= \sum_i \phi_i(x)\sigma(P_i)(x,\xi),
\end{aligned}$$

where we use $\psi_i \equiv 1$ on $\mathrm{supp}(\phi_i)$, $\partial_x^\alpha \phi(x) = 0$ and $\partial_x^\alpha \psi = 0$ on $\mathrm{supp}(\phi)$ for $\alpha \neq 0$. Thus symbols can be calculated locally.

Recall that the *Wodzicki residue* of a ΨDO P on sections of a bundle $E \longrightarrow M^n$ is

$$\int_{S^*M} \mathrm{tr}\ \sigma_{-n}(P)(x,\xi)d\xi dx,$$

where S^*M is the unit cosphere bundle for some metric. The Wodzicki residue is independent of choice of local coordinates, and up to scaling is the unique trace on the algebra of ΨDOs (see e.g. [9]). It will be used in Part II to define characteristic classes on LM.

If the U_i are diffeomorphic to precompact open balls in $\mathbb{R}^n$, $\sigma(P_i)$ extends smoothly to ∂U_i after possible shrinking of the U_i. Let $V_1 = U_1, V_i = U_i - \cup_{j=1}^{i-1} U_j$. As with any differential form, letting ϕ_1 equal one on "more and more" of U_1 and letting the other ϕ_i equal one on "a little more than" V_i, we get

$$\begin{aligned}
\int_{S^*M} \mathrm{tr}\ \sigma_{-n}(P)(x,\xi)d\xi dx &= \sum_i \int_{U_i} \phi_i(x)\, \mathrm{tr}\ \sigma_{-n}(P_i)(x,\xi)d\xi dx \\
&= \sum_i \int_{V_i} \mathrm{tr}\ \sigma_{-n}(P_i)(x,\xi)d\xi dx;
\end{aligned}$$

the invariance of the Wodzicki residue makes the right hand side well defined.

Therefore, for Wodzicki residue calculations we can sum up the integrals of the locally defined symbols. In particular, for a bundle E over S^1 with ΨDO P, we can find a closed cover $I_i = [a_i, a_{i+1}]$ with $E|_{I_i}$ trivial, and then

$$\int_{S^*S^1} \mathrm{tr}\ \sigma_{-1}(P)(x,\xi)d\xi d\theta = \sum_i \int_{(a_i,a_{i+1})} \mathrm{tr}\ \sigma_{-1}(P_i)(x,\xi)d\xi dx. \tag{2.2}$$

2.3. **Computing the Levi-Civita Connection.** The H^s metric makes LM a Riemannian manifold. The H^s Levi-Civita connection on LM is determined by the six term formula

$$\begin{aligned} 2\langle \nabla^s_Y X, Z\rangle_s &= X\langle Y, Z\rangle_s + Y\langle X, Z\rangle_s - Z\langle X, Y\rangle_s \\ &\quad + \langle [X,Y], Z\rangle_s + \langle [Z,X], Y\rangle_s - \langle [Y,Z], X\rangle_s. \end{aligned} \tag{2.3}$$

Recall that

$$[X,Y]^a = X(Y^a)\partial_a - Y(X^a)\partial_a \equiv \delta_X(Y) - \delta_Y(X) \tag{2.4}$$

in local coordinates on a finite dimensional manifold. Note that $X^i\partial_i(Y^a) = X(Y^a) = (\delta_X Y)^a$ in this notation.

(2.4) continues to hold for vector fields on LM, even though the index a does not refer to coordinates on LM. To see this, one checks that the coordinate-free proof that $L_X Y(f) = [X,Y](f)$ for $f \in C^\infty(M)$ (e.g. [25, p. 70]) carries over to functions on LM. In brief, the usual proof involves a map $H(s,t)$ of a neighborhood of the origin in $\mathbb{R}^2$ into M, where s, t are parameters for the flows of X, Y, resp. For LM, we have a map $H(s,t,\theta)$, where θ is the loop parameter. The usual proof involves involve only s, t differentiations, so θ is unaffected. The point is that the Y^i are local functions on the (s,t,θ) parameter space, whereas the Y^i are not local functions on M at points where loops cross or self-intersect.

Fix a loop γ and choose a cover $\{(a_i, b_i)\}$ of S^1 such that there is a coordinate cover $\{U_i\}$ of an open neighborhood of $\mathrm{Im}(\gamma)$ in M with

$$\gamma([a_i, b_i]) \subset U_i. \tag{2.5}$$

With these covers fixed, (2.5) holds for all loops near γ. Let $\{\phi_i\}$ be a partition of unity on S^1 subordinate to the cover $\{(a_i, b_i)\}$, and let $g_{ab} = g^{(i)}_{ab}$ be the metric tensor on U_i. The first term on the right hand side of (2.3) is

$$X\langle Y, Z\rangle_s = X\left(\sum_i \int_{(a_i,b_i)} \phi_i \cdot g^{(i)}_{ab}[(1+\Delta)^s Y]^a Z^b\right). \tag{2.6}$$

Since ϕ_j is independent of γ, we have

$$\begin{aligned} X\langle Y, Z\rangle_s &= \sum_i \int_{(a_i,b_i)} \phi_i \cdot \delta_X g^{(i)}_{ab}[(1+\Delta)^s Y]^a Z^b \\ &\quad + \sum_i \int_{(a_i,b_i)} \phi_i \cdot g^{(i)}_{ab}([\delta_X(1+\Delta)^s]Y)^a Z^b \\ &\quad + \sum_i \int_{(a_i,b_i)} \phi_i \cdot g^{(i)}_{ab}((1+\Delta)^s \delta_X Y)^a \cdot Z^b \\ &\quad + \sum_i \int_{(a_i,b_i)} \phi_i \cdot g^{(i)}_{ab}((1+\Delta)^s Y)^a \cdot \delta_X Z^b \end{aligned} \tag{2.7}$$

We will abbreviate terms like $\sum_i \int_{(a_i,b_i)} \phi_i \cdot \delta_X g^{(i)}_{ab}[(1+\Delta)^s Y]^a Z^b$ by $\int_{S^1} \delta_X g_{ab}[(1+\Delta)^s Y]^a Z^b$, and terms like $[(\delta_X(1+\Delta)^s)(Y)]^a$ by $\delta_X(1+\Delta)^s Y^a$. For example,

$$\langle Y, Z\rangle_s = \sum_i \int_{(a_i,b_i)} \phi_i \cdot g^{(i)}_{ab}((1+\Delta)^s Y)^a Z^b = \int_{S^1} g_{ab}((1+\Delta)^s Y)^a Z^b.$$

Collecting all terms from the six term formula similar to the last two terms in (2.7) gives

$$\begin{aligned} &\int_{S^1} g_{ab}(1+\Delta)^s \left[\delta_X Y^a \cdot Z^b + Y^a \cdot \delta_X Z^b + \delta_Y X^a \cdot Z^b + X^a \cdot \delta_Y Z^b - \delta_Z X^a \cdot Y^b \right. \\ &\qquad \left. - X^a \cdot \delta_Z Y^b + (\delta_X Y - \delta_Y X)^a \cdot Z^b + (\delta_Z X - \delta_X Z)^a \cdot Y^b - (\delta_Y Z - \delta_Z Y)^a \cdot X^b\right] \\ &= 2\int_{S^1} g_{ab}(1+\Delta)^s \delta_X Y^a \cdot Z^b. \end{aligned} \tag{2.8}$$

The three terms in the six term formula corresponding to the first term on the right hand side of (2.7) contribute

$$\begin{aligned} &\int_{S^1} \delta_X g_{ab}\cdot(1+\Delta)^s Y^a \cdot Z^b + \delta_Y g_{ab}\cdot(1+\Delta)^s X^a \cdot Z^b - \delta_Z g_{ab}\cdot(1+\Delta)^s X^a \cdot Y^b \\ &= \int_{S^1} \delta_X g_{ab}\cdot(1+\Delta)^s Y^a \cdot Z^b + \delta_Y g_{ab}\cdot(1+\Delta)^s X^a \cdot Z^b \\ &\qquad - Z^t \partial_t g_{ab}\cdot(1+\Delta)^s X^a \cdot Y^b \\ &= \int_{S^1} \delta_X g_{ab}\cdot(1+\Delta)^s Y^a \cdot Z^b + \delta_Y g_{ab}\cdot(1+\Delta)^s X^a \cdot Z^b \\ &\qquad - g_{ab} Z^b g^{at} \partial_t g_{ef}\cdot(1+\Delta)^s X^e \cdot Y^f \end{aligned} \tag{2.9}$$

The three terms in the six term formula corresponding to the second term on the right hand side of (2.7) contribute

$$\int_{S^1} g_{ab}\left[\delta_X(1+\Delta)^s Y^a \cdot Z^b + \delta_Y(1+\Delta)^s X^a \cdot Z^b - \delta_Z(1+\Delta)^s X^a \cdot Y^b\right]. \tag{2.10}$$

The last term in (2.10) is linear in Z, and so is of the form

$$\langle \delta_Z(1+\Delta)^s X, Y\rangle_s = \langle A_X(Y), Z\rangle_s \tag{2.11}$$

for some $A_X(Y) \in T_\gamma LM$.

By (2.7) – (2.11),

$$\begin{aligned} &2\langle \nabla_X Y, Z\rangle_s \\ &= \int_{S^1} \left(2g_{ab}(1+\Delta)^s \delta_X Y^a \cdot Z^b + \delta_X g_{ab}\cdot(1+\Delta)^s Y^a \cdot Z^b + g_{ab}\delta_X(1+\Delta)^s Y^a \cdot Z^b \right. \\ &\qquad + \delta_Y g_{ab}(1+\Delta)^s X^a \cdot Z^b + g_{ab}\delta_Y(1+\Delta)^s X^a \cdot Z^b \\ &\qquad \left. - g_{ab}[g^{at}\partial_t g_{ef}\cdot(1+\Delta)^s X^e \cdot Y^f + A_X(Y)^a] Z^b\right). \end{aligned} \tag{2.12}$$

The second term on the right hand side of (2.12) is

$$\delta_X g_{ab}\cdot(1+\Delta)^s Y^a\cdot Z^b = g_{ab}Z^b g^{af}\delta_X g_{ef}\cdot(1+\Delta)^s Y^e,$$

so for $g_{ab}=g_{ab}^{(i)}$, we get

$$\begin{aligned}
&\int_{S^1}\delta_X g_{ab}((1+\Delta)^s Y^a)Z^b\\
&= \sum_i\int_{(a_i,b_i)}\phi_i\delta_X g_{ab}((1+\Delta)^s Y)^a\cdot Z^b \qquad (2.13)\\
&= \sum_i\int_{(a_i,b_i)}\phi_i g_{ab}\left[(1+\Delta)^s(1+\Delta)^{-s}(g^{tf}\delta_X g_{ef}((1+\Delta)^s Y)^e\partial_t)\right]^a Z^b.
\end{aligned}$$

In the last line, we take a diffeomorphism of (a_i,b_i) to $\mathbb{R}$, and compute the various ΨDOs in these local charts as in §2.2.

The last expression in (2.13) is not of the form $\langle W,Z\rangle_s$, since $(1+\Delta)^{-s}(g^{tf,(i)}\delta_X g_{ef}^{(i)}((1+\Delta)^s Y)^e\partial_t)$ depends on i. In fact, all the other terms on right hand side of (2.12) are of the form $\int_{S^1}\phi_i g_{ab}(f^i)^a Z^b$ for some locally defined vector fields f^i, since e.g. $\delta_X Y^a$ also depends on coordinate choices along γ. Since the left hand side of (2.12) is global, the apparently local terms on the right hand side must sum to a global expression. With this understanding, we have

Theorem 2.1. *The Levi-Civita connection $\nabla=\nabla^{(s)}$ for the H^s-metric on LM is given by*

$$\begin{aligned}
(\nabla_X Y)^a &= \delta_X(Y^a)+\frac{1}{2}(1+\Delta)^{-s}\left[g^{af}\delta_X g_{ef}\cdot(1+\Delta)^s Y^e+\delta_X(1+\Delta)^s Y^a\right]\\
&\quad+\frac{1}{2}(1+\Delta)^{-s}\left[g^{af}\delta_Y g_{ef}\cdot(1+\Delta)^s X^e+\delta_Y(1+\Delta)^s X^a\right] \qquad (2.14)\\
&\quad-\frac{1}{2}(1+\Delta)^{-s}\left[g^{at}\partial_t g_{ef}\cdot(1+\Delta)^s X^e\cdot Y^f+A_X(Y)^a\right],
\end{aligned}$$

with A_XY defined by (2.11).

In the theorem, $\delta_X(1+\Delta)^s Y^a$ is shorthand for $[(\delta_X(1+\Delta)^s)(Y)]^a$, and similarly for the other terms.

2.4. **Integer Sobolev Parameters.** If s is a positive integer, it is easy to understand the terms on the right hand side of Theorem 2.1.

Of the six terms on the right hand side of Theorem 2.1 involving the ΨDO $(1+\Delta)^{-s}$, the first, third and fifth are standard ΨDOs acting on Y.

For the second and fourth terms, we have to analyze the variation of $(1+\Delta)^s$. Let $Z\in T_\gamma LM=H^s(\gamma^*TM)$. If $f:M\longrightarrow\mathbb{R}$ is a (locally defined) function on M, then $\delta_Z f(x)=Z^i\partial_i f(x)$ in local coordinates near $x=\gamma(\theta)$, since $\delta_Z f(x)=(d/dt)|_{t=0}f(\alpha(t))$, where $\alpha(0)=x,\dot\alpha(0)=Z_{\gamma(\theta)}$.

The situation is different for (locally defined) functions on $S^1 \times LM$, such as $\dot\gamma^\nu$. Let $\tilde\gamma : [0,2\pi]\times(-\varepsilon,\varepsilon) \longrightarrow M$ be a smooth map with $\tilde\gamma(\theta,0)=\gamma(\theta)$, and $\frac{d}{d\tau}|_{\tau=0}\tilde\gamma(\theta,\tau) = Z(\theta)$. Since (θ,τ) are coordinate functions on $S^1\times(-\varepsilon,\varepsilon)$, we have

$$Z(\dot\gamma^\nu) = \partial_\tau^Z(\dot\gamma^\nu) = \frac{\partial}{\partial\tau}\Big|_{\tau=0}\left(\frac{\partial}{\partial\theta}(\tilde\gamma(\theta,\tau)^\nu\right) = \frac{\partial}{\partial\theta}\frac{\partial}{\partial\tau}\Big|_{\tau=0}\tilde\gamma(\theta,\tau)^\nu = \partial_\theta Z^\nu \equiv \dot Z^\nu. \tag{2.15}$$

The covariant derivative along γ is the operator on $\tau\in\Gamma(\gamma^*TM)$ given by

$$\begin{aligned}\frac{D\tau}{dt} &= (\gamma^*\nabla^M)_{\partial_\theta}(\tau) = \partial_\theta\tau + (\gamma^*\omega^M)(\partial\theta)(\tau)\\ &= \partial_\theta(\tau^i)\partial_i + \dot\gamma^t\tau^r\Gamma^j_{tr}\partial_j,\end{aligned}$$

where ∇^M is the Levi-Civita connection on M, ω^M is the connection one-form in local coordinates $\{\partial_i\}$ on M, s is a section of γ^*TM, and Γ^j_{tr} are the Christoffel symbols. For $\Delta = (\frac{D}{d\gamma})^*\frac{D}{d\gamma}$, an integration by parts gives

$$(\Delta Y)^k = -\partial_\theta^2 Y^k - 2\Gamma^k_{\nu\mu}\dot\gamma^\nu\partial_\theta Y^\mu - \left(\partial_\theta\Gamma^k_{\nu\delta}\dot\gamma^\nu + \Gamma^k_{\nu\delta}\ddot\gamma^\nu + \Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot\gamma^\nu\right)Y^\delta.$$

Therefore, by (2.15)

$$\begin{aligned}((\delta_Z\Delta)Y)^k &= (-2Z^i\partial_i\Gamma^k_{\nu\mu}\dot\gamma^\nu - 2\Gamma^k_{\nu\mu}u\dot Z^\nu)\dot Y^\mu - \Big(Z^i\partial_i\dot\Gamma^k_{\nu\delta}\dot\gamma^\nu + \dot\Gamma^k_{\nu\delta}\dot Z^\nu\\ &\quad + Z^i\partial_i\Gamma^k_{\nu\delta}\ddot\gamma^\nu + \Gamma^k_{\nu\delta}\ddot Z^\nu + Z^i\partial_i\Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot\gamma^\nu + \Gamma^k_{\nu\mu}Z^i\partial_i\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot\gamma^\nu\\ &\quad + \Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot Z^\varepsilon\dot\gamma^\nu + \Gamma^k_{\nu\mu}\Gamma^\mu_{\varepsilon\delta}\dot\gamma^\varepsilon\dot Z^\nu\Big)Y^\delta.\end{aligned} \tag{2.16}$$

Thus $(\delta_Z\Delta)Y$, resp. $\delta_Z(1+\Delta)^sY$, is a second order differential operators in Z and a first, resp. $2s-1$, order differential operator in Y.

In summary, the second term in (2.1) is order -1 in Y. Similarly, the fourth term $(1+\Delta)^{-s}[\delta_Y(1+\Delta)^sX]$ in (2.1) is order $-2s+2$ in Y.

We now treat the last term in Theorem 2.1. By (2.11),

$$\langle A_XY, Z\rangle_0 = \int_{S^1} g_{ab}\delta_Z(1+\Delta)^sX^aY^b = \sum_i\int_{(a_i,b_i)}\phi_i g^{(i)}_{ab}\delta_Z(1+\Delta)^sX^aY^b. \tag{2.17}$$

Applying integration by parts to the right hand side of (2.17), we obtain the L^2 inner product of Z with a second order differential operator acting on Y; this operator is A_XY. Applying $(1+\Delta)^{-s}$, we get the H^s inner product of Z with a ΨDO of order $-2s+2$.

In particular, for $s>\frac{3}{2}$, this term does not contribute to the 0 or -1 order symbol of the connection one-form.

We summarize this section:

Proposition 2.2. *The term A_XY in Theorem 2.1 is a second order differential operator acting on Y. For $s\in\mathbb{Z}^+$, the last six terms on the right hand side of Theorem 2.1 are ΨDOs in Y of order* $0, -1, -2s, -2s+2, -2s, -2s+2$*, respectively.*

Remark 2.1. For $s \geq 2$, only the first two terms have order 0 or -1. This is crucial for the Chern-Simons theory calculations in §6, as the Wodzicki residue for ΨDOs over the circle depends on σ_{-1}. The case $s = 1$ is more difficult to treat directly. In §6, we instead compute Wodzicki-Chern-Simons classes for $s \gg 0$ and continue the formulas to $s = 1$.

2.5. **The Levi-Civita Connection One-Form.** In local coordinates on a manifold N, the Levi-Civita connection on TN is $\nabla = d + \omega$. For a vector field Y and a tangent vector X, $d_X(Y) = \delta_X Y$ in the notation of §2.3, and $\omega(X)(Y)$ is the Levi-Civita connection one-form. In Theorem 2.1, the right hand side is not in the form $d + \omega$, since $\delta_X(Y^a)$ is computed in local coordinates on M, not on $N = LM$.

Nevertheless, we make the following definition:

Definition 2.1. *The* Levi-Civita connection one-form *for the H^s metric on LM is the sum of the last six terms on the right hand side of (2.14) and is denoted $\omega = \omega^{(s)}$.*

In this section, we prove a technical local lemma that justifies using this Levi-Civita connection one-form to compute symbols as in §4 below. We also extend the local lemma to a global lemma used in §6 to extend our definition of Wodzicki-Chern-Simons classes from parallelizable to stably parallelizable manifolds.

As just mentioned, for a vector field Y on LM and a tangent vector $X \in T_\gamma LM$, $(\delta_X^{LM} Y)^a \neq \delta_X(Y^a)$ in local coordinates along a portion of γ. Indeed, in local coordinates $(\psi_\alpha)_{\alpha=1}^\infty$ of LM near γ, $\delta_X^{LM} Y$ can only mean $X(Y^\alpha)\psi_\alpha$, while Y^a are compnents with respect to a finite set of coordinates of M near some $\gamma(\theta_0)$. Of course, we can still substitute $\nabla = \delta + \omega$ from Theorem 2.1 into $\Omega(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ to compute the curvature on LM; as usual, we obtain $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$, where $d\omega$ is defined by the Cartan formula in the a-coordinates. This is very useful for computing symbols of Ω, whereas a decomposition $\nabla = d_{LM} + \omega_{LM}$ in coordinates on LM would be useless for symbol computations.

For Chern-Simons theory in Part II, we have to compute terms of the form $\chi^*\theta$, where χ is a local section of the frame bundle FLM and θ is a Lie algebra valued connection one-form on FLM. We have to compare $\chi^*\theta$ for LM with the M-coordinate expression $\omega^{(s)}$, as we can only compute symbols for ω-like expressions.

The following lemma relates $\chi^*\theta$ to ω.

Lemma 2.3. *Given $\gamma_0 \in LM$, there exists an open neighborhood $V \subset LM$ of γ_0, a local frame $\chi : V \longrightarrow FLM$, and local coordinates $\{U_i\}$ on M covering Im(γ_0) such that $\chi^*\theta_\gamma(X)(Y) = \omega_\gamma(X)(Y)$ on each U_i, for all $\gamma \in V$.*

Proof: For any local frame χ, $\chi^*\theta$ is the connection one-form in the χ trivialization of TLM, so $\nabla_X Y = X(Y^\alpha)\psi_\alpha + \chi^*\theta(X)(Y)$, where $\chi(\gamma) = (\psi_\alpha)$ for $\gamma \in V$. Let $\{\rho_i\}$ be a partition of unity for a cover $\{U_i\}$ as in (2.5), and let $\rho_i Y = Y_i^\alpha \psi_\alpha = Y_i^\alpha \psi_\alpha^j \partial_j$.

Then

$$\begin{aligned}
\nabla_X Y &= \nabla_X\left(\sum_i \rho_i Y\right) = \sum_i \nabla_X(Y_i^\alpha\psi_\alpha) = \sum_i X(Y_i^\alpha)\psi_\alpha + \chi^*\theta(X)(Y)\\
&= \sum_i X(Y_i^\alpha)\psi_\alpha^j\partial_j + \chi^*\theta(X)(Y)\\
&= \sum_i X(Y_i^\alpha\psi_\alpha^j)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j\\
&= X\left(\sum_i(\rho_i Y)^j\right)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j\\
&= X(Y^j)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j\\
&= \delta_X(Y^j)\partial_j + \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j.
\end{aligned}$$

Since $\nabla_X Y = \delta_X(Y^j)\partial_j + \omega(X)(Y)$, we get

$$\omega(X)(Y) = \chi^*\theta(X)(Y) - \sum_i Y_i^\alpha X(\psi_\alpha^j)\partial_j.$$

The lemma follows if we can find χ and U_i (with coordinates) such that $X(\psi_\alpha^j)(\gamma)(\theta) = 0$ whenever $\gamma(\theta)\in U_i$.

Let (ψ_α^0) be a basis of the $H^s\gamma_0^*TM$, and let $\{e_i\}$ be a global frame of γ_0^*TM. Fix $p=\gamma_0(\theta_0)$. For short vectors $V_p=\{v\in T_pM : |v|<\epsilon\}$, $\exp_p V$ is a coordinate neighborhood U, with coordinate vectors $\partial_i = d\exp_v e_i$ (thinking of $e_i\in T_vT_pM$). At p, $\partial_i = e_i$. $\{\tilde e_i = d\exp_v e_i\}$ is a global frame of γ^*TM for γ close to γ_0. We can trivialize TLM on some neighborhood V of γ_0 by writing $\psi_\alpha^0 = \psi_\alpha^k e_k$ and setting

$$H^s\gamma_0^*TM\times V \xrightarrow{\approx} TLM|_V,\quad (\psi_\alpha,\gamma)\mapsto \psi_k^\alpha\tilde e_k.$$

For the section $\chi:\gamma\mapsto(\psi_\alpha)\equiv(\psi_k^\alpha\tilde e_k)$, we have $\psi_\alpha = \psi_k^\alpha\tilde e_k = \psi_k^\alpha\partial_k$ near p, and so $X(\psi_\alpha) = X(\psi_\alpha^k)\partial_k = 0$. □

Remark 2.2. (i) The metric on M used to define the exponential coordinates and the local frame χ in the proof need not be the fixed metric on M.

(ii) If M is parallelizable with global frame $\{e_i\}$, this frame also trivializes γ^*TM for all $\gamma\in LM$: cf. Definition 6.4 of regularized Wodzicki-Chern-Simons classes.

To end this section, we check that the local section in Lemma 2.3 can be extended to a global section if the frame bundle FLM is trivial, the setup of Part II.

Lemma 2.4. *Assume FLM is trivial. Let $V\subset LM$ be an open set with a local section $\chi: V\longrightarrow FLM$. There exists an open subset $V'\subset V$, and a global section $\tilde\chi: LM\longrightarrow FLM$ with $\tilde\chi|_{V'}=\chi$.*

Proof: Let χ_1 be a global section of FLM. There exists a gauge transformation $g : V \longrightarrow Aut(\mathcal{R})$ with $g \circ \chi_1 = \chi$. It suffices to extend g to $g_1 : LM \longrightarrow \mathcal{G}$. Let V' be an open subset of V with $C = \overline{V'} \subset V$. By [7], $g|_C$ has a continuous extension g_2 from the metric space LM to the locally convex vector space $Hom(\mathcal{R})$. Identifying $\mathcal{R}$ with γ^*TM at a loop γ and composing g_2 with the pointwise exponential map on $GL(n, \mathbb{C})$ gives a continuous extension $g_3 : LM \longrightarrow \mathcal{G}$.

For a cover $\{U_\alpha\}$ of $\mathcal{G}$, set $V_\alpha = g_3^{-1}(U_\alpha)$. Since LM is a Hilbert manifold, each component function of $g_3|_{V_\alpha - C}$ can be uniformly approximated by a smooth function $g_{4,\alpha}$ [3]. Since LM admits a partition of unity, these local approximations can be glued to a smooth function g_1 which agrees with g on V. □

3. The Levi-Civita Connection for Noninteger Sobolev Parameters

If $s \notin \mathbb{Z}^+$, we have to analyze the fourth and sixth terms $A_XY, \delta_Y(1+\Delta)^sX$ in (2.14) more carefully. The calculations leads to an extension of the structure group of the frame bundle of LM, which weakens the corresponding theory of characteristic classes.

In §3.1, we check that $\delta_Z(1+\Delta)^s$ is a ΨDO for any value of s. In §3.2, we analyze A_XY and $\delta_Y(1+\Delta)^sX$, noting that the integration by parts used in §2.4 no longer terminates after a finite number of steps. This leads to an extended notion of Wodzicki residue in §3.3, and forces us in §3.4 to extend of the frame bundle of LM from a gauge bundle to a $GL(\mathcal{H})$ bundle for a particular Hilbert space $\mathcal{H}$.

This is the most technical section of the paper. Its purpose is to justify later calculations of Wodzicki-type characteristic classes for noninteger parameters, but the reader may prefer to just assume the integer theory extends to the noninteger case.

3.1. The Variation of $(1+\Delta)^s$.

Lemma 3.1. *For* $\mathrm{Re}(s) \neq 0$, $\delta_Z(1+\Delta)^sY$ *is a* ΨDO *of order* $2s-1$ *in* Y.

Proof: $\delta_Z(1+\Delta)^s$ is a limit of differences of ΨDOs. The result follows, since the algebra of ΨDO is closed in the Fréchet topology of all C^k seminorms of symbols and smoothing terms on compact sets.

Since we will need the symbol of $\delta_Z(1+\Delta)^s$, we give a second proof. First assume $\mathrm{Re}(s) < 0$. As in the construction of $(1+\Delta)^s$, we will compute what the symbol asymptotics of $\delta_Z(1+\Delta)^s$ should be, and then construct an operator with these asymptotics. From the functional calculus for unbounded operators, we have

$$\begin{aligned}\delta_Z(1+\Delta)^s &= \delta_Z\left(\frac{i}{2\pi}\int_\Gamma \lambda^s(1+\Delta-\lambda)^{-1}d\lambda\right)\\ &= \frac{i}{2\pi}\int_\Gamma \lambda^s\delta_Z(1+\Delta-\lambda)^{-1}d\lambda\\ &= -\frac{i}{2\pi}\int_\Gamma \lambda^s(1+\Delta-\lambda)^{-1}(\delta_Z\Delta)(1+\Delta-\lambda)^{-1}d\lambda,\end{aligned}$$

where Γ is a contour around the spectrum of $1+\Delta$, and the hypothesis on s justifies the exchange of δ_Z and the integral. The operator $A=(1+\Delta-\lambda)^{-1}\delta_Z\Delta(1+\Delta-\lambda)^{-1}$ is a ΨDO of order -3 with top order symbol

$$\begin{aligned}\sigma_{-3}(A)(\theta,\xi)^\ell_j &= (\xi^2-\lambda)^{-1}\delta^\ell_k(-2Z^i\partial_i\Gamma^k_{\nu\mu}\dot\gamma^\nu-2\Gamma^k_{\nu\mu}\dot Z^\nu)\xi(\xi^2-\lambda)^{-1}\delta^\mu_j\\ &= (-2Z^i\partial_i\Gamma^\ell_{\nu j}\dot\gamma^\nu-2\Gamma^\ell_{\nu j}\dot Z^\nu)\xi(\xi^2-\lambda)^{-2}.\end{aligned}$$

Thus the top order symbol of $\delta_Z(1+\Delta)^s$ should be

$$\begin{aligned}\sigma_{2s-1}(\delta_Z(1+\Delta)^s)(\theta,\xi)^\ell_j &= -\frac{i}{2\pi}\int_\Gamma \lambda^s(-2Z^i\partial_i\Gamma^\ell_{\nu j}\dot\gamma^\nu-2\Gamma^\ell_{\nu j}\dot Z^\nu)\xi(\xi^2-\lambda)^{-2}d\lambda\\ &= \frac{i}{2\pi}\int_\Gamma s\lambda^{s-1}(-2Z^i\partial_i\Gamma^\ell_{\nu j}\dot\gamma^\nu-2\Gamma^\ell_{\nu j}\dot Z^\nu)\xi(\xi^2-\lambda)^{-1}d\lambda\\ &= s(-2Z^i\partial_i\Gamma^\ell_{\nu j}\dot\gamma^\nu-2\Gamma^\ell_{\nu j}\dot Z^\nu)\xi(\xi^2-\lambda)^{s-1}.\end{aligned}$$

Similarly, all the terms in the symbol asymptotics for A are of the form $B^\ell_j\xi^n(\xi^2-\lambda)^m$ for some matrices $B^\ell_j=B^\ell_j(n,m)$. This produces a symbol sequence $\sigma\sim\sum_k\sigma_{2s-k}$, and there exists a ΨDO P with $\sigma(P)=\sigma$. (As in §2.2, we first produce operators P_i on a coordinate cover U_i of S^1, and then set $P=\sum_i\phi_iP_i\psi_i$.) The construction depends on the choice of local coordinates covering γ as in (2.5), the partition of unity and cutoff functions as above, and a cutoff function in ξ; as usual, different choices change the operator by a smoothing operator. Standard estimates show that $P-\delta_Z(1+\Delta)^s$ is a smoothing operator, so $\delta_Z(1+\Delta)^s$ is a ΨDO.

For $\mathrm{Re}(s)>0$, motivated by differentiating $(1+\Delta)^{-s}\circ(1+\Delta)^s=\mathrm{Id}$, we set

$$\delta_Z(1+\Delta)^s=-(1+\Delta)^s\circ\delta_Z(1+\Delta)^{-s}\circ(1+\Delta)^s.\qquad\square$$

Remark 3.1. (i) For $s\in\mathbb{Z}^+$, $\delta_Z(1+\Delta)^s$ differs from the usual definition by a smoothing operator.

(ii) For all s, the proof shows that $\sigma(\delta_z(1+\Delta)^s)=\delta_Z(\sigma((1+\Delta)^s))$.

(iii) Since $\delta_Z\Delta$ is second order in Z, a symbol calculus calculation shows that $\sigma_{-k}(A)$ has at most $k-2$ derivatives of Z. It follows that for $\mathrm{Re}(s)<0$, $\sigma_{2s-k}(\delta_Z(1+\Delta)^s)$ has at most k derivatives of Z, and the same result then holds for $\mathrm{Re}(s)>0$.

3.2. **Analyzing the Difficult Terms.** We continue to assume $s \notin \mathbb{Z}^+$.

We first analyze the term A_XY defined by (2.11). By Lemma 3.1 and Remark 3.1(iii), $\delta_Z(1+\Delta)^s$ is a ΨDO with explicitly computable symbol $\sigma \sim \sum_{k\in\mathbb{Z}^+}\sigma_{2s-k}$, and σ_{2s-k} contains at most k derivatives of Z. Fix ℓ and let $P = P_{Z,\ell}$ be a ΨDO with symbol $\sum_{k=1}^{\ell}\sigma_{2s-k}$. As in the proof of Lemma 3.1, $P = \sum_i \phi_i P_i \psi_i$ with respect to some cover $\{(a_i, b_i)\}$ of S^1, and the symbol expansion of σ depends on the cover. $Q = \delta_Z(1+\Delta)^s - P$ is an operator of order at most $2s-\ell-1$. Suppressing the i dependence of the symbols, we have

$$\begin{aligned}
&\int_{S^1} g_{ab}\delta_Z(1+\Delta)^s X^aY^b \\
&= \int_{S^1} g_{ab}P(X)^aY^b + \int_{S^1} g_{ab}Q(X)^aY^b \qquad (3.1)\\
&= \sum_i \int_{(a_i,b_i)} \phi_i g^{(i)}_{ab}\left(\sum_j \int_{T^*(a_j,b_j)} e^{i(\theta-\theta')\cdot\xi}\phi_j(\theta)\right.\\
&\qquad \left.\cdot\sum_{k=1}^{\ell}(\sigma_{2s-k})^a_c(\theta,\xi)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^b d\theta\\
&\qquad + \sum_i \int_{(a_i,b_i)} \phi_i g^{(i)}_{ab}(QX)^aY^b.
\end{aligned}$$

In the next to last term in (3.1), consider a term $(Z^d)^{(k)}A^a_{dc}$ in a fixed σ_{2s-k_0} with $k \leq k_0$ derivatives in Z. In order to write this term as an inner product with Z, we have to perform k integration by parts in θ. There are several special cases:

(1) If k θ-derivatives act on ϕ_i, replace this term with $\phi_i\frac{\phi_i^{(k)}}{\phi_i}$, noting that this function extends by zero to $\{\phi_i = 0\}$. This gives the expression

$$\begin{aligned}
&\sum_i \int_{(a_i,b_i)} \phi_i\frac{\phi_i^{(k)}}{\phi_i}g_{ab}\left(\sum_j\int_{T^*(a_j,b_j)} e^{i(\theta-\theta')\cdot\xi}Z^dA^a_{dc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^bd\theta\\
&= \sum_i \int_{(a_i,b_i)} \phi_i\frac{\phi_i^{(k)}}{\phi_i}g_{de}g^{er}g_{ab}\\
&\qquad \cdot\left(\sum_j\int_{T^*(a_j,b_j)} e^{i(\theta-\theta')\cdot\xi}Z^dA^a_{rc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^bd\theta,
\end{aligned}$$

which locally is the L^2 inner product of $Z = Z^d\partial_d$ with

$$\frac{\phi_i^{(k)}}{\phi_i}g^{er}g_{ab}\left(\sum_j\int_{T^*(a_j,b_j)} e^{i(\theta-\theta')\cdot\xi}A^a_{rc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^b\partial_e.$$

This is the (local) H^s inner product of Z with

$$(1+\Delta)^{-s}\left[\frac{\phi_i^{(k)}}{\phi_i}g^{er}g_{ab}\left(\sum_j\int_{T^*(a_j,b_j)}e^{i(\theta-\theta')\cdot\xi}A^a_{rc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^b\partial_e\right].$$

This term is a ΨDO of order $-2s$ as an operator on Y.

(2) If k θ-derivatives act on g_{ab}, we get the local H^s inner product of Z with

$$(1+\Delta)^{-s}\left[(g_{ab})^{(k)}g^{er}\left(\sum_j\int_{T^*(a_j,b_j)}e^{i(\theta-\theta')\cdot\xi}A^a_{rc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^b\partial_e\right],$$

which is of order $-2s$ as an operator on Y.

(3) If k θ-derivatives act on $e^{i\theta\cdot\xi}$, we get the local H^s inner product of Z with

$$(1+\Delta)^{-s}\left[g^{er}g_{ab}\left(\sum_j\int_{T^*(a_j,b_j)}(i\xi)^k e^{i(\theta-\theta')\cdot\xi}A^a_{rc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^b\partial_e\right],$$

which is of order $-2s$ as an operator on Y.

(4) If k θ-derivatives act on $A^a_{dc}(\theta,\xi)\phi_j(\theta)$, we get the local H^s inner product of Z with

$$(1+\Delta)^{-s}\left[g^{er}g_{ab}\left(\sum_j\int_{T^*(a_j,b_j)}e^{i(\theta-\theta')\cdot\xi}\left(A^a_{rc}(\theta,\xi)\phi_j(\theta)\right)^{(k)}\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^b\partial_e\right],$$

which is of order $-2s$ as an operator on Y.

(5) If k θ-derivatives act on Y^b, we get the local H^s inner product of Z with

$$(1+\Delta)^{-s}\left[g^{er}g_{ab}\left(\sum_j\int_{T^*(a_j,b_j)}e^{i(\theta-\theta')\cdot\xi}A^a_{rc}(\theta,\xi)\phi_j(\theta)\psi_j(\theta')X^c(\theta')d\xi d\theta'\right)Y^{b,(k)}\partial_e\right],$$

which is of order $-2s+k$ as an operator on Y.

In general, each of the k integration by parts will act as in one of the previous cases, so the final result can be written as an H^s inner product with Z.

In summary, for fixed ℓ, A_XY involves the operator Q in (3.1) which we are ignoring, and an operator $(1+\Delta)^{-s}P'_{X,\ell}$ of order at most $-2s+\ell$ defined by

$$\int_{S^1}g_{ab}P_{Z,\ell}(X)^aY^b=\int_{S^1}g_{ab}P'_{X,\ell}(Y)^aZ^b=\langle(1+\Delta)^{-s}P'_{X,\ell}(Y),Z\rangle_s \tag{3.2}$$

The term $(1+\Delta)^{-s}\delta_Y(1+\Delta)^sX$ is easier to treat. For fixed Y, let $\sigma(\delta_Y(1+\Delta)^s)\sim\sum_k\sigma_{2s-k}$ as an operator on X. As before, let $P_{Y,\ell}$ be an operator with symbol $\sum_k^\ell\sigma_{2s-k}$. Omitting partitions of unity and cutoff functions, by Remark 3.1(iii) a typical term in $(1+\Delta)^sP_{Y,\ell}$ is of the form

$$\int_{T^*S^1}e^{i(\theta-\theta')\cdot\xi}Y^{d,(k)}(\theta)A^a_{dc}(\theta,\xi)X^c(\theta')d\theta' d\xi.$$

Pulling $Y^{d,(k)}$ out of the integral shows that for fixed X,

$$(1+\Delta)^{-s}P_{Y,\ell} \tag{3.3}$$

is a ΨDO in Y of order at most $-2s+\ell$.

In the next section, we address the issue that ℓ can be arbitrarily large.

3.3. **A Framework for the Difficult Terms.** Let $\mathcal{H}$ be the Hilbert space $H^s\gamma^*TM$ for a fixed s and γ. Let $GL(\mathcal{H})$ be the group of bounded invertible linear operators on $\mathcal{H}$; inverses of elements are bounded by the closed graph theorem. $GL(\mathcal{H})$ has the subset topology of the norm topology on $\mathcal{B}(\mathcal{H})$, the bounded linear operators on $\mathcal{H}$. $GL(\mathcal{H})$ is an infinite dimensional Banach Lie group, as a group which is an open subset of the infinite dimensional Hilbert manifold $\mathcal{B}(\mathcal{H})$ [18, p. 59], and has Lie algebra $\mathcal{B}(\mathcal{H})$. Let ΨDO, ΨDO_0^* denote the algebra of classical ΨDOs and the group of invertible zeroth order ΨDOs, respectively, where all ΨDOs act on $\mathcal{H}$. Note that $\Psi\text{DO}_0^* \subset GL(\mathcal{H})$.

Remark 3.2. The inclusions of ΨDO_0^*, ΨDO into $GL(\mathcal{H}), \mathcal{B}(\mathcal{H})$ are trivially continuous in the subset topology. For the Fréchet topology on ΨDO, the inclusion is not continuous.

Lemma 3.2. *$(1+\Delta)^{-s}A_XY$ and $(1+\Delta)^{-s}\delta_Y(1+\Delta)^sX$ (as operators on Y) are in $\mathcal{B}(\mathcal{H})$.*

Proof: If $Y \longrightarrow 0$ in H^s, then for fixed X and for all $Z\in H^s$,

$$\langle A_X(Y), Z\rangle_0 = \langle Y, \delta_Z(1+\Delta)^sX\rangle_0 \longrightarrow 0,$$

since $\delta_Z(1+\Delta)^sX \in H^{-s+1}\subset H^{-s}$, which pairs with H^s. This implies that $A_X(Y)\longrightarrow 0$ in H^{-s}. Thus $(1+\Delta)^{-s}A_X(Y)\longrightarrow 0$ in H^s.

For $(1+\Delta)^{-s}\delta_Y(1+\Delta)^sX$, we extend X, Z to $\tilde{X}, \tilde{Z}$ near γ as follows. Extend X arbitrarily to $\tilde{X} = \tilde{X}(r,\psi)$, where (r,ψ) are polar coordinates on $T_\gamma LM$ (i.e. on each fiber of the trivial bundle γ^*TM) and $r\approx 0$. (Set $\tilde{X} = 0$ if $X_\gamma = 0$.) Let $\epsilon(\psi)$ be a smooth positive function on the unit sphere of $T_\gamma LM$ in the H^s metric, and set $\tilde{X}_\epsilon(r,\psi) = \tilde{X}(\epsilon(\psi)r,\psi)$. For $\epsilon(\psi)$ small enough, $|\tilde{X}_\epsilon(r,\psi) - X|_{H^s}\longrightarrow 0$ uniformly in ψ as $r\longrightarrow 0$. For $s>\frac{3}{2}$, $|\tilde{X}_\epsilon(r,\psi)-X|_{C^1}\longrightarrow 0$ uniformly by the Sobolev embedding theorem. Thus $\delta_Y\tilde{X}_\epsilon \longrightarrow 0$ uniformly in C^0 as $Y\longrightarrow 0$ in H^s. We similarly extend Z arbitrarily to $\tilde{Z}_1$ and then to $\tilde{Z}_{1,\epsilon}$.

Set $\tilde{Z}_\epsilon = \tilde{Z}_{1,\epsilon} + (\langle Z, X\rangle_s - \langle \tilde{Z}_{1,\epsilon}, \tilde{X}_\epsilon\rangle_s)\frac{\tilde{X}_\epsilon}{\|\tilde{X}_\epsilon\|_s^2}$. Then $\delta_Y\tilde{Z}_\epsilon\longrightarrow 0$ uniformly in C^0 as $Y\longrightarrow 0$ in H^s and $Y\langle \tilde{X}_\epsilon, \tilde{Z}_\epsilon\rangle_s = 0$ for all Y. Dropping the tildes and epsilons, we have

$$\begin{aligned} 0 \ &= \ Y\langle X, Z\rangle_s = \int_{S^1}\delta_Y g_{ab}\cdot(1+\Delta)^sX^a\cdot Z^b + \int_{S^1} g_{ab}\delta_Y(1+\Delta)^sX^a\cdot Z^b \\ &\quad + \int_{S^1} g_{ab}(1+\Delta)^s\delta_YX^a\cdot Z^b + \int_{S^1} g_{ab}(1+\Delta)^sX^a\delta_YZ^b. \end{aligned} \tag{3.4}$$

Since $\delta_Y g_{ab} \longrightarrow 0$ uniformly as $Y \longrightarrow 0$ in H^s, by the estimates above and (3.4), $\langle \delta_Y(1+\Delta)^s X, Z\rangle_0 \longrightarrow 0$ for all $Z \in H^s$. Thus $\delta_Y(1+\Delta)^s X \longrightarrow 0$ in H^{-s}, and $(1+\Delta)^{-s}\delta_Y(1+\Delta)^s X \longrightarrow 0$ in H^s. □

This justifies the next definition.

Definition 3.1. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be the subspace spanned by* ΨDO *(considered as a vector space), $(1+\Delta)^{-s}A_X$ for $X \in T_\gamma LM$, and $(1+\Delta)^{-s}\delta.(1+\Delta)^s X$ for $X \in T_\gamma LM$.*

Elements of $\mathcal{A}$ have a *ΨDO expansion* given by an $\alpha \in \mathbb{R}$ and a sequence of ΨDOs $P_\ell, \ell \in Z^+$, $\text{ord}(P_\ell) \leq \alpha + \ell$. This expansion is linear in $A \in \mathcal{A}$ and is defined as follows: (i) if $P \in \Psi\text{DO}$, set $P_k = P$ for all k, and $\alpha = \text{ord}(P)$; (ii) for $(1+\Delta)^{-s}A_X$, set $\alpha = -2s$ and $P_\ell = (1+\Delta)^{-s}P'_{X,\ell}$ in (3.2); (iii) for $(1+\Delta)^{-s}\delta.(1+\Delta)^s X$, set $\alpha = -2s$ and $P_\ell = (1+\Delta)^{-s}P_{Y,\ell}$ in (3.3).

Note that $X \mapsto A_X$ in (ii) and $X \mapsto (Y \mapsto (1+\Delta)^{-s}\delta_Y(1+\Delta)^s X)$ in (iii) are elements of $\Lambda^1(LM, \mathcal{A})$,

Definition 3.2. *For $A \in \mathcal{A}$, we define the* extended Wodzicki residue *to be*

$$\lim_{\ell \longrightarrow \infty} \frac{1}{\ell} \sum_{k=0}^{\ell} \int_{S^*S^1} \text{tr }\ \sigma_{-1}(A_k) d\xi d\theta,$$

provided the limit exists.

Lemma 3.3. *The extended Wodzicki residue exists for all* ΨDO*s on $\mathcal{H}$ and equals the usual Wodzicki residue. In cases (ii), (iii), the extended Wodzicki residue vanishes if s is not a half-integer.*

Proof: The first statement follows from the definition. In cases (ii), (iii), P_ℓ has order $-2s + \ell \neq -1$ for s not a half integer. □

In summary:

Corollary 3.4. *For $s > \frac{3}{2}$ and not a half-integer, only the first and second terms in the connection one-form (2.14) contribute to its extended Wodzicki residue, and the extended and ordinary Wodzicki residues agree.*

The restriction on s follows as in the case of integer s (cf. Remark 2.1).

Remark 3.3. (i) Our Chern-Simons classes will have a linear dependence on s, so we will extend our results to half integer values by continuity.

(ii) An alternative approach to this section would be to couple the Sobolev parameter s to the "cutoff" parameter ℓ, so that the $\text{ord}(P_\ell) = -2s + \ell$ is always less than -1. This would artificially force the fourth and sixth terms of (2.14) to have no contribution to the Wodzicki residue of the connection one-form.

3.4. **Extensions of the Frame Bundle of LM.** We recall the relationship between the Levi-Civita connection one-form θ on the frame bundle FN of a manifold N and local expressions for the Levi-Civita connection on TN. Let $\chi : N \longrightarrow FN$ be a local section of the frame bundle. A metric connection ∇ on TN with local connection one-form ω determines a connection $\theta_{FN} \in \Lambda^1(FN, \mathfrak{o}(n))$ on FN by *(i)* θ_{FN} is the Maurer-Cartan one-form on each fiber, and *(ii)* $\theta_{FN}(Y_u) = \omega(X_p)$, for $Y_u = \chi_* X_p$ [23, Ch. 8, Vol. II], or equivalently

$$\chi^* \theta_{FN} = \omega. \tag{3.5}$$

This applies to $N = LM$. The frame bundle $FLM \longrightarrow LM$ is constructed as in the finite dimensional case. The fiber over γ is isomorphic to the gauge group $\mathcal{G}$ of $\mathcal{R}$ and fibers are glued by the transition functions for TLM. Thus the frame bundle is topologically a $\mathcal{G}$-bundle.

However, for $s \in \mathbb{Z}^+$, the connection form and hence the curvature form take values in $\Psi\mathrm{DO}_{\leq 0}$, the algebra of ΨDOs of order at most zero. For $s \notin \mathbb{Z}$, these forms take values in the bounded operators $\mathcal{B}(\mathcal{H})$, on $\mathcal{H} = H^s \gamma^* TM$. These forms should take values in the Lie algebra of the structure group. In the first case, we should take as structure group $\Psi\mathrm{DO}_0^*$, the group of invertible zeroth order (i.e. elliptic) ΨDOs, since the Lie algebra is $\Psi\mathrm{DO}_{\leq 0}$. In the second case, we will take $GL(\mathcal{H})$ as the structure group. This leads to extended frame bundles, also denoted FLM. The transition functions are unchanged, since $\mathcal{G} \subset \Psi\mathrm{DO}_0^* \subset GL(\mathcal{H})$. Thus (FLM, θ^s) as a geometric bundle (i.e. as a bundle with connection θ^s associated to ∇^s) is either a $\Psi\mathrm{DO}_0^*$-bundle or a $\mathcal{B}(\mathcal{H})$-bundle.

In summary, we have

$$\begin{array}{ccccccccc}
\mathcal{G} & \longrightarrow & FLM & \Psi\mathrm{DO}_0^* & \longrightarrow & (FLM, \theta^s) & GL(\mathcal{H}) & \longrightarrow & (FLM, \theta^s) \\
 & & \downarrow & & & \downarrow & & & \downarrow \\
 & & LM & & & LM & & & LM \\
 & & & & (s \in \mathbb{Z}^+) & & & (s \notin \mathbb{Z}^+) &
\end{array}$$

As in the previous section, we restrict attention to $s > 3/2$.

Remark 3.4. For $s \in \mathbb{Z}^+$, if we extend the structure group of the frame bundle with connection from $\Psi\mathrm{DO}_0^*$ to $GL(\mathcal{H})$, the frame bundle becomes trivial by Kuiper's theorem. This would allow us to define Chern-Simons forms on LM for all M by the procedures of §6.4 However, there is a potential loss of information if we pass to the larger frame bundle. For $s \notin \mathbb{Z}^+$, we are forced to extend the structure group.

The situation is similar to the following examples. Let $E \longrightarrow S^1$ be the $GL(1, \mathbb{R})$ (real line) bundle with gluing functions (multiplication by) 1 at $1 \in S^1$ and 2 at $-1 \in S^1$. E is trivial as a $GL(1, \mathbb{R})$-bundle, with global section $\theta \mapsto f(\theta), f(-\pi) = 1, f(0) = 0, f(\pi) = 2$. However, as a $GL(1, \mathbb{Q})^+$-bundle, E is nontrivial, as a global section is locally constant. As a second example, let $E \longrightarrow M$ be a nontrivial $GL(n, \mathbb{C})$-bundle.

Embed $\mathbb{C}^n$ into a Hilbert space $\mathcal{H}$, and extend E to an $GL(\mathcal{H})$-bundle $\mathcal{E}$ with fiber $\mathcal{H}$ and with the same transition functions. Then $\mathcal{E}$ is trivial.

4. Local Symbol Calculations

In this section, we compute the 0 and -1 order symbols of the connection one-form and the curvature two-form of the H^s Levi-Civita connection, under the assumption that $s > 3/2$ is not a half integer. The dependence of these symbols on s is linear, so we can extract the parameter independent part (see Definition 6.4 of regularized Wodzicki-Chern-Simons classes).

Since the connection one-form takes values in $\Psi\mathrm{DO}_{\leq 0}$, the curvature takes values in the same algebra. The main result is Theorem 4.2, which states that the order of the curvature is at most -1. This is crucial for the Chern-Simons theory in Part II.

Throughout this section, we denote the last six terms on the right hand side of (2.14) by a_X through f_X, so their sum is the Levi-Civita connection form ω_X as an operator on Y.

4.1. The 0 and -1 Order Symbols of the Connection Form.

The only term in (2.14) contributing to the zeroth order symbol of ω_X is a_X. Hence

$$\sigma_0(\omega_X)^a_e = \frac{1}{2} g^{af} X^i \partial_i g_{ef}. \tag{4.1}$$

From the identity $\partial_m g_{ab} = \Gamma^n_{am} g_{nb} + \Gamma^n_{bm} g_{an}$, we get $g^{af}\partial_i g_{ef} = \Gamma^n_{ei} g_{nf} g^{af} + \Gamma^n_{fi} g_{en} g^{af}$. Thus

Lemma 4.1. *The Levi-Civita connection form ω_X is a zeroth order ΨDO with zeroth order symbol*

$$\sigma_0(\omega_X)^a_e = \frac{1}{2} g^{af} X^i \partial_i g_{ef} = \frac{1}{2}\left(\Gamma^a_{ei} + g^{af} g_{en} \Gamma^n_{fi}\ \right) X^i. \tag{4.2}$$

We now compute the -1 order symbol of ω_X, which involves contributions from a_X and b_X. Denote the kth order symbol of an operator P by $[P]^k$. To simplify notation, set

$$\sigma(\Delta)^j_k = \xi^2 \delta^j_k + i h^j_k \xi + r^j_k.$$

The contribution from a_X is

$$\begin{aligned}
([a_X]^{-1})^a_e &= \frac{1}{2}\Big([(1+\Delta)^{-s}]^{-2s-1}[g^{af}\delta_X g_{ef}(1+\Delta)^s]^{2s} \\
&\quad + [(1+\Delta)^{-s}]^{-2s}[g^{af}\delta_X g_{ef}(1+\Delta)^s]^{2s-1} \\
&\quad - i\frac{\partial}{\partial \xi}([(1+\Delta)^{-s}]^{-2s})\frac{\partial}{\partial \theta}[g^{af}\delta_X g_{ef}(1+\Delta)^s]^{2s}\Big) \\
&= is\xi(\xi^2)^{-1}\left(\partial_\theta(g^{af}\delta_X g_{fe}) + \frac{1}{2}(g^{af}\delta_X g_{mf} h^m_e - h^a_k g^{kf}\delta_X g_{ef})\right).
\end{aligned}$$

The contribution from b_X is

$$([b_X]^{-1})^a_e = \frac{1}{2}([(1+\Delta)^{-s}\delta_X(1+\Delta)^s]^{-1})^a_e = \frac{1}{2}is\xi(\xi^2)^{-1}\delta_X h^a_e.$$

From these expressions, we obtain

Lemma 4.2. *The -1 order symbol of the Levi-Civita connection form ω_X is given by*

$$\begin{aligned}([\omega_X]^{-1})^a_e &= is\xi(\xi^2)^{-1}\partial_\theta(g^{af}\delta_X g_{fe}) \\ &\quad + \frac{1}{2}is\xi(\xi^2)^{-1}\left(\delta_X h^a_e + g^{af}\delta_X g_{mf}h^m_e - h^a_k g^{kf}\delta_X g_{ef}\right), \quad (4.3)\end{aligned}$$

where $h^a_k = -2\Gamma^a_{\nu k}\dot{\gamma}^\nu$ and $\delta_X = X^i\partial_i$.

4.2. **Symbols of the Curvature Form.** In this section we compute the symbols of the curvature

$$\Omega = \Omega^s = \Omega(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \quad (4.4)$$

To use Theorem 2.1, by Lemma 2.3 we have to choose exponential normal coordinates on local neighborhoods of $\mathrm{Im}(\gamma)$. Since the top order symbol of an operator is tensorial, we can center the normal coordinates at a fixed point $\gamma(\theta)$, which allows us to drop terms with first order derivatives of the metric or with an undifferentiated Christoffel symbol. In particular, we can assume that the first order symbol of Δ is zero.

- *The zeroth order symbol of the curvature*

Combine (4.4) with (2.14). The only zeroth order terms in (4.4) are

$$[\nabla_X\nabla_Y]^0 = [\delta_X a_Y]^0,\ [\nabla_Y\nabla_X]^0 = [\delta_Y a_X]^0,\ [\nabla_{[X,Y]}]^0 = 0, \quad (4.5)$$

as the terms multiplied by a_X or a_Y have vanishing zeroth order terms in normal coordinates. (In (4.5) we abuse notation by omitting terms like $\delta_X\delta_Y$ on the right hand sides, since $\delta_X\delta_Y - \delta_Y\delta_X - \delta_{[X,Y]} = 0$.)

The following crucial result extends [11] for loop groups.

Theorem 4.3. *For $s > 3/2$ and not a half integer, the zeroth order symbol of the curvature vanishes. Thus Ω^s is a* ΨDO *of order at most -1.*

PROOF: From the computation of a_X in (4.1), and using (4.4), (4.5), we get

$$\sigma_0(\Omega(X,Y)) = \frac{1}{2}g^{af}X^iY^j\left(\partial_i\partial_j g_{ef} - \partial_i\partial_j g_{ef}\right) \quad (4.6)$$

in normal coordinates. Recall that in normal coordinates $\partial_j\partial_i g_{ef} = \frac{1}{3}\left(R_{iejf} - R_{ifej}\right)$. Hence,

$$\partial_i\partial_j g_{ef} = \frac{1}{3}\left(R_{jeif} - R_{jfei}\right) = \partial_j\partial_i g_{ef},$$

i.e. (4.6) vanishes at $\gamma(\theta)$.

Proposition 2.2 implies that the order of Ω^s is $\max\{-2s+2,-1\} = -1$. □

• *The* -1 *order symbol for the curvature tensor*

We explicitly compute the -1 order symbol of Ω^s. As before, we use normal coordinates to find the contribution from each of the terms on the right hand side of (4.4). The contribution from the first term is given by

$$[\nabla_X \nabla_Y]^{-1} = [\delta_X \nabla_Y]^{-1} + [a_X \nabla_Y]^{-1} + [b_X \nabla_Y]^{-1}. \tag{4.7}$$

In fact, the contribution from the last two terms of (4.7) is zero. We have

$$[a_X \nabla_Y]^{-1} = \sum_{s+t=-1} [a_X]^s [\nabla_Y]^t = [a_X]^0 [\nabla_Y]^{-1} + [a_X]^{-1} [\nabla_Y]^0.$$

However, in normal coordinates $[a_X]^0 = [a_Y]^0 = 0$, so this contribution vanishes. Similarly,

$$[b_X \nabla_Y]^{-1} = \sum_{s+t=-1} [b_X]^s [\nabla_Y]^t = [b_X]^{-1} [\nabla_Y]^0 = [b_X]^{-1} [a_Y]^0 = 0.$$

Hence the only contribution to (4.7) comes from $[\delta_X \nabla_Y]^{-1} = [\delta_X \omega_Y]^{-1}$. From (4.3) and using $h^a_j = 0$, we get

$$\begin{aligned}\frac{([\delta_X \nabla_Y]^{-1})^a_e}{is\xi(\xi^2)^{-1}} &= \delta_X \left(\partial_\theta (g^{af} \delta_Y g_{fe}) + \frac{1}{2} \left(\delta_Y h^a_e + g^{af} \delta_Y g_{mf} h^m_e - h^a_k g^{kf} \delta_Y g_{ef} \right) \right) \\ &= \delta_X (\partial_\theta (g^{af} \delta_Y g_{fe})) + \frac{1}{2} \delta_X \delta_Y h^a_e.\end{aligned}$$

Call these last two terms A^X, B^X. For A^X, we obtain

$$\begin{aligned}(A^X)^a_e &= \delta_X \left(\partial_\theta (g^{af} \delta_Y g_{fe}) \right) = g^{af} \delta_X \partial_\theta \delta_Y g_{ef} = g^{af} X^p \partial_p (\partial_\theta (Y^q \partial_q g_{ef})) \\ &= g^{af} X^p (\partial_\theta (Y^q) \partial_p \partial_q g_{ef} + \partial_p (Y^q) \partial_\theta \partial_q g_{ef} + Y^q \partial_p \partial_\theta \partial_q g_{ef}).\end{aligned} \tag{4.8}$$

For B^X, we obtain

$$\begin{aligned}(B^X)^a_e &= \frac{1}{2} \delta_X \delta_Y h^a_e = \delta_X \delta_Y (\dot{\gamma}^t \Gamma^a_{te}) \\ &= X^p \partial_p (Y^q \partial_q (\dot{\gamma}^t \Gamma^a_{te})) = X^p \partial_p (Y^q (\partial_q \dot{\gamma}^t \Gamma^a_{te} + \dot{\gamma}^t \partial_q (\Gamma^a_{te}))) \\ &= X^p \dot{\gamma}^t \partial_p (Y^q) \partial_q \Gamma^a_{te} + \xi X^p Y^q (\partial_q (\dot{\gamma}^t) \partial_p \Gamma^a_{te} + \partial_p \dot{\gamma}^t \partial_q \Gamma^a_{te} + \dot{\gamma}^t \partial_p \partial_q \Gamma^a_{te}).\end{aligned} \tag{4.9}$$

The second term on the right hand side of (4.4) has -1 order terms A^Y and B^Y given by switching X and Y in (4.8) and (4.9).

Finally, from (4.3), (4.4), the last contribution to the -1 order symbol of the curvature tensor is given by ($is\xi(\xi^2)^{-1}$ times)

$$\frac{([\nabla_{[X,Y]}]^{-1})^a_e}{is\xi(\xi^2)^{-1}} = \frac{1}{is\xi(\xi^2)^{-1}} ([\omega_{[X,Y]}]^{-1})^a_e = \partial_\theta (g^{af} \delta_{[X,Y]} g_{fe}) + \frac{1}{2} \delta_{[X,Y]} h^a_e. \tag{4.10}$$

Let $A^{[X,Y]}$, $B^{[X,Y]}$ be the last two terms in (4.10). The notation $X \leftrightarrow Y$ means "the previous term(s) with X and Y switched." We obtain

$$\begin{aligned} A^{[X,Y]} &= \partial_\theta(g^{af}\delta_{[X,Y]}g_{fe}) = g^{af}\partial_\theta(X^p\partial_p(Y^q\partial_q g_{fe}) - X \leftrightarrow Y) \\ &= g^{af}\partial_\theta(X^p(\partial_p(Y^q)\partial_q g_{fe} + X^pY^q\partial_p\partial_q g_{fe}) - X \leftrightarrow Y) \\ &= g^{af}\left(X^p\partial_p(Y^q)\partial_\theta\partial_q g_{fe}\right) - X \leftrightarrow Y \\ &= g^{af}\left(X^p\partial_p(Y^q)\partial_\theta\partial_q g_{fe}\right) - X \leftrightarrow Y \end{aligned}$$

The first term on the right hand side of the last equation cancels with the third term of A^X, and the second term cancels with the third term of A^Y.

For the remaining term $B^{[X,Y]}$, we have

$$\begin{aligned} B^{[X,Y]} &= \frac{1}{2}\delta_{[X,Y]}h^a_e = \partial_\theta\gamma^t\delta_{[X,Y]}\Gamma^a_{te} \\ &= \partial_\theta\gamma^t X^p\partial_p Y^q\partial_q\Gamma^a_{te} - X \leftrightarrow Y \end{aligned}$$

The first term on the right hand side cancels the first term in B^X, and the second term cancels the corresponding term in B^Y.

Therefore, for $\tilde{\sigma}_{-1}(\Omega) \equiv \frac{1}{is\xi(\xi^2)^{-1}}([\Omega(X,Y)]^{-1})$, we obtain

$$\begin{aligned} (\tilde{\sigma}_{-1}(\Omega))^a_e &= A^X - A^Y + B^X - B^Y - A^{[X,Y]} - B^{[X,Y]} \\ &= g^{af}X^p(\partial_\theta Y^q\partial_p\partial_q g_{ef} + Y^q\partial_p\partial_\theta\partial_q g_{ef}) - X \leftrightarrow Y \\ &\qquad + X^pY^q(\partial_q\partial_\theta\gamma^t\partial_p\Gamma^a_{te} + \partial_p\partial_\theta\gamma^t\partial_q\Gamma^a_{te} + \partial_\theta\gamma^t\partial_p\partial_q\Gamma^a_{te}) - X \leftrightarrow Y. \end{aligned}$$

Applying $\delta_X\dot{\gamma}^t = \dot{X}^t$ from (2.15) gives

$$\begin{aligned} &(\tilde{\sigma}_{-1}(\Omega))^a_e \\ &= g^{af}\left(X^p(\partial_\theta Y^q\partial_p\partial_q g_{ef} + Y^q\partial_\theta\gamma^t\partial_p\partial_t\partial_q g_{ef}) + Y^q\partial_\theta X^p\partial_p\partial_q g_{ef}\right) - X \leftrightarrow Y \\ &\qquad + X^pY^q(\partial_q\partial_\theta\gamma^t\partial_p\Gamma^a_{te} + \partial_p\partial_\theta\gamma^t\partial_q\Gamma^a_{te} + \partial_\theta\gamma^t\partial_p\partial_q\Gamma^a_{te}) - X \leftrightarrow Y \\ &= g^{af}X^pY^q\partial_\theta\gamma^t\left(\partial_p\partial_t\partial_q g_{ef} - \partial_q\partial_t\partial_p g_{ef}\right) + X^pY^q\partial_\theta\gamma^t\left(\partial_p\partial_q\Gamma^a_{te} - \partial_q\partial_p\Gamma^a_{te}\right). \end{aligned}$$

Since $\partial_j\partial_i g_{ef} = \frac{1}{3}\left(R_{iejf} - R_{ifej}\right)$ and $\partial_k\Gamma^a_{ml} = \frac{1}{3}\left(R_{kml}{}^a - R_{klm}{}^a\right)$, we have

$$\begin{aligned} (\tilde{\sigma}_{-1}(\Omega))^a_e &= \frac{1}{3}g^{af}X^pY^q\partial_\theta\gamma^t\left(\partial_p\left(R_{qetf} - R_{qfet}\right) - \partial_q\left(R_{petf} - R_{pfet}\right)\right) \\ &\qquad + \frac{1}{3}X^pY^q\partial_\theta\gamma^t\left(\partial_p\left(R_{qte}{}^a - R_{qet}{}^a\right) - \partial_q\left(R_{pte}{}^a - R_{pet}{}^a\right)\right) \\ &= \frac{1}{3}g^{af}X^pY^q\partial_\theta\gamma^t\left(\partial_p\left(R_{qetf} - R_{qfet}\right) - \partial_q\left(R_{petf} - R_{pfet}\right)\right) \\ &\qquad + \frac{1}{3}X^pY^qg^{af}\partial_\theta\gamma^t\left(\partial_p\left(R_{qtef} + R_{qet\ f}\right) - \partial_q\left(R_{ptef} + R_{petf}\right)\right). \end{aligned}$$

Let $\partial_p(.)$ denote all the terms with p-derivatives in $(\tilde{\sigma}_{-1}(\Omega))^a_e$. Using the symmetries of the curvature tensor and the Bianchi identity, we get

$$\begin{aligned}\partial_p(.) &= \partial_p\left(2R_{qetf} - R_{qfet} + R_{qtef}\right)\\ &= \partial_p\left(-2R_{qeft} + R_{fqet} + R_{efqt}\right)\\ &= -3\partial_p R_{qeft} = 3R_{fteq;p}.\end{aligned}$$

There is a similar expression for the q-derivative terms. We finally obtain

Theorem 4.4. *The curvature $\Omega^s(X,Y)$ of the H^s connection is a ΨDO of order at most -1 for $s > \frac{3}{2}$ and s not a half integer. The -1 order symbol at a loop γ is*

$$(\sigma_{-1}[\Omega^s(X,Y)])^a_e = is\xi(\xi^2)^{-1}X^pY^q\dot{\gamma}^\ell(R^a{}_{\ell eq;p} - R^a{}_{\ell ep;q}), \tag{4.11}$$

where $R^q{}_{\ell eq;p}(\gamma(\theta))$ are the components of the covariant derivative of the curvature tensor on M.

Remark 4.1. (i) Locally symmetric spaces are characterized by the vanishing of the covariant derivatives of the curvature tensor, so for these spaces the order of the curvature is at most -2.

(ii) This theorem extends to $Maps(N,M)$, where N has a Riemannian metric $(h_{\mu\nu})$ [24]. Ω^s has order at most -1 and

$$(\sigma_{-1}[\Omega(X,Y)])^a_e = is(\xi^2)^{-1}\xi_\nu h^{\mu\nu}X^pY^q\partial_\mu\gamma^t\left(R^a{}_{teq;p} - R^a{}_{tep;q}\right). \tag{4.12}$$

5. The Loop Group Case

In this section, we relate our work to Freed's work on based loop groups ΩG [11]. We find a particular representation of the loop algebra that controls the order of the curvature of the H^1 metric on ΩG.

$\Omega G \subset LG$ with base point e.g. $e \in G$ has tangent space $T_\gamma\Omega G = \{X \in T_\gamma LG : X(0) = X(2\pi) = 0\}$ in some Sobolev topology. Instead of using $D^2/d\dot{\gamma}^2$ to define the Sobolev spaces, the usual choice is $\Delta_{S^1} = -d^2/d\theta^2$ coupled to the identity operator on the Lie algebra $\mathfrak{g}$. Since this operator has no kernel on $T_\gamma\Omega M$, $1+\Delta$ is replaced by Δ. These changes in the H^s inner product do not alter the spaces of Sobolev sections, but they do change the Levi-Civita connection. In any case, for X, Y, Z left invariant vector fields, the first three terms on the right hand side of (2.3) vanish. Under the standing assumption that G has a left invariant, Ad-invariant inner product, one obtains

$$2\nabla^{(s)}_X Y = [X,Y] + \Delta^{-s}[X, \Delta^s Y] + \Delta^{-s}[Y, \Delta^s X]$$

[11].

It is an interesting question to compute the order of the curvature operator as a function of s. For the H^s inner product on the free loop space LG defined via $1 + D^2/\partial\dot{\gamma}^2$, the order is at most -2, using Remark 4.1(i) and considering G as a symmetric space.

For the more standard H^s inner product on ΩG, the following case appears to be very special.

Proposition 5.1. *The curvature of the Levi-Civita connection for the H^1 inner product on ΩG is a* ΨDO *of order $-\infty$.*

PROOF: We give two proofs.

By [11], the H^1 curvature operator $\Omega = \Omega^{(1)}$ satisfies

$$(\Omega(X,Y)Z,W)_1 = \left(\int_{S^1}[Y,\dot{Z}], \int_{S^1}[X,\dot{W}]\right)_{\mathfrak{g}} - (X \leftrightarrow Y),$$

where $\dot{Z} = \partial_\theta Z$ as usual, and the inner product is with respect to the Ad-invariant form on the Lie algebra $\mathfrak{g}$. We want to write the right hand side of this equation as an H^1 inner product with W, in order to recognize $\Omega(X,Y)$ as a ΨDO.

Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{g}$, considered as a left-invariant frame of TG. Let $c_{ij}^k = ([e_i,e_j],e_k)_{\mathfrak{g}}$ be the structure constants of $\mathfrak{g}$. (The Levi-Civita connection on left invariant vector fields for the left-invariant metric is given by $\nabla_X Y = \frac{1}{2}[X,Y]$, so the structure constants are twice the Christoffel symbols.) For $X = X^i e_i = X^i(\theta)e_i, Y = Y^j e_j$, etc., integration by parts gives

$$(\Omega(X,Y)Z,W)_1 = \left(\int_{S^1}\dot{Y}^i Z^j d\theta\right)\left(\int_{S^1}\dot{X}^\ell W^m d\theta\right) c_{ij}^k c_{\ell m}^n \delta_{kn} - (X \leftrightarrow Y).$$

Since

$$\int_{S^1} c_{\ell m}^n \dot{X}^\ell W^m = \int_{S^1}\left(\delta^{mc}c_{\ell c}^n \dot{X}^\ell e_m), W^b e_b\right)_{\mathfrak{g}} = \left(\Delta^{-1}(\delta^{mc}c_{\ell c}^n \dot{X}^\ell e_m), W\right)_1,$$

we get

$$\begin{aligned}(\Omega(X,Y)Z,W)_1 &= \left(\left[\int_{S^1}\dot{Y}^i Z^j\right] c_{ij}^k \delta_{kn}\delta^{mc}c_{\ell c}^n \Delta^{-1}(\dot{X}^\ell e_m), W\right)_1\\ &= \left(\left[\int_{S^1} a_j^k(\theta,\theta')Z^j(\theta')d\theta'\right] e_k, W\right)_1,\end{aligned}$$

with

$$a_j^k(\theta,\theta') = \dot{Y}^i(\theta')c_{ij}^s \delta_{sn}\left(\Delta^{-1}(\dot{X}^\ell e_m)\right)^k(\theta).$$

We now show that $Z \mapsto \left(\int_{S^1} a_j^k(\theta,\theta')Z^j(\theta')d\theta'\right) e_k$ is a smoothing operator. Applying Fourier transform and Fourier inversion to Z^j yields

$$\begin{aligned}\int_{S^1} a_j^k(\theta,\theta')Z^j(\theta')d\theta' &= \int_{S^1\times\mathbb{R}\times S^1} a_j^k(\theta,\theta')e^{i(\theta'-\theta'')\cdot\xi}Z^j(\theta'')d\theta'' d\xi d\theta'\\ &= \int_{S^1\times\mathbb{R}\times S^1}\left[a_j^k(\theta,\theta')e^{-i(\theta-\theta')\cdot\xi}\right] e^{i(\theta-\theta'')\cdot\xi}Z^j(\theta'')d\theta'' d\xi d\theta',\end{aligned}$$

so $\Omega(X,Y)Z$ is a ΨDO with symbol

$$b_j^k(\theta,\xi) = \int_{S^1} a_j^k(\theta,\theta')e^{i(\theta-\theta')\cdot\xi}d\theta'$$

(with the usual mixing of local and global notation.)

For fixed θ, the integral is the Fourier transform of $\dot{Y}^i(\theta')$, the only term in $a_j^k(\theta,\theta')$ depending on θ'. Since the Fourier transform is taken in a local chart with respect to a partition of unity, and since in each chart $\dot{Y}^i$ times the partition of unity function is compactly supported, the Fourier transform of a_j^k in each chart is rapidly decreasing. Thus $b_j^k(\theta,\xi)$ is the product of a rapidly decreasing function with $e^{i\theta\cdot\xi}$, and hence is of order $-\infty$.

We now give a second proof. Recall that for all s

$$\nabla_X Y = \frac{1}{2}[X,Y] - \frac{1}{2}\Delta^{-s}[\Delta^s X, Y] + \frac{1}{2}\Delta^{-s}[X,\Delta^s Y].$$

Label the terms on the right hand side (1) – (3). As an operator on Y for fixed X, the symbol of (1) is $\sigma((1))_\mu^a = \frac{1}{2}X^e c_{\varepsilon\mu}^a$. Abbreviating $(\xi^2)^{-s}$ by ξ^{-2s}, we have

$$\begin{aligned}\sigma((2))_\mu^a &\sim -\frac{1}{2}c_{\varepsilon\mu}^a\left[\xi^{-2s}\Delta^s X^\varepsilon + \frac{1}{i}\xi^{-2s-1}\partial_\theta\Delta^s X^\varepsilon\right.\\ &\qquad\left. + \sum_{\ell=2}^\infty \frac{(-2s)(-2s-1)\ldots(-2s-\ell+1)}{i^\ell \ell!}\xi^{-2s-\ell}\partial_\theta^\ell\Delta^s X^\varepsilon\right]\\ \sigma((3))_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a\left[X^\varepsilon + \sum_{\ell=1}^\infty \frac{(-2s)(-2s-1)\ldots(-2s-\ell+1)}{i^\ell \ell!}\xi^{-\ell}\partial_\theta^\ell X^\varepsilon\right].\end{aligned}$$

Thus

$$\begin{aligned}\sigma(\nabla_X)_\mu^a &\sim \frac{1}{2}c_{\varepsilon\mu}^a\left[2X^\varepsilon - \xi^{-2s}\Delta^s X^\varepsilon - \frac{1}{i}\xi^{-2s-1}\partial_\theta\Delta^s X^\varepsilon\right.\\ &\qquad - \sum_{\ell=2}^\infty \frac{(-2s)(-2s-1)\ldots(-2s-\ell+1)}{i^\ell \ell!}\xi^{-2s-\ell}\partial_\theta^\ell\Delta^s X^\varepsilon \qquad (5.1)\\ &\qquad\left. + \sum_{\ell=1}^\infty \frac{(-2s)(-2s-1)\ldots(-2s-\ell+1)}{i^\ell \ell!}\xi^{-\ell}\partial_\theta^\ell X^\varepsilon.\right].\end{aligned}$$

Set $s=1$ in (5.1), and replace ℓ by $\ell-2$ in the first infinite sum. Since $\Delta = -\partial_\theta^2$, a little algebra gives

$$\sigma(\nabla_X)_\mu^a \sim c_{\varepsilon\mu}^a\sum_{\ell=0}^\infty\frac{(-1)^\ell}{i^\ell}\partial_\theta^\ell X^\varepsilon\xi^{-\ell} = \text{ad}\left(\sum_{\ell=0}^\infty\frac{(-1)^\ell}{i^\ell}\partial_\theta^\ell X\xi^{-\ell}e_k\right), \qquad (5.2)$$

where $\{e_k\}$ is an orthonormal basis of $\mathfrak{g}$ as before.

Denote the infinite sum in the last term of (5.2) by $W(X,\theta,\xi)$. The map $X \mapsto W(X,\theta,\xi)$ takes the Lie algebra of left invariant vector fields on LG to the Lie algebra $L\mathfrak{g}[[\xi^{-1}]]$, the space of formal ΨDOs of nonpositive integer order on the trivial bundle $S^1 \times \mathfrak{g} \longrightarrow S^1$, where the Lie bracket on the target involves multiplication of power series and bracketing in $\mathfrak{g}$. We claim that this map is a Lie algebra homomorphism. Assuming this, we see that

$$\begin{aligned}\sigma\left(\Omega(X,Y)\right) &= \sigma\left([\nabla_X,\nabla_Y]-\nabla_{[X,Y]}\right) \sim \sigma\left([\operatorname{ad} W(X), \operatorname{ad} W(Y)] - \operatorname{ad} W([X,Y])\right)\\ &= \sigma\left(\operatorname{ad}\left([W(X),W(Y)]\right) - \operatorname{ad} W([X,Y])\right) = 0,\end{aligned}$$

which proves that $\Omega(X,Y)$ is a smoothing operator.

To prove the claim, set $X = x^a_n e^{in\theta}e_a$, $Y = y^b_m e^{im\theta}e_b$. Then

$$\begin{aligned}W([X,Y]) &= W(x^n y^m e^{i(n+m)\theta} c^k_{ab} e_k) = \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{i^\ell} c^k_{ab}\partial^\ell_\theta\left(x^a_n y^b_m e^{i(n+m)\theta}\right)\xi^{-\ell} e_k\\ [W(X),W(Y)] &= \sum_{\ell=0}^{\infty}\sum_{p+q=\ell}\frac{(-1)^{p+q}}{i^{p+q}}\partial^p_\theta\left(x^a_n e^{in\theta}\right)\partial^q_\theta\left(y^b_m e^{im\theta}\right)\xi^{-(p+q)}c^k_{ab}e_k,\end{aligned}$$

and these two sums are clearly equal. □

It would be interesting to understand how the map W fits into the representation theory of the loop algebra $L\mathfrak{g}$.

Part II. Characteristic Classes on LM

In this part of the paper, we construct a general theory of Chern-Simons classes on infinite rank bundles including the frame/tangent bundle of loop spaces, following the construction of primary characteristic classes in [20]. The primary classes vanish on loop spaces, which forces the consideration of secondary classes. We discuss the metric and frame dependence of these "Wodzicki-Chern-Simons" (WCS) classes, and give an example of a nontrivial WCS class. The key ingredient is to replace the ordinary matrix trace in the Chern-Weil theory of invariant polynomials on finite dimensional Lie groups with the Wodzicki residue on invertible ΨDOs.

In §6, the general theory is developed for stably parallelizable manifolds, and we prove a vanishing result for the Pontrjagin classes of LM and more general spaces of maps. In §7, we show that the WCS forms are closed in dimensions three and above, and the resulting cohomology classes are metric independent in dimensions above three. The three dimensional WCS class is an invariant for conformal families of Einstein metrics, and the first WCS form is closed for Einstein metrics. In §8, we use computer calculations to check that certain metrics on $SO(3)\times S^1$ have nontrivial three dimensional WCS class.

6. Chern-Simons Classes on Loop Spaces

We begin this section with a review of Chern-Weil and Chern-Simons theory in finite dimensions (§6.1), following the seminal paper [6]. In §6.2, we discuss Chern-Simons theory on a class of infinite rank bundles including the frame bundles of loop spaces. Since the geometric structure group of these bundles is $\Psi\mathrm{DO}_0^*$, we need traces on the Lie algebra $\Psi\mathrm{DO}_{\leq 0}$ to define invariant polynomials. There are essentially two possible traces, one given by taking the zeroth order symbol and one given by the Wodzicki residue.

In §6.3, we discuss the zeroth order symbol theory. Chern-Simons forms are always defined, but we have no examples of nontrivial Chern-Simons classes. In §6.4, we consider the richer Wodzicki-Chern-Simons theory. WCS classes are defined for the loop space of a stably parallelizable manifold and are independent of the frame/trivialization of the stabilized tangent bundle. As a result, the WCS classes are real, not just $\mathbb{R}/\mathbb{Z}$ classes. We can spot the part of the WCS class which is independent of the Sobolev parameter, and this enables us to define regularized WCS classes. In §6.5, we prove that the corresponding Wodzicki-Pontrjagin classes vanish for the tangent bundle to $Maps(N, M)$ for Riemannian manifolds N, M.

6.1. Chern-Weil and Chern-Simons Theory for Finite Dimensional Bundles.

We first review the Chern-Weil construction. Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$, and let $G \longrightarrow E \longrightarrow M$ be a principal G-bundle over a manifold M Set $\mathfrak{g}^l = \mathfrak{g}^{\otimes l}$ and let

$$I^l(G) = \{P : \mathfrak{g}^l \longrightarrow \mathbb{R} \mid P \text{ symmetric, multilinear, Ad-invariant}\}$$

be the Ad-invariant polynomials on $\mathfrak{g}$.

Remark 6.1. For classical Lie groups G, $I^l(G)$ is generated by the polarization of the Newton polynomials $\mathrm{Tr}(A^l)$, where Tr is the usual trace on finite dimensional matrices.

For $\phi \in \Lambda^k(E, \mathfrak{g}^l)$, $P \in I^l(G)$, set $P(\phi) = P \circ \phi \in \Lambda^k(E)$. Two key properties are:

- (The *commutativity property*) For $\phi \in \Lambda^k(E, \mathfrak{g}^l)$,

$$d(P(\phi)) = P(d\phi). \tag{6.1}$$

- (The *infinitesimal invariance property*) For $\psi_i \in \Lambda^{k_i}(E, \mathfrak{g})$, $\phi \in \Lambda^1(E, \mathfrak{g})$ and $P \in I^l(G)$,

$$\sum_{i=1}^{l} (-1)^{k_1+\cdots+k_i} P(\psi_1 \wedge \cdots \wedge [\psi_i, \phi] \wedge \ldots \psi_l) = 0. \tag{6.2}$$

Theorem 6.1 (The Chern-Weil Homomorphism [15])**.** *Let $E \longrightarrow M$ have a connection θ with curvature $\Omega_E \in \Lambda^2(E, \mathfrak{g})$. For $P \in I^l(G)$, $P(\Omega_E)$ is a closed invariant real* *form on E, and so determines a closed form $P(\Omega_M) \in \Lambda^{2l}(M, \mathbb{R})$. The Chern-Weil map*

$$\oplus_{l=1} I^l(G) \longrightarrow H^*(M, \mathbb{R}),\ \ P \mapsto [P(\Omega_M)]$$

is a well-defined algebra homomorphism.

$[P(\Omega_M)]$ is called the *characteristic class* of P.

We now review Chern-Simons theory. A crucial observation is that $P(\Omega_E)$ is exact, although in general $P(\Omega_M)$ is not.

Proposition 6.2. [6, Prop. 3.2] *Let G be a finite dimensional Lie group. For a G-bundle $E \longrightarrow M$ with connection θ and curvature $\Omega = \Omega_E$, and for $P \in I^l(G)$, set*

$$\phi_t = t\Omega + \frac{1}{2}(t^2 - t)[\theta, \theta],\ \ TP(\theta) = l\int_0^1 P(\theta \wedge \phi_t^{l-1})dt.$$

Then $dTP(\theta) = P(\Omega) \in \Lambda^{2l}(E)$.

Proof. We recall the proof for later purposes. Set $f(t) = P(\phi_t^l)$, so $P(\Omega) = \int_0^1 f'(t)dt$. We show $f'(t) = l \cdot dP(\theta \wedge \phi_t^{l-1})$ by computing each side. First, we have

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(P(\phi_t^l)\right) = P\left(\frac{d}{dt}\phi_t^l\right) = lP\left(\frac{d}{dt}\phi_t \wedge \phi^{l-1}\right) \\ &= lP(\Omega \wedge \phi^{l-1}) + l\left(t - \frac{1}{2}\right)P\left([\theta, \theta] \wedge \phi_t^{l-1}\right), \end{aligned} \tag{6.3}$$

where we have used the commutativity property (6.1). On the other hand, we have

$$\begin{aligned} l \cdot dP(\theta \wedge \phi_t^{l-1}) &= lP(d\theta \wedge \phi_t^{l-1}) - l(l-1)P(\theta \wedge d\phi_t \wedge \phi_t^{l-2}) \qquad (6.4) \\ &= lP(\Omega \wedge \phi_t^{l-1}) - \frac{1}{2}lP([\theta, \theta] \wedge \phi_t^{l-1}) - l(l-1)P(\theta \wedge d\phi_t \wedge \phi_t^{l-2}), \end{aligned}$$

by (6.1) and the structure equation $\Omega = d\theta + \frac{1}{2}[\theta, \theta]$. Since $d\phi_t = t[\phi_t, \theta]$, the last term in (6.4) equals

$$l(l-1)P(\theta \wedge d\phi_t \wedge \phi_t^{l-2}) = l(l-1)P(\theta \wedge t[\phi_t, \theta] \wedge \phi_t^{l-2}).$$

Using the invariance property (6.2) with $\phi = \theta$, $\psi_1 = \theta$ and $\psi_k = \psi_t, k = 2, \ldots, l-1$, we obtain

$$l(l-1)P(\theta \wedge t[\phi_t, \theta] \wedge \phi_t^{l-2}) = -ltP([\theta, \theta] \wedge \phi_t^{l-1}).$$

This implies (6.4) equals (6.3). □

Setting $E = EG, M = BG$ in Theorem 6.1 gives the universal Chern-Weil homomorphism

$$W : I^l(G) \longrightarrow H^{2l}(BG, \mathbb{R}).$$

We write $P \in I_0^l(G)$ if $W(P) \in H^{2l}(BG, \mathbb{Z})$. For this subalgebra of polynomials, we obtain more information on $TP(\theta)$.

Theorem 6.3. [6, Prop. 3.15]. *Let $E \xrightarrow{\pi} B$ be a G-bundle with connection θ. For $P \in I_0^l(G)$, let $\widetilde{TP(\theta)}$ be the mod $\mathbb{Z}$ reduction of the real cochain $TP(\theta)$. Then there exists a cochain $U \in C^{2l-1}(B, \mathbb{R}/\mathbb{Z})$ such that*

$$\widetilde{TP(\theta)} = \pi^* U + \text{coboundary}.$$

In Theorem 6.10 below, we rework the proof of this theorem in our context.

Corollary 6.4. [6, Thm. 3.16] *Assume $P \in I_0^l(G)$ and $P(\Omega_E) = 0$. Then there exists $CS_P(\theta) \in H^{2l-1}(B, \mathbb{R}/\mathbb{Z})$ such that*

$$\left[\widetilde{TP(\theta)}\right] = \pi^*(CS_P(\theta)).$$

Proof. Choose $U \in C^{2l-1}(B, \mathbb{R}/\mathbb{Z})$ as in Theorem 6.3. Since $P(\Omega_E^l) = 0$, Proposition 6.2 implies $\delta\widetilde{TP(\theta)} = d\widetilde{TP(\theta)} = 0$. By Theorem 6.3, $\pi^* U$ and $\widetilde{TP(\theta)}$ are cohomologous. Set $CS_P(\theta) = [U]$. □

Notice that the secondary class or *Chern-Simons class* $CS_P(\theta)$, is defined only when the characteristic form $P(\Omega_E)$ vanishes. The proof of Theorem 6.10 shows that $CS_P(\theta)$ is independent of the choice of U.

The following corollary will be taken as the definition of Chern-Simons classes for trivial $\Psi\mathrm{DO}_0^*$-bundles (see Definition 6.2).

Corollary 6.5. *Let $(E, \theta) \xrightarrow{\pi} B$ be a trivial G-bundle with connection, and let χ be a global section. For $P \in I_0^l(G)$,*

$$CS_P(\theta) = \chi^*[\widetilde{TP(\theta)}].$$

Proof. This follows from Corollary 6.4 and $\pi\chi = \mathrm{Id}$. □

If we do not reduce coefficients to $\mathbb{R}/\mathbb{Z}$, this corollary fails; cf. Prop. 6.13.

6.2. **Chern-Simons Theory on Loop Spaces.** In [20], Chern forms are defined on complex vector/principal bundles with structure group $\Psi\mathrm{DO}_0^*$ and with $\Psi\mathrm{DO}_0^*$-connections, where the ΨDOs act on sections of a finite rank hermitian bundle $E \longrightarrow N$ over a closed manifold (e.g. $\gamma^* TM \otimes \mathbb{C} \longrightarrow S^1$ for loop spaces). The key technical point is to find suitable polynomials $P \in I^l(\Psi\mathrm{DO}_0^*)$. We single out two analogs of the Newton polynomials $\mathrm{Tr}(A^l)$: for $A \in \Psi\mathrm{DO}_0^*$, define

$$P_l^{(0)}(A) = k(l) \int_{S^*N} \mathrm{tr}\ \sigma_0(A^l)(x, \xi)\ d\xi dx. \tag{6.5}$$

Here S^*N is the unit cosphere bundle of N and $k(l) = (2\pi i)^{-l}(\mathrm{Vol}\, S^*N)^{-1}$. Note that $d_l = (2\pi i)^{-l}$ is the normalizing constant such that $d_l[\mathrm{tr}((\Omega^u)^l)] \in H^{2l}(BU(n), \mathbb{Z})$ for a connection θ^u on $EU(n) \longrightarrow BU(n)$[1]. In [19], $P_l^{(0)}$ is called a *Leading Order Symbol Trace*.

[1]We often omit this normalizing constant in the rest of the paper.

The second analog is

$$P_l^W(A) = k(l)\int_{S^*N} \text{tr}\ \sigma_{-n}(A^l)(x,\xi)\ d\xi dx. \tag{6.6}$$

$P_l^W(A)$ is a multiple of the Wodzicki residue of A^l. As usual, $P_l^{(0)}, P_l^W$ determine polynomials by polarization.

For both $P_l^{(0)}, P_l^W$, the commutativity and invariance properties hold because (6.5) and (6.6) are tracial [20]: $\text{Tr}[\sigma_0([A,B])] = 0$ for $A, B \in \Psi\text{DO}_{\leq 0}$, and the Wodzicki residue vanishes on commutators. Thus $P_l^{(0)}, P_l^W$ are in both $I^l(\mathcal{G}), I^l(\Psi\text{DO}_0^*)$ (although trivially $P_l^W = 0$ on the gauge group $\mathcal{G}$).

The proof of Proposition 6.2 to carries over to ΨDO_0^*-bundles with connections. In particular, it applies to the H^s-frame bundle of loop space for $s \in \mathbb{Z}^+$, and then to general $s > 3/2$, s not a half-integer, by §§2.5 – 2.6. Thus, we have

Proposition 6.6. *For a ΨDO_0^*-bundle with connection $(\mathcal{E},\theta) \xrightarrow{\pi} \mathcal{B}$, and for $P \in I^l(\Psi\text{DO}_0^*)$, set*

$$\begin{aligned} \phi_t &= t\Omega + \frac{1}{2}(t^2 - t)[\theta,\theta], \\ TP(\theta) &= l\int_0^1 P(\theta \wedge \phi_t^{l-1})dt \end{aligned} \tag{6.7}$$

Then $dTP(\theta) = P(\Omega^l)$. We can replace ΨDO_0^* by $\mathcal{G}$.

Remark 6.2. By Chern-Weil theory, $P_l^{(0)}(\Omega), P_l^W(\Omega)$ are closed forms with cohomology class independent of the connection θ. The cohomology classes for $P_l^{(0)}, P_l^W$ are the components of the so-called *leading order Chern character* and the *Wodzicki-Chern character*. Using Newton's formulas, the Chern characters define Chern classes $c_l^{(0)}$, c_l^W as well as Pontrjagin classes for real bundles. Examples of nontrivial leading order Chern classes are given in [20], and examples of nontrivial Wodzicki-Chern classes are given in [21].

6.3. **Leading Order Chern-Weil and Chern-Simons Theory.** In this section, we show that Theorem 6.3 extends to the $\mathbb{R}/\mathbb{Z}$ secondary classes associated to the characteristic forms $P = P_l^{(0)}$ built from the leading order symbol of a connection on a $\mathcal{G}$-bundle. We also show that leading order Chern classes are essentially pullbacks of finite dimensional Chern classes, and hence contain limited new information.

For FLM, only the $L^2 = H^0$ Levi-Civita connection is a $\mathcal{G}$-connection. One easily checks that the L^2 connection one-form $\omega_\gamma^{(0)}(X)(\theta) = \omega_{\gamma(\theta)}^M(X_{\gamma(\theta)})$ on LM acts pointwise, as does the curvature two-form. Thus $P_l^{(0)}(\Omega^{(0)})_\gamma = \frac{1}{4\pi}\int_\gamma \text{tr}\, P_l(\Omega^M)\ d\theta$, so the theory of leading order characteristic forms is a straightforward generalization of the finite dimensional case. Considering constant loops, we see that secondary classes are only defined for flat metrics on M. In contrast, we will see in §6.4 that Wodzicki-Chern-Simons classes are defined for all metrics.

As we now explain, the case of gauge bundles over $Maps(N, M)$ which arise from finite rank bundles over M is similar. This class of bundles includes $TMaps(N, M)$. For most of this section, we assume that $\mathcal{G}$ is the gauge group of an oriented real bundle $E \longrightarrow M$, but the arguments carry over to e.g. hermitian bundles.

By [2], $B\mathcal{G} = C^\infty_{(0)}(M, BSO(n)) = \{f : M \longrightarrow BSO(n) | f^*ESO(n) \simeq E\}$. For N closed and connected, let $\mathrm{ev} : C^\infty(N, M) \times N \longrightarrow M$ be the evaluation map $\mathrm{ev}(f, n) = f(n)$. The bundle $\mathrm{ev}^* E$ determines an infinite rank bundle $\pi_* \mathrm{ev}^* E \longrightarrow C^\infty(N, M)$, where $\pi_* \mathrm{ev}^* E|_f = \Gamma(f^*E \longrightarrow N)$, with Γ denoting some Sobolev space of sections. (Here $\pi : C^\infty(N, M) \times N \longrightarrow C^\infty(N, M)$ is the projection.) For $n \in N$, define $\mathrm{ev}_n : C^\infty(N, M) \longrightarrow M$ by $\mathrm{ev}_n(f) = f(n)$.

It is well known that connections push down under π_*. For the gauge group case, this gives the following:

Lemma 6.7. *The universal bundle $E\mathcal{G} \longrightarrow B\mathcal{G}$ is isomorphic to $\pi_* \mathrm{ev}^* ESO(n)$. $E\mathcal{G}$ has a universal connection $\theta^{E\mathcal{G}}$ defined on $s \in \Gamma(E\mathcal{G})$ by*

$$(\theta^{E\mathcal{G}}_Z s)(\gamma)(\alpha) = \left((\mathrm{ev}^* \theta^u)_{(Z,0)} u_s\right)(\gamma, \alpha).$$

Here θ^u is the universal connection on $ESO(n) \longrightarrow BSO(n)$, and $u_s : C^\infty(N, M) \times N \longrightarrow \mathrm{ev}^* ESO(n)$ is defined by $u_s(f, n) = s(f)(n)$.

Proof. See [19, §4]. □

Corollary 6.8. *The curvature $\Omega^{E\mathcal{G}}$ of $\theta^{E\mathcal{G}}$ satisfies*

$$\Omega^{E\mathcal{G}}(Z, W)s(f)(n) = \mathrm{ev}^* \Omega^u((Z, 0), (W, 0))u_s(f, n).$$

Proof. This follows from

$$\Omega^{E\mathcal{G}}(Z, W)s(f)(n) = [\nabla^{E\mathcal{G}}_Z \nabla^{E\mathcal{G}}_W - \nabla^{E\mathcal{G}}_W \nabla^{E\mathcal{G}}_Z - \nabla^{E\mathcal{G}}_{[Z,W]}]s(f)(n)$$

and the previous lemma. □

Lemma 6.9. *Let $\mathcal{G}$ be the group of gauge transformations acting on sections of a finite rank bundle $E \longrightarrow M$. Then $P^{(0)}_l \in I^l_0(\mathcal{G})$.*

Proof. For all $n_0 \in N$, the maps ev_{n_0} are homotopic, so the cohomology class

$$\left[P^{(0)}_l(\mathrm{ev}^*_{n_0} \Omega^u)\right] \in H^{2k}(B\mathcal{G} \times \{n_0\}, \mathbb{R}) \cong H^{2k}(B\mathcal{G}, \mathbb{R})$$

is independent of n_0. Thus

$$\begin{aligned}
\left[\frac{d_l}{\mathrm{vol}\ S^*N} \int_{S^*N} \mathrm{tr}\ \sigma_0((\Omega^{E\mathcal{G}})^l) d\xi dx\right] &= \frac{d_l}{\mathrm{vol}\ S^*N} \int_{S^*N} \left[\mathrm{tr}\ \sigma_0\left((\mathrm{ev}^*_{n_0} \Omega^u)^l\right)\right] d\xi dx, \\
&= \left[d_l\, \mathrm{ev}^*_{n_0} \mathrm{tr}\ \sigma_0\left((\Omega^u)^l\right)\right], \qquad (6.8) \\
&= \mathrm{ev}^*_{n_0}\left[d_l\, \mathrm{tr}\left((\Omega^u)^l\right)\right],
\end{aligned}$$

since Ω^u is a multiplication operator. By the choice of d_l, the last term in (6.8) lies in $\mathrm{ev}_{n_0}^* H^{2l}(BSO(n),\mathbb{Z}) \subset H^{2l}(B\mathcal{G}),\mathbb{Z})$. Thus

$$W(P_l^{(0)}) = [P_l^{(0)}(\Omega^{E\mathcal{G}})] \in H^{2l}(B\mathcal{G},\mathbb{Z}).$$

□

Remark 6.3. Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a $\mathcal{G}$-bundle with connection, where $\mathcal{G}$ is the gauge group of the rank n hermitian bundle $E \longrightarrow N$, and let $f : \mathcal{B} \longrightarrow B\mathcal{G}$ be a geometric classifying map. The argument above shows that the l^{th} leading order Chern class equals $f^*\mathrm{ev}_{n_0}^* c_l(EU(n))$. Thus all leading order Chern classes are pullbacks of finite dimensional Chern classes, although the effect of $\mathrm{ev}_{n_0}^*$ may be difficult to compute. (This argument was developed with S. Paycha.)

As in [6], we have

Theorem 6.10. *Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a $\mathcal{G}$-bundle with connection θ and assume $P_l(\Omega) \equiv 0$. Let $\widetilde{TP(\theta)}$ be the mod $\mathbb{Z}$ reduction of $TP(\theta)$. Then there exists a cochain $U \in C^{2l-1}(\mathcal{B},\mathbb{R}/\mathbb{Z})$ such that*

$$\widetilde{TP(\theta)} = \pi^*(U) + \text{ coboundary}.$$

Proof. By Lemma 6.7, $E\mathcal{G} \longrightarrow B\mathcal{G}$ has a universal connection $\hat{\theta}$ (with curvature $\hat{\Omega}$). Thus there exists a geometric classifying map $\phi : \mathcal{B} \longrightarrow B\mathcal{G}$: i.e. $(\mathcal{E},\theta) \simeq (\phi^*E\mathcal{G},\phi^*\hat{\theta})$. By Lemma 6.9, $P \in I_0^l(\mathcal{G})$, so its mod $\mathbb{Z}$ reduction is zero. From the Bockstein sequence

$$\cdots \longrightarrow H^i(B\mathcal{G},\mathbb{Z}) \longrightarrow H^i(B\mathcal{G},\mathbb{R}) \xrightarrow{\text{mod } \mathbb{Z}} H^i(B\mathcal{G},\mathbb{R}/\mathbb{Z}) \longrightarrow H^{i+1}(B\mathcal{G},\mathbb{Z}) \longrightarrow \cdots$$

we deduce that $P(\hat{\Omega})$ represents an integral class in $B\mathcal{G}$. Thus $\widetilde{P(\hat{\Omega})}$ as a cochain vanishes on all cycles in $B\mathcal{G}$, and hence is an $\mathbb{R}/\mathbb{Z}$ coboundary, i.e. there exists $\bar{u} \in C^{2l-1}(B\mathcal{G},\mathbb{R}/\mathbb{Z})$ such that $\delta\bar{u} = \widetilde{P(\hat{\Omega})}$. We have

$$\delta\pi^*(\bar{u}) = \pi^*(\delta u) = \pi^*(\widetilde{P(\hat{\Omega})}) = d\widetilde{TP(\hat{\theta})} = \delta(\widetilde{TP(\hat{\theta})}).$$

The acyclicity of $E\mathcal{G}$ implies $\widetilde{TP(\hat{\theta})} = \pi^*(\bar{u}) + \text{coboundary}$. Now set $U = \phi^*(\bar{u})$. □

Definition 6.1. *Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a $\mathcal{G}$-bundle with connection θ and curvature Ω, and assume $P_l^{(0)}(\Omega) \equiv 0$. In the notation of Theorem 6.10, define the* Chern-Simons class *$CS_{2l-1}^{(0)}(\theta) \in H^{2l-1}(B,\mathbb{R}/\mathbb{Z})$ by*

$$CS_{2l-1}^{(0)}(\theta) = [U].$$

We can also define leading order Chern-Simons classes for $\Psi\mathrm{DO}_0^*$-bundles with connection. If $\Psi\mathrm{DO}_0^*$ acts on $E \longrightarrow N$, the top order symbol is a homomorphism $\sigma_0 : \Psi\mathrm{DO}_0^* \longrightarrow \mathcal{G}$, where $\mathcal{G}$ is the gauge group of $\pi^*E \longrightarrow S^*N$. A $\Psi\mathrm{DO}_0^*$-bundle $\mathcal{E}$ has an associated $\mathcal{G}$-bundle $\mathcal{E}'$ with transition functions $\sigma_0(A)$, for A a transition function of $\mathcal{E}$. A connection θ with curvature Ω on $\mathcal{E}$ gives rise to a connection $\theta' = \sigma_0(\theta)$ on $\mathcal{E}'$ with curvature $\sigma_0(\Omega)$. Since $P_l^{(0)}(\Omega) = P_l^{(0)}(\sigma_0(\Omega))$, we define $CS_{2l-1}^{(0)}(\theta) = CS_{2l-1}^{(0)}(\theta')$.

The homomorphism σ_0 may lose information from the original $\Psi\mathrm{DO}_0^*$-bundle. This indirect definition is forced on us, because we do not know if $E\Psi\mathrm{DO}_0^* \longrightarrow B\Psi\mathrm{DO}_0^*$ admits a universal connection.

This lack of a Narasimhan-Ramanan theorem prevents us from defining Chern-Simons classes on arbitrary $\Psi\mathrm{DO}_0^*$-bundles using the Wodzicki residue. In the next section, we will define Wodzicki-Chern-Simons class for FLM when M is stably parallelizable.

6.4. **Wodzicki-Chern-Simons Classes.** In this section, we extend the classical definition of Chern-Simons classes to P_l^W for trivial $\Psi\mathrm{DO}_0^*$-bundles. This allows us to define Wodzicki-Chern-Simons classes for loop spaces of stably parallelizable manifolds.

We use the construction of Corollary 6.5 to define secondary classes.

Definition 6.2. *Let $(\mathcal{E},\theta) \longrightarrow \mathcal{B}$ be a trivial $\Psi\mathrm{DO}_0^*$-bundles with connection θ, curvature Ω and global section $\chi : \mathcal{B} \longrightarrow \mathcal{E}$. Let P be an Ad-invariant degree l polynomial on $\Psi\mathrm{DO}_{\leq 0}$. Assume that $P(\Omega) \equiv 0$. The* Chern-Simons class $CS_{2l-1}^P(\theta,\chi) \in H^{2l-1}(\mathcal{B},\mathbb{R})$ *is*

$$CS_{2l-1}^P(\theta,\chi) = \chi^*\left[TP(\theta)\right].$$

For the case of a trivial frame bundle $FLM \longrightarrow LM$ for a Riemannian manifold (M,g) and $P = P_l^W$ in (6.6), the corresponding Chern-Simons class is denoted

$$CS_{2l-1}^W(\theta^s(g),\chi)$$

and is called the s^{th} Wodzicki-Chern-Simons (WCS) class *of LM with respect to g.*

Remark 6.4. In finite dimensions, the form $TP_{4l-3}(\theta)$ vanishes because the trace of the product of an odd number of skew-symmetric matrices is zero. Thus the usual indexing is $CS_l \in H^{4l-1}(M,\mathbb{R}/\mathbb{Z})$. On LM, the connection and curvature forms are skew-symmetric ΨDOs, but their symbols need not be skew-symmetric. Therefore, we have to allow for the existence of WCS classes in all odd dimensions.

For the rest of this section, we specialize to the frame bundle FLM.

Remark 6.5. (i) We have not taken the mod $\mathbb{Z}$ reduction as in finite dimensions, but the *a priori* dependence of the WCS class on χ will be removed in Proposition 6.13.

(ii) As in Remark 3.4, we can always extend the structure group to $GL(\mathcal{H})$ so that a global section χ exists whether or not the original $\Psi\mathrm{DO}_0^*$ bundle is trivial. This yields a general definition of a Wodzicki-Chern-Simons form, but with a possible loss of information.

Theorem 6.11. *On LM, $P_\ell^W(\Omega^s) \equiv 0$ for all $\ell \geq 2$. Thus WCS classes are defined whenever LM is parallelizable.*

PROOF: This follows immediately from Theorem 4.4. □

Lemma 6.12. *If M is parallelizable, then LM is parallelizable.*

Proof. Let $\phi : TM \longrightarrow M \times \mathbb{R}^n$ be a trivialization of TM. For $X_\gamma \in T_\gamma LM = \Gamma(\gamma^* TM)$, define

$$\Psi : TLM \longrightarrow LM \times \Gamma(S^1 \times \mathbb{R}^n \longrightarrow S^1)$$
$$X_\gamma \longmapsto (\gamma, \alpha \mapsto \pi_2(\phi(X_\gamma(\alpha)))),$$

where $\pi_2 : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the projection. It is easy to check that α is a smooth trivialization of TLM in the H^s norm. □

The proof extends to $Maps(N, M)$.

To investigate the dependence of the WCS class on the frame, we recall the Cartan homotopy formula [17, 26]. For $A_0, A_1 \in \Lambda^1(M, \mathfrak{g})$ for a manifold M and a Lie algebra $\mathfrak{g}$, set $A_t = A_0 + t(A_1 - A_0), \Omega_t = dA_t + A_t \wedge A_t$. Define l_t from the algebra F generated by the symbols A_t, Ω_t to $\Lambda^*(M \times [0,1], \mathfrak{g})$ by $l_t A_t = 0,\ l_t \Omega_t = (A_1 - A_0)dt$, with l_t extended as a signed derivation to F. For a polynomial $S(A, \Omega)$, the Cartan homotopy formula is

$$S(A_1, \Omega_1) - S(A_0, \Omega_0) = (dk + kd)S(A_t, \Omega_t), \tag{6.9}$$

where

$$kS(A_t, \Omega_t) = \int_0^1 l_t S(A_t, \Omega_t).$$

This formalism implies $kP(\Omega_t) = TP(A_0)$ on the total space of a bundle $E \longrightarrow M$. In fact, the Cartan homotopy formula is just the standard Cartan formula [23, Ch. 8, Vol. I] applied to polynomials of A, Ω.

A frame $\chi : M \longrightarrow FM$ determines a "loopified" frame $L\chi : LM \longrightarrow FLM$ by $L\chi(\gamma)(\theta) = \chi(\gamma(\theta))$. Denote $L\chi$ just by χ.

Proposition 6.13. *On a parallelizable manifold M, the WCS class $CS_{2l-1}^W(\theta, \chi) \in H^{2l-1}(LM, \mathbb{R})$ is independent of the choice of loopified frame $\chi : LM \longrightarrow FLM$.*

Proof. Consider loopified frames $\chi_1, \chi_0 : LM \longrightarrow FLM$. The pullbacks of the connection θ on FLM are related by

$$\chi_1^*(\theta) = g^{-1}\chi_0^*(\theta)g + g^{-1}dg,\ \ g(\gamma) : \chi_0(\gamma) \mapsto \chi_1(\gamma),$$

where g is the loopified gauge transformation taking $\chi_0(m)$ to $\chi_1(m)$. For the family $A_t = tg^{-1}\chi_0^*(\theta)g + g^{-1}dg$ and $S(A,\Omega) = TP(A,\Omega) = \chi^*TP_l^W(\theta)$, (6.9) yields

$$TP(A_1) - TP(g^{-1}dg) = (dk + kd)TP(A_t) = d\alpha + kP(\Omega_t) = d\alpha + TP(A_0),$$

with $\alpha = kTP(A_t)$. Hence,

$$CS^W_{2l-1}(\theta,\chi_1) - CS^W_{2l-1}(\theta,\chi_0) = [TP(A_1)] - [TP(A_0)] = [TP(g^{-1}dg)].$$

Since the gauge transformation g is a multiplication operator on TLM, the Wodzicki residues of the connection $g^{-1}dg$ and its curvature vanish. Thus $TP(g^{-1}dg) = 0$. □

Remark 6.6. (i) For a principal bundle with compact structure group, $[TP(g^{-1}dg)]$ is an integer class, called the instanton number in [10]. This shows that the $\mathbb{R}/\mathbb{Z}$ reduction of $\chi^*TP(\theta)$ is frame independent. A more topological proof, valid for compact structure groups only, is in [6, (6.2)].

(ii) Assume the mod $\mathbb{Z}$ reduction of $CS_2^W(\theta) = CS_2^W(\theta,\chi) \in H^3(LM,\mathbb{R})$ vanishes for a loopified frame on LM for M parallelizable. The Bockstein sequence gives a (non-unique) class in $\alpha \in H^3(LM,\mathbb{Z})$ mapping onto $CS_2^W(\theta)$. α has a representative, a gerbe with connection, whose curvature is $\chi^*TP(\theta)$ [14]. Analogously, for finite dimensional parallelizable manifolds, there is a gerbe associated to a vanishing three dimensional Chern-Simons class. This gerbe functions as a tertiary characteristic class associated to a connection and a framing.

The definitions and results of this section extend to stably parallelizable manifolds such as S^n, i.e. manifolds M with $TM \oplus \varepsilon^k = \varepsilon^r$ for trivial bundles $\varepsilon^k, \varepsilon^r$. We have $TM \oplus \varepsilon^k = T(M \times \mathbb{R}^k)$. For a Riemannian metric g on M and the standard metric g_0 on $\mathbb{R}^k$, put the product metric $\tilde{g} = g \oplus g_0$ on $M \times \mathbb{R}^k$. By Lemma 6.12, $FL(M \times \mathbb{R}^k)$ has a global section $\tilde{\chi}$.

Definition 6.3. *Let (M,g) be a stably parallelizable Riemannian manifold. The s^{th}* Wodzicki-Chern-Simons (WCS) class *of LM with respect to g is $CS^W_{2l-1}(\theta^s(\tilde{g}),\tilde{\chi}) \in H^{2l-1}(LM,\mathbb{R})$.*

Here we work with $\mathbb{R}^k$ with its standard chart to define $(1+\Delta)^s$ on $M \times \mathbb{R}^k$ and the H^s metrics on $M \times \mathbb{R}^k$. The definition uses that $L(M \times \mathbb{R}^k)$ is diffeomorphic to $LM \times L\mathbb{R}^k$, so $H^*(LM) = H^*(L(M \times \mathbb{R}^k))$ by the de Rham theorem for loop spaces [4]. There is an ambiguity in the definition, in that one can replace ε^k by ε^s for $s > k$. However, taking the standard frame on $\mathbb{R}^{s-k}$ and its loopification χ^{s-k}, one easily checks that the $CS^W_{2l-1}(\theta^s(g \oplus g_0^s), \tilde{\chi}^k \times \chi^{s-k})$ is independent of s.

For M stably parallelizable, by Remark 2.2(ii) we can take a global frame so that Lemma 2.3 applies, which allows us to use the symbol calculations of §4. By Lemma 4.1, Lemma 4.2 and Theorem 4.4, the Wodzicki residue of a wedge of connection and curvature forms is a constant multiple of $s > 3/2$. Therefore, the WCS class also depends linearly on s. This motivates the following definition:

Definition 6.4. *The* regularized WCS class $CS^{\rm reg}_{2l-1}(\theta,\chi)$ *is*

$$s^{-1}CS^W_{2l-1}(\theta^s,\chi).$$

The regularized WCS class captures the s-independent part of this theory. It differs *a priori* from a direct definition of the WCS class at $s=1$, as the symbol calculations are not valid at this parameter.

6.5. **Vanishing Results for Wodzicki-Chern Classes.** The tangent space TLM to a loop space fits into the framework of the Families Index Theorem. In this section, we show that the infinite rank bundles appearing in this framework have vanishing Wodzicki-Chern classes, generalizing [16]. For loop spaces, this follows from Theorem 4.4. This vanishing indicates that WCS classes could be interesting objects in the more general Families Index Theorem setup.

Recall this setup: there is a fibration $Z \longrightarrow M \stackrel{\pi}{\longrightarrow} B$ of closed manifolds and a finite rank bundle $E\longrightarrow M$, inducing an infinite rank bundle $\mathcal{E}=\pi_*E\longrightarrow B$. For the fibration $N\longrightarrow N\times Maps(N,M)\longrightarrow Maps(N,M)$ and $E={\rm ev}^*TM$, $\mathcal{E}$ is $TMaps(N,M)$.

Theorem 6.14. *If* $\mathcal{E}\longrightarrow\mathcal{B}$ *satisfies* $\mathcal{E}=\pi_*E$ *as above, then the Wodzicki-Chern classes* $c^W_k(\mathcal{E})$ *vanish for all* k.

Proof. As in Lemma 6.7, $\mathcal{E}$ admits a connection whose curvature Ω is a multiplication operator. Ω^l is also a multiplication operator, and hence $c^W_k(\Omega)\equiv 0$. □

For a real infinite rank bundle, Wodzicki-Pontrjagin classes are defined as in finite dimensions: $p^W_k(\mathcal{E})=(-1)^kc^W_{2k}(\mathcal{E}\otimes\mathbb{C})$.

Corollary 6.15. *The Wodzicki-Pontrjagin classes of* $TMaps(N,M)$ *and of all naturally associated bundles vanish.*

Proof. Pick an element f_0 in a fixed path component A_0 of $Maps(N,M)$. For $f\in A_0$, $T_fMaps(N,M)\simeq\Gamma(f^*TM\longrightarrow N)\simeq\Gamma(f_0^*TM\longrightarrow N)$ with the second isomorphism noncanonical. Thus over each component, $TMaps(N,M)$ is of the form $\pi_*\,{\rm ev}^*\,TM$, and the previous Theorem applies. The vanishing of the Wodzicki-Pontrjagin classes of associated bundles (such as exterior powers of the tangent bundle) follows as in finite dimensions, since there is a universal geometric bundle. □

In general, the Wodzicki-Chern classes are an obstruction to the reduction of the structure group of a $\Psi{\rm DO}^*_0$ bundle.

Proposition 6.16. *Let* $\mathcal{E}\longrightarrow\mathcal{B}$ *be an infinite rank* $\Psi{\rm DO}_0^*$*-bundle, for* $\Psi{\rm DO}_0^*$ *acting on* $E\longrightarrow N^n$. *If* $\mathcal{E}$ *admits a reduction to the structure group* $\mathcal{G}(E)$, *then* $c^W_k(\mathcal{E})=0$ *for all* k. *If* $\mathcal{E}$ *admits a* $\Psi{\rm DO}_0^*$*-connection whose curvature has order* $-k$, *then* $0= c_{[n/k]}(\mathcal{E})=c_{[n/k]+1}(\mathcal{E})=\dots$

PROOF: For such an $\mathcal{E}$, there exists a connection one-form with values in $Lie(\mathcal{G}) = Hom(E)$, the Lie algebra of multiplication operators. Thus the Wodzicki residue of powers of the curvature vanishes. For the second statement, the powers of the curvature of the connection has $ord(\Omega^\ell) < -n$ for $\ell \geq [n/k]$, and so the Wodzicki residue vanishes in this range. □

7. Invariance of Wodzicki-Chern-Simons Classes

In this section we show that the WCS classes $CS^W_{2k-1} \in H^{2k-1}(LM, \mathbb{R}), k \geq 3$, are smooth invariants on loop spaces of parallelizable manifolds, and give results for CS^W_1, CS^W_3 related to conformal geometry. In contrast, Chern-Simons classes are conformal invariants in finite dimensions.

7.1. Conformal Invariance in Finite Dimensions.

We begin with a discussion of the finite dimensional case $TM \longrightarrow M$. We avoid using the Narasimhan-Ramanan universal bundle theorem, as these techniques do not carry over to infinite dimensions.

We recall results from [6] about the conformal geometry of Chern-Simons classes. Let $\hat{g} = e^{2f}g$ be a conformal change of the metric g on M, and pick a g-orthonormal frame χ of TM. For $\chi = (e_1, \ldots, e_n)$, $A = A^g = \chi^*\theta^g$ is a global one-form defined by $\nabla^g_{e_i} e_j = A^k_j(e_i)e_k$, where θ^g, ∇^g are the Levi-Civita connection one-form on FM and the Levi-Civita connection on TM for g, respectively. $\hat{\chi} = (e^{-f}e_1, \ldots, e^{-f}e_n)$ is $\hat{g}$-orthonormal. Let Ω_t be the curvature of $\nabla_t = (1-t)\nabla^g + t\nabla^{\hat{g}}$, the Levi-Civita connection for the metric $g_t = e^{2tf}g$.

The invariant polynomials are generated by $P(A) = \mathrm{Tr}(A^\ell)$ for ℓ even. [6, Thm. 4.5] shows that $TP(\hat{\theta}) - TP(\theta)$ is exact on FM, so $TP(\hat{\chi}^*\hat{\theta}) - TP(\hat{\chi}^*\theta)$ is exact on M. Thus

$$\begin{aligned} TP(\hat{\chi}^*\hat{\theta}) - TP(\chi^*\theta) &= [TP(\hat{\chi}^*\hat{\theta}) - TP(\hat{\chi}^*\theta)] + [TP(\hat{\chi}^*\theta) - TP(\chi^*\theta)] \\ &= \text{exact } + [TP(\hat{\chi}^*\theta) - TP(\chi^*\theta)]. \end{aligned} \tag{7.1}$$

Since the last term is an integral class [6, (6.2)], $[TP(A)] \in H^3(M; \mathbb{R}/\mathbb{Z})$ is conformally invariant whenever $P(\Omega) \equiv 0$, a conformally invariant condition.

For M parallelizable, we have a slightly stronger result with real coefficients.

Proposition 7.1. *Let M be parallelizable. For the Levi-Civita connections A_0, A_1 of two conformally related metrics and for $P(\Omega) = \mathrm{Tr}(\Omega^\ell)$, $TP(A_1) - TP(A_0)$ is exact. Thus, if $P(\Omega_0) \equiv 0$, then the class $[TP(A_0)] \in H^{2\ell-1}(M, \mathbb{R})$ is a conformal invariant.*

PROOF: Let $A_t = \chi_t^*\theta_t$ be the connection one-form for ∇_t in the χ_t frame. For the projection $\pi : M' = M \times [0,1] \longrightarrow M$, the bundle $\pi^*TM \longrightarrow M'$ is trivial via the frame χ_t over $M_t = M \times \{t\}$, and comes with a connection $\nabla = d + A = d_{M'} + A_t$ with respect to this trivialization. The curvature Ω of ∇ satisfies $\Omega|_{M_t} = \Omega_t + \partial_t A_t dt$.

For $i_t : M \longrightarrow M', i_t(m) = (m,t)$, the Cartan homotopy formula gives

$$\begin{aligned} TP(A_1) - TP(A_0) &= i_1^*TP(A) - i_0^*TP(A) = d_M\mathcal{I}TP(A) + \mathcal{I}d_{M'}TP(A) \\ &= \text{exact } + \mathcal{I}d_{M'}TP(A), \end{aligned} \tag{7.2}$$

where $\mathcal{I}\omega = -\int_0^1 (-1)^{\deg(\omega)-1} \imath_{\partial_t}\omega\ dt$.

We claim the last term in (7.2) vanishes. We have

$$\mathcal{I}d_{M'}TP(A) = \mathcal{I}P(\Omega) = \ell\int_0^1 \mathrm{Tr}(\Omega_t^{\ell-1}\wedge\partial_t A_t)dt. \tag{7.3}$$

It follows from the definition of A_j^k that

$$(\partial_t A_t)_j^k = (e_j(f)\delta_i^k - e_k(f)\delta_i^j)e^i = (e_j(f)\delta_i^k - \delta^{mk}e_m(f)\delta_j^\ell\delta_{i\ell})e^i, \tag{7.4}$$

where $\{e^i\}$ are the dual one-forms. Fix t, and use tildes to denote e.g. $(\Omega_t)_j^i = \tilde{R}^i{}_{jk\ell}\tilde{e}^k\wedge\tilde{e}^\ell, \tilde{e}^k = e^{-f}e^k$. Then

$$\begin{aligned} &\mathrm{Tr}(\tilde{\Omega}^{\ell-1}\wedge(e_j(f)\delta_i^k - \delta^{mk}e_m(f)\delta_j^\ell\delta_{i\ell})e^i) \\ &= \tilde{R}^{i_1}{}_{j_1k_1r_1}\tilde{e}^{k_1}\tilde{e}^{r_1}\wedge\tilde{R}^{j_1}{}_{j_2k_2r_2}\tilde{e}^{k_2}\tilde{e}^{r_2}\wedge\ldots \\ &\qquad \wedge\tilde{R}^{j_{\ell-2}}{}_{j_{\ell-1}k_{\ell-1}r_{\ell-1}}\tilde{e}^{k_1}\tilde{e}^{r_1}\wedge(e_{i_1}(f)\delta_i^{j_{\ell-1}} - \delta^{m,j_{\ell-1}}e_m(f)\delta_{i_1}^\ell\delta_{i\ell})e^{-f}\tilde{e}^i \\ &= \tilde{R}^{j_{\ell-2}}{}_{i,k_{\ell-1},r_{\ell-1}}\tilde{e}^{k_{\ell-1}}\tilde{\wedge}e^{r_{\ell-1}}\wedge\tilde{e}^i\wedge\Big[e_{i_1}(f)e^{-f}\tilde{R}^{i_1}{}_{j_1k_1r_1}\tilde{e}^{k_1}\tilde{e}^{r_1}\wedge\tilde{R}^{j_1}{}_{j_2k_2r_2}\tilde{e}^{k_2}\tilde{e}^{r_2}\wedge\ldots \\ &\qquad \wedge\tilde{R}^{j_{\ell-3}}{}_{i,k_{\ell-2},r_{\ell-2}}\tilde{e}^{k_{\ell-2}}\tilde{e}^{k_{\ell-2}}\wedge e^{r_{\ell-2}}\Big] \\ &\qquad +\tilde{R}_{j_1ik_1r_1}\tilde{e}^{k_1}\wedge\tilde{e}^{r_{\ell-1}}\wedge\tilde{e}^i\Big[e^{-f}\tilde{R}^{j_1}{}_{j_2k_2r_2}\tilde{e}^{k_2}\tilde{e}^{r_2}\wedge\ldots\wedge\tilde{R}^{j_{\ell-2}}{}_{j_{\ell-1}k_{\ell-1}r_{\ell-1}}\tilde{e}^{k_1}\tilde{e}^{r_1}\Big] \\ &= 0, \end{aligned}$$

because $\tilde{R}^i{}_{jk\ell}\tilde{e}^j\wedge\tilde{e}^k\wedge\tilde{e}^\ell = 0$ by the Bianchi identity. This vanishing and (7.3), (7.4) show that the last term in (7.2) is zero. □

The proof carries over to stably parallelizable manifolds with product metric as in Definition 6.3.

7.2. **Metric Invariance in Infinite Dimensions.** We modify the previous Proposition to show that on LM the WCS classes $CS_{2\ell-1}^W$, $\ell\geq 3$, are independent of the metric on M.

As in (6.7), we set

$$TP_\ell^W(\theta) = \ell\int_0^1\int_{S^*S^1}\mathrm{tr}\ \sigma_{-1}(P_\ell(\theta\wedge\phi_t^{l-1}))dt,$$

for P_ℓ the polarization of $\mathrm{Tr}(A^\ell)$.

Theorem 7.2. *Let M be stably parallelizable with global section χ of $F(M \times \mathbb{R}^k)$ for some k. Let $\theta = \theta^s$ be the Levi-Civita one-form for the H^s metric on $L(M \times \mathbb{R}^k)$ associated to a metric on M. For $\ell \geq 3$ and for fixed s, $CS^W_{2\ell-1}(\theta^s) = [\chi^* TP_\ell(\theta^s)] \in H^{2\ell-1}(LM, \mathbb{R})$ is independent of the metric on M. In particular, the regularized WCS class is independent of the metric.*

PROOF: By Proposition 6.13, $CS^W_{2\ell-1}$ is independent of the frame χ. Let $g, \hat{g}$ be metrics on M. For notational simplicity, we assume that M is parallelizable. We need to show that $TP^W_\ell(\chi^*\hat{\theta}) - TP^W_\ell(\chi^*\theta)$ is exact. By the proof of Proposition 7.1 (with $\chi_t = \chi$ for all t), specifically by (7.2), (7.3), it suffices to show $\mathcal{I}d_{M'}TP^W(A) = 0$. This is equivalent to

$$0 = \operatorname{tr} \sigma_{-1}(\Omega_t^{\ell-1} \wedge \partial_t A_t) = \operatorname{tr}[\sigma_{-1}((\Omega_t)^{\ell-1}) \wedge \sigma_0(\partial_t A_t)], \tag{7.5}$$

for A_t the connection one-forms for the line of metrics joining $g, \hat{g}$. By Theorem 4.3, $\sigma_{-1}((\Omega_t)^{\ell-1}) = 0$ automatically for $\ell > 2$. Here A_t can be the connection one-forms for any family of metrics on M. □

For $\ell = 2$, we know that $\mathcal{I}d_{M'}TP(A)$ in (7.2) is exact for conformal families of metrics. Since $\sigma_0(\partial_t A_t) = \partial_t \sigma_0(A_t)$, reworking (7.4) in local coordinates gives

$$\sigma_0(\partial_t A_t)^k_j = \delta^k_j \partial_i f.$$

By (4.11),

$$\begin{aligned}
\operatorname{tr}[\sigma_{-1}(\Omega_t) \wedge \sigma_0(\partial_t A_t)] &= \dot{\gamma}^r \partial_\ell f ({R^j}_{rjq;p} - {R^j}_{rjp;q}) dx^p \wedge dx^q \wedge dx^\ell \\
&= \dot{\gamma}^r (\mathrm{Ric}_{rq;p} - \mathrm{Ric}_{rp;q}) dx^p \wedge dx^q \wedge df \\
&= 2\dot{\gamma}^r \mathrm{Ric}_{rq;p} dx^p \wedge dx^q \wedge df \\
&= 2\imath_{\dot{\gamma}} \nabla \mathrm{Ric} \wedge df,
\end{aligned}$$

where we drop the dependence on t in the notation.

If the conformal family of metrics are all Einstein, then $\nabla\mathrm{Ric} = \lambda \nabla g = 0$. This gives a conformally invariant class:

Proposition 7.3. *Let $g_t, t \in [0, 1]$ be a conformal family of Einstein metrics on a stably parallelizable manifold. Then $CS^W_3(\theta_0) = CS^W_3(\theta_1) \in H^3(LM, \mathbb{R})$.*

Finally, for $l = 1$ and $P^W_1(A) = \int_{S^*S^1} \operatorname{tr}\ \sigma_{-1}(A)$, the form $\chi^* TP^W_1(\theta) \in \Lambda^1(LM)$ has

$$d\chi^* TP^W_1(\theta) = dTP^W_1(\chi^*\theta) = \int_{S^*S^1} \operatorname{tr}\ \sigma_{-1}(\Omega).$$

By Theorem 4.4, the last expression vanishes on locally symmetric or Einstein spaces.

Proposition 7.4. *The form $\chi^* TP^W_1(\theta)$ for $P^W_1(A) = \int_{S^*S^1} \operatorname{tr}\ \sigma_{-1}(A)$ defines a class $CS^W_1(\theta) \in H^1(LM, \mathbb{R})$ for locally symmetric spaces and for Einstein metrics.*

8. **Nontrivial Wodzicki-Chern-Simons Classes**

In this section, we find metrics on $M = SO(3) \times S^1$ for which the WCS class $CS_3^W(\theta^s) \in H^3(LM, \mathbb{R})$ is nontrivial.

Since de Rham's theorem is valid on LM [4], it suffices to find $[a] \in H_3(LM; \mathbb{R})$ with

$$\langle \chi^* TP(\theta^s), [a] \rangle \neq 0, \tag{8.1}$$

for $P(\Omega) = P_2(\Omega) = \frac{d_2}{(2\pi)^2}\,\mathrm{Tr}(\Omega^2)$ and any frame χ of LM.

We first calculate the homology of $L(SO(3) \times S^1)$. For $\gamma_1 \in \Omega SO(3)$, the based loops on $SO(3)$, $\eta \in SO(3)$, $\gamma_2 \in \Omega S^1$, and $e^{2\pi i t_0} \in S^1$ define

$$\begin{aligned} \phi : \Omega SO(3) \times SO(3) \times \Omega S^1 \times S^1 &\longrightarrow L(SO(3) \times S^1) \\ (\gamma_1, \eta, \gamma_2, e^{2\pi i t_0}) &\mapsto (\eta\gamma_1, e^{2\pi i t_0}\gamma_2). \end{aligned} \tag{8.2}$$

Since LG is diffeomorphic to $\Omega G \times G$ for Lie groups, ϕ is a diffeomorphism.

$\Omega SO(3)$ has two diffeomorphic path components, and by [5], $H_2(\Omega SO(3), \mathbb{Z})$ has two generators τ_1, τ_2. ΩS^1 is homotopy equivalent to $\mathbb{Z}$, since based loops are homotopic iff they have the same winding number. This gives:

Lemma 8.1. *The generators for $H_3(LM; \mathbb{Z}) \simeq H_3(\Omega SO(3) \times SO(3) \times \Omega S^1 \times S^1; \mathbb{Z})$ are*

$$\begin{aligned} \omega_0^{(n)} &= 1 \otimes [SO(3)] \otimes [n] \otimes 1,\ \omega_1^{(n)} = \tau_1 \otimes 1 \otimes 1 \otimes [n] \otimes [S^1], \\ \omega_2^{(n)} &= \tau_2 \otimes 1 \otimes 1 \otimes [n] \otimes [S^1], \end{aligned} \tag{8.3}$$

where $[\ \cdot\ ]$ denotes the fundamental class.

Define $z_n : SO(3) \longrightarrow \Omega SO(3) \times SO(3) \times \Omega S^1 \times S^1$ by

$$z_n(p) = (t \mapsto Id_{SO(3)}, p, e^{2\pi t} \mapsto e^{2\pi i n t}, Id_{S^1}) \equiv (\gamma_0, p, \gamma_1, Id_{S^1}), \tag{8.4}$$

where $Id_{SO(3)}, Id_{S^1}$ denote the identity maps on $\mathbb{R}^3$, $\mathbb{R}$. By the Lemma, $[a] = (\phi \circ z_n)_*[SO(3)] \in H_3(L(SO(3) \times S^1), \mathbb{Z})$ is nontrivial. It is represented by the 3-cycle which assigns to each point $A \in SO(3)$ the loop which goes n times around $\{A\} \times S^1$. Thus $CS_3^W(\theta^s)$ is nonzero if

$$\langle \chi^* TP(\theta), (\phi \circ z_n)_*[SO(3)] \rangle = \int_{SO(3)} (\phi \circ z_n)^* \chi^* TP(\theta) \neq 0. \tag{8.5}$$

To begin the computation, we fix exponential coordinates (x, y, z) in $T_{np}S^3$, for $np = (0, 0, 0, 1)$. of S^3. ("Exponential" refers to the standard metric, but we will use these coordinates for all metrics below.) Let $\pi : S^3 \longrightarrow SO(3)$ be the double cover map, and let S^2 be the standard equator in S^3. The exponential coordinates are valid on $SO(3) \setminus \pi(S^2)$, i.e. for $x^2+y^2+z^2 \leq (\pi/2)^2$. In these coordinates, we take a metric on $SO(3)\times S^1$ of the form

$$g = \begin{pmatrix} F(x,y,z,\alpha) & 0 & 0 & 0 \\ 0 & G(x,y,z,\alpha) & 0 & 0 \\ 0 & 0 & H(x,y,z,\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{8.6}$$

where F, G, and H are smooth functions of (x,y,z) and α is the angular variable on S^1. The metric extends to a metric on all of $SO(3)$ if for all (x,y,z) with $x^2+y^2+z^2 = (\pi/2)^2$,

$$\partial^k_{i_1\ldots i_k} F(x,y,z,\alpha) = (-1)^k \partial^k_{i_1\ldots i_k} F(-x,-y,-z,\alpha), \tag{8.7}$$

where the partial derivatives are in the (x,y,z) coordinates, and similarly for G, H.

For $l=2$, (6.7) gives

$$\begin{aligned} \chi^*TP(\theta) &= 2\int_0^1 P_2^W\left(\chi^*\theta\wedge t\chi^*\Omega + \frac{1}{2}(t^2-t)[\chi^*\theta,\chi^*\theta]\right) dt \\ &= P_2^W(\chi^*\theta\wedge\chi^*\Omega) - \frac{1}{6}P_2^W(\chi^*\theta\wedge\chi^*\theta\wedge\chi^*\theta) \\ &= -\frac{i}{8\pi^3}\int_{S^*S^1} \mathrm{Tr}\left[\sigma_{-1}(\chi^*\theta\wedge\chi^*\Omega)\right] d\xi d\alpha \\ &\quad +\frac{i}{48\pi^3}\int_{S^*S^1} \mathrm{Tr}\left[\sigma_{-1}(\chi^*\theta\wedge\chi^*\theta\wedge\chi^*\theta)\right] d\xi d\alpha, \end{aligned} \tag{8.8}$$

where $\theta=\theta^s, \Omega=\Omega^s$. By the symbol calculus for ΨDOs and Theorem 4.3, we have

$$\mathrm{Tr}\left[\sigma_{-1}(\chi^*\theta\wedge\chi^*\theta\wedge\chi^*\theta)\right] = 3\,\mathrm{Tr}\left[\sigma_{-1}(\chi^*\theta)\wedge\sigma_0(\chi^*\theta)\wedge\sigma_0(\chi^*\theta)\right], \tag{8.9}$$

$$\mathrm{Tr}\left[\sigma_{-1}(\chi^*\theta\wedge\chi^*\Omega)\right] = \mathrm{Tr}\left[\sigma_0(\chi^*\theta)\wedge\sigma_{-1}(\chi^*\Omega)\right]. \tag{8.10}$$

As in §3.4, we may replace $\chi^*\theta$ by ω^s and $\chi^*\Omega$ by Ω^s.

First, we show that the contribution to (8.5) from (8.9) vanishes. The left hand side of (8.8) is a global three-form on LM, as is the next to last term, the Wodzicki residue of the ΨDO $\omega^s\wedge\Omega^s$. Therefore, the last term of (8.8) (i.e. the integral of (8.9)) can be computed in any coordinates, viz. the "symbol friendly" coordinates and frame in Lemmas 2.3, 2.4. (Take γ_0 in the proof of Lemma 2.3 to be the loop in a based at np. The chart consisting of exponential coordinates on $SO(3)$ and the standard exponential map on S^1 covers all the loops in the cycle a except for a set of measure zero, so this chart suffices to compute (8.5). Strictly speaking, the exponential map on S^1 is only a chart map to S^1 minus a point, but the chart computations extend to the point.)

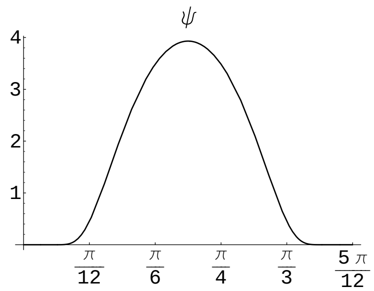

FIGURE 1. The bump function used in the computations.

Computing $\sigma_0(\omega^s)$ in (8.9) for the metric (8.6), we get that $\sigma_0(\omega^s)$ is the diagonal matrix

$$\begin{pmatrix} \frac{1}{F}X^i\partial_i F & 0 & 0 & 0 \\ 0 & \frac{1}{G}X^i\partial_i G & 0 & 0 \\ 0 & 0 & \frac{1}{H}X^i\partial_i H & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad i = 1, \dots, 4. \tag{8.11}$$

Since two of the terms in (8.9) are this diagonal matrix, it follows easily that (8.9) vanishes on each loop in the cycle a.

This leaves the contribution of (8.10) to (8.8). Since this term is the top order symbol of $\omega^s \wedge \Omega^s$, it can be computed in symbol friendly coordinates. By Theorem 4.4, the -1 order symbol of the curvature is

$$(\sigma_{-1}[\Omega(X,Y)])^a_e = is\xi^{-1}X^pY^q\partial_\mu\gamma^t \left(R^a{}_{teq;p} - R^a{}_{tep;q}\right).$$

This symbol has a fairly complicated explicit expression for the metric (8.6). To simplify the computations, we choose the following form for the diagonal elements:

$$\begin{aligned} F(\rho,\theta,\phi,\alpha) &= \lambda(\alpha)\left((\pi/2)^3 + xyx\psi(x)\psi(y)\psi(z)\right), \\ G(\rho,\theta,\phi,\alpha) &= \mu(\alpha), \\ H(\rho,\theta,\phi,\alpha) &= \left(\pi/2 + \psi(x)\psi(y)\psi(z)\right), \end{aligned} \tag{8.12}$$

where $\lambda(\alpha), \mu(\alpha)$ are positive smooth periodic functions of $\alpha \in [0, 2\pi]$, and $\psi(t)$ is the smooth bump function on the interval $[0, 5\pi/12]$ shown in Figure 1 and given explicitly by

$$\psi(t) = 10\exp\left(-\frac{1}{5t^2}\right)\exp\left(-\frac{1}{5(t-\frac{5\pi}{12})^2}\right).$$

In particular, F, G, H obey condition (8.7). With the help of Mathematica to compute (8.10) and its integral in (8.5), one finds that (8.5) is a product of two integrals:

$\int_{SO(3)} g(x,y,z)$ and $\int_{S^1} f^{(n)}(\alpha)$. Here

$$f^{(n)}(\alpha) = n\sqrt{\frac{\mu(\alpha)}{\lambda(\alpha)}}\lambda'(\alpha), \tag{8.13}$$

and $g(x,y,z)$ has a long explicit expression in F, G, H which can be loaded as the file FiLoTrace1 in the notebook `http://math.bu.edu/people/sr/notebook.nb`. One can gain some sense of $g(x,y,z)$ by computing a few slices (Figure 2).

The presence of a bump function in the metric rules out a closed form expression for $\int_{SO(3)} g(x,y,z)$. Within our computing limits, a numerical integration gives $\int_{SO(3)} g(x,y,z) = -3.89289$ for the metric given by (8.12). (The Mathematica file listed above includes this computation.) The key point is that the $SO(3)$ integral is nonzero.

Thus CS_3^W is nonzero if $\int_{S^1} f^{(n)}(\alpha)$ is nonzero. Figure 3 shows plots of (8.13) for different choices of $\lambda(\alpha), \mu(\alpha)$. The number above each of the plots in the third column is a numerical computation of $\int_{S^1} f^{(1)}(\alpha)$, and $\int_{S^1} f^{(n)}(\alpha)$ just multiplies this number by n. Since these values are nonzero up to a high degree of precision, for these (and many other) choices of λ, μ, ν,

$$CS_3^W(\theta) \in H^3(L(SO(3)\times S^1), \mathbb{R})$$

is almost certainly nontrivial.

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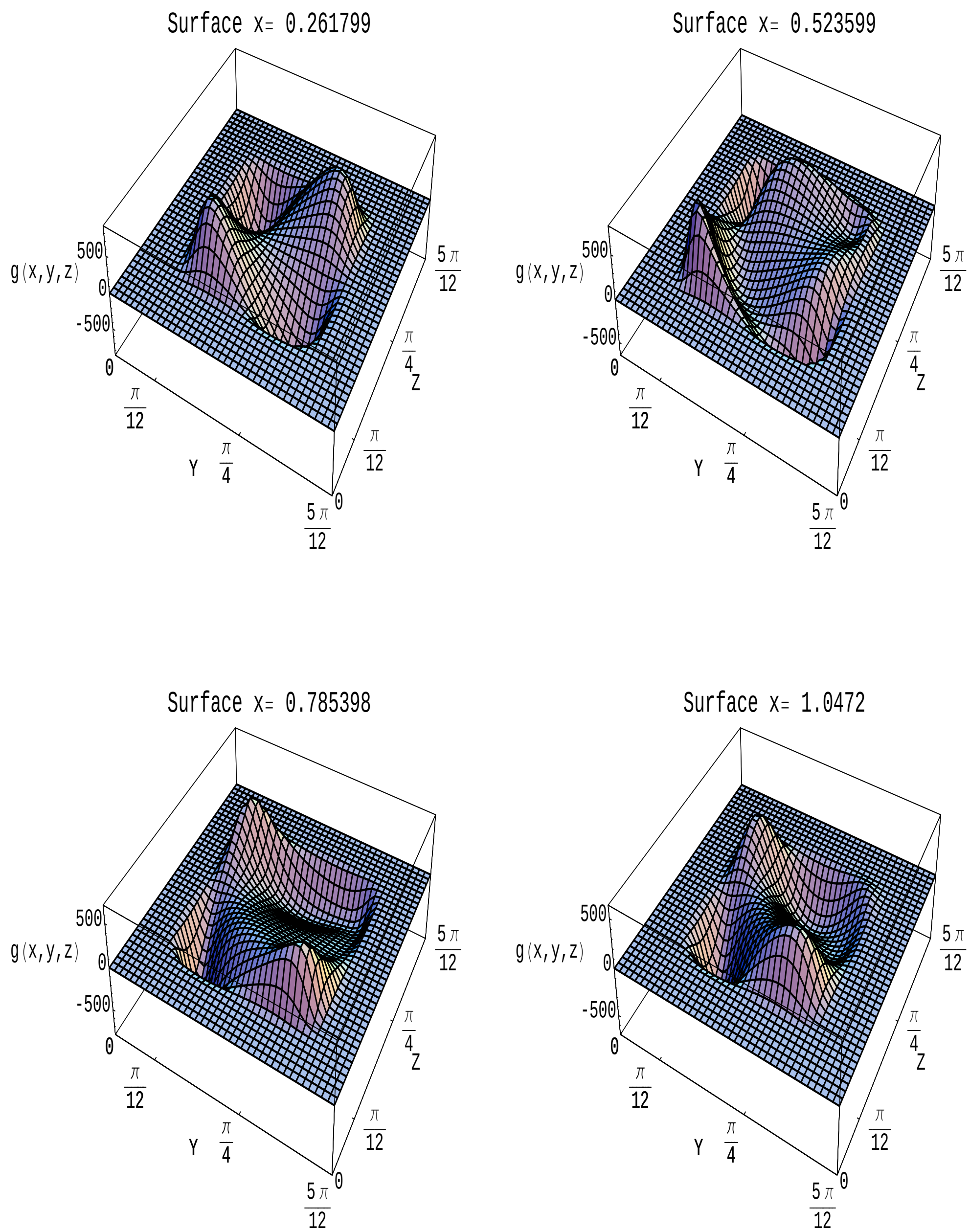

FIGURE 2. Slices of the function $g(x, y, z)$ for fixed x.

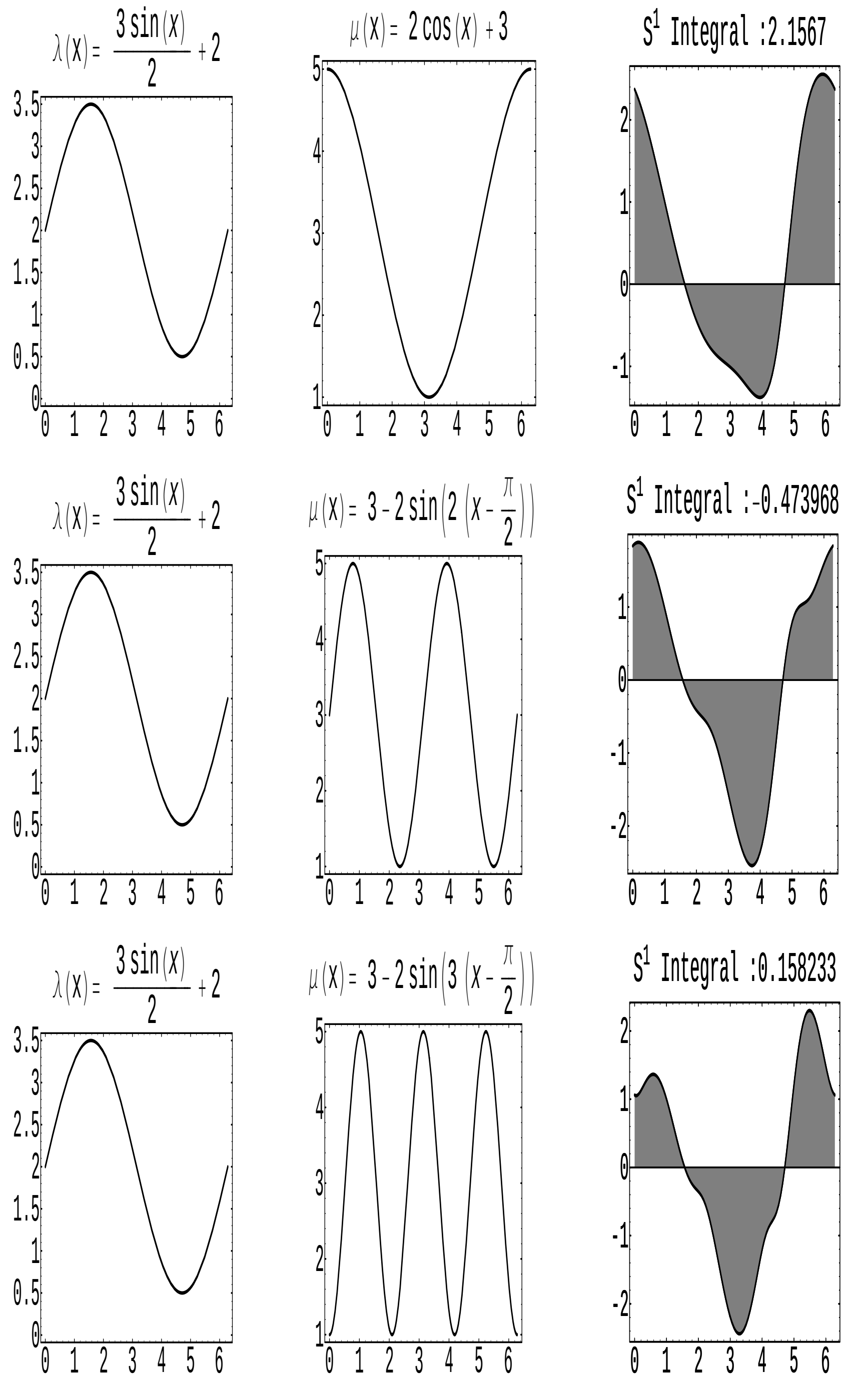

FIGURE 3. Examples of nonvanishing Chern-Simons classes. The shaded areas in the right column are the S^1 integral contributions to the Chern-Simons class evaluated on the test cycle $(\phi \circ z_1)_*[SO(3)]$. The number above the plot is the numerically computed area.

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