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Positivity of Lyapunov exponents for a continuum Matrix-valued Anderson model

by

Hakim Boumaza

Hakim Boumaza
Institut de Mathématique de Jussieu
Université Paris 7 Denis Diderot

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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Hakim Boumaza
Institut de Mathématique de Jussieu
Université Paris 7 Denis Diderot
2 place Jussieu
75251 Paris, France
boumaza@math.jussieu.fr

Abstract

In this article, we study a continuum matrix-valued Anderson-type model. Both leading Lyapounov exponents of this model are proved to be positive and distincts for all energies in $(2, +\infty)$ except those in a discrete set, which leads to absence of absolutely continuous spectrum in $(2, +\infty)$. This result is an improvement of the result proved in [3]. The methods, based upon a result by Breuillard and Gelander [4] on generating dense Lie subgroups in semisimple Lie groups, and a criterion by Goldsheid and Margulis [7], allow for singular Bernoulli distributions.

1 Introduction

In this article we will study the question of separability of Lyapounov exponents for a continuous matrix-valued Anderson-Bernoulli model of the form :

$$H_{AB}(\omega) = -\frac{d^2}{dx^2} I_2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{n\in\mathbb{Z}} \begin{pmatrix} \omega_1^{(n)} \chi_{[0,1]}(x-n) & 0 \\ 0 & \omega_2^{(n)} \chi_{[0,1]}(x-n) \end{pmatrix} \quad (1)$$

acting on $L^2(\mathbb{R}, \mathbb{C}^2)$. This question is coming from a more general problem on Anderson-Bernoulli models. Indeed, localization for Anderson models in dimension $d \geq 2$ is still an open problem if one look for arbitrary disorder, especially for Bernoulli randomness. A possible approach to try to understand localization for $d = 2$ is to discretize one direction. It leads to consider one-dimensional continuous Schrödinger operators, no longer scalar-valued, but now $N \times N$ matrix-valued. Before trying to understand how to handle with $N \times N$ matrix-valued continuous Schrödinger operators, we start with the model (1) corresponding to $N = 2$.

What is already well understood is the case of dimension one scalar-valued continuous Schrödinger operators with arbitrary randomness including Bernoulli distributions (see [6]) and discrete matrix-valued Schrödinger operators also including the Bernoulli case (see [7] and [10]). We aim at combining existing techniques for these cases to prove that for our model (1), the Lyapounov exponents are all positive and distincts for all energies except those in a discrete set, at least for energies in $(2, +\infty)$ (see *Th.* 3).

It is already proved in [3] that for model (1), the Lyapounov exponents are separable for all energies except those in a countable set, the critical energies. But the techniques used in [3] didn't allow us to avoid the case of an everywhere dense countable set of critical energies. Due to Kotani's theory (see [11]) this result will imply absence of absolutley continuous spectrum in the interval $(2, +\infty)$. But we keep in mind that we want to be able to use our result to prove Anderson localization and not only the absence of absolutely continuous spectrum. The separability of Lyapounov exponents can be view as a first step in order to follow a multiscale analysis scheme. The next step would be to prove some regularity on the integrated density of states, like local Hölder-continuity and then to prove a Wegner estimate and an Initial Length Scale estimate to start the multiscale analysis (see [13]). To prove the local Hölder-continuity of the integrated density of states, we need to have the separability of the Lyapounov exponents on intervals (see [5] or [6]). But, if like in [3] we can get an everywhere dense countable set of critical energies, we will not be able to prove local Hölder-continuity of the integrated density of states. That is why we need to improve the result of [3].

Our approach of the separability of Lyapounov exponents is based upon an abstract criterion in terms of the group generated by the random transfer matrices. This criterion has been provided by Gol'dsheid and Margulis in [7]. It is exactly this criterion which allowed to prove Anderson localization for discrete strips (see [10]). This criterion is also interesting because it allow for singularly distributed random parameters, including Bernoulli distributions.

We had the same approach in [3], what changes here is the way to apply the criterion of Gol'dsheid and Margulis. To apply this criterion we have to prove that a certain group is Zariski-dense in the symplectic group $\mathrm{Sp}_2(\mathbb{R})$. In [3] we were constructing explicitely a family of ten matrices linearly independant in the Lie algebra $\mathfrak{sp}_2(\mathbb{R})$ of $\mathrm{Sp}_2(\mathbb{R})$. This construction was only possible by considering an everywhere dense countable set of critical energies. By using a result of group theory by Breuillard and Gelander (see [4]), we are here able to prove that the group involved in Gol'dsheid/Margulis's criterion is dense in $\mathrm{Sp}_2(\mathbb{R})$ for all energies in $(2, +\infty)$, except those in a discrete set.

We start at Section 2 with a presentation of the necessary background on products of *i.i.d* symplectic matrices and with a statement of the criterion of Gol'dsheid and Margulis. We also present the result of Breuillard and Gelander in this section. Then, in Section 3 we precise the assumptions made on the model (1) and we explicit the transfer matrices associated to this model. In Section 4 we give the proof of our main result, the Theorem 3 by following the steps given by the assumptions of Theorem 2 by Breuillard and Gelander.

We finish by mentioniong that different methods have been used to prove localization properties for random operators on strips in [9]. They are based upon the use of spectral averaging techniques which did not allow to handle with singular distributions of the random parameters. So even if the methods used in [9] (which only considers discrete strips) have potential to be applicable to continuum models, one difference between these methods and the ones used here is that, like in [3], we handle singular distributions, in particular Bernoulli distributions.

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2 Criterion of separability of Lyapunov exponents

We will first review some results about Lyapounov exponents and how to prove their separability. These results hold for general sequences of *i.i.d* random symplectic matrices. Even if we will only use them for symplectic matrices in $\mathcal{M}_4(\mathbb{R})$, we will write these results for symplectic matrices in $\mathcal{M}_{2N}(\mathbb{R})$ for arbitrary N.

Let N be a positive integer. Let $\mathrm{Sp}_N(\mathbb{R})$ denote the group of $2N \times 2N$ real symplectic matrices. It is the subgroup of $\mathrm{GL}_{2N}(\mathbb{R})$ of matrices M satisfying

$$^tMJM = J,$$

where J is the matrix of order $2N$ defined by $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. Here, I is the identity matrix of order N.

We define the Lyapunov exponents.

Definition 1. *Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random matrices in* $\mathrm{Sp}_N(\mathbb{R})$ *with*

$$\mathbb{E}(\log^+ ||A_1^\omega||) < \infty.$$

The Lyapunov exponents $\gamma_1, \ldots, \gamma_{2N}$ associated with $(A_n^\omega)_{n\in\mathbb{N}}$ are defined inductively by

$$\sum_{i=1}^{p} \gamma_i = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}(\log || \wedge^p (A_n^\omega \ldots A_1^\omega)||).$$

Here, $\wedge^p(A_n^\omega \ldots A_1^\omega)$ denote the p-th exterior power of the matrix $(A_n^\omega \ldots A_1^\omega)$, acting on the p-th exterior power of $\mathbb{R}^{2N}$. For more details about these p-th exterior powers, see [2].

One has $\gamma_1 \geq \ldots \geq \gamma_{2N}$ and, due to symplecticity of the random matrices $(A_n)_{n\in\mathbb{N}}$, the symmetry property $\gamma_{2N-i+1} = -\gamma_i$, $\forall i \in \{1, \ldots, N\}$ (see [2] p.89, Prop 3.2).

We say that the Lyapounov exponents associated to a sequence $(A_n^\omega)_{n\in\mathbb{N}}$ of i.i.d. random matrices are *separable* when they are all distinct :

$$\gamma_1 > \gamma_2 > \ldots > \gamma_{2N}$$

We can now give a criterion of separability of the Lyapounov exponents. For the definitions of L_p-strong irreducibility and p-contractivity we refer to [2], definitions $A.IV.3.3$ and $A.IV.1.1$, respectively.

Let μ be a probability measure on $\mathrm{Sp}_N(\mathbb{R})$. We denote by G_μ the smallest closed subgroup of $\mathrm{Sp}_N(\mathbb{R})$ which contains the topological support of μ, supp μ.

Now we can set forth the main result on separability of Lyapunov exponents, which is a generalization of Furstenberg's theorem to the case $N > 1$.

Proposition 1. *Let $(A_n^\omega)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random symplectic matrices of order $2N$ and p be an integer in $\{1,\dots,N\}$. We denote by μ the common distribution of the A_n^ω. Suppose that G_μ is p-contracting and L_p-strongly irreducible and that $\mathbb{E}(\log ||A_1^\omega||) < \infty$. Then the following holds :*

(i) $\gamma_p > \gamma_{p+1}$

(ii) For any non zero x in L_p :

$$\lim_{n\to\infty} \frac{1}{n} \log || \wedge^p A_n^\omega \dots A_1^\omega x|| = \sum_{i=1}^{p} \gamma_i .$$

A proof of this proposition can be found in [2] at proposition 3.4. As a corollary we have that if G_μ is p-contracting and L_p-strongly irreducible for all $p \in \{1,\dots,N\}$ and if $\mathbb{E}(\log ||A_1^\omega||) < \infty$, then $\gamma_1 > \gamma_2 > \ldots > \gamma_N > 0$ (using the symmetry property of Lyapunov exponents).

For explicit models like the model (1) we study here, it can be quite difficult to check p-contractivity and L_p-strong irreducibility for all p. To avoid this difficulty, we will use the Gol'dsheid-Margulis theory presented in [7] which gives us an algebraic criterion to verify these assumptions. The idea is that if the group G_μ is large enough in an algebraic sense then it is p-contractive and L_p-strongly irreducible for all p.

We first recall the definition of the Zariski topology on $\mathcal{M}_{2\mathrm{N}}(\mathbb{R})$. We identify $\mathcal{M}_{2\mathrm{N}}(\mathbb{R})$ to $\mathbb{R}^{(2N)^2}$ by viewing a matrice as its coefficients. Then for $S \subset \mathbb{R}[X_1,\dots,X_{(2N)^2}]$, we set :

$$V(S) = \{x \in \mathbb{R}^{(2N)^2} \mid \forall P \in S,\ P(x) = 0\}$$

So, $V(S)$ is the set of common zeros of the polynomials of S. These sets $V(S)$ are the closed sets of a topology on $\mathbb{R}^{(2N)^2}$, we call it the Zariski topology. Then, on any subset of $\mathcal{M}_{2\mathrm{N}}(\mathbb{R})$ we can define the Zariski topology as the topology induced by the Zariski topology on $\mathcal{M}_{2\mathrm{N}}(\mathbb{R})$. In particular we define this way the Zariski topology on $\mathrm{Sp}_N(\mathbb{R})$.

We can now define the Zariski closure of a subset G of $\mathrm{Sp}_N(\mathbb{R})$. It is the smallest closed subset for the Zariski topology that contains G. We denote it by $\mathrm{Cl_Z}(G)$. In other words, if G is a subset of $\mathrm{Sp}_N(\mathbb{R})$, its Zariski closure $\mathrm{Cl_Z}(G)$ is the set of the zeros of polynomials vanishing on G. A subset $G' \subset G$ is said to be Zariski-dense in G if $\mathrm{Cl_Z}(G') = \mathrm{Cl_Z}(G)$, *i.e.* each polynomial vanishing on G' vanishes on G.

Being Zariski-dense is the meaning of being large enough for a subgroup of $\mathrm{Sp}_N(\mathbb{R})$ to be p-contractive and L_p-strongly irreducible for all p. More precisely, from the results of Gol'dsheid and Margulis one gets :

Theorem 1 (Gol'dsheid-Margulis criterion, [7]**).** *If G_μ is Zariski dense in $\mathrm{Sp}_N(\mathbb{R})$, then for all p, G_μ is p-contractive and L_p-strong irreducible.*

Proof. It is explain in [3] how to get that criterion from the results of Gol'dhseid and Margulis stated in [7]. □

As we can see in [3], it is not easy to check directly the Zariski-denseness of the group G_{μ_E} introduced there. In fact, in [3] we were reconstructing explicitely the Zariski closure of G_{μ_E}. But this construction was possible only for energies not in dense countable subset of $\mathbb{R}$. We will now give a way to prove more systematically the Zariski-denseness of a subgroup of $\mathrm{Sp}_N(\mathbb{R})$. It is based on the following result of Breuillard and Gelander :

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Theorem 2 (**Breuillard and Gelander**, [4]). *Let G be a semisimple Lie group, whose Lie algebra is $\mathfrak{g}$. Then there is an identity neighborhood $\mathcal{O} \subset G$, on which $\log = \exp^{-1}$ is a well defined diffeomorphism, such that $g_1, \dots g_m \in \mathcal{O}$ generate a dense subgroup whenever $\log(g_1), \dots \log(g_m)$ generate $\mathfrak{g}$.*

We will use this theorem in the following to prove that the subgroup generated by the transfer matrices associated to our operator is dense in $\mathrm{Sp}_N(\mathbb{R})$ and thus Zariski-dense in $\mathrm{Sp}_N(\mathbb{R})$.

In the next section we will precise the assumptions on model (1) and give the statement of our main result.

3 A matrix-valued continuum Anderson model

Let

$$H_{AB}(\omega) = -\frac{d^2}{dx^2} + V_0 + \sum_{n\in\mathbb{Z}} \begin{pmatrix} \omega_1^{(n)} \chi_{[0,1]}(x-n) & 0 \\ 0 & \omega_2^{(n)} \chi_{[0,1]}(x-n) \end{pmatrix} \quad (2)$$

be a random Schrödinger operator acting in $L^2(\mathbb{R}, \mathbb{C}^2)$. $\chi_{[0,1]}$ is the characteristic function of the interval $[0,1]$, V_0 is the constant-coefficient multiplication operator by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and $(\omega_1^{(n)})_{n\in\mathbb{Z}}$, $(\omega_2^{(n)})_{n\in\mathbb{Z}}$ are two sequences of i.i.d. random variables (also independent from each other) with common distribution ν such that $\{0,1\} \subset \operatorname{supp} \nu$.

This operator is a bounded perturbation of $(-\frac{d^2}{dx^2}) \oplus (-\frac{d^2}{dx^2})$ and thus self-adjoint on the Sobolev space $H^2(\mathbb{R}, \mathbb{C}^2)$.

For the operator $H_{AB}(\omega)$ we have the following result :

Theorem 3. *Let $\gamma_1(E)$ and $\gamma_2(E)$ be the positive Lyapounov exponents associated to $H_{AB}(\omega)$. It exists a discrete set $\mathcal{S}_B \subset \mathbb{R}$ such that for all $E \in (2, +\infty) \setminus \mathcal{S}_B$, $\gamma_1(E) > \gamma_2(E) > 0$. In particular, $H_{AB}(\omega)$ has no absolutely continuous spectrum in the interval $(2, +\infty)$.*

We will first precise some notations. We consider the differential system :

$$H_{AB} u = Eu, \ E \in \mathbb{R}. \quad (3)$$

For a solution $u = (u_1, u_2)$ of this system we define the transfer matrices $(A_n^{\omega^{(n)}}(E))_{n\in\mathbb{Z}}$ from n to $n+1$ by the relation

$$\begin{pmatrix} u_1(n+1) \\ u_2(n+1) \\ u_1'(n+1) \\ u_2'(n+1) \end{pmatrix} = A_n^{\omega^{(n)}}(E) \begin{pmatrix} u_1(n) \\ u_2(n) \\ u_1'(n) \\ u_2'(n) \end{pmatrix}.$$

The sequence of $(A_n^{\omega^{(n)}}(E))_{n\in\mathbb{Z}}$ is a sequence of *i.i.d.* random matrices in the symplectic group $\mathrm{Sp}_2(\mathbb{R})$. This sequence will determine the Lyapunov exponents at energy E. In order to use proposition 1, we need to define a measure on $\mathrm{Sp}_2(\mathbb{R})$ adapted to the sequence $(A_n^{\omega^{(n)}}(E))_{n\in\mathbb{Z}}$. The distribution μ_E is given by :

$$\mu_E(\Gamma) = \nu(\{\omega^{(0)} = (\omega_1^{(0)}, \omega_2^{(0)}) \in (\operatorname{supp}\nu)^2 \mid A_0^{\omega^{(0)}}(E) \in \Gamma\})$$

for any Borel subset Γ of $\mathrm{Sp}_2(\mathbb{R})$. This distribution is defined only by $A_0^{\omega^{(0)}}(E)$ because the matrices $A_n^{\omega^{(n)}}(E)$ are *i.i.d.*.

We then consider G_{μ_E} the smallest closed subgroup of $\mathrm{Sp}_2(\mathbb{R})$ generated by the support of μ_E. As $\{0,1\} \subset \operatorname{supp}\nu$, we also have that :

$$\{A_0^{(0,0)}(E), A_0^{(1,0)}(E), A_0^{(0,1)}(E), A_0^{(1,1)}(E)\} \subset G_{\mu_E}\,.$$

We want to work with an explicit form of these four transfer matrices. First, we set :

$$M_{\omega^{(0)}} = \begin{pmatrix} \omega_1^{(0)} & 1 \\ 1 & \omega_2^{(0)} \end{pmatrix} \tag{4}$$

We begin by writing $A_0^{\omega^{(0)}}(E)$ as an exponential. To do this we associate to the second order differential system (3) the following first order differential system :

$$Y' = \begin{pmatrix} 0 & I_2 \\ M_{\omega^{(0)}} - E & 0 \end{pmatrix} Y \tag{5}$$

with $Y \in \mathcal{M}_4(\mathbb{R})$. If Y is the solution with initial condition $Y(0) = I_4$, then $A_0^{\omega^{(0)}}(E) = Y(1)$. If we solve the system (5), we get :

$$A_0^{\omega^{(0)}}(E) = \exp\left(\begin{pmatrix} 0 & I_2 \\ M_{\omega^{(0)}} - E & 0 \end{pmatrix}\right) \tag{6}$$

To compute this exponential, we have to compute the successive powers of $M_{\omega^{(0)}}$. To do this, we diagonalize the real symmetric matrix $M_{\omega^{(0)}}$ in an orthonormal basis :

$$M_{\omega^{(0)}} = \begin{pmatrix} \omega_1^{(0)} & 1 \\ 1 & \omega_2^{(0)} \end{pmatrix} = S_{\omega^{(0)}} \begin{pmatrix} \lambda_1^{\omega^{(0)}} & 0 \\ 0 & \lambda_2^{\omega^{(0)}} \end{pmatrix} S_{\omega^{(0)}}^{-1},$$

where the matrices $S_{\omega^{(0)}}$ are orthogonal and the eigenvalues of $M_{\omega^{(0)}}$, $\lambda_2^{\omega^{(0)}} \leq \lambda_1^{\omega^{(0)}}$, are real. We can compute these eigenvalues and the corresponding matrices $S_{\omega^{(0)}}$ for the different values of $\omega^{(0)} \in \{0,1\}^2$. We get :

$$S_{(0,0)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ \lambda_1^{(0,0)} = 1,\ \lambda_2^{(0,0)} = -1, \tag{7}$$

$$S_{(1,1)} = S_{(0,0)},\ \lambda_1^{(1,1)} = 2,\ \lambda_2^{(1,1)} = 0, \tag{8}$$

$$S_{(1,0)} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \\ \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \end{pmatrix},\ \lambda_1^{(1,0)} = \frac{1+\sqrt{5}}{2},\ \lambda_2^{(1,0)} = \frac{1-\sqrt{5}}{2}, \tag{9}$$

$$S_{(0,1)} = \begin{pmatrix} \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \\ \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \end{pmatrix}, \ \lambda_1^{(0,1)} = \frac{1+\sqrt{5}}{2}, \ \lambda_2^{(0,1)} = \frac{1-\sqrt{5}}{2}. \tag{10}$$

We also define the matrices define by blocks :

$$R_{\omega(0)} = \begin{pmatrix} S_{\omega(0)} & 0 \\ 0 & S_{\omega(0)} \end{pmatrix}$$

Let $E > 2$ to be larger than all eigenvalues of all $M_{\omega(0)}$. With the abbreviation $r_i = r_i(E, \omega^{(0)}) := \sqrt{E - \lambda_i^{\omega^{(0)}}}$ for $i = 1, 2$, the transfer matrices become

$$A_0^{\omega^{(0)}}(E) = R_{\omega(0)} \begin{pmatrix} \cos(r_1) & 0 & \frac{1}{r_1}\sin(r_1) & 0 \\ 0 & \cos(r_2) & 0 & \frac{1}{r_2}\sin(r_2) \\ -r_1\sin(r_1) & 0 & \cos(r_1) & 0 \\ 0 & -r_2\sin(r_2) & 0 & \cos(r_2) \end{pmatrix} R_{\omega(0)}^{-1}. \tag{11}$$

We can now turn to the proof of the Theorem 3.

4 Proof of Theorem 3

We will show in a last part of this section that the Theorem 3 can be easily deduced from the following proposition :

Proposition 2. *There exists a discrete set $\mathcal{S}_B$ such that for all $E \in (2, \infty) \setminus \mathcal{S}_B$, G_{μ_E} is dense (and therefore Zariski-dense) in $\mathrm{Sp}_2(\mathbb{R})$.*

To prove this proposition, we will follow the Theorem 2.

4.1 Elements of G_{μ_E} in $\mathcal{O}$

To apply Theorem 2 we need to work with elements in the neighborhood $\mathcal{O}$ of the identity. We will work with the four matrices $A_0^{(0,0)}(E)$, $A_0^{(1,0)}(E)$, $A_0^{(0,1)}(E)$ and $A_0^{(1,1)}(E)$ which are in G_{μ_E}. We will prove that by taking a suitable power of each of these matrices we find four matrices in G_{μ_E} which lies in an arbitrary small neighborhood of the identity and thus in $\mathcal{O}$. For this we will use a simultaneous diophantine approximation result.

Theorem 4 (Dirichlet, [12]). *Let $\alpha_1, \ldots, \alpha_N$ be real numbers and $M > 1$ an integer. Then it exists $y, x_1, \ldots, x_N$ in $\mathbb{Z}$ such that $1 \leq y \leq M$ and :*

$$\forall i \in \{1, \ldots, N\}, \ |\alpha_i y - x_i| < M^{-\frac{1}{N}}$$

From this theorem we deduce the proposition :

Proposition 3. *Let $E \in (2, +\infty)$. For all $\omega^{(0)} \in \{0,1\}^2$, it exists $m_\omega(E) \in \mathbb{N}^*$ such that :*

$$(A_0^{\omega^{(0)}}(E))^{m_\omega(E)} \in \mathcal{O}$$

Proof. We fix $\omega^{(0)} \in \{0,1\}^2$. Let $M > 1$ be an integer. We apply Theorem 4 with $\alpha_1 = \frac{r_1}{2\pi}$ and $\alpha_2 = \frac{r_2}{2\pi}$. Then it exists $y \in \mathbb{Z}$, $1 \leq y \leq M$ and $x_1, x_2 \in \mathbb{Z}$ such that :

$$\left|\frac{r_1}{2\pi}y - x_1\right| < M^{-\frac{1}{2}}, \ \left|\frac{r_2}{2\pi}y - x_2\right| < M^{-\frac{1}{2}}$$

which be can be written as :

$$|r_1 y - 2x_1\pi| < 2\pi M^{-\frac{1}{2}}, \ |r_2 y - 2x_2\pi| < 2\pi M^{-\frac{1}{2}} \tag{12}$$

Then we have :

$$(A_0^{\omega^{(0)}}(E))^y = R_{\omega^{(0)}} \begin{pmatrix} \cos(yr_1) & 0 & \frac{1}{r_1}\sin(yr_1) & 0 \\ 0 & \cos(yr_2) & 0 & \frac{1}{r_2}\sin(yr_2) \\ -r_1\sin(yr_1) & 0 & \cos(yr_1) & 0 \\ 0 & -r_2\sin(yr_2) & 0 & \cos(yr_2) \end{pmatrix} R_{\omega^{(0)}}^{-1}$$

$$= R_{\omega^{(0)}} \begin{pmatrix} \cos(yr_1 - 2x_1\pi) & 0 & \frac{1}{r_1}\sin(yr_1 - 2x_1\pi) & 0 \\ 0 & \cos(yr_2 - 2x_2\pi) & 0 & \frac{1}{r_2}\sin(yr_2 - 2x_2\pi) \\ -r_1\sin(yr_1 - 2x_1\pi) & 0 & \cos(yr_1 - 2x_1\pi) & 0 \\ 0 & -r_2\sin(yr_2 - 2x_2\pi) & 0 & \cos(yr_2 - 2x_2\pi) \end{pmatrix} R_{\omega^{(0)}}^{-1}$$

by 2π-periodicity of the functions sinus and cosinus.

Let $\varepsilon > 0$. If we choose M large enough, $M^{-\frac{1}{2}}$ will be small enough to get :

$$\left|\left| \begin{pmatrix} \cos(yr_1 - 2x_1\pi) & 0 & \frac{1}{r_1}\sin(yr_1 - 2x_1\pi) & 0 \\ 0 & \cos(yr_2 - 2x_2\pi) & 0 & \frac{1}{r_2}\sin(yr_2(E) - 2x_2\pi) \\ -r_1\sin(yr_1 - 2x_1\pi) & 0 & \cos(yr_1 - 2x_1\pi) & 0 \\ 0 & -r_2\sin(yr_2 - 2x_2\pi) & 0 & \cos(yr_2 - 2x_2\pi) \end{pmatrix} - I_4 \right|\right| < \varepsilon$$

We recall that the matrices $S_{\omega^{(0)}}$ being orthogonals, so are the matrices $R_{\omega^{(0)}}$. Then conjugating by $R_{\omega^{(0)}}$ does not change the norm :

$$||(A_0^{\omega^{(0)}}(E))^y - I_4|| < \varepsilon$$

As $\mathcal{O}$ depend only on the semisimple group $\mathrm{Sp}_2(\mathbb{R})$, we can choose ε such that $B(I_4, \varepsilon) \subset \mathcal{O}$. So if we set $y = m_\omega(E)$, we have $1 \leq m_\omega(E) \leq M$ and :

$$(A_0^{\omega^{(0)}}(E))^{m_\omega(E)} \in \mathcal{O}$$

□

Remark. It is important to precise here that the neighborhood does not depend on E and $\omega^{(0)}$. So the integer $M > 1$ does also not depend E and $\omega^{(0)}$. It will be important in a next step of the proof to be able to say that even if the integer $m_\omega(E)$ depend on E and $\omega^{(0)}$, it is always in a an interval of integers $\{1, \ldots, M\}$ independant of E and $\omega^{(0)}$.

To apply Theorem 2, we need to show that the logarithms of the matrices $(A_0^{\omega^{(0)}}(E))^{m_\omega(E)}$ are generating a Lie algebra equal to $\mathfrak{sp}_2(\mathbb{R})$, the Lie algebra of $\mathrm{Sp}_2(\mathbb{R})$. A first difficulty is to compute the logarithm of $(A_0^{\omega^{(0)}}(E))^{m_\omega(E)}$ which is in $\log \mathcal{O}$.

4.2 Computation of the logarithm of $(A_0^{\omega}(E))^{m_\omega(E)}$

We fix $\omega^{(0)} \in \{0,1\}^2$ and we assume that $E > 2$. To compute the logarithm of $(A_0^{\omega}(E))^{m_\omega(E)}$, we start from its expression :

$$(A_0^{\omega^{(0)}}(E))^{m_\omega(E)} = R_{\omega^{(0)}} \begin{pmatrix} \cos(m_\omega(E)r_1) & 0 & \frac{1}{r_1}\sin(m_\omega(E)r_1) & 0 \\ 0 & \cos(m_\omega(E)r_2) & 0 & \frac{1}{r_2}\sin(m_\omega(E)r_2) \\ -r_1\sin(m_\omega(E)r_1) & 0 & \cos(m_\omega(E)r_1) & 0 \\ 0 & -r_2\sin(m_\omega(E)r_2) & 0 & \cos(m_\omega(E)r_2) \end{pmatrix} R_{\omega^{(0)}}^{-1} \tag{13}$$

We can always permutate the vectors of the orthonormal basis define by the columns of $R_{\omega^{(0)}}$. So it exists a matrix $P_{\omega^{(0)}}$ of permtutation (and thus orthogonal) such that :

$$(A_0^{\omega^{(0)}}(E))^{m_\omega(E)} = R_{\omega^{(0)}} P_{\omega^{(0)}} \begin{pmatrix} \cos(m_\omega(E)r_1) & \frac{1}{r_1}\sin(m_\omega(E)r_1) & 0 & 0 \\ -r_1\sin(m_\omega(E)r_1) & \cos(m_\omega(E)r_1) & 0 & 0 \\ 0 & 0 & \cos(m_\omega(E)r_2) & \frac{1}{r_2}\sin(m_\omega(E)r_2) \\ 0 & 0 & -r_2\sin(m_\omega(E)r_2) & \cos(m_\omega(E)r_2) \end{pmatrix} P_{\omega^{(0)}}^{-1} R_{\omega^{(0)}}^{-1} \tag{14}$$

Recall that we can choose $m_\omega(E)$ such that $(A_0^{\omega}(E))^{m_\omega(E)}$ is arbitrary close to the identity in $\mathrm{Sp}_2(\mathbb{R})$. Particularly we can assume that :

$$||(A_0^{\omega^{(0)}}(E))^{m_\omega(E)} - I_4|| < 1 \ .$$

So we can use the power series of the logarithm :

$$\log((A_0^{\omega^{(0)}}(E))^{m_\omega(E)}) = \sum_{k\geq 1} \frac{(-1)^{k+1}}{k} ((A_0^{\omega^{(0)}}(E))^{m_\omega(E)} - I_4)^k \tag{15}$$

To simplify our computations we will also use complex form for the sinus and the cosinus. We set :

$$Q_{\omega^{(0)}} = \begin{pmatrix} -\frac{\mathrm{i}}{r_1} & \frac{\mathrm{i}}{r_1} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\mathrm{i}}{r_2} & \frac{\mathrm{i}}{r_2} \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{16}$$

and so :

$$Q_{\omega^{(0)}}^{-1} = \frac{1}{2} \begin{pmatrix} \mathrm{i}r_1 & 1 & 0 & 0 \\ -\mathrm{i}r_1 & 1 & 0 & 0 \\ 0 & 0 & \mathrm{i}r_2 & 1 \\ 0 & 0 & -\mathrm{i}r_2 & 1 \end{pmatrix} \tag{17}$$

Then we have :

$$(A_0^{\omega^{(0)}}(E))^{m_\omega(E)} - I_4 =$$

$$R_{\omega^{(0)}} P_{\omega^{(0)}} Q_{\omega^{(0)}} \begin{pmatrix} e^{\mathrm{i}m_\omega(E)r_1} - 1 & 0 & 0 & 0 \\ 0 & e^{-\mathrm{i}m_\omega(E)r_1} - 1 & 0 & 0 \\ 0 & 0 & e^{\mathrm{i}m_\omega(E)r_2} - 1 & 0 \\ 0 & 0 & 0 & e^{-\mathrm{i}m_\omega(E)r_2} - 1 \end{pmatrix} Q_{\omega^{(0)}}^{-1} P_{\omega^{(0)}}^{-1} R_{\omega^{(0)}}^{-1}$$

So by using (15) we only have to compute :

$$\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}(e^{\pm \mathrm{i}m_\omega(E)r_l}-1)^k$$

Let Ln be the main determination of the complex logarithm define on $\mathbb{C}\setminus\mathbb{R}_-$. We want to write :

$$\sum_{k\geq 1}\frac{(-1)^{k+1}}{k}(e^{\pm \mathrm{i}m_\omega(E)r_l}-1)^k=\mathrm{Ln}(e^{\pm \mathrm{i}m_\omega(E)r_l}),\ l=1,2 \tag{18}$$

To do this, we have to assume that $r_l=\sqrt{E-\lambda_l^{\omega(0)}}\notin(\pi+2\pi\mathbb{Z})$. So we introduce the discrete set $\mathcal{S}_1$ of the $E>2$ of the form $E=-\lambda_l^{\omega(0)}+\pi+2j\pi$ for $j\in\mathbb{Z}$, $l\in\{1,2\}$ and $\omega^{(0)}\in\{0,1\}^2$. If we choose $E\in(2,+\infty)\setminus\mathcal{S}_1$ we can write :

$$\log((A_0^{\omega^{(0)}}(E))^{m_\omega(E)})=$$

$$R_{\omega(0)}P_{\omega(0)}Q_{\omega(0)}\begin{pmatrix}\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1}) & 0 & 0 & 0\\ 0 & \mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1}) & 0 & 0\\ 0 & 0 & \mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_2}) & 0\\ 0 & 0 & 0 & \mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_2})\end{pmatrix}Q^{-1}_{\omega(0)}P^{-1}_{\omega(0)}R^{-1}_{\omega(0)}$$

So we are left with computing $\mathrm{Ln}(e^{\pm\mathrm{i}m_\omega(E)r_l})$. We can do it only for $l=1$, the computation will be the same for $l=2$. We have :

$$\begin{aligned}\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1}) &= \mathrm{i}\mathrm{Arg}(e^{\mathrm{i}m_\omega(E)r_1})\\ &= \mathrm{i}\mathrm{Arcsin}(\sin(m_\omega(E)r_1)) && (19)\\ &= \mathrm{i}\left(m_\omega(E)r_1-\pi\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)\right)(-1)^{\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)} && (20)\end{aligned}$$

where Fl in (20) is the floor function. We recall that by (12), $m_\omega(E)r_1$ can be choosen arbitrary close to a multiple of 2π. So we can assume that $\frac{m_\omega(E)r_1}{\pi}$ is arbitrary close to an even number. It suffices to choose M such that $2M^{-\frac{1}{2}}<\frac{1}{2}$ to have that $\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)$ is even and more precisely equal to $2x_1$. Thus (20) become :

$$\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})=\mathrm{i}\left(m_\omega(E)r_1-\pi\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)\right) \tag{21}$$

And we have the corresponding equation for the conjugate logarithm :

$$\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1})=\mathrm{i}\left(-m_\omega(E)r_1-\pi\mathsf{Fl}\left(-\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)\right)(-1)^{\mathsf{Fl}\left(-\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)} \tag{22}$$

We have :

$$\begin{pmatrix}-\frac{\mathrm{i}}{r_1} & \frac{\mathrm{i}}{r_1}\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1}) & 0\\ 0 & \mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1})\end{pmatrix}\cdot\frac{1}{2}\begin{pmatrix}\mathrm{i}r_1 & 1\\ -\mathrm{i}r_1 & 1\end{pmatrix}= \tag{23}$$

$$\frac{1}{2}\begin{pmatrix}\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})+\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1}) & -\frac{\mathrm{i}}{r_1}\left(\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})-\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1})\right)\\ \mathrm{i}r_1\left(\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})-\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1})\right) & \mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})+\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1})\end{pmatrix}$$

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By (21) and (22) we have :

$$\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})+\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1}) = -\mathrm{i}\pi\left(\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)+\mathsf{Fl}\left(-\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)\right) \quad (24)$$

and :

$$\forall x\in\mathbb{R},\ \mathsf{Fl}\left(x+\frac{1}{2}\right)+\mathsf{Fl}\left(\frac{1}{2}-x\right)=\begin{cases}1 & \text{if } x=\frac{1}{2}+n,\ n\in\mathbb{Z}\\ 0 & \text{otherwise}\end{cases} \quad (25)$$

But as we can assume that $\frac{m_\omega(E)r_l}{\pi}$ is arbitrary close to an even number, we can assume that for $l=1,2$, $\frac{m_\omega(E)r_l}{\pi}$ is not of the form $\frac{1}{2}+n,\ n\in\mathbb{Z}$. So we have :

$$\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})+\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1})=0 \quad (26)$$

and :

$$\begin{aligned}\mathrm{Ln}(e^{\mathrm{i}m_\omega(E)r_1})-\mathrm{Ln}(e^{-\mathrm{i}m_\omega(E)r_1}) &= \mathrm{i}\left(2m_\omega(E)r_1-\pi\left(\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}-\frac{1}{2}\right)-\mathsf{Fl}\left(-\frac{m_\omega(E)r_1}{\pi}+\frac{1}{2}\right)\right)\right)\\ &= \mathrm{i}\left(2m_\omega(E)r_1-2\pi\mathsf{Fl}\left(\frac{m_\omega(E)r_1}{\pi}-\frac{1}{2}\right)\right) \quad (27)\end{aligned}$$

So we can set for $l=1,2$:

$$x_l=x_l(E,\omega):=\frac{1}{2}\mathsf{Fl}\left(\frac{m_\omega(E)r_l}{\pi}-\frac{1}{2}\right) \quad (28)$$

By putting the expressions (26) and (27) in the matrix (23), and by doing the same for the block corresponding to r_2, we get :

$$\begin{aligned}\log((A_0^{\omega^{(0)}}(E))^{m_\omega(E)}) &= R_{\omega^{(0)}}P_{\omega^{(0)}}\times\\ &\begin{pmatrix}0 & m_\omega(E)-\frac{2\pi x_1}{r_1} & 0 & 0\\ -m_\omega(E)r_1^2+2\pi r_1x_1 & 0 & 0 & 0\\ 0 & 0 & 0 & m_\omega(E)-\frac{2\pi x_2}{r_2}\\ 0 & 0 & -m_\omega(E)r_2^2+2\pi r_2x_2 & 0\end{pmatrix}P_{\omega^{(0)}}^{-1}R_{\omega^{(0)}}^{-1}\\ &= R_{\omega^{(0)}}\begin{pmatrix}0 & 0 & m_\omega(E)-\frac{2\pi x_1}{r_1} & 0\\ 0 & 0 & 0 & m_\omega(E)-\frac{2\pi x_2}{r_2}\\ -m_\omega(E)r_1^2+2\pi r_1x_1 & 0 & 0 & 0\\ 0 & -m_\omega(E)r_2^2+2\pi r_2x_2 & 0 & 0\end{pmatrix}R_{\omega^{(0)}}^{-1}\end{aligned}$$

We set :

$$LA_{\omega^{(0)}}:=\log((A_0^{\omega^{(0)}}(E))^{m_\omega(E)}) \quad (29)$$

So we can summarize the computations we have done in this section. For all $E\in(2,+\infty)\setminus\mathcal{S}_1$:

$$LA_{\omega^{(0)}}=R_{\omega^{(0)}}\begin{pmatrix}0 & 0 & m_\omega(E)-\frac{2\pi x_1}{r_1} & 0\\ 0 & 0 & 0 & m_\omega(E)-\frac{2\pi x_2}{r_2}\\ -m_\omega(E)r_1^2+2\pi r_1x_1 & 0 & 0 & 0\\ 0 & -m_\omega(E)r_2^2+2\pi r_2x_2 & 0 & 0\end{pmatrix}R_{\omega^{(0)}}^{-1} \quad (30)$$

We have now to prove that the four matrices $LA_{\omega^{(0)}}$, for $\omega^{(0)}\in\{0,1\}^2$, are generating the whole Lie algebra $\mathfrak{sp}_2(\mathbb{R})$.

4.3 The Lie algebra $\mathfrak{LA}_2(E)$

For $E \in (2,+\infty) \setminus \mathcal{S}_1$, we denote by $\mathfrak{LA}_2(E)$ the Lie algebra generated by the $LA_{\omega^{(0)}}$ for $\omega^{(0)} \in \{0,1\}^2$. It is a subalgebra of $\mathfrak{sp}_2(\mathbb{R})$. We will use the expressions of $\lambda_i^{\omega(0)}$ and S_ω computed at the section 3.

4.3.1 Notations

We set :

$$a_1 := x_1(E,(0,0)) = \mathsf{Fl}\left(\frac{m_{(0,0)}(E)\sqrt{E-1}}{\pi} + \frac{1}{2}\right)$$

$$a_2 := x_2(E,(0,0)) = \mathsf{Fl}\left(\frac{m_{(0,0)}(E)\sqrt{E+1}}{\pi} + \frac{1}{2}\right)$$

$$b_1 := x_1(E,(1,0)) = \mathsf{Fl}\left(\frac{m_{(1,0)}(E)\sqrt{E-\frac{1+\sqrt{5}}{2}}}{\pi} + \frac{1}{2}\right)$$

$$b_2 := x_2(E,(1,0)) = \mathsf{Fl}\left(\frac{m_{(1,0)}(E)\sqrt{E-\frac{1-\sqrt{5}}{2}}}{\pi} + \frac{1}{2}\right)$$

and

$$c_1 := x_1(E,(0,1)) = \mathsf{Fl}\left(\frac{m_{(0,1)}(E)\sqrt{E-\frac{1+\sqrt{5}}{2}}}{\pi} + \frac{1}{2}\right)$$

$$c_2 := x_2(E,(0,1)) = \mathsf{Fl}\left(\frac{m_{(0,1)}(E)\sqrt{E-\frac{1-\sqrt{5}}{2}}}{\pi} + \frac{1}{2}\right)$$

$$d_1 := x_1(E,(1,1)) = \mathsf{Fl}\left(\frac{m_{(1,1)}(E)\sqrt{E}}{\pi} + \frac{1}{2}\right)$$

$$d_2 := x_2(E,(1,1)) = \mathsf{Fl}\left(\frac{m_{(1,1)}(E)\sqrt{E-2}}{\pi} + \frac{1}{2}\right)$$

For $M \in \mathcal{M}_4(\mathbb{R})$, we will denote by $M[i,j]$ the coefficient of M at the line i and at the column j. We also set :

$$r_1^{00} := \sqrt{E-1} \quad , \quad r_2^{00} := \sqrt{E+1}$$
$$r_1^{11} := \sqrt{E-2} \quad , \quad r_2^{11} := \sqrt{E}$$
$$r_1^{10} = r_1^{01} := \sqrt{E-\frac{1+\sqrt{5}}{2}} \quad , \quad r_2^{10} = r_2^{01} := \sqrt{E-\frac{1-\sqrt{5}}{2}}$$

At last we set :

$$\begin{aligned} D_1(E) &:= \sqrt{E-1}\sqrt{E+1}\sqrt{E-\frac{1+\sqrt{5}}{2}}\sqrt{E-\frac{1-\sqrt{5}}{2}} \\ D_2(E) &:= \sqrt{E}\sqrt{E-2}\sqrt{E-\frac{1+\sqrt{5}}{2}}\sqrt{E-\frac{1-\sqrt{5}}{2}} \end{aligned}$$

To prove that $\mathfrak{L}\mathfrak{A}_2(E) = \mathfrak{sp}_2(\mathbb{R})$, we will build a family of 10 matrices linearly independant in $\mathfrak{L}\mathfrak{A}_2(E)$. First we will consider the subspace generated by the Lie brackets $[LA_{\omega(0)}, LA_{\tilde{\omega}(0)}]$.

4.3.2 The subspace V_1 generated by the $[LA_{\omega(0)}, LA_{\tilde{\omega}(0)}]$

A direct computation shows that the Lie bracket $[LA_{\omega(0)}, LA_{\tilde{\omega}(0)}]$ is always of the form :

$$\begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix},\ A \in \mathcal{M}_2(\mathbb{R}) \tag{31}$$

We denote by V_1 the subpsace of $\mathfrak{sp}_2(\mathbb{R})$ of dimension 4 of these matrices. We will show that out of a discrete set of energies E, the four Lie brackets $[LA_{(1,0)}, LA_{(0,0)}]$, $[LA_{(1,0)}, LA_{(1,1)}]$, $[LA_{(0,1)}, LA_{(0,0)}]$ and $[LA_{(0,1)}, LA_{(0,0)}]$ are generating V_1.

Expression of $[LA_{(1,0)}, LA_{(0,0)}]$. We give the expressions of the coefficients. By (31) it suffices to give the coefficients corresponding to the first 2×2 block.

$$\begin{aligned}[LA_{(1,0)}, LA_{(0,0)}][1,1] &= -\frac{1}{4\sqrt{5}D_1(E)}\Big[(-\pi(a_1r_2^{00}+a_2r_1^{00})+2m_{00}r_1^{00}r_2^{00})(\pi b_1(1+\sqrt{5})r_2^{10} \\ &\quad -\pi b_2(1-\sqrt{5})r_1^{10}-2\sqrt{5}m_{10}r_1^{10}r_2^{10})\Big]\end{aligned}$$

$$[LA_{(1,0)}, LA_{(0,0)}][1,2] = \frac{\pi^2 E}{2\sqrt{5}D_1(E)}\left[(b_1r_2^{10}-b_2r_1^{10})(a_1r_2^{00}-a_2r_1^{00})\right]$$

$$\begin{aligned}[LA_{(1,0)}, LA_{(0,0)}][2,1] &= -\frac{\pi}{4\sqrt{5}D_1(E)}\Big[\pi(a_2r_1^{00}-5a_1r_2^{00}+4m_{00}r_1^{00}r_2^{00})(b_1r_2^{10}+b_2r_1^{10})+ \\ &\quad (a_1r_2^{00}-a_2r_1^{00})(2\sqrt{5}m_{10}r_1^{10}r_2^{10}+2\pi E(b_1r_2^{10}-b_2r_1^{10}))\Big]\end{aligned}$$

$$[LA_{(1,0)}, LA_{(0,0)}][2,2] = \frac{\pi^2}{2\sqrt{5}D_1(E)}\left[(b_1r_2^{10}-b_2r_1^{10})(a_1r_2^{00}-a_2r_1^{00})\right]$$

Expression of $[LA_{(0,1)}, LA_{(0,0)}]$. We have :

$$\begin{aligned}[LA_{(0,1)}, LA_{(0,0)}][1,1] &= -\frac{1}{20D_1(E)}\Big[(10\sqrt{5}\pi m_{00}r_1^{00}r_2^{00}-\sqrt{5}\pi^2(a_2r_1^{00}+3a_1r_2^{00}))(c_1r_2^{10}-c_2r_1^{10}) \\ &\quad +5(\pi^2(a_1r_2^{00}-3a_2r_1^{00})+2\pi m_{00}r_1^{00}r_2^{00})(c_1r_2^{10}+c_2r_1^{10}) \\ &\quad -10(\pi m_{01}(a_1r_2^{00}-3a_2r_1^{00})+2m_{00}m_{01}r_1^{00}r_2^{00})r_1^{10}r_2^{10}\Big]\end{aligned}$$

$$\begin{aligned}[LA_{(0,1)}, LA_{(0,0)}][1,2] &= -\frac{1}{2\sqrt{5}D_1(E)}\Big[(\pi^2(a_1r_2^{00}-3a_2r_1^{00})+\pi^2E(a_1r_2^{00}-a_2r_1^{00}) \\ &\quad +(2+2\sqrt{5})\pi m_{00}r_1^{00}r_2^{00})(c_1r_2^{10}-c_2r_1^{10})-\sqrt{5}\pi^2(a_1r_2^{00}+a_2r_1^{00})(c_1r_2^{10}+c_2r_1^{10}) \\ &\quad +2\sqrt{5}(\pi m_{01}(a_1r_2^{00}+a_2r_1^{00})-2m_{00}m_{01}r_1^{00}r_2^{00})r_1^{10}r_2^{10}\Big]\end{aligned}$$

$$\begin{aligned}
[LA_{(0,1)}, LA_{(0,0)}][2,1] &= -\frac{1}{20D_1(E)}\left[(5\pi^2(a_1r_2^{00}+3a_2r_1^{00})-20\pi m_{00}r_1^{00}r_2^{00})(c_1r_2^{10}+c_2r_1^{10})\right.\\
&+\sqrt{5}\pi^2(2E-5)(a_1r_2^{00}-a_2r_1^{00})(c_1r_2^{10}-c_2r_1^{10})\\
&\left.-10(\pi m_{01}(a_1r_2^{00}+3a_2r_1^{00})-4m_{00}m_{01}r_1^{00}r_2^{00})r_1^{10}r_2^{10}\right]
\end{aligned}$$

$$\begin{aligned}
[LA_{(0,1)}, LA_{(0,0)}][2,2] &= -\frac{\pi}{10D_1(E)}\left[5\pi(a_1r_2^{00}-a_2r_1^{00})(c_1r_2^{10}+c_2r_1^{10})\right.\\
&+2\sqrt{5}(\pi(a_1r_2^{00}+a_2r_1^{00})+2m_{00}r_1^{00}r_2^{00})(c_1r_2^{10}-c_2r_1^{10})\\
&\left.+10m_{01}(a_1r_2^{00}+a_2r_1^{00})r_1^{10}r_2^{10}\right]
\end{aligned}$$

Expression of $[LA_{(1,0)}, LA_{(1,1)}]$. We have :

$$\begin{aligned}
[LA_{(1,0)}, LA_{(1,1)}][1,1] &= -\frac{\pi}{10D_2(E)}\left[2\sqrt{5}(2m_{11}r_1^{11}r_2^{11}-\pi(d_1r_2^{11}+d_2r_1^{11}))(b_1r_2^{10}-b_2r_1^{10})\right.\\
&\left.+5\pi(d_2r_1^{11}-d_1r_2^{11})(b_1r_2^{10}+b_2r_1^{10})+10m_{10}(d_1r_2^{11}-d_2r_1^{11})r_1^{10}r_2^{10}\right]
\end{aligned}$$

$$\begin{aligned}
[LA_{(1,0)}, LA_{(1,1)}][1,2] &= \frac{1}{20D_2(E)}\left[\sqrt{5}\pi^2(d_1r_2^{11}-d_2r_1^{11})(2E-3)(b_1r_2^{10}-b_2r_1^{10})\right.\\
&+(5\pi^2(d_1r_2^{11}+3d_2r_1^{11})-20\pi m_{11}r_1^{11}r_2^{11})(b_1r_2^{10}+b_2r_1^{10})\\
&\left.+(40m_{11}m_{10}r_1^{11}r_2^{11}-10\pi m_{10}(d_1r_2^{11}+3d_2r_1^{11}))r_1^{10}r_2^{10}\right]
\end{aligned}$$

$$\begin{aligned}
[LA_{(1,0)}, LA_{(1,1)}][2,1] &= -\frac{1}{10D_2(E)}\left[(2\pi^2\sqrt{5}(2d_2r_1^{11}-d_1r_2^{11})+\pi^2\sqrt{5}E(d_1r_2^{11}-d_2r_1^{11})\right.\\
&-2\pi\sqrt{5}m_{11}r_1^{11}r_2^{11})(b_1r_2^{10}-b_2r_1^{10})+(10\pi m_{11}r_1^{11}r_2^{11}-5\pi^2(d_1r_2^{11}+d_2r_1^{11}))\\
&\left.(b_1r_2^{10}+b_2r_1^{10})+(10\pi m_{10}(d_1r_2^{11}+d_2r_1^{11})-20m_{11}m_{00}r_1^{11}r_2^{11})r_1^{10}r_2^{10}\right]
\end{aligned}$$

$$\begin{aligned}
[LA_{(1,0)}, LA_{(1,1)}][2,2] &= -\frac{1}{20D_2(E)}\left[(10\pi\sqrt{5}m_{11}r_1^{11}r_2^{11}-\pi^2\sqrt{5}(3d_1r_2^{11}+7d_2r_1^{11}))(b_1r_2^{10}-b_2r_1^{10})\right.\\
&+(5\pi^2(3d_2r_1^{11}-d_1r_2^{11})-10\pi m_{11}r_1^{11}r_2^{11})(b_1r_2^{10}+b_2r_1^{10})\\
&\left.+(10\pi m_{10}(d_1r_2^{11}-3d_2r_1^{11})+20m_{11}m_{00}r_1^{11}r_2^{11})r_1^{10}r_2^{10}\right]
\end{aligned}$$

Expression of $[LA_{(0,1)}, LA_{(1,1)}]$. We have :

$$[LA_{(0,1)}, LA_{(1,1)}][1,1] = \frac{\pi^2}{2\sqrt{5}D_2(E)}\left[(d_1r_2^{11}-d_2r_1^{11})(c_1r_2^{10}-c_2r_1^{10})\right]$$

$$[LA_{(0,1)}, LA_{(1,1)}][1,2] = \frac{\pi}{4\sqrt{5}D_2(E)} \Big[(\pi(d_1r_2^{11} + 3d_2r_1^{11}) + 2\pi E(d_1r_2^{11} - d_2r_1^{11}) - 4m_{11}r_1^{11}r_2^{11}) (c_1r_2^{10} - c_2r_1^{10}) + \sqrt{5}\pi(d_2r_1^{11} - d_1r_2^{11})(c_1r_2^{10} + c_2r_1^{10}) +2\sqrt{5}m_{01}(d_1r_2^{11} - d_2r_1^{11})r_1^{10}r_2^{10}\Big]$$

$$[LA_{(0,1)}, LA_{(1,1)}][2,1] = -\frac{\pi^2}{2\sqrt{5}D_2(E)} \left[(E-1)(d_1r_2^{11} - d_2r_1^{11})(c_1r_2^{10} - c_2r_1^{10})\right]$$

$$[LA_{(0,1)}, LA_{(1,1)}][2,2] = -\frac{1}{4\sqrt{5}D_2(E)} \Big[(2m_{11}r_1^{11}r_2^{11} - \pi(d_1r_2^{11} + d_2r_1^{11}))(2\sqrt{5}m_{01}r_1^{10}r_2^{10} +\pi(c_1r_2^{10} - c_2r_1^{10}) - \sqrt{5}\pi(c_1r_2^{10} + c_2r_1^{10}))\Big]$$

We can then consider the determinant of these coefficients :

$$\begin{vmatrix} [LA_{(1,0)}, LA_{(0,0)}][1,1] & [LA_{(0,1)}, LA_{(0,0)}][1,1] & [LA_{(1,0)}, LA_{(1,1)}][1,1] & [LA_{(0,1)}, LA_{(1,1)}][1,1] \\ [LA_{(1,0)}, LA_{(0,0)}][1,2] & [LA_{(0,1)}, LA_{(0,0)}][1,2] & [LA_{(1,0)}, LA_{(1,1)}][1,2] & [LA_{(0,1)}, LA_{(1,1)}][1,2] \\ [LA_{(1,0)}, LA_{(0,0)}][2,1] & [LA_{(0,1)}, LA_{(0,0)}][2,1] & [LA_{(1,0)}, LA_{(1,1)}][2,1] & [LA_{(0,1)}, LA_{(1,1)}][2,1] \\ [LA_{(1,0)}, LA_{(0,0)}][2,2] & [LA_{(0,1)}, LA_{(0,0)}][2,2] & [LA_{(1,0)}, LA_{(1,1)}][2,2] & [LA_{(0,1)}, LA_{(1,1)}][2,2] \end{vmatrix}$$

$$= f_1(E) = \tilde{f}_1(a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, m_{00}, m_{01}, m_{10}, m_{11}, E) \qquad (32)$$

where $\tilde{f}_1(X_1, \ldots, X_{12}, Y)$ is polynomial in $X_1, \ldots, X_{12}$ and analytic in Y. Indeed, the determinant (32) is a rationnal fraction in the r_i^{jk} which are analytic functions in E not vanishing on the interval $(2, +\infty)$.

We have to precise that the coefficients $a_1, \ldots, d_2$ and $m_{00}, \ldots, m_{11}$ depend also on E and are not analytic in E. So f_1 is not analytic in E. We will now explain how to avoid this difficulty.

We recall that for all E and ω, $1 \leq m_\omega(E) \leq M$ with M independant of E and ω. Thus $m_\omega(E)$ only take a finite number of values in the set $\{1, \ldots, M\}$.

Then we consider the sequence of intervals $I_2 =]2,3]$, $I_3 = [3,4]$, and for all $k \geq 3$, $I_k = [k, k+1]$. These intervals are recovering $(2, +\infty)$. We fix $k \geq 2$ and we assume that $E \in I_k$. Then the integers

$$x_i^\omega(E) = \mathsf{FI}\left(\frac{m_\omega(E)\sqrt{E - \lambda_i^\omega}}{\pi} + \frac{1}{2}\right)$$

are bounded by a constant depending only on M and I_k. Indeed, the eigenvalues λ_i^ω are all in the fixed interval $[-2, 2]$, $m_\omega(E)$ take its values in $\{1, \ldots, M\}$ and $E \in I_k$. So the integers $x_i^\omega(E)$ take only a finite number of values in a set $\{0, \ldots, N_k\}$.

To study the zeros of the function f_1 on I_k, we have only to study the zeros of a finite number of analytics functions :

$$\tilde{f}_{1,p,l} \; : \; E \mapsto \tilde{f}_1(p_1,\ldots,p_8,l_1,\ldots,l_4,E)$$

for $p_i \in \{0,\ldots,N_k\}$ and $l_j \in \{1,\ldots M\}$. We have to show that the functions $\tilde{f}_{1,p,l}$ are not identically vanishing on I_k. In fact, the only case were it does not hold is when all the x_i^{ω} are zero. Indeed, $\tilde{f}_1(0,\ldots,0,X_9,\ldots,X_{12},Y)$ is identically zero. But if we look at the values of x_i^{ω} for $E>2$ and $m_{\omega}(E)\geq 1$, we get that $a_2 \geq 1$. We can compute the term of the determinant (32) involving only a_2. We get :

$$\frac{m_{10}^2 m_{01}^2 m_{11}^2 \pi^2 a_2^2}{E+1} \geq \frac{\pi^2}{E+1} > 0$$

And by observing all the coefficients of the determinant (32), this term is the only one involving E only by this power of $E+1=(r_2^{00})^2$ and no other power of the r_j^{kl}. So this term cannot be canceled uniformly in E by another term of the development of the determinant (32), whatever values taken by the integers $a_1,b_1,\ldots,d_2$ and $m_{00},\ldots,m_{11}$. So the only case where $\tilde{f}_{1,p,l}$ could identically vanish does not happen. We set :

$$J_1 = (\{0,\ldots,N_k\}\times\{1,\ldots,N_k\}\times\{0,\ldots,N_k\}^6)\times\{1,\ldots M\}^4$$

Then, as $(a_1,\ldots,m_{11})\in J_1$ the set of zeros of f_1 in I_k is included in the following finite union of discrete sets :

$$\{E\in I_k \mid f_1(E)=0\} \subset \bigcup_{(p,l)\in J_1} \{E\in I_k \mid \tilde{f}_{1,p,l}(E)=0\}$$

Thus this set is also discrete in I_k. We finally get that :

$$\{E\in]2,+\infty[\mid f_1(E)=0\} = \bigcup_{k\geq 2} \{E\in I_k \mid f_1(E)=0\}$$

is discrete in $(2,+\infty)$. We set :

$$\mathcal{S}_2 = \{E\in(2,+\infty) \mid f_1(E)=0\} \tag{33}$$

Let $E\in(2,+\infty)\setminus(\mathcal{S}_1\cup\mathcal{S}_2)$. As the determinant (32) is not zero, it implies that the four matrices $[LA_{(1,0)},LA_{(0,0)}]$, $[LA_{(1,0)},LA_{(1,1)}]$, $[LA_{(0,1)},LA_{(0,0)}]$, $[LA_{(0,1)},LA_{(0,0)}]$ are linearly independant in the subspace $V_1\subset \mathfrak{sp}_2(\mathbb{R})$ of 4. Thus, they generate V_1. We deduce that :

$$\forall E\in(2,+\infty)\setminus(\mathcal{S}_1\cup\mathcal{S}_2),\ V_1\subset \mathfrak{L}\mathfrak{A}_2(E) \tag{34}$$

We now have to find another family of six matrices linearly independant in a supplementary subspace of V_1 in $\mathfrak{sp}_2(\mathbb{R})$.

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4.3.3 Construction of the orthogonal of V_1 in $\mathfrak{sp}_2(\mathbb{R})$

We begin by giving the expressions of the matrices $LA_{(1,0)} - LA_{(0,0)}$, $LA_{(1,0)} - LA_{(1,1)}$ and $LA_{(0,1)} - LA_{(0,0)}$. Looking at the form of $LA_{\omega^{(0)}}$ given by (30) we already know that all these differences are of the form :

$$\begin{pmatrix} 0 & 0 & e & g \\ 0 & 0 & g & f \\ a & c & 0 & 0 \\ c & b & 0 & 0 \end{pmatrix},\ (a,b,c,e,f,g) \in \mathbb{R}^6 \tag{35}$$

We denote by V_2 the subspace of $\mathfrak{sp}_2(\mathbb{R})$ of the matrices of the form (35). V_2 is of dimension 6. We have that $\mathfrak{sp}_2(\mathbb{R}) = V_1 \oplus V_2$. The form (35) allows us to only compute the coefficients $[3,1]$, $[3,2]$, $[4,2]$, $[1,3]$,$[1,4]$ and $[2,4]$ of the matrices $LA_{(1,0)} - LA_{(0,0)}$, $LA_{(1,0)} - LA_{(1,1)}$ and $LA_{(0,1)} - LA_{(0,0)}$.

Expression of $LA_{(1,0)} - LA_{(0,0)}$. We have :

$$(LA_{(1,0)}-LA_{(0,0)})[3,1] = m_{10}(1-E)+m_{00}E-\frac{\pi}{2}(a_1r_2^{00}+a_2r_1^{00})+\frac{\pi}{2\sqrt{5}}(b_1r_2^{10}-b_2r_1^{10})+\frac{\pi}{2}(b_1r_2^{10}+b_2r_1^{10})$$

$$(LA_{(1,0)} - LA_{(0,0)})[3,2] = m_{10} - m_{00} + \frac{\pi}{2}(a_2r_1^{00} - a_1r_2^{00}) + \frac{\pi}{\sqrt{5}}(b_1r_2^{10} - b_2r_1^{10})$$

$$(LA_{(1,0)}-LA_{(0,0)})[4,2] = (m_{00}-m_{10})E-\frac{\pi}{2}(a_1r_2^{00}+a_2r_1^{00})-\frac{\pi}{2\sqrt{5}}(b_1r_2^{10}-b_2r_1^{10})+\frac{\pi}{2}(b_1r_2^{10}+b_2r_1^{10})$$

$$(LA_{(1,0)}-LA_{(0,0)})[1,3] = m_{10}-m_{00}+\frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}-\frac{a_2}{r_1^{00}}\right)+\frac{\pi}{2\sqrt{5}}\left(\frac{b_2}{r_1^{10}}-\frac{b_1}{r_2^{10}}\right)-\frac{\pi}{2}\left(\frac{b_1}{r_2^{10}}+\frac{b_2}{r_1^{10}}\right)$$

$$(LA_{(1,0)} - LA_{(0,0)})[1,4] = \frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}-\frac{a_2}{r_1^{00}}\right) + \frac{\pi}{\sqrt{5}}\left(\frac{b_2}{r_1^{10}}-\frac{b_1}{r_2^{10}}\right)$$

$$(LA_{(1,0)}-LA_{(0,0)})[2,4] = m_{10}-m_{00}+\frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}+\frac{a_2}{r_1^{00}}\right)+\frac{\pi}{2\sqrt{5}}\left(\frac{b_1}{r_2^{10}}-\frac{b_2}{r_1^{10}}\right)-\frac{\pi}{2}\left(\frac{b_1}{r_2^{10}}+\frac{b_2}{r_1^{10}}\right)$$

Expression of $LA_{(1,0)} - LA_{(1,1)}$. We have :

$$(LA_{(1,0)}-LA_{(1,1)})[3,1] = m_{10}+(m_{11}-m_{10})E-\frac{\pi}{2}(d_1r_2^{11}+d_2r_1^{11})+\frac{\pi}{2\sqrt{5}}(b_1r_2^{10}-b_2r_1^{10})+\frac{\pi}{2}(b_1r_2^{10}+b_2r_1^{10})$$

$$(LA_{(1,0)} - LA_{(1,1)})[3,2] = m_{10} + m_{11} + \frac{\pi}{2}(d_2r_1^{11} - d_1r_2^{11}) + \frac{\pi}{\sqrt{5}}(b_1r_2^{10} - b_2r_1^{10})$$

$$(LA_{(1,0)}-LA_{(1,1)})[4,2]=(m_{11}-m_{10})E-\frac{\pi}{2}(d_1r_2^{11}+d_2r_1^{11})-\frac{\pi}{2\sqrt{5}}(b_1r_2^{10}-b_2r_1^{10})+\frac{\pi}{2}(b_1r_2^{10}+b_2r_1^{10})$$

$$(LA_{(1,0)}-LA_{(1,1)})[1,3]=m_{10}-m_{11}+\frac{\pi}{2}\left(\frac{d_1}{r_2^{00}}+\frac{d_2}{r_1^{00}}\right)-\frac{\pi}{2\sqrt{5}}\left(\frac{b_1}{r_2^{10}}-\frac{b_2}{r_1^{10}}\right)-\frac{\pi}{2}\left(\frac{b_1}{r_2^{10}}+\frac{b_2}{r_1^{10}}\right)$$

$$(LA_{(1,0)}-LA_{(1,1)})[1,4]=\frac{\pi}{2}\left(\frac{d_1}{r_2^{00}}-\frac{d_2}{r_1^{00}}\right)-\frac{\pi}{\sqrt{5}}\left(\frac{b_1}{r_2^{10}}-\frac{b_2}{r_1^{10}}\right)$$

$$(LA_{(1,0)}-LA_{(1,1)})[2,4]=m_{10}-m_{11}+\frac{\pi}{2}\left(\frac{d_1}{r_2^{00}}+\frac{d_2}{r_1^{00}}\right)+\frac{\pi}{2\sqrt{5}}\left(\frac{b_1}{r_2^{10}}-\frac{b_2}{r_1^{10}}\right)-\frac{\pi}{2}\left(\frac{b_1}{r_2^{10}}+\frac{b_2}{r_1^{10}}\right)$$

Expression of $LA_{(0,1)}-LA_{(0,0)}$. We have :

$$(LA_{(0,1)}-LA_{(0,0)})[3,1]=m_{01}+(m_{00}-m_{01})E-\frac{\pi}{2}(a_1r_2^{00}+a_2r_1^{00})+\frac{\pi}{2\sqrt{5}}(c_1r_2^{10}-c_2r_1^{10})+\frac{\pi}{2}(c_1r_2^{10}+c_2r_1^{10})$$

$$(LA_{(0,1)}-LA_{(0,0)})[3,2]=-(m_{00}+m_{01})+\frac{\pi}{2}(a_2r_1^{00}-a_1r_2^{00})+\frac{\pi}{\sqrt{5}}(c_2r_1^{10}-c_1r_2^{10})$$

$$(LA_{(0,1)}-LA_{(0,0)})[4,2]=(m_{00}-m_{01})E-\frac{\pi}{2}(a_1r_2^{00}+a_2r_1^{00})+\frac{\pi}{2\sqrt{5}}(c_1r_2^{10}-c_2r_1^{10})-\frac{\pi}{2}(c_1r_2^{10}+c_2r_1^{10})$$

$$(LA_{(0,1)}-LA_{(0,0)})[1,3]=m_{01}-m_{00}+\frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}+\frac{a_2}{r_1^{00}}\right)+\frac{\pi}{2\sqrt{5}}\left(\frac{c_2}{r_1^{10}}-\frac{c_1}{r_2^{10}}\right)-\frac{\pi}{2}\left(\frac{c_1}{r_2^{10}}+\frac{c_2}{r_1^{10}}\right)$$

$$(LA_{(0,1)}-LA_{(0,0)})[1,4]=\frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}-\frac{a_2}{r_1^{00}}\right)-\frac{\pi}{\sqrt{5}}\left(\frac{c_2}{r_1^{10}}-\frac{c_1}{r_2^{10}}\right)$$

$$(LA_{(0,1)}-LA_{(0,0)})[2,4]=m_{01}-m_{00}+\frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}+\frac{a_2}{r_1^{00}}\right)-\frac{\pi}{2\sqrt{5}}\left(\frac{c_2}{r_1^{10}}-\frac{c_1}{r_2^{10}}\right)-\frac{\pi}{2}\left(\frac{c_1}{r_2^{10}}+\frac{c_2}{r_1^{10}}\right)$$

Now we assume that $E\in(2,+\infty)\setminus(\mathcal{S}_1\cup\mathcal{S}_2)$. Then $V_1\subset\mathfrak{L}\mathfrak{A}_2(E)$ and in particular the following matrices are in $\mathfrak{L}\mathfrak{A}_2(E)$:

$$Z_1=\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix},\ Z_2=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&-1\end{pmatrix},\ Z_3=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&-1&0\end{pmatrix}\qquad(36)$$

So we can consider the three matrices of $\mathfrak{L}\mathfrak{A}_2(E)$, $[LA_{(1,0)} - LA_{(0,0)}, Z_1]$, $[LA_{(1,0)} - LA_{(1,1)}, Z_2]$ and $[LA_{(0,1)} - LA_{(0,0)}, Z_3]$. We can verify that in general the Lie bracket of an element of V_1 and an element of V_2 is still in V_2. So, to write this three matrices we will only have to explicit six of their coefficients.

Expression of $[LA_{(1,0)} - LA_{(0,0)}, Z_1]$. We have :

$$[LA_{(1,0)}-LA_{(0,0)}, Z_1][3,1] = 2m_{10}+2(m_{00}-m_{10})E-\pi(a_1r_2^{00}+a_2r_1^{00})+\pi(b_1r_2^{10}+b_2r_1^{10})+\frac{\pi}{\sqrt{5}}(b_1r_2^{10}-b_2r_1^{10})$$

$$[LA_{(1,0)} - LA_{(0,0)}, Z_1][3,2] = m_{10} - m_{00} + \pi(a_2r_1^{00} - a_1r_2^{00}) + \frac{\pi}{\sqrt{5}}(b_1r_2^{10} - b_2r_1^{10})$$

$$[LA_{(1,0)} - LA_{(0,0)}, Z_1][4,2] = 0$$

$$[LA_{(1,0)} - LA_{(0,0)}, Z_1][1,3] = 2(m_{00} - m_{10}) - \pi\left(\frac{a_1}{r_2^{00}} + \frac{a_2}{r_1^{00}}\right) + \pi\left(\frac{b_1}{r_2^{10}} + \frac{b_2}{r_1^{10}}\right) + \frac{\pi}{\sqrt{5}}\left(\frac{b_1}{r_2^{10}} - \frac{b_2}{r_1^{10}}\right)$$

$$[LA_{(1,0)} - LA_{(0,0)}, Z_1][1,4] = \frac{\pi}{2}\left(\frac{a_2}{r_1^{00}} - \frac{a_1}{r_2^{00}}\right) + \frac{\pi}{\sqrt{5}}\left(\frac{b_1}{r_2^{10}} - \frac{b_2}{r_1^{10}}\right)$$

$$[LA_{(1,0)} - LA_{(0,0)}, Z_1][2,4] = 0$$

Expression of $[LA_{(1,0)} - LA_{(1,1)}, Z_2]$. We have :

$$[LA_{(1,0)} - LA_{(1,1)}, Z_2][3,1] = 0$$

$$[LA_{(1,0)} - LA_{(1,1)}, Z_2][3,2] = m_{10} + m_{11} + \frac{\pi}{2}(d_2r_1^{11} - d_1r_2^{11}) + \frac{\pi}{\sqrt{5}}(b_1r_2^{10} - b_2r_1^{10})$$

$$[LA_{(1,0)}-LA_{(1,1)}, Z_2][4,2] = 2(m_{11}-m_{10})E-2m_{11}-\pi(d_1r_2^{11}+d_2r_1^{11})+\pi(b_1r_2^{10}+b_2r_1^{10})-\frac{\pi}{\sqrt{5}}(b_1r_2^{10}-b_2r_1^{10})$$

$$[LA_{(1,0)} - LA_{(1,1)}, Z_2][1,3] = 0$$

$$[LA_{(1,0)} - LA_{(1,1)}, Z_2][1,4] = \frac{\pi}{2}\left(\frac{d_2}{r_1^{11}} - \frac{d_1}{r_2^{11}}\right) + \frac{\pi}{\sqrt{5}}\left(\frac{b_1}{r_2^{10}} - \frac{b_2}{r_1^{10}}\right)$$

$$[LA_{(1,0)}-LA_{(1,1)},Z_2][2,4]=2(m_{11}-m_{10})-\pi\left(\frac{d_1}{r_2^{11}}+\frac{d_2}{r_1^{11}}\right)+\pi\left(\frac{b_1}{r_2^{10}}+\frac{b_2}{r_1^{10}}\right)-\frac{\pi}{\sqrt{5}}\left(\frac{b_1}{r_2^{10}}-\frac{b_2}{r_1^{10}}\right)$$

Expression of $[LA_{(0,1)}-LA_{(0,0)},Z_3]$. We have :

$$[LA_{(0,1)}-LA_{(0,0)},Z_3][3,1]=0$$

$$[LA_{(0,1)}-LA_{(0,0)},Z_3][3,2]=m_{01}+(m_{00}-m_{01})E-\frac{\pi}{2}(a_1r_2^{00}+a_2r_1^{00})+\frac{\pi}{2}(c_1r_2^{10}+c_2r_1^{10})+\frac{\pi}{2\sqrt{5}}(c_1r_2^{10}-c_2r_1^{10})$$

$$[LA_{(0,1)}-LA_{(0,0)},Z_3][4,2]=-2(m_{00}+m_{01})+\pi(a_2r_1^{00}-a_1r_2^{00})-\frac{2\pi}{\sqrt{5}}(c_1r_2^{10}-c_2r_1^{10})$$

$$[LA_{(0,1)}-LA_{(0,0)},Z_3][1,3]=\pi\left(\frac{a_2}{r_1^{00}}-\frac{a_1}{r_2^{00}}\right)-\frac{2\pi}{\sqrt{5}}\left(\frac{c_1}{r_2^{10}}-\frac{c_2}{r_1^{10}}\right)$$

$$[LA_{(0,1)}-LA_{(0,0)},Z_3][1,4]=m_{00}-m_{01}-\frac{\pi}{2}\left(\frac{a_1}{r_2^{00}}+\frac{a_2}{r_1^{00}}\right)+\frac{\pi}{2}\left(\frac{c_1}{r_2^{10}}+\frac{c_2}{r_1^{10}}\right)-\frac{\pi}{2\sqrt{5}}\left(\frac{c_1}{r_2^{10}}-\frac{c_2}{r_1^{10}}\right)$$

$$[LA_{(0,1)}-LA_{(0,0)},Z_3][2,4]=0$$

It remains to verifiy that these six matrices are linearly independant, at least for all $E\in(2,+\infty)$ except those in a discrete set. We denote by $f_2(E)$ the determinant of the 6×6 matrix whose columns are representing the 6 matrices we just compute. Each column is made of the 6 coefficients we compute for each matrix. We also set :

$$f_2(E)=\tilde{f}_2(a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2,m_{00},m_{01},m_{10},m_{11},E) \quad (37)$$

where $\tilde{f}_2(X_1,\dots,X_{12},Y)$ is polynomial in the coefficients $X_1,\dots,X_{12}$ and analytic in Y.

We define the functions $\tilde{f}_{2,p,l}$ as we have define the functions $\tilde{f}_{1,p,l}$. We can show that the $\tilde{f}_{2,p,l}$ do not vanish identically on I_k. More precisely we can look at the term in the development of the determinant (37) involving only a_2 :

$$\frac{m_{10}(m_{11}-m_{10})\pi^2a_2^2}{4(E+1)^3}\Big[\pi a_2m_{11}\Big(10\sqrt{E+1}E^3-8(E+1)^{3/2}E^3-9(E+1)^{7/2}+(E+1)^{5/2}+28\sqrt{E+1}E^2$$
$$+14(E+1)^{3/2}E-2(E+1)^{3/2}E^2-11(E+1)^{5/2}E+8(E+1)^{7/2}E+26\sqrt{E+1}E+8(E+1)^{3/2}+8\sqrt{E+1}\Big)$$
$$+\pi a_2m_{10}\Big(10(E+1)^{5/2}+2(E+1)^{7/2}+8(E+1)^{3/2}E^3+14(E+1)^{5/2}E-8(E+1)^{7/2}E$$
$$-29\sqrt{E+1}E^2-(E+1)^{3/2}E+10(E+1)^{3/2}E^2-28\sqrt{E+1}E-3(E+1)^{3/2}-9\sqrt{E+1}-10\sqrt{E+1}E^3\Big)$$
$$+m_{10}m_{11}(16E^4+32E^3-16E^2-64E-32)\Big]$$

This term is different from 0 for $a_2 \geq 1$, $m_{10} \geq 1$, $m_{11} \geq 1$ and $m_{10} \neq m_{11}$. But we can always assume that these two integers are distinct. Indeed, in the proof of the proposition 3, we can choose $2m_{10}$ instead of m_{10} and we have just to also multiply by 2 the integers x_1^{10} and x_1^{10}. And of course m_{10} and $2m_{10}$ cannot be both equal to m_{11}.

We also have that the term we just compute is the only one in the development of the determinant (37) involving exactly those powers of E and $E+1$ at the numerator and at the denominator. So this term cannot be canceled uniformly in E by another term of the development of the determinant (37). As before, the functions $\tilde{f}_{2,p,l}$ are not identically vanishing on I_k whenever $(p,l) \in J_2$ with :

$$J_2 = (\{0,\ldots,N_k\} \times \{1,\ldots,N_k\} \times \{0,\ldots,N_k\}^6) \times (\{1,\ldots M\}^4 \setminus \{(l_1,l_2,l_3,l_4) \mid l_3 = l_4\})$$

And as we have justified that $(a_1,\ldots,m_{11}) \in J_2$, we have :

$$\{E \in I_k \mid f_2(E) = 0\} \subset \bigcup_{(p,l)\in J_2} \{E \in I_k \mid \tilde{f}_{2,p,l}(E) = 0\}$$

So the zeros of f_2 are a discrete set in $(2,+\infty)$. If we set :

$$\mathcal{S}_3 = \{E \in]2,+\infty[\mid f_2(E) = 0\},$$

$\mathcal{S}_3$ is discrete and for $E \in (2,+\infty) \setminus (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3)$, $f_2(E) \neq 0$. So for these energies, the matrices $LA_{(1,0)} - LA_{(0,0)}$, $LA_{(1,0)} - LA_{(1,1)}$, $LA_{(0,1)} - LA_{(0,0)}$, $[LA_{(1,0)} - LA_{(0,0)}, Z_1]$, $[LA_{(1,0)} - LA_{(1,1)}, Z_2]$ and $[LA_{(0,1)} - LA_{(0,0)}, Z_3]$ are linearly independant in V_2 of dimension 6. So, for all $E \in (2,+\infty) \setminus (\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3)$, $V_2 \subset \mathfrak{L}\mathfrak{A}_2(E)$.

We set $\mathcal{S}_B = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$. We fix $E \in (2,+\infty) \setminus \mathcal{S}_B$. We have $V_1 \subset \mathfrak{L}\mathfrak{A}_2(E)$ and $V_2 \subset \mathfrak{L}\mathfrak{A}_2(E)$. As $V_1 \oplus V_2 = \mathfrak{sp}_2(\mathbb{R})$, we get :

$$\forall E \in (2,+\infty) \setminus \mathcal{S}_B,\ \mathfrak{sp}_2(\mathbb{R}) \subset \mathfrak{L}\mathfrak{A}_2(E)$$

And we have proven :

$$\forall E \in (2,+\infty) \setminus \mathcal{S}_B,\ \mathfrak{sp}_2(\mathbb{R}) = \mathfrak{L}\mathfrak{A}_2(E)$$

This is ending our study of the Lie algebra $\mathfrak{L}\mathfrak{A}_2(E)$. We have proven that for $E \in (2,+\infty)\setminus\mathcal{S}_B$, we can apply theorem 2 to the four matrices $(A_0^{(0,0)}(E))^{m_{00}(E)}$, $(A_0^{(1,0)}(E))^{m_{10}(E)}$, $(A_0^{(0,1)}(E))^{m_{01}(E)}$ and $(A_0^{(1,1)}(E))^{m_{11}(E)}$. Indeed, they are all in $\mathcal{O}$ and their logarithms are generating the whole Lie algebra $\mathfrak{sp}_2(\mathbb{R})$. So it achieves the proof of proposition 2.

4.4 End of the proof of Theorem 3

We have to explain how we deduce Theorem 3 from proposition 2. We fix $E \in (2,+\infty)\setminus\mathcal{S}_B$. By proposition 2, G_{μ_E} is dense in $\mathrm{Sp}_2(\mathbb{R})$ and therefore Zariski-dense in it. So, applying Theorem 1, we get that G_{μ_E} is p-contractive and L_p-strong irreducible for all p. Then applying the corollary of proposition 1 we get the separability of the Lyapounov exponents of the operator $H_{AB}(\omega)$ and the positivity of the two leading exponents. Thus we have the first part of Theorem 3 :

$$\forall E \in (2,+\infty) \setminus \mathcal{S}_B,\ \gamma_1(E) > \gamma_2(E) > 0$$

We can deduce the absence of absolutely continuous spectrum in $(2, +\infty)$ for $H_{AB}(\omega)$. For this we refer to Kotani theory in [11]. We have to precise that [11] considers $\mathbb{R}$-ergodic systems, while our model is $\mathbb{Z}$-ergodic. But we can use the suspension method provided in [8] to extend Kotani-theory to $\mathbb{Z}$-ergodic operators. So, non-vanishing of all lyapunov exponents for all energies except those in a discrete set allows to conclude absence of absolutely continuous spectrum via Theorem 7.2 of [11]. And so the second part of Theorem 3 is proved.

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Department of Mathematics
Faculty of Science and Technology
Keio University

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