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Uncertainty principle for the Fourier-Jacobi transform

by

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Abstract

We obtain a uncertainty principle for the Fourier-Jacobi transform $\hat{f}_{\alpha,\beta}(\lambda)$. When $|\beta| \leq \alpha + 1$, as in the Euclidean case, an analogues of the uncertainty principle holds, because there is no discrete part in the Parseval formula. Moreover, we can obtain a new type of a uncertainty inequality: the L^2-norm of $\hat{f}_{\alpha,\beta}(\lambda)\lambda$ is estimated below by the L^2-norm of $(\alpha+\beta+1)f(x)(\cosh x)^{-1}$. Otherwise, the discrete part of f appears in the Parseval formula and it influences the uncertainty principle.

1. Notation. Let $\alpha, \beta \in \mathbb{C}$, $\Re\alpha > -1$ and $\rho = \alpha + \beta + 1$. For $\lambda \in \mathbb{C}$, let $\phi_\lambda(x)$ denote the Jacobi function of the first kind, that is, the unique solution of $(L + \lambda^2 + \rho^2)f = 0$ satisfying $f(0) = 1$ and $f'(0) = 0$, where

$$L = \Delta(x)^{-1}\frac{d}{dx}\Big(\Delta(x)\frac{d}{dx}\Big) \tag{1}$$

and $\Delta(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$. For $\lambda \neq -i, -2i, -3i, \ldots$, let $\Phi_\lambda(x)$ denote the Jacobi function of the second kind which satisfies

$$2\pi^{1/2}\Gamma(\alpha+1)^{-1}\phi_\lambda(x) = C(\lambda)\Phi_\lambda(x) + C(-\lambda)\Phi_{-\lambda}(x), \tag{2}$$

where $C(\lambda)$ is Harish-Chandra's C-function. Then the following estimates are well-known (cf. [2, 3]): For $x \geq 0$ and $\lambda \in \mathbb{C}$ with $|\Im\lambda| \leq \rho$

$$|\phi_\lambda(x)| \leq 1 \tag{3}$$

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and for $\delta > 0$ there exist a positive constant K_δ such that for $x \geq \delta$, $\lambda \in \mathbb{R}$

$$|\Phi_\lambda(x)| \leq K_\delta e^{-\rho x}, \tag{4}$$

and there exists a positive constant K such that for $\lambda \in \mathbb{R}$

$$|C(-\lambda)|^{-1} \leq K(1+|\lambda|)^{\alpha+1/2}. \tag{5}$$

Let $f \in C^\infty_{c,e}(\mathbb{R})$, the space of all even C^∞ functions on $\mathbb{R}$ with compact support. Then the Fourier-Jacobi transform $\hat{f}(\lambda)$ is defined as

$$\hat{f}(\lambda) = \frac{\pi^{1/2}}{\Gamma(\alpha+1)} \int_0^\infty f(x)\phi_\lambda(x)\Delta(x)dx. \tag{6}$$

This transform $f \to \hat{f}$ satisfies analogous properties of the classical cosine Fourier transform; the inversion formula, the Paley-Wiener theorem, and the Plancherel formula are obtained in [2, 3]. For convenience we suppose that $\alpha, \beta \in \mathbb{R}$ in the following. We define

$$D_{\alpha,\beta} = \{i(|\beta| - \alpha - 1 - 2m); m = 0, 1, 2, \cdots, |\beta| - \alpha - 1 - 2m > 0\}.$$

Then the inversion formula is given as follows. For $f \in C^\infty_{c,e}(\mathbb{R})$,

$$f(x) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda + \sum_{\mu \in D_{\alpha,\beta}} \hat{f}(\mu)\phi_\mu(x)d(\mu),$$

where $d(\mu) = -i\mathrm{Res}_{\lambda=\mu}(C(\lambda)C(-\lambda)^{-1})$. We denote this decomposition as

$$f = f_P + {}^\circ f \tag{7}$$

and we call f_P and ${}^\circ f$ the principal and discrete part of f respectively. We here recall that for each $\mu \in D_{\alpha,\beta}$, there exists a positive constant $K(\mu)$ such that

$$|\phi_\mu(x)| \leq K(\mu)e^{-(\rho+|\mu|)x}. \tag{8}$$

We denote by $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ a function on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\boldsymbol{F}(\nu) = \begin{cases} F(\lambda) & \text{if } \nu = \lambda \in \mathbb{R}_+ \\ a_\mu & \text{if } \nu = \mu \in D_{\alpha,\beta}. \end{cases}$$

We put $\overline{\boldsymbol{F}}(\nu) = (\overline{F(\lambda)}, \{\overline{a_\mu}\})$ and define the product of $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\mu\})$ and $\boldsymbol{G}(\nu) = (G(\lambda), \{b_\mu\})$ as

$$(\boldsymbol{FG})(\nu) = (F(\lambda)G(\lambda), \{a_\mu b_\mu\}).$$

For a function $h(\lambda)$ on $\mathbb{C}$, we define $\boldsymbol{F}(\nu)h(\nu)$ by regarding $h(\nu)$ as a function on $D_{\alpha,\beta}$. Let $d\nu$ denote the measure on $\mathbb{R}_+ \cup D_{\alpha,\beta}$ defined by

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \boldsymbol{F}(\nu) d\nu = \frac{1}{2\pi} \int_0^\infty F(\lambda)|C(\lambda)|^{-2} d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu d(\mu).$$

For $f \in C_{c,e}^\infty(\mathbb{R})$, we put

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\mu)\}).$$

Then the Parseval formula for the Fourier-Jacobi transform on $C_{c,e}^\infty(\mathbb{R})$ can be stated as follows (see [3, Theorem 2.4]):

$$\int_0^\infty f(x)\overline{g(x)}\Delta(x)dx = \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} \hat{\boldsymbol{f}}(\nu)\overline{\hat{\boldsymbol{g}}(\nu)} d\nu \tag{9}$$

for $f, g \in C_{c,e}^\infty(\mathbb{R})$. This map $f \to \hat{\boldsymbol{f}}$, $f \in C_{c,e}^\infty(\mathbb{R})$, is extended to an isometry between $L^2(\Delta) = L^2(\mathbb{R}_+, \Delta(x)dx)$ and $L^2(\nu) = L^2(\mathbb{R}_+ \cup D_{\alpha,\beta}, d\nu)$. Actually, each function f in $L^2(\Delta)$ is of the form $f = f_P + {}^\circ f$ (see (7)) and their L^2-norms are given as

$$\int_0^\infty |f_P(x)|^2 \Delta(x) = \frac{1}{2\pi} \int_0^\infty |\hat{f}(\lambda)|^2 |C(\lambda)|^{-2} d\lambda,$$

$$\int_0^\infty |{}^\circ f(x)|^2 \Delta(x) dx = \sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 d(\mu) \quad \text{if } {}^\circ f(x) = \textstyle\sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu).$$

Therefore, if we put $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$, then $\|f\|_{L^2(\Delta)} = \|\hat{\boldsymbol{f}}\|_{L^2(\nu)}$ and

$$f(x) = \frac{1}{2\pi} \int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2} d\lambda + \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu). \tag{10}$$

We define

$$B(x) = \int_0^x \Delta(t)dt, \ x \geq 0 \tag{11}$$

and put

$$\theta(x) = \frac{B(x)}{\Delta(x)} \quad \text{and} \quad \Theta(\lambda) = (\lambda^2 + \rho^2)^{1/2}.$$

2. Main theorem. We keep the notation in §1 and prove the following.

Theorem 2.1. *Let $\alpha > -1$, $\beta \in \mathbb{R}$. For $f \in L^2(\Delta)$, we suppose that $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. Then*

$$\|f\theta\|^2_{L^2(\Delta)} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \Theta(\nu)^2 d\nu \geq \frac{1}{4} \|f\|^4_{L^2(\Delta)}, \tag{12}$$

where the equality holds if and only if f is of the form

$$f(x) = ce^{\gamma \int_0^x \theta(t)dt}$$

for some $c, \gamma \in \mathbb{C}$.

Proof. Without loss of generality we may supppose that $f \in C^\infty_{c,e}(\mathbb{R})$ and f is real valued. Since

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\mu)\}).$$

and $(-Lf)^\wedge(\lambda) = \hat{f}(\lambda)(\lambda^2 + \rho^2) = \hat{f}(\lambda)\Theta(\lambda)^2$, the Parseval formula (9) yields that

$$\begin{aligned} \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \Theta(\nu)^2 d\nu &= \int_0^\infty f(x)(-Lf)(x)\Delta(x)dx \\ &= \int_0^\infty (f'(x))^2 \Delta(x)dx. \end{aligned}$$

Hence it follows that

$$\begin{aligned} & \int_0^\infty f(x)^2\theta(x)^2\Delta(x)dx \int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \Theta(\nu)^2 d\nu \\ = & \int_0^\infty f(x)^2\theta(x)^2\Delta(x)dx \int_0^\infty (f'(x))^2\Delta(x)dx \\ \geq & \Big(\int_0^\infty f(x)f'(x)\theta(x)\Delta(x)dx \Big)^2 \\ = & \frac{1}{4}\Big(\int_0^\infty (f(x)^2)'B(x)dx \Big)^2 = \frac{1}{4}\Big(\int_0^\infty f(x)^2\Delta(x)dx \Big)^2. \end{aligned}$$

Here we used the fact that $B' = \Delta$ (see (11)). Clearly, the equality holds if and only if $f\theta = cf'$ for some $c \in \mathbb{R}$, that is, $f'/f = c^{-1}\theta$. This means that $\log(f) = c^{-1}\int_0^x \theta(t)dt + C$ and thus, the desired result follows. ■

We recall that $\Theta^2(\lambda) = \lambda^2 + \rho^2$. Then (12) and the Parseval formula imply the following.

Corollary 2.2. *Let f be the same as in Theorem 2.1.*

$$\|f\theta\|^2_{L^2(\Delta)}\int_{\mathbb{R}_+\cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2\nu^2 d\nu \geq \frac{1}{4}\|f\|^2_{L^2(\Delta)}\int_0^\infty |f(x)|^2(1-4\rho^2\theta^2)\Delta(x)dx. \quad (13)$$

We shall estimate θ and $1-4\rho^2\theta^2$. Since $\alpha > -1$, it follows that

$$\begin{aligned} B(x) &= \int_0^x (2\sinh s)^{2\alpha+1}(2\cosh s)^{2\beta+1} ds \\ &= 2^{2\rho}\int_0^{\sinh x} t^{2\alpha+1}(1+t^2)^\beta dt \\ &= 2^{2\rho}(\sinh x)^{2\alpha+2}\int_0^1 t^{2\alpha+1}(1+(\sinh x)^2t^2)^\beta dt \\ &= 2^{2\rho-1}(\sinh x)^{2\alpha+2}(\cosh x)^{2\beta}\int_0^1 (1-s)^\alpha(1-(\tanh x)^2 s)^\beta ds \\ &= 2^{2\rho-1}(\sinh x)^{2\alpha+2}(\cosh x)^{2\beta}\frac{1}{\alpha+1}F(1,-\beta,2+\alpha;(\tanh x)^2) \end{aligned}$$

and thus,

$$\theta(x) = \frac{1}{2(\alpha+1)}F(1,-\beta,2+\alpha;(\tanh x)^2)\tanh x. \quad (14)$$

Lemma 2.3. *Let $k = 0, 1, 2, \cdots$ and $0 \leq x \leq 1$. We suppose that $k-(\alpha+1) < \beta \leq k+(2k+1)\alpha$. Then $x^{2k+1}F(k+1, k-\beta, k+2+\alpha, x^2)$ is increasing and*

$$0 \leq x^{2k+1}F(k+1, k-\beta, k+2+\alpha; x^2) \leq \frac{\Gamma(k+2+\alpha)\Gamma(\rho-k)}{\Gamma(1+\alpha)\Gamma(\rho+1)}.$$

Proof. When $k-(\alpha+1) < \beta \le k$, it follows that $F(k+1, k-\beta, k+2+\alpha; x)$ is increasing on $0 \le x \le 1$. Hence $H(x) = H_k(\alpha,\beta,x) = x^{2k+1}F(k+1, k-\beta, k+2+\alpha; x^2)$ is dominated by $H(1) = \Gamma(k+2+\alpha)\Gamma(\rho-k)/\Gamma(1+\alpha)\Gamma(\rho+1)$. Let $k < \beta \le k+(2k+1)\alpha$. We shall prove that $H(x)$ is increasing and $H(x) \le H(1)$ as before. In order to prove that $H(x)$ is increasing, we shall show that its derivative is positive. We note that

$$\begin{aligned} H'(x) &= (1+2k)x^{-1}H_k(\alpha,\beta,x) + \frac{2(1+k)(k-\beta)x^{-1}}{2+k+\alpha}H_{k+1}(\alpha,\beta,x) \quad (15) \\ &= (1+2k)x^{-1}H_k(\alpha,\beta,x) \\ &\quad +2(1+k+\alpha)x^{-1}\Big(H_k(\alpha-1,\beta,x) - H_k(\alpha,\beta,x)\Big) \\ &= x^{2k}K(x^2), \end{aligned}$$

where $K(x) = (1+2k)F(1+k, k-\beta, 2+k+\alpha; x) + 2(1+k+\alpha)(F(1+k, k-\beta, 1+k+\alpha; x) - F(1+k, k-\beta, 2+k+\alpha, x))$. Then

$$\begin{aligned} K'(x) &= (1+k)(k-\beta)x^{-(2k+3)}\Big(\frac{(1+2k)}{2+k+\alpha}H_{k+1}(\alpha,\beta,x) \\ &\quad +2(1+k+\alpha)\Big(\frac{H_{k+1}(\alpha-1,\beta,x)}{1+k+\alpha} - \frac{H_{k+1}(\alpha,\beta,x)}{2+k+\alpha}\Big)\Big). \end{aligned}$$

Since $\beta > k$, $x^{-(2k+3)}H_{k+1}(\alpha,\beta,x) = F(2+k, 1+k-\beta, 3+k+\alpha; x) \le F(2+k, 1+k-\beta, 2+k+\alpha; x) = x^{-(2k+3)}H_{k+1}(\alpha-1,\beta,x)$ and $1/(1+k+\alpha) - 1/(2+k+\alpha) > 0$, it follows that $K'(x) < 0$ and thus, $K(x)$ is decreasing. Therefore, $H'(x)$ is decreasing and

$$H'(x) \ge H'(1) = (k+(2k+1)\alpha-\beta)\frac{\Gamma(2+k+\alpha)\Gamma(\alpha+\beta-k)}{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)} \ge 0$$

under the assumption on β. Hence $H(x)$ is increasing. ■

Lemma 2.4. *Let notation be as above and $\rho > 0$. If $-(\alpha+1) < \beta \le \alpha$, then*

$$0 \le \theta(x) \le 1/2\rho$$

and if $\alpha < \beta$ and k is an integer such that $k-1+(2k-1)\alpha < \beta \le k+(2k+1)\alpha$, then

$$0 \le \theta(x) \le \frac{1}{2\rho}\frac{(2k)!!}{(2k-1)!!}\frac{\Gamma(\beta+1)\Gamma(\rho-k)}{\Gamma(\beta-k+1)\Gamma(\rho)} = O(\sqrt{\beta}).$$

Proof. Clearly, $\theta(x) \geq 0$ from Euler's integral expression of hypergeometric functions. The first assertion follows from (9) and Lemma 2.3 with $k = 0$. We suppose that $\alpha < \beta \leq 1 + 3\alpha$, that is, the case of $k = 1$ in Lemma 2.3. Since β is out of the range when $k = 0$, we couldn't conclude that $H(x) = xF(1, -\beta, 2+\alpha; x^2)$ is increasing on $0 \leq x \leq 1$. Let $x = x_0$ be the maximaum point of $H(x)$. Since

$$H'(x) = \Big(F(1, -\beta, \alpha+2, x^2) - \frac{2\beta}{2+\alpha} x^2 F(2, 1-\beta, \alpha+3, x^2)\Big)$$

and $H'(x_0) = 0$, it follows that

$$H(x_0) = \frac{2\beta}{2+\alpha} x_0^3 F(2, 1-\beta, \alpha+3, x_0^2).$$

Since $\alpha < \beta \leq 1 + 3\alpha$, applying Lemma 2.3 with $k = 1$, we see that

$$\begin{aligned} \theta(x) \leq \frac{H(x_0)}{2(\alpha+1)} &\leq \frac{1}{2(\alpha+1)} \frac{2\beta}{2+\alpha} \frac{\Gamma(3+\alpha)\Gamma(\rho-1)}{\Gamma(1+\alpha)\Gamma(\rho+1)} \\ &= \frac{1}{2\rho} \frac{2!!}{1!!} \frac{\Gamma(1+\beta)\Gamma(\rho-1)}{\Gamma(\beta)\Gamma(\rho)}. \end{aligned}$$

When $1 + 3\alpha < \beta \leq 2 + 5\alpha$, we couldn't apply Lemma 2.3 in the above argument to conclude that $x^3 F(2, 1-\beta, \alpha+3, x^2)$ is increasing on $0 \leq x \leq 1$. Hence, we shall consider its derivative and the maximum point again. Then we can apply Lemma 2.3 with $k = 2$ to the derivative. Generally, when $(k-1) + (2k-1)\alpha < \beta \leq k + (2k+1)\alpha$, $H_l(\alpha, \beta, x) = x^{2l+1} F(l+1, l-\beta, l+2+\alpha; x^2)$, $0 \leq l \leq k-1$, are not increasing and $H_k(\alpha, \beta, x)$ is increasing. Then it follows from (9) that

$$H_{l-1}(\alpha, \beta, x_{l-1}) = \frac{2l(\beta - l + 1)}{(2l-1)(1+l+\alpha)} H_l(\alpha, \beta, x_{l-1}),$$

where x_{l-1} is the maximum point of $H_{l-1}(\alpha, \beta, x)$ and thus,

$$\begin{aligned} \theta(x) &\leq \frac{1}{2(\alpha+1)} \prod_{l=1}^{k} \frac{2l(\beta - l + 1)}{(2l-1)(1+l+\alpha)} \frac{\Gamma(k+2+\alpha)\Gamma(\rho-k)}{\Gamma(1+\alpha)\Gamma(\rho+1)} \\ &= \frac{1}{2\rho} \frac{(2k)!!}{(2k-1)!!} \frac{\Gamma(\beta+1)\Gamma(\rho-k)}{\Gamma(\beta-k+1)\Gamma(\rho)}. \end{aligned}$$

The asymptotic behavior of $\theta(x)$ follows from Wallis' formula. ■

Lemma 2.5. *Let* $\Upsilon(x) = 1 - 4\rho^2\theta(x)^2$. *If* $-(\alpha+1) < \beta \leq 0$, *then* $\Upsilon(x) \geq (\cosh x)^{-2}$. *Generally,*

$$\Upsilon(x) = \begin{cases} O((\cosh x)^{-2}) & \text{if } x \to \infty, \\ O(1) & \text{if } x \to 0 \end{cases}$$

and if $\beta \leq \alpha$, *then* $\Upsilon(x) \geq 0$.

Proof. Since $F(1, -\beta, 2+\alpha; 0) = 1$ and $F(1, -\beta, 2+\alpha; 1) = (\alpha+1)/\rho$, the asymptotic behaviour easily follows. If $-(\alpha+1) < \beta \leq 0$, then $F(1, -\beta, 2+\alpha; x)$ is increasing with respect to x. Hence $\theta(x) \leq F(1, -\beta, 2+\alpha; 1) \tanh x$ $/2(\alpha+1) \leq (1/2\rho) \tanh x$ and thus, $\Upsilon(x) \geq (\cosh x)^{-2}$. If $0 < \beta \leq \alpha$, then $\Upsilon(x) \geq 0$ from Lemma 2.4. ■

We put

$$\varepsilon_k = \varepsilon_k(\alpha, \beta) = \frac{(2k-1)!!}{(2k)!!} \frac{\Gamma(\beta - k + 1)\Gamma(\rho)}{\Gamma(\beta+1)\Gamma(\rho - k)}. \tag{16}$$

Lemma 2.4 implies that, if $k - 1 + (2k-1)\alpha < \beta \leq k + (2k+1)\alpha$, then

$$\theta(x) \leq (2\rho\epsilon_k)^{-1}. \tag{17}$$

The following assertion follows from Theorem 2.1, Corollary 2.2, Lemma 2.4 and Lemma 2.5.

Corollary 2.6. *Let* $\rho > 0$ *and* f *be the same as in Theorem 2.1. If* $-(\alpha+1) < \beta \leq \alpha$, *then* $f = f_P$,

$$\int_0^\infty |\hat{f}(\lambda)|^2 \Theta(\lambda)^2 |C(\lambda)|^{-2} d\lambda \geq \rho^2 \|f\|^2_{L^2(\Delta)}$$

and

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \geq \rho^2 \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx,$$

where $\Upsilon(x) = 1 - 4\rho^2\theta(x)^2 \geq 0$, *and if* $k - 1 + (2k-1)\alpha < \beta \leq k + (2k+1)\alpha$, *then*

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \Theta(\nu)^2 d\nu \geq \rho^2 \varepsilon_k^2 \|f\|^2_{L^2(\Delta)},$$

and

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \geq \rho^2 \varepsilon_k^2 \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx. \tag{18}$$

The shapes of $\theta(t)$ with $1/2\rho$ and $\Upsilon(t)$, $t = \tanh x$, are respectively given as follows.

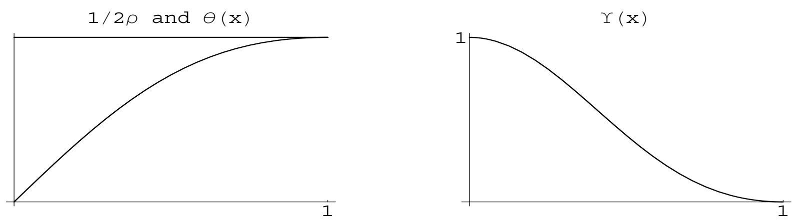

Figure 1: The case of $\beta \leq \alpha$.

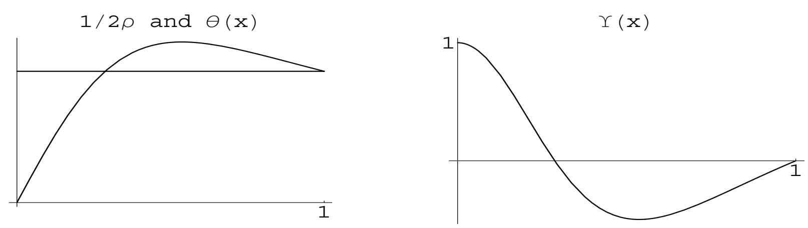

Figure 2: The case of $\beta > \alpha$.

3. Uncertainty. We shall apply the inequalities obtained in the previous section to deduce some information on the concentration of f and $\hat{f}$. Let f be a non-zero function in $L^2(\Delta)$ satisfying $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. We recall that

$$f = f_P + {}^\circ f, \quad {}^\circ f(x) = \sum_{\mu \in D_{\alpha,\beta}} a_\mu \phi_\mu(x) d(\mu)$$

and $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{a_\mu\})$.

Definition 3.1. *Let* $0 < \epsilon < 1/4\rho^2$ *and* $M > 0$.

(1) We say that a function $f(x)$ on $\mathbb{R}_+$ is (θ, ϵ)-concentrated at $x = 0$ if

$$\|f\theta\|_{L^2(\Delta)} \leq \epsilon \|f\|_{L^2(\Delta)}$$

and is (θ, ϵ)-nonconcentrated at $x = 0$ if the reverse holds.

(2) We say that a function $\hat{f}(\lambda)$ on $\mathbb{R}_+$ is (λ, ϵ)-concentrated at $\lambda = 0$ if

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \leq \epsilon \|f\|^2_{L^2(\Delta)}$$

and is (λ, ϵ)-nonconcentrated at $\lambda = 0$ if the reverse holds.

(3) We say that a function $f(x)$ on $\mathbb{R}_+$ is (μ, ϵ)-concentrated at $x = 0$ if

$$\sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \leq \epsilon \|f\|^2_{L^2(\Delta)}.$$

(4) We say that a function $f(x)$ on $\mathbb{R}_+$ is (Υ, ϵ)-nonconcentrated at $x = 0$ if

$$\left| \int_0^\infty |f(x)|^2 \Upsilon(x) \Delta(x) dx \right| \leq \epsilon \|f\|^2_{L^2(\Delta)}.$$

(5) We say that a function $f(x)$ on $\mathbb{R}_+$ is (x_0, ϵ)-bounded if

$$|f(x)| \leq \epsilon \|f\|_{L^2(\Delta)} \text{ if } x \geq x_0.$$

Now we suppose that $f(x)$ is (θ, ϵ)-concentrated at $x = 0$. Since

$$\begin{aligned} &\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\boldsymbol{\hat{f}}(\nu)|^2 \Theta(\nu)^2 d\nu \\ = &\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) + \rho^2 \|f\|^2_{L^2(\Delta)}, \end{aligned}$$

it follows from (12) that

$$\int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda - \sum_{D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \geq (1/4\epsilon - \rho^2) \|f\|^2_{L^2(\Delta)}.$$

Therefore, $\hat{f}(\nu)$ is $(\lambda, 1/4\epsilon - \rho^2)$-nonconcentrated at $\lambda = 0$.

Conversely, we suppose that $\hat{f}(\nu)$ is (ν, ϵ)-concentrated at $\lambda = 0$. Then it follows from (17) that, if $k-1+(2k-1)\alpha < \beta \le k+(2k+1)\alpha$, then

$$\int_0^\infty |f_P(x)|^2 \Upsilon(x) \Delta(x) dx \ge (1-\varepsilon_k^{-2}) \|f_P\|^2_{L^2(\Delta)}.$$

Here we recall that $1-\varepsilon_k^{-2} \le 0$. Moreover, letting $A = \displaystyle\int_0^\infty |f_P(x)|^2 \Upsilon(x)| \Delta(x)dx$ and $B = \|f_P\|^2_{L^2(\Delta)}$, we see from (13) for $f = f_P$ that

$$(B-A)\epsilon B \ge \rho^2 AB$$

and thus,

$$A \le \frac{\epsilon B}{\rho^2+\epsilon} \le \frac{\epsilon}{\rho^2} B. \tag{19}$$

Therefore, $f_P(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where

$$\delta = \max\{\varepsilon_k^{-2} - 1, \rho^{-2}\epsilon\}.$$

Moreover, let $x_0 = 1$. Then it follows from (2), (3), and (4) that for $x \ge 1$,

$$\begin{aligned}
|f_P(x)| &\le \Big| \int_0^\infty \hat{f}(\lambda) \Phi_\lambda(x) C(\lambda)^{-1} d\lambda \Big| \\
&\le e^{-\rho x} K_{x_0} \Big(\int_0^{\sqrt{\epsilon}} |\hat{f}(\lambda)||C(-\lambda)|^{-1} d\lambda + \int_{\sqrt{\epsilon}}^\infty |\hat{f}(\lambda)||C(-\lambda)|^{-1} d\lambda \Big) \\
&\le e^{-\rho x} K_{x_0} \Big(\epsilon^{1/4} \|f_P\|_{L^2(\Delta)} \\
&\qquad + \Big(\int_{\sqrt{\epsilon}}^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda \Big)^{1/2} \Big(\int_{\sqrt{\epsilon}}^\infty \lambda^{-2} d\lambda \Big)^{1/2} \Big) \\
&\le 2e^{-\rho x} K_{x_0} \epsilon^{1/4} \|f_P\|_{L^2(\Delta)}.
\end{aligned}$$

Hence we have the following.

Theorem 3.2 *Let $\rho > 0$ and $f \in L^2(\Delta)$ satisfy $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. Let k be an integer such that $k-1+(2k-1)\alpha < \beta \le k+(2k+1)\alpha$, where $k = 0$ if $\beta \le \alpha$. We define ε_k by (16) where $\epsilon_0 = 1$. If $f(x)$ is (θ, ϵ)-concentrated at $x = 0$, then $\hat{f}(\lambda)$ is $(\lambda, 1/4\epsilon - \rho^2)$-nonconcentrated at $\lambda = 0$. Conversely, if $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$, then $f_P(x)$ is (Υ, δ)-nonconcentrated* *at* $x=0$*, where* $\delta=\max\{\epsilon_k^{-2}-1,\rho^{-2}\epsilon\}$*, and there exists a positive constant* c *such that* $f_P(x)$ *is* $(1,c\epsilon^{1/4})$*-bounded.*

When $\beta\le\alpha$, we recall that $f=f_P$ and $\varepsilon_0=1$ $(k=0)$. Hence, the above theorem implies that $f(x)$ is $(\Upsilon,\rho^{-2}\epsilon)$-nonconcentrated at $x=0$ and $(1,c\epsilon^{1/4})$-bounded. Therefore, $f(x)$ is spread if ϵ goes to 0. However, when $\beta>\alpha$, then $\varepsilon_k<1$ and it is not clear that $f(x)$ is spread if ϵ goes to 0. This implies that the discrete part of f influences the uncertaintity.

We now suppose that $\beta>\alpha$, $\hat{f}(\lambda)$ is (λ,ϵ)-concentrated at $\lambda=0$ and $f(x)$ is (μ,ϵ_d)-concentrated at $x=0$. We shall prove that $f(x)$ is spread if ϵ and ϵ_d go to 0. As in the previous argument, let $A=\int_0^\infty|f(x)|^2\Upsilon(x)|\Delta(x)dx$ and $B=\|f\|^2_{L^2(\Delta)}$. Then it follows from (13) that

$$(B-A)(\epsilon+\epsilon_d)B\ge\rho^2AB$$

and thus, $A\le\rho^{-2}(\epsilon+\epsilon_d)B$. Let $x_0>0$ the point such that $\Upsilon(x_0)=0$ and $x\ge x_0$. As before, it follows that

$$|f_P(x)|\le e^{-\rho x}K_{x_0}\epsilon^{1/4}\|f_P\|_{L^2(\Delta)}.$$

On the other hand, it follows from (8) that

$$\begin{aligned}|{}^\circ f(x)| &\le \sum_{\mu\in D_{\alpha,\beta}}|a_\mu||\phi_\mu(x)|d(\mu)\\ &\le e^{-\rho x}\Big(\sum_{\mu\in D_{\alpha,\beta}}e^{-2|\mu|x_0}|\mu|^{-2}d(\mu)\Big)^{1/2}\epsilon_d^{1/2}\|{}^\circ f\|_{L^2(\Delta)}.\end{aligned}$$

Hence, for $x\ge x_0$, we see that there exist a positive constant c such that

$$|f(x)e^{\rho x}| \;\le\; c(\epsilon^{1/4}+\epsilon_d^{1/2})\|f\|_{L^2(\Delta)}.$$

Therefore, it follows that

$$\begin{aligned}\int_0^\infty|f(x)|^2\Upsilon(x)\Delta(x)dx &\ge c\int_{x_0}^\infty|f(x)e^{\rho x}|^2\Upsilon(x)dx\\ &\ge c^2(\epsilon^{1/4}+\epsilon_d^{1/2})^2\|f\|^2_{L^2(\Delta)}\int_{x_0}^\infty\Upsilon(x)dx\\ &= c_\Upsilon(\epsilon^{1/4}+\epsilon_d^{1/2})^2\|f\|^2_{L^2(\Delta)}.\end{aligned}$$

Here $c_\Upsilon < 0$, because $\displaystyle\int_{x_0}^{\infty} \Upsilon(x)dx < 0$.

Theorem 3.3 *Let $\beta > \alpha$ and $\alpha > -1$. Let $f \in L^2(\Delta)$ satisfy $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. We suppose that $\hat{f}(\lambda)$ is (λ, ϵ)-concentrated at $\lambda = 0$ and $f(x)$ is (μ, ϵ_d)-concentrated at $x = 0$. Then there exists constants $c_\Upsilon < 0, c > 0$ such that $f(x)$ is (Υ, δ)-nonconcentrated at $x = 0$, where $\delta = \max\{-c_\Upsilon(\epsilon^{1/4} + \epsilon_d^{1/2}), \rho^{-2}(\epsilon + \epsilon_d)\}$, and is $(x_0, c(\epsilon^{1/4} + \epsilon_d^{1/2}))$-bounded.*

We suppose that f is supported on $[R, \infty)$. Then there exists a constant $0 < \varepsilon(R) \le 1$ such that

$$0 \le \theta(x) \le \frac{1}{2\rho\varepsilon(R)}$$

and $\varepsilon(R) \to 1$ if $R \to \infty$. Then it follows from (13) that

$$\int_{\mathbb{R}_+ \cup D_{\alpha,\beta}} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \ge \rho^2 \int_0^\infty |f(x)|^2 (\varepsilon(R)^2 - 1)\Delta(x)dx.$$

Then we obtain the following.

Proposition 3.4. *Let $\rho > 0$ and suppose that $f \in L^2(\Delta)$ satisfies $f\theta \in L^2(\Delta)$ and $\hat{\boldsymbol{f}}\Theta \in L^2(\nu)$. We suppose that f is supported on $[R, \infty)$. Then*

$$\sum_{\mu \in D_{\alpha,\beta}} |a_\mu|^2 |\mu|^2 d(\mu) \le \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 |C(\lambda)|^{-2} d\lambda + \rho^2(1 - \varepsilon(R)^2)\|f\|^2_{L^2(\Delta)}.$$

Remark 3.5. When $\beta = 0$, we can calculate more precisely. In this case $\theta = (2\rho)^{-1}\tanh x$ and $1 - 4\rho^2\theta^2 = (\cosh x)^{-2}$. Therefore, (12) and (13) became

$$\|f(x)\tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)(\lambda^2 + \rho^2)^{1/2}\|^2_{L^2(|C|^{-2})} \ge \rho^2 \|f\|^4_{L^2(\Delta)},$$

where the equality holds if and only if f is of the form $c(\cosh x)^\gamma$, and

$$\|f(x)\tanh x\|^2_{L^2(\Delta)} \|\hat{f}(\lambda)\lambda\|^2_{L^2(|C|^{-2})} \ge \rho^2 \|f\|^2_{L^2(\Delta)} \|f(x)(\cosh x)^{-1}\|^2_{L^2(\Delta)}.$$

Since the Jacobi transform of $(\cosh \lambda)^\gamma$ is explicitly calcurated in [1], we can directly check the equality condition for these inequalities.

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