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A distribution for a pair of unit vectors generated by Brownian motion

by

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Abstract

We propose a bivariate model for a pair of dependent unit vectors which is generated by Brownian motion. Both marginals have uniform distributions on the sphere, while conditionals obey the exit distributions. Some properties of the proposed model, including parameter estimation and pivotal statistic, are investigated. Further study is given to the bivariate circular case by transforming variables and parameters into the form of complex numbers. Some desirable properties, such as multiplicative property and infinite divisibility, hold for this submodel. As a related topic, the proposed distribution is generalized in order that both marginals have the exit distributions. We also construct distributions on the plane and on the cylinder by applying bilinear fractional transformations to the proposed bivariate circular model.

Key words and phrases: bivariate circular distribution, copula, directional statistics, exit distribution, wrapped Cauchy distribution.

1 Introduction

In some scientific fields, observations are described as pairs of d-dimensional unit vectors. In meteorology, for example, wind directions at the weather station in Milwaukee at 6 a.m. and noon (Johnson and Wehrly, 1977) are considered a data of this type for $d = 2$. Another example is directions of magnetic field, $d = 3$, in a rock sample before and after some laboratory treatment (Stephens, 1979).

For the analysis of these data, some stochastic models have been proposed in the literature. Mardia (1975) provided a class of distributions for two unit vectors using the principle of maximum entropy subject to constraints on certain moments. Wehrly and Johnson (1980) proposed a family of bivariate circular distributions, $d = 2$, having specified marginals. Their submodel with the von Mises marginals

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was studied by Shieh and Johnson (2005). Saw (1983) introduced bivariate families for pairs of dependent unit vectors, one of which is an offset distribution of the multivariate normal distribution with some restrictions on parameters. Rivest (1988) provided another model for dependent two unit vectors which is a generalization of the Fisher-von Mises distribution. A general class of bivariate distributions with exponential conditionals was proposed and discussed by Arnold and Strauss (1991) and Arnold *et al.* (1999), and a special case of their model defined on the two-dimensional torus were considered by SenGupta (2004). A recent work by Alfonsi and Brigo (2005) suggested new families of copulas based on periodic functions.

The main purpose of the paper is to introduce a new distribution for a pair of dependent unit vectors which is generated by $\mathbb{R}^d$-valued Brownian motion. To our knowledge, no distributions on this manifold have been proposed by applying the Brownian motion. In this paper, a new approach is taken to provide a tractable model. This method enables us to define a distribution with uniform marginals, which can be viewed as a spherical equivalent of copula, as consequences of a natural phenomenon and to derive some desirable properties.

Section 2 suggests a model for two dependent unit vectors and Section 3 investigates properties of the proposed model, including parameter estimation and pivotal statistic. In Section 4 we focus on the bivariate circular case of the model and discuss its detailed properties. It is shown that some desirable properties, such as multiplicative property and infinite divisibility, hold for the submodel. In Section 5 generalizations of the proposed model are discussed. Also, related models on $\mathbb{R}^2$ and on the cylinder are constructed by applying bilinear fractional transformations to the proposed model.

2 A Model for a Pair of Unit Vectors

2.1 Definition of the proposed model

Let $\{B_t\,;\,t \geq 0\}$ be $\mathbb{R}^d$-valued Brownian motion. Starting at $B_0 = 0$, this Brownian particle will eventually hit a d-sphere with radius ρ ($\in (0,1)$), and let τ_1 be the minimum time at which the particle exits the sphere, i.e. $\tau_1 = \inf\{t\,;\,\|B_t\| = \rho\}$ where $\|\cdot\|$ is the Euclidean norm. After leaving the sphere with radius ρ, the particle will hit a unit sphere first at the time τ_2, meaning $\tau_2 = \inf\{t\,;\,\|B_t\| = 1\}$. Then the proposed model is defined by the joint distribution of a random vector

$$\left(Q\frac{B_{\tau_1}}{\|B_{\tau_1}\|}, B_{\tau_2} \right),$$

where Q is a member of $O(d)$, the group of orthogonal transformations in $\mathbb{R}^d$. It is remarked here that the reason for multiplying Q by $B_{\tau_1}/\|B_{\tau_1}\|$ is to make the model more flexible without losing its tractability.

2.2 Probability density function

For convenience, write $(U, V) = (QB_{\tau_1}/\|B_{\tau_1}\|, B_{\tau_2})$. It is clear that (U, V) is a random vector of which each variable takes values on the unit sphere. The joint distribution of (U, V) has density

$$c(u, v) = \frac{1}{A_{d-1}^2} \frac{1 - \rho^2}{(1 - 2\rho u' Q v + \rho^2)^{d/2}}, \quad u, v \in S^{d-1}, \tag{2.1}$$

where $\rho \in [0, 1)$, $Q \in O(d)$, $S^{d-1} = \{x \in \mathbb{R}^d; \|x\| = 1\}$, x' a transpose of x, and A_{d-1} a surface area of S^{d-1}, i.e. $A_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$. The domain of ρ is extended to include $\rho = 0$ so that the model includes the uniform distribution. For the derivation of the density (2.1), see Appendix A.

The parameter ρ influences the dependence between U and V. When $\rho = 0$, U and V are independent and distributed as the uniform distribution on the sphere, i.e. $c(u, v) = 1/A_{d-1}^2$ on $u, v \in S^{d-1}$. As ρ tends to 1, it is shown that $P(\|U - QV\| < \varepsilon) \to 1$ for any $\varepsilon > 0$.

As is clear from the form (2.1), $c(u, v)$ is a function of $u'Qv$, the inner product of u and Qv. From this fact, we easily find that the density (2.1) takes maximum (resp. minimum) values for each v at $u = Qv$ (resp. $u = -Qv$). Thus the parameter Q controls the mode of the density. It is known that the orthogonal transformation Q consists of two transformations, namely, rotation and reflection. In particular, when $d = 2$, these transformations can be expressed as

$$v \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} v \quad \text{and} \quad v \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v,$$

where $0 \le \theta < 2\pi$. If $\det Q = 1$, this transformation consists of only rotation. Otherwise, the transformation is made up of both transformations.

3 Properties and Inference of the Proposed Model

3.1 Marginals and conditionals

One important feature of the proposed model is that it has well-known marginals and conditionals. Suppose $(U, V) \sim BS_d(\rho Q)$. Density for this random vector, (2.1), is $O(d)$-symmetric in the sense of Rivest (1984, Example 1). It follows then that the marginals of U and V are uniform distributions on S^{d-1} with density

$$f(x) = \frac{1}{A_{d-1}}, \quad x \in S^{d-1}.$$

Hence, model (2.1) can be viewed as a copula on $S^{d-1} \times S^{d-1}$. A difference between this special copula and usual one is the periodicity of the variables, which this one assumes and usual one does not.

Let U_j be the jth element of U, i.e. $U = (U_1, \ldots, U_j, \ldots, U_d)'$. It is known that the marginal of U_j has a distribution with density

$$f(u_j) = \frac{(1 - {u_j}^2)^{(d-3)/2}}{B\{\frac{1}{2}(d-1), \frac{1}{2}\}}, \quad -1 < u_j < 1,$$

where $B(\cdot, \cdot)$ is a beta function. This model represents U-shaped ($d = 2$), uniform ($d = 3$), or unimodal ($d \geq 4$). Note that $\frac{1}{2}(U_j + 1)$ has a beta distribution on $(0, 1)$, $\text{Beta}\{\frac{1}{2}(d-1), \frac{1}{2}(d-1)\}$.

Both conditional distributions of U given $V = v$ and V given $U = u$ are the exit distributions for the sphere. The terminology *exit distribution* is taken from Durrett (1984, Section 1.10), and the exit distribution on S^{d-1}, $\text{Exit}_d(\eta)$, is of the form

$$f(x) = \frac{1}{A_{d-1}} \frac{1 - \|\eta\|^2}{\|x - \eta\|^d}, \quad x \in S^{d-1}, \tag{3.1}$$

where $\eta \in \{\zeta \in \mathbb{R}^d;\, \|\zeta\| < 1\}$. This model is unimodal and rotationally symmetric about $x = \eta/\|\eta\|$. The concentration of the model is controlled by $\|\eta\|$. In particular, when $\|\eta\| = 0$, the model reduces to a uniform distribution. As $\|\eta\| \to 1$, the model approaches a point distribution with singularity at $x = \eta$. It is clear that the conditionals of model (2.1) are $U|(V = v) \sim \text{Exit}_d(\rho Q v)$ and $V|(U = u) \sim \text{Exit}_d(\rho Q' u)$.

It is worth remarked that the conditioal of $W \equiv v'Q'U$ given $V = v$ has a family discussed by Leipnik (1947) and McCullagh (1989). As used in the latter paper, write $X \sim H'(\theta, \nu)$ if density for a random vector X is

$$f(x) = \frac{1 - \theta^2}{B(\nu + \frac{1}{2}, \frac{1}{2})} \frac{(1 - x^2)^{\nu - 1/2}}{(1 - 2\theta x + \theta^2)^{\nu + 1}}, \quad -1 < x < 1,$$

where $-1 < \theta < 1$ and $\nu > -\frac{1}{2}$. Then it is easy to see that $W|(V = v) \sim H'\{\rho, \frac{1}{2}(d-2)\}$ holds for this conditional. For future reference, we here introduce a property of this model:

$$X \sim H'(\theta, \nu) \quad \Longrightarrow \quad E(X) = \theta. \tag{3.2}$$

3.2 Some properties

This subsection investigates some properties of model (2.1). The property we introduce first is that this distribution is closed under orthogonal transformations:

$$(U, V) \sim BS_d(\rho Q) \implies (Q_1 U, Q_2 V) \sim BS_d(\rho Q_1 Q Q_2'), \quad Q_1, Q_2 \in O(d).$$

The next result is obtainable by applying the known result written, for example, in Durrett (1984).

Theorem 1 *Let f be harmonic on the open unit ball in $\mathbb{R}^d$ and be continuous on the closed unit ball in $\mathbb{R}^d$. Suppose that (U, V) is distributed as $BS_d(\rho Q)$. Then $E\{f(V) \,|\, U = u\} = f(\rho Q' u)$ and $E\{f(U) \,|\, V = v\} = f(\rho Q v)$.*

Using this fact, it is easy to show that $E\{f(U)\} = E\{f(V)\} = f(0)$.

Rivest (1984, Proposition 1) showed that calculation of moments is simplified to some extent for a class of $O(d)$-symmetric distributions. This fact is helpful to get the moments and correlation coefficient of our model, which are given in the theorem below.

Theorem 2 *Suppose (U, V) has density (2.1). Then*

$$E(U) = E(V) = 0, \quad E(UU') = E(VV') = d^{-1}I,$$

$$E(UV') = d^{-1}\rho Q. \tag{3.3}$$

Jupp and Mardia (1980) coefficient of correlation, r^2, is thus

$$r^2 \equiv \mathrm{tr}(\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}) = d\rho^2,$$

where $\Sigma_{11} = E(UU') - E(U)E(U')$, $\Sigma_{12} = E(UV') - E(U)E(V')$, $\Sigma_{21} = \Sigma_{12}'$, and $\Sigma_{22} = E(VV') - E(V)E(V')$.

Note that simplicity of these moments and the correlation coefficient. See Appendix B for the proof.

The following result is useful to construct a pivotal statistic for (ρ, Q), which is discussed in Section 3.5. The proof is given in Appendix B.

Theorem 3 *If $(U, V) \sim BS_d(\rho Q)$, then $U'QV \sim H'\{\rho, \frac{1}{2}(d-2)\}$.*

3.3 Random vector generator

To generate a random vector having density (2.1), it is profitable to use the idea of the tangent-normal decomposition.

Let W be a random variable from $H'\{\rho, \frac{1}{2}(d-2)\}$, and let $d(X;\zeta) = (I - \zeta\zeta')X/\|(I-\zeta\zeta')X\|$ where $\zeta \in S^{d-1}$ and X a random vector having a uniform distribution on S^{d-1}. In other words, $d(X;\zeta)$ has a uniform distribution on the $(d-1)$-sphere, $S_\perp$, in $\mathbb{R}^d$ defined by $S_\perp = \{\eta \in \mathbb{R}^d;\ \|\eta\| = 1,\ \zeta'\eta = 0\}$. Then the conditional of U given $V = v$ can be decomposed into

$$U \,|\, (V = v) \ \stackrel{d}{=}\ WQv + (1 - W^2)^{1/2} d(X; Qv).$$

The generation of the variate from (2.1) consists of the following three steps: (i) Generate a random vector V which has a uniform distribution on S^{d-1}. This is achieved by using the method proposed by Tashiro (1977). (ii) Then generate W, which has $H'\{\rho, \frac{1}{2}(d-2)\}$, as stated in Section 4 of McCullagh (1989). (iii) Finally, a random vector $d(X; Qv)$ distributed as a uniform distribution on $S_\perp$ is obtained in a similar manner as in Step (i), and one gets a variate from the conditional of U given $V = v$ as described in the preceding paragraph. Then the joint distribution of (U, V) is $BS_d(\rho Q)$.

3.4 Parameter estimation

It is often a difficult work to look into the parameter estimation for multivariate distributions. As for our model, however, one can discuss parameter estimation under some conditions. The focus of this subsection is to consider parameter estimation based on method of moments and maximum likelihood.

First, the method of moments estimator based on (3.3) is constructed. Assume that (U_j, V_j) $(j = 1, \ldots, n\,(\geq 2))$ is a random sample from density (2.1) with unknown parameters ρ and Q. Under a condition, $\mathrm{rank}(\sum_{j=1}^n U_j V_j') = d$, one can construct an estimator for the parameters based on the moment $E(UV')$. This is done by equating the theoretical and sample moments. Thus we get

$$\hat{\rho} = d \left| \det \left(\frac{1}{n} \sum_{j=1}^n U_j V_j' \right) \right|^{1/d} \quad \text{and} \quad \hat{Q} = \frac{d}{n\hat{\rho}} \sum_{j=1}^n U_j V_j'. \tag{3.4}$$

Next, we consider the maximum likelihood estimation. Let (U_j, V_j) $(j = 1, \ldots, n)$ be an iid sample from $BS_d(\rho Q)$, where Q is known and ρ unknown. The log-likelihood for ρ is given by

$$l(\rho) = C + n \log(1 - \rho^2) - \frac{d}{2} \sum_{j=1}^n \log(1 - 2\rho u_j' Q v_j + \rho^2), \tag{3.5}$$

where C is a constant not depend on ρ. The derivative with respect to ρ is

$$\frac{\partial l}{\partial \rho} = \frac{-2n\rho}{1 - \rho^2} + d \sum_{j=1}^n \frac{x_j - \rho}{1 - 2\rho x_j + \rho^2},$$

where $x_j = u_j' Q v_j \in [-1, 1]$. From this expression, we find that the maximization of (3.5) with respect to ρ is essentially the same as that of $H'\{\rho, \frac{1}{2}(d-2)\}$ with respect to ρ.

3.5 Pivotal statistic

Suppose (U, V) is a random vector having $BS_d(\rho Q)$. Define a random variable

$$T(\rho, Q) = \frac{U'QV - \rho}{1 - 2\rho U'QV + \rho^2},$$

It is easy to see that $0 < T(\rho, Q) < 1$ a.s. for any ρ and Q. As shown in Theorem 3, $U'QV \sim H'\{\rho, \frac{1}{2}(d-2)\}$. Then by using equations (15.1.13) and (15.3.1) of Abramowitz and Stegun (1970), one obtains

$$E\left\{T(\rho, Q)^r\right\} = \frac{B\{r + \frac{1}{2}(d-1), \frac{1}{2}\}}{B\{\frac{1}{2}(d-1), \frac{1}{2}\}}.$$

Since these moments are equal to those of a beta distribution $\mathrm{Beta}\{\frac{1}{2}(d-1), \frac{1}{2}\}$, it follows that $T(\rho, Q)$ is a pivotal statistic for (ρ, Q) having $\mathrm{Beta}\{\frac{1}{2}(d-1), \frac{1}{2}\}$ almost surely. Because we know the exact distribution of $T(\rho, Q)$, the confidence intervals for the parameters based on $T(\rho, Q)$ can be obtained in the usual way.

4 Bivariate Circular Case

4.1 Transformation of random vectors and parameters

So far we have considered properties of model (2.1) for the general dimensional case. The theme of this section is to discuss the bivariate circular case, $d = 2$, of the proposed model which possesses some unique properties.

Suppose $(U, V) \sim BS_2(\rho Q)$. Then its density is expressed as

$$c(u, v) = \frac{1}{4\pi^2} \frac{1 - \rho^2}{1 - 2\rho u' Q v + \rho^2}, \quad u, v \in S^1.$$

For further discussion, it is advantageous to change the random variables and parameters by taking

$$(Z_U, Z_V) = (U_1 + iU_2, V_1 + iV_2) \quad \text{and} \quad \psi = \rho e^{i\theta},$$

where $U = (U_1, U_2)'$, $V = (V_1, V_2)'$, and θ a constant satisfying

$$Q = \begin{pmatrix} \cos\theta & -\det Q \sin\theta \\ \sin\theta & \det Q \cos\theta \end{pmatrix}, \quad 0 \le \theta < 2\pi.$$

Then it follows that $|\psi| < 1$ and $Z_U, Z_V \in \Omega$ where $\Omega = \{z \in \mathbb{C}; |z| = 1\}$. The density for (Z_U, Z_V) is given by

$$c(z_u, z_v) = \frac{1}{4\pi^2} \frac{1 - |\psi|^2}{|1 - \psi z_v z_u^{-\det Q}|^2}, \quad z_u, z_v \in \Omega. \tag{4.1}$$

If (Z_U, Z_V) has the density (4.1) with $\det Q = 1$, we write $(Z_U, Z_V) \sim BC_+(\psi)$. Similarly, write $(Z_U, Z_V) \sim BC_-(\psi)$ if (Z_U, Z_V) has the density (4.1) with $\det Q = -1$.

Note that this transformation basically does not change the distribution. We just express random variables and parameters in the form of complex numbers for the sake of further investigation of the properties.

As already stated in Section 2.2, marginals of Z_U and Z_V are the circular uniform, whereas both conditionals of Z_U given $Z_V = z_v$ and Z_V given $Z_U = z_u$ are the exit distributions for the circle or, more commonly, the wrapped Cauchy distributions. For brevity, we here introduce a notation $C^*(\phi)$ which is derived from McCullagh (1996) and denotes the wrapped Cauchy distribution with density

$$f(z) = \frac{1}{2\pi} \frac{1 - |\phi|^2}{|z - \phi|^2}, \quad z \in \Omega; \ |\phi| < 1.$$

The relationship, $|\phi| = \|\xi\|$ and $\arg(\phi) = \arg(\xi_1 + i\xi_2)$ where $\xi = (\xi_1, \xi_2)'$, holds between parameters of model (3.1) and those of the density above via a transformation $Z = X_1 + iX_2$. See McCullagh (1996) and Mardia and Jupp (2000, pp.51-52) for further properties of the wrapped Cauchy distribution. For model (4.1), it is easy to show that $Z_U | (Z_V = z_v) \sim C^*(\psi z_v)$ and $Z_V | (Z_U = z_u) \sim C^*(\overline{\psi} z_u)$.

4.2 Some properties

To investigate more properties of the model, it is favorable to calculate its moments. Assume that (Z_U, Z_V) has $BC_+(\psi)$. Then the moments for (Z_U, Z_V) are obtained by applying Theorem 11.13 of Rudin (1987) as

$$E\left({Z_U}^j {Z_V}^k\right) = \begin{cases} \psi^j, & j = -k, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for } j, k \in \mathbb{Z}. \tag{4.2}$$

Similarly, we get moments for BC_-. According to the theory of Fourier series expansion, one can recover the density from these moments if the density f satisfies $f \in L^2(\Omega \times \Omega)$. See Dym and McKean (1972, Section 1.10) for details.

Using these results, the following properties are established. First, model BC_+ has the multiplicative property:

$$\begin{aligned}(Z_{U1}, Z_{V1}) \sim BC_+(\psi_1) \perp (Z_{U2}, Z_{V2}) \sim BC_+(\psi_2) \\ \Longrightarrow (Z_{U1}Z_{U2}, Z_{V1}Z_{V2}) \sim BC_+(\psi_1\psi_2).\end{aligned} \tag{4.3}$$

Likewise, it is shown that model BC_- also has the multiplicative property. However, the convolution of BC_+ and BC_- is the uniform distribution, i.e.

$$\begin{aligned}(Z_{U1}, Z_{V1}) \sim BC_+(\psi_1) \perp (Z_{U2}, Z_{V2}) \sim BC_-(\psi_2) \\ \Longrightarrow (Z_{U1}Z_{U2}, Z_{V1}Z_{V2}) \sim BC_+(0).\end{aligned}$$

In addition, the following property holds for the proposed model,

$$(Z_U, Z_V) \sim BC_\pm(\psi) \implies ({Z_U}^n, {Z_V}^n) \sim BC_\pm(\psi^n) \text{ for any } n \in \mathbb{N}.$$

As n tends to infinity, $({Z_U}^n, {Z_V}^n)$ approaches a uniform distribution on the torus.

Furthermore, model (4.1) is infinitely divisible with respect to multiplication. This is proved as follows. Let $(Z_U, Z_V) \sim BC_\pm(\psi)$. Then for any positive integer n, the assumption that (Z_{Uj}, Z_{Vj}) $(j = 1, \ldots, n)$ is an iid sample from $BC_\pm(\sqrt[n]{\psi})$ yields

$$\left(\prod_{j=1}^n Z_{Uj}, \prod_{j=1}^n Z_{Vj}\right) \stackrel{d}{=} (Z_U, Z_V). \tag{4.4}$$

Remark. By putting $(Z_{Uj}, Z_{Vj}) = (\exp\{i\Theta_{Uj}\}, \exp\{i\Theta_{Vj}\})$ where $0 \leq \Theta_{Uj}, \Theta_{Vj} < 2\pi$, the left hand side of the above equation can be expressed as

$$\left(\exp\left\{i\sum_{j=1}^n \Theta_{Uj}\right\}, \exp\left\{i\sum_{j=1}^n \Theta_{Vj}\right\}\right).$$

This expression clearly shows that the product of point on the circle Z_{Uj} is essentially equivalent to the sum of angle Θ_{Uj}. Therefore property (4.4) can be called, as usual, infinite divisibility with respect to summation when the variables are transformed into angles. Similarly, the multiplicative property (4.3) could be also called additive property from this point of view.

4.3 Random vector generator

For the generation of a random vector having $BC_+(\psi)$, one certain way is to produce $\mathbb{R}^2$-valued Brownian motion and record points where the Brownian particle hits circles with radiuses of ρ and 1. However, this algorithm is a bit troublesome because we need to generate the Brownian motion at least by the time at which the particle hits the unit circle. Another way discussed in Section 3.3 could be useful, but is still more tiresome than the method proposed below. The focus of this subsection is to discuss an algorithm to generate $BC_+(\psi)$ which, we suppose, is easier than the aforementioned methods.

To obtain the random vector, we use the fact that the marginal of Z_U is circular uniform and the conditional of Z_V given $Z_U = z_u$ is the wrapped Cauchy $C^*(\overline{\psi} z_u)$. For the generation of a variate from a wrapped Cauchy distribution, we apply the result by McCullagh (1996) about the Möbius transformation of the circular uniform, i.e.

$$Z \sim C^*(0) \implies \frac{Z + \beta}{1 + \overline{\beta} Z} \sim C^*(\beta), \quad |\beta| < 1. \tag{4.5}$$

An algorithm for generating the random vector from $BC_+(\psi)$ consists of the following steps:

Step 1: Generate uniform $(0, 1)$ random numbers U_1 and U_2.

Step 2: Put $Z_U = \exp(2\pi i U_1)$ and $Z_T = \exp(2\pi i U_2)$.

Step 3: Take $Z_V = \dfrac{\overline{\psi} Z_U + Z_T}{1 + \psi \overline{Z_U} Z_T}$.

Then the joint distribution of (Z_U, Z_V) has $BC_+(\psi)$. It is clear in Step 2 that Z_U and Z_T are independently distributed as the circular uniform distributions. In Step 3, because of the property (4.5), the conditional of Z_V given $Z_U = z_u$ obeys $C^*(\overline{\psi} z_u)$. Therefore we get $(Z_U, Z_V) \sim BC_+(\psi)$.

A random vector from $BC_-(\psi)$ is obtained in a very similar manner.

4.4 Parameter estimation

This subsection provides parameter estimation of model $BC_+(\psi)$ based on maximum likelihood and method of moments using (4.2). Although we discuss the parameter estimation only for $BC_+(\psi)$ here, it is possible to get the estimates of the parameters for $BC_-(\psi)$ by a straightforward modification of the result below.

Firstly, we consider the method of moments estimation using (4.2). Assume (Z_U, Z_V) has $BC_+(\psi)$, and, as discussed in Section 4.2, its theoretical moments are given by (4.2). Suppose (Z_{Uj}, Z_{Vj}) $(j = 1, \ldots, n)$ is a random sample from $BC_+(\psi)$. The method of moments estimator is obtained by equating the theoretical and sample moments. Thus we get

$$\hat{\psi} = \frac{1}{n} \sum_{j=1}^{n} Z_{Uj} \overline{Z_{Vj}}.$$

Remark that this estimator is the same as (3.4), which is the method of moments estimator based on (3.3), if $\mathrm{rank}(\sum_{j=1}^{n} U_j V_j') = 2$.

Secondly, we discuss the maximum likelihood estimation. For a single observation, $n = 1$, it is obvious that the maximum likelihood estimator coincides with the method of moments estimator, i.e. $\hat{\psi} = Z_{U1}\overline{Z_{V1}}$. For $n \geq 2$, the estimates are obtained numerically. Note that the likelihood function can be written as

$$L(\psi) \propto \prod_{j=1}^{n} \frac{1 - |\psi|^2}{|z_{uj}\overline{z_{vj}} - \psi|^2}.$$

This expression suggests that the maximum likelihood estimation for $BC_+(\psi)$ essentially coincides with that for the wrapped Cauchy distribution $C^*(\psi)$. Therefore we can get estimates by applying the algorithm by Kent and Tyler (1988) who discussed the maximum likelihood estimation for the wrapped Cauchy distribution.

5 Related Models

5.1 Generalizations of model (2.1)

As described in Section 2.1, model (2.1) is generated by the Brownian motion starting at $B_0 = 0$. In this subsection we briefly discuss a distribution which is generated by Brownian motion starting at $B_0 = \xi$ ($\|\xi\| < \rho$), instead of $B_0 = 0$. Define a random vector $(U, V) = (QB_{\tau_1}/\|B_{\tau_1}\|, B_{\tau_2})$ in the same manner as in Section 2.1 except for the starting point of the Brownian motion. Then the density for (U, V) is given by

$$f(u, v) = \frac{1}{A_{d-1}^2} \frac{1 - \rho^2}{(1 - 2\rho u'Qv + \rho^2)^{d/2}} \frac{\rho^2 - \|\xi\|^2}{(\rho^2 - 2\rho U'Q\xi + \|\xi\|^2)^{d/2}}, \quad u, v \in S^{d-1}. \tag{5.1}$$

The marginals and conditional of V given $U = u$ are the exit distributions:

$$U \sim \mathrm{Exit}_d(\rho^{-1}Q\xi), \quad V \sim \mathrm{Exit}_d(\xi) \quad \text{and} \quad V|(U = u) \sim \mathrm{Exit}_d\left(\rho Q'u\right).$$

The conditional of U given $V = v$ is not of the familiar form. This conditional can be unimodal or bimodal and has skewed form except for some special cases such as $v = \pm\xi/\|\xi\|$. It is remarked that the bivariate circular case of model (5.1) is a submodel of the distribution briefly discussed by Kato *et al.* (2006) as a model related to a circular–circular regression model.

Another generalization is to use the method discussed in Saw (1983). This method enables us to derive a distribution with prescribed marginals.

In the bivariate circular case, it would be also promising to apply the Möbius transformation to each variable. Let $(Z_U, Z_V) \sim BC_+(\psi)$ and define a random vector

$$(\tilde{Z}_U, \tilde{Z}_V) = \left(\frac{Z_U + \alpha_1}{1 + \overline{\alpha_1} Z_U}, \frac{Z_V + \alpha_2}{1 + \overline{\alpha_2} Z_V} \right), \quad |\alpha_1|, |\alpha_2| < 1.$$

Then, because of property (4.5), the marginals of $\tilde{Z}_U$ and $\tilde{Z}_V$ have the wrapped Cauchy distributions $C^*(\alpha_1)$ and $C^*(\alpha_2)$, respectively. Another benefit of this extension is that its density has simple and exact form, including the normalizing constant without any special functions.

5.2 Related distributions on $\mathbb{R}^2$ and on the cylinder

In the past sections, we have dealt with distributions for two directional observations. In this subsection, we provide models on different manifolds, namely, $\mathbb{R}^2$ and the cylinder.

By applying bilinear fractional transformations to model (4.1), a distribution on $\mathbb{R}^2$ is constructed. Let (Z_U, Z_V) be distributed as $BC_-(\psi)$. Define a random vector (X, Y) as

$$X = i \frac{1 - Z_U}{1 + Z_U} \quad \text{and} \quad Y = i \frac{1 - Z_V}{1 + Z_V}.$$

Clearly, (X, Y) takes values on $\mathbb{R}^2$. It is straightforward to show that the joint density for (X, Y) is

$$f(x, y) = \frac{1}{\pi^2} \frac{\text{Im}(\theta)}{|x + y + \theta(1 - xy)|^2}, \quad x, y \in \mathbb{R}, \tag{5.2}$$

where $\theta = i(1 - \psi)/(1 + \psi)$. Since $|\psi| < 1$, it is evident that $\text{Im}(\theta) > 0$.

This model has the following properties:

$$X \sim C(i), \quad Y \sim C(i),$$

$$X|(Y = y) \sim C\left(\frac{\theta + y}{1 - \theta y}\right), \quad Y|(X = x) \sim C\left(\frac{\theta + x}{1 - \theta x}\right),$$

where a notation $C(\phi)$ is derived from McCullagh (1992) and denotes a Cauchy distribution on the real line with location parameter $\text{Re}(\phi)$ and scale parameter $\text{Im}(\phi)$. Thus the marginals and conditionals are real Cauchy family. Further properties of model (5.2), including the parameter estimation, are obtainable by the inverse transformations $Z_U = (1 + iX)/(1 - iX)$ and $Z_V = (1 + iY)/(1 - iY)$, which map the real line onto the unit circle in the complex plane.

A related distribution on the cylinder $\Omega \times \mathbb{R}$ is obtained in a similar fashion. Let (Z_U, Z_V) be distributed as $BC_+(\psi)$. Define a random vector

$$(Z_\Theta, X) = \left(Z_U, i \frac{1 - Z_V}{1 + Z_V} \right).$$

Then the marginals and conditionals of (Z_Θ, X) are

$$Z_\Theta \sim C^*(0), \quad X \sim C(i),$$

$$Z_\Theta|(X = x) \sim C^*\left(\frac{1+ix}{1-ix}\psi\right), \quad X|(Z_\Theta = z_\theta) \sim C\left(-i\frac{1-\overline{z_\theta}\psi}{1+\overline{z_\theta}\psi}\right).$$

Hence the marginals are circular uniform and standard Cauchy, while the conditionals are the wrapped Cauchy and linear Cauchy distributions.

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Appendices

A Derivation of probability density function (2.1)

Let $c(u,v)$ be the joint density of $(U,V) = (QB_{\tau_1}/\|B_{\tau_1}\|, B_{\tau_2})$ which is defined in the same way as in Section 2.1. Note that the density can be expressed as

$$c(u,v) = f_U(u) g_{V|U}(v|u), \quad u, v \in S^{d-1},$$

where f_U is a density for the marginal of U and $g_{V|U}$ that for the conditional of V given $U = u$. Clearly, the marginal of U is distributed as the uniform distribution and thus $f_U(u) = 1/A_{d-1}$. Because of the Markov property of the Brownian motion, the conditional of V given $U = u$ is essentially equivalent to the exit distribution for the sphere generated by Brownian motion starting at $B_0 = \rho Q' u$. (See Durrett (1984, Section 1.10)). The density for the exit distribution for the sphere is known to be

$$g_{V|U}(v|u) = \frac{1}{A_{d-1}} \frac{1-\rho^2}{\|v - \rho Q' u\|^d}, \quad v \in S^{d-1}.$$

Thus we get the density (2.1).

Density (5.1) is obtained by a straightforward modification of the calculation above.

B Proof of Theorems 2 and 3

Proof of Theorem 2. Since the marginals of U and V are uniformaly distributed over the sphere, it is evident that $E(U) = E(V) = 0$ and $E(UU') = E(VV') = d^{-1} I$.

We show that $E(UV') = d^{-1}\rho I$. Because model (2.1) is $O(d)$-symmetric in the sense of Rivest (1984), calculation of $E(UV')$ is simplified, by using Proposition 1 of his paper, to

$$E(UV') = \mathrm{diag}\{E(R_j S_j)\}Q,$$

where $(R, S) \sim BS_d(\rho I)$, $R = (R_1, \ldots, R_d)'$, $S = (S_1, \ldots, S_d)'$. Consider the integral

$$E(R_1S_1) = \int_{S^{d-1}\times S^{d-1}} r_1 s_1 c(r,s) dr ds = \int_{S^{d-1}} \frac{r_1}{A_{d-1}} \int_{S^{d-1}} \frac{s_1}{A_{d-1}} \frac{1-\rho^2}{\|s-\rho r\|^d} ds dr.$$

Transforming S into $\tilde{S} = PS$ where P is a $d \times d$ orthogonal matrix such that $P = (r, p_2, \ldots, p_d)'$, $p_j = (p_{j1}, \ldots, p_{jd})' \in \mathbb{R}^d$, we have

$$\begin{aligned}
&\int_{S^{d-1}} \frac{r_1}{A_{d-1}} \int_{S^{d-1}} \frac{s_1}{A_{d-1}} \frac{1-\rho^2}{\|s-\rho r\|^d} ds dr \\
&= \int_{S^{d-1}} \frac{r_1}{A_{d-1}} \int_{S^{d-1}} \frac{r_1\tilde{s}_1 + \sum_{j=2}^{d} p_{j1}\tilde{s}_j}{A_{d-1}} \frac{1-\rho^2}{(1-2\rho\tilde{s}_1+\rho^2)^{d/2}} d\tilde{s} dr \\
&= \int_{S^{d-1}} \frac{\tilde{s}_1}{dA_{d-1}} \frac{1-\rho^2}{(1-2\rho\tilde{s_1}+\rho^2)^{d/2}} d\tilde{s}.
\end{aligned}$$

The last equality comes from $E(R) = 0$ and $E(R_1^2) = d^{-1}$. Then, because of property (3.2), the above equation can be expressed as

$$\begin{aligned}
&\int_{S^{d-1}} \frac{\tilde{s}_1}{dA_{d-1}} \frac{1-\rho^2}{(1-2\rho\tilde{s_1}+\rho^2)^{d/2}} d\tilde{s} \\
&= \frac{1-\rho^2}{dA_{d-1}} \frac{2\pi^{(d-1)/2}}{\Gamma\{\frac{1}{2}(d-1)\}} \int_0^{\pi} \frac{\cos\theta \sin^{d-2}\theta}{(1-2\rho\cos\theta+\rho^2)^{d/2}} d\theta \\
&= \frac{1-\rho^2}{dB\{\frac{1}{2}(d-1), \frac{1}{2}\}} \int_{-1}^{1} \frac{t\,(1-t^2)^{(d-3)/2}}{(1-2\rho t+\rho^2)^{d/2}} dt \\
&= \frac{\rho}{d}.
\end{aligned}$$

The other elements, $E(R_jS_j)$ $(2 \le j \le d)$, are calculated by similar means. □

Proof of Theorem 3. Next, we prove that $T \equiv U'QV \sim H'\{\rho, \frac{1}{2}(d-2)\}$. The distribution function of T, F_T, is given by

$$\begin{aligned}
F_T(t) &= P(T \le t) = E_V\{P(U'Qv \le t \,|\, V = v)\} \\
&= E_{\tilde{V}}\left\{P(U'\tilde{v} \le t \,|\, \tilde{V} = \tilde{v})\right\},
\end{aligned}$$

where $\tilde{V} = QV$. Then transform $\tilde{U} = PU$ where $P \in O(d)$ such that $P = (\tilde{v}, p_2, \ldots, p_d)'$, $p_j \in \mathbb{R}^d$, and one obtains

$$E_{\tilde{V}}\left\{P(U'\tilde{v} \le t \,|\, \tilde{V} = \tilde{v})\right\} = \int_{S_{d-1}} \frac{1}{A_{d-1}} \int_{\substack{\tilde{U}_1 \le t \\ \tilde{U} \in S_{d-1}}} \frac{1}{A_{d-1}} \frac{1-\rho^2}{1-2\rho\tilde{u}_1+\rho^2} d\tilde{u} d\tilde{v}$$

$$= \frac{1}{A_{d-1}} \frac{2\pi^{(d-1)/2}}{\Gamma\{\frac{1}{2}(d-1)\}} \int_{\substack{\cos\theta \le t \\ 0\le\theta<\pi}} \frac{(1-\rho^2)\sin^{d-2}\theta}{(1-2\rho\cos\theta+\rho^2)^{d/2}} d\theta$$

$$= \frac{1-\rho^2}{B\{\frac{1}{2}(d-1),\frac{1}{2}\}} \int_{-1}^{t} \frac{(1-x^2)^{(d-3)/2}}{(1-2\rho x+\rho^2)^{d/2}} dx.$$

Thus

$$f_T(t) = \frac{dF_T}{dt}(t) = \frac{1-\rho^2}{B\{\frac{1}{2}(d-1),\frac{1}{2}\}} \frac{(1-t^2)^{(d-3)/2}}{(1-2\rho t+\rho^2)^{d/2}}. \quad \square$$

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