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Geometric objects in an approach to quantum geometry

by

Hideki Omori, Yoshiaki Maeda,
Naoya Miyazaki, Akira Yoshioka

Hideki Omori Tokyo University of Science	Yoshiaki Maeda Keio University
Naoya Miyazaki Keio University	Akira Yoshioka Tokyo University of Science

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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Geometric objects in an approach to quantum geometry

Hideki Omori
Department of Mathematics, Tokyo University of Science,
Noda, Chiba, 278-8510, Japan,
omori@ma.noda.tus.ac.jp

Yoshiaki Maeda *
Department of Mathematics, Faculty of Science and Technology,
Keio University, Hiyoshi, Yokohama, 223-8825, Japan,
maeda@math.keio.ac.jp;

Naoya Miyazaki †
Department of Mathematics, Faculty of Economics,
Keio University, Hiyoshi, Yokohama, 223-8521, Japan,
miyazaki@math.hc.keio.ac.jp;

Akira Yoshioka ‡
Department of Mathematics, Tokyo University of Science,
Kagurazaka, Tokyo, 102-8601, Japan,
yoshioka@rs.kagu.tus.ac.jp;

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Abstract Ideas from deformation quantization applied to algebra with one generator lead to the construction of non-linear flat connection, whose parallel sections have algebraic significance. The moduli space of parallel sections is studied as an example of bundle-like objects with discordant (sogo) transition functions, which suggests a method to treat families of meromorphic functions with smoothly varying branch points.

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1 Introduction

The aim of this paper is to show that deformation quantization provides us with a new geometric idea going beyond classical geometry. In fact, there have been several attempts to describe "quantum objects" in a geometric way (cf. [3], [5], [6]), although no treatment has been accepted as definitive. Motivated by these attempts, we produce a description of objects which arise from the deformation of algebras, as one approach to describing quantum mechanics mathematically is via deformation quantization, which is a deformation of Poisson algebras. Through the construction of the star exponential functions of the quadratic forms in the complex Weyl algebra, we found several strange phenomena which cannot be treated as classical geometric objects (cf. [9], [11], [12], [13]). Our main concern is to understand how to handle these objects geometrically, and we consider our results are a step toward quantum geometry. However, similar questions arise even for deformations of commutative algebras, as in the case of deformation quantizations. For this reason, in this paper we deal with the simplest case of the deformation of the associative commutative algebra of polynomials of one variable.

In §2.1, we construct an algebra $\mathbb{C}_*[\zeta]$ whose elements are elements of $\mathbb{C}[\zeta]$ parametrized by the indeterminate κ. Motivated by deformation quantization, we introduce associative commutative products on $\mathbb{C}[\zeta]$ parametrized by a complex number κ (cf. Definition 2.1), which gives both a deformation of the canonical product and a representation parameterized by κ of $\mathbb{C}$.

Our standpoint formulated in § 2.1 is to view elements in the abstract algebra $\mathbb{C}_*[\zeta]$ as a family of elements. The deformation parameter κ is viewed as an indeterminate.

One method of treating this family of elements as geometric objects is to introduce the notion of infinitesimal intertwiners, which play the role of a connection. In fact, elements of $\mathbb{C}_*[\zeta]$ can be viewed as parallel sections with respect to this connection. These elements are called q-number polynomials.

In § 2.2 and § 2.3, we extend this setting to a class of transcendental elements such as exponential functions. In this setting, the notion of densely defined multi-valued parallel sections appears crucially. We also call these q-number functions in analogy with [1]. However, the only geometrical setting possible is to extend the infinitesimal intertwiners to a linear connection on a trivial bundle over $\mathbb{C}$ with a certain Fréchet space of entire functions.

In § 3 we investigate the moduli space of densely defined parallel sections consisting of exponential functions of quadratic forms. We show that the moduli space is not an ordinary bundle, as it contains fuzzy transition functions. This has similarities to the theory of gerbes (cf. [2], [8]).

However, our construction has a different flavor from the differential geometric point of view, since gerbes are classified by the Dixmier-Douady classes in the third cohomology over $\mathbb{Z}$, while our example is constructed on the 2-sphere or the complex plane. We prefer call this fuzzy object a *pile*, although $\mathbb{Z}_2$-gerbes have been proposed as similar notions (cf.[14]).

We run into a similar situation in quantizing non-integral closed 2-forms on manifolds. As for integral symplectic forms on symplectic manifolds, we can construct a prequantum bundle, which is a line bundle with connection whose curvature is given by the symplectic form. We attempt the prequantization of a non-integral closed 2-form by mimicing our examples describing the moduli space of densely defined multi-valued parallel sections. We note that Melrose [7] proposed a method handling a type of prequantization of non-integral closed 2-forms, which seems closely related to our approach.

In § 5, we give an simple example for treating solution spaces of ordinary differential equation with movable branch singularities. We introduce an associative product on the space of parallel sections of exponential functions of quadratic forms, but this product is "broken" in the sense that for every κ, there is a singular set on which the product diverges. Thanks to the movable singularities, this broken product defines an associative product by treating κ as an indeterminate. This computation provides a novel aspect of the noncommutative calculus. We also hope that our attempt will help with the study for solutions of ordinary differential equations with movable branch singularities.

In the end, our work seems to extend the notion of *points* as established elements of a fixed set to a more flexible notion of elements.

2 Deformation of a commutative product

We construct an algebra $\mathbb{C}_*[\zeta]$ whose elements are elements of $\mathbb{C}[\zeta]$ parametrized by the indeterminate κ. For convenience, we denote by $*$ the product on the algebra $\mathbb{C}_*[\zeta]$. The algebra $\mathbb{C}_*[\zeta]$ is isomorphic to the algebra $\mathbb{C}[\zeta]$ of polynomials in ζ over $\mathbb{C}$, but we will view $\mathbb{C}_*[\zeta]$ as a family of algebras which are mutually *isomorphic*.

2.1 A deformation of commutative product on $\mathbb{C}[\zeta]$

We denote the set of polynomials of ζ viewed as a linear space by $\mathcal{P}(\mathbb{C})$. We introduce a family of product $*_\kappa$ on $\mathcal{P}(\mathbb{C})$ parametrized by $\kappa \in \mathbb{C}$ as follows.

Definition 2.1. *For every $f, g \in \mathcal{P}(\mathbb{C})$, we set*

$$f *_\kappa g = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\big(\frac{\kappa}{2}\big)^{\ell} \partial_\zeta^\ell f(\zeta) \cdot \partial_\zeta^\ell g(\zeta).$$

Then $(\mathcal{P}(\mathbb{C}), *_\kappa)$ is an associative commutative algebra for every $\kappa \in \mathbb{C}$. Since putting $\kappa = 0$ gives the algebra $\mathbb{C}[\zeta]$, the family of algebras $\{(\mathcal{P}(\mathbb{C}), *_\kappa)\}_{\kappa \in \mathbb{C}}$ gives a deformation of $\mathbb{C}[\zeta]$ within associative commutative algebras. We note the following.

Lemma 2.2. *For every $\kappa, \kappa' \in \mathbb{C}$, the algebras $(\mathcal{P}(\mathbb{C}), *_\kappa)$ and $(\mathcal{P}(\mathbb{C}), *_{\kappa'})$ are* *mutually isomorphic. Namely, the mapping* $T_\kappa^{\kappa'}:\mathcal{P}(\mathbb{C})\to\mathcal{P}(\mathbb{C})$ *given by*

$$T_\kappa^{\kappa'}(f)=\Big(\exp\frac{1}{4}(\kappa'-\kappa)\partial_\zeta^2\Big)f(\zeta)=\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\big(\frac{1}{4}(\kappa'-\kappa)\big)^{\ell}(\partial_\zeta^{2\ell})f(\zeta) \tag{1}$$

satisfies $T_\kappa^{\kappa'}(f*_\kappa g)=T_\kappa^{\kappa'}(f)*_{\kappa'}T_\kappa^{\kappa'}(g)$.

Definition 2.3. *The isomorphism* $T_\kappa^{\kappa'}$ *given by* (1) *is called the intertwiner between the algebras* $(\mathcal{P}(\mathbb{C}),*_\kappa)$ *and* $(\mathcal{P}(\mathbb{C}),*_{\kappa'})$.

Taking the derivative in κ' for $T_\kappa^{\kappa'}$ defines an *infinitesimal intertwiner*. Namely, for $\kappa\in\mathbb{C}$ we set

$$t_\kappa(u)(f)=\frac{d}{ds}\Big|_{s=0}T_\kappa^{\kappa+su}(f)=\frac{1}{4}u\partial_\zeta^2 f. \tag{2}$$

The infinitesimal intertwiner gives a realization of $\mathbb{C}_*[\zeta]$ as follows. Let $\pi:\mathbb{C}\times\mathcal{P}(\mathbb{C})\to\mathbb{C}$ be the trivial bundle over $\mathbb{C}$, and $\Gamma(\mathbb{C}\times P(\mathbb{C}))$ the set of sections of this bundle. Using the infinitesimal intertwiner defined by (2), we introduce a connection ∇ on $\Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$: For a smooth curve $c(s)$ in $\mathbb{C}$ and $\gamma\in\Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$, we set

$$\nabla_{\dot c}\gamma(s)=\frac{d}{ds}_{|s=0}\gamma(c(s))-t_{c(s)}(\dot c(s))(\gamma(c(s)),\quad\text{where }\dot c(s)=\frac{d}{ds}c(s). \tag{3}$$

Definition 2.4. *A section* $\gamma\in\Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$ *is parallel if* $\nabla\gamma=0$. *We denote by* $\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$ *the set of all parallel sections* $\gamma\in\Gamma(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$.

Let us consider an element $f_*\in\mathbb{C}_*[\zeta]$. Corresponding to the unique expression of an element $f_*\in\mathbb{C}_*[\zeta]$ as

$$f_*=\sum a_j\underbrace{\zeta*\cdots*\zeta}_{j\text{-times}}\quad(\text{finite sum}),\,a_j\in\mathbb{C},$$

we set the element $f_\kappa\in\mathcal{P}(\mathbb{C})$ for $\kappa\in\mathbb{C}$ by

$$f_\kappa=\sum a_j\underbrace{\zeta*_\kappa\cdots*_\kappa\zeta}_{j\text{-times}}\quad(\text{finite sum}),\,a_j\in\mathbb{C}.$$

The section $\gamma_{f_*}(\kappa)=f_\kappa$ gives a parallel section of the bundle $\pi:\mathbb{C}\times\mathcal{P}(\mathbb{C})\to\mathbb{C}$.

Using the product formula $*_\kappa$, we define a product $*$ on $\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C}))$ by

$$(\gamma_1*\gamma_2)(\kappa)=\gamma_1(\kappa)*_\kappa\gamma_2(\kappa),\quad\gamma_1,\ \gamma_2\in\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})). \tag{4}$$

Lemma 2.5. $(\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})),*)$ *is an associative commutative algebra.*

This procedure gives an identification of the algebra $(\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})),*)$ with $\mathbb{C}_*[\zeta]$. Elements of $(\mathcal{S}(\mathbb{C}\times\mathcal{P}(\mathbb{C})),*)$ will be called q-number polynomials. Although the space of parallel sections could also be defined as space of leaves of a foliation, we attempt to give examples via deformations as alternative geometric objects.

2.2 Strange exponential functions

We now extend this procedure to exponential functions. For $f_* \in \mathbb{C}_*[\zeta]$, we want to describe the star exponential functions $\exp_* f_*$ (singular), which may be highly transcendental elements.

Let $\mathcal{E}(\mathbb{C})$ be the set of all entire functions on $\mathbb{C}$. For $p > 0$, we set

$$(5) \qquad \mathcal{E}_p(\mathbb{C}) \quad = \quad \{f \in \mathcal{E}(\mathbb{C}) \mid ||f||_{p,\delta} = \sup_{\zeta\in\mathbb{C}} e^{-\delta|\zeta|^p}|f(\zeta)| < \infty, \forall\delta > 0\},$$

and also set $\mathcal{E}_{p+}(\mathbb{C}) = \cap_{q>p}\, \mathcal{E}_q(\mathbb{C})$. Then $(\mathcal{E}_p(\mathbb{C}), *_\kappa)$ is a Fréchet commutative associative algebra for $p \le 2$ (cf. [11]). Recalling the intertwiner $T_\kappa^{\kappa'}$ given by (1), we have the following [12]:

Lemma 2.6. *Let $p \le 2$. The intertwiner $T_\kappa^{\kappa'}$ in* (1) *canonically extends to a map $T_\kappa^{\kappa'} : \mathcal{E}_p(\mathbb{C}) \to \mathcal{E}_p(\mathbb{C})$ satisfying*

$$(6) \qquad T_\kappa^{\kappa'}(f *_\kappa g) = T_\kappa^{\kappa'}(f) *_{\kappa'} T_\kappa^{\kappa'}(g) \quad \textit{for every } f,\ g \in \mathcal{E}_p(\mathbb{C}).$$

We note that while the product $*_\kappa$ does not give an associative commutative product and the intertwiner $T_\kappa^{\kappa'}$ does not extend to $\mathcal{E}_p(\mathbb{C})$ for $p \ge 2$, the notion of the connection ∇ is still defined.

Namely, we consider the trivial bundle $\pi : \mathbb{C}\times\mathcal{E}(\mathbb{C}) \to \mathbb{C}$ over $\mathbb{C}$ with the fiber $\mathcal{E}(\mathbb{C})$, and the set of sections $\Gamma(\mathbb{C}\times\mathcal{E}(\mathbb{C}))$. For $\gamma\in\Gamma(\mathbb{C}\times\mathcal{E}(\mathbb{C}))$, we define a covariant derivative $\nabla_{\dot{c}}\gamma$ as the natural extension of (3). It is easily seen that ∇ is well defined for $\Gamma(\mathbb{C}\times\mathcal{E}_p(\mathbb{C}))$ and $\Gamma(\mathbb{C}\times\mathcal{E}_{p+}(\mathbb{C}))$ for every $p\ge 0$. As before, we denote by $\mathcal{S}(\mathbb{C}\times\mathcal{E}_p(\mathbb{C}))$, $\mathcal{S}(\mathbb{C}\times\mathcal{E}_{p+}(\mathbb{C}))$ the sets of parallel sections.

We wish to treat the star exponential function $\exp_* f_*$ for $f_* \in \mathbb{C}_*[\zeta]$. As in §2.1, we have the realization $\{f_\kappa\}_{\kappa\in\mathbb{C}}$ of $f_*\in\mathbb{C}_*[\zeta]$, where $f_\kappa\in\mathcal{P}(\mathbb{C})$. Fixing the $*_\kappa$ product gives the star exponential functions of $f_\kappa \in \mathcal{P}(\mathbb{C})$ with respect to $*_\kappa$as follows. We consider the following evolution equation

$$(7) \qquad \begin{cases} \partial_t F_\kappa(t) = f_\kappa(\zeta) *_\kappa F_\kappa(t), \\ F_\kappa(0) = g_\kappa. \end{cases}$$

If (7) has a real analytic solution in t, then this solution is unique. Thus, we may set $\exp_{*_\kappa} f_\kappa = F_\kappa(1)$ when (7) has an analytic solution with $F_\kappa(0) = 1$.

By letting $\kappa\in$ vary in $\mathbb{C}$, the totality of the star exponential functions $\{\exp_{*_\kappa} f_\kappa\}_{\kappa\in\mathbb{C}}$ may be viewed as a natural representation of the star exponential function $\exp_* f_*$.

As an example, we consider the linear function $f(\zeta) = a\zeta$, where $a \in \mathbb{C}$. Then the evolution equation (7) is expressed as

$$(8) \qquad \begin{cases} \partial_t F_\kappa(t) = a\zeta F_\kappa + \frac{\kappa}{2} a\partial_\zeta F_\kappa, \\ F_\kappa(0) = 1. \end{cases}$$

By a direct computation, we have

Lemma 2.7. *The equation* (8) *has the solution* $F_\kappa(t){=}\exp(at\zeta{+}\frac{\kappa}{4}a^2t^2)$*. Thus, we may set*

$$\exp_{*_\kappa} t\zeta = \exp(t\zeta + \frac{\kappa}{4}t^2) \tag{9}$$

which is contained in $\mathcal{E}_{1+}(\mathbb{C})$ *for every* $\kappa \in \mathbb{C}$.

Since the intertwiner $T_\kappa^{\kappa'}$ is defined on $\mathcal{E}_{1+}(\mathbb{C})$, and $T_\kappa^{\kappa'}(\exp_{*_\kappa} a\zeta) = \exp_{*_{\kappa'}} a\zeta$, we see that $\{\exp_{*_\kappa} a\zeta\}_{\kappa\in\mathbb{C}}$ is an element of $\mathcal{S}(\mathbb{C}{\times}\mathcal{E}_{1+}(\mathbb{C}))$. As in §2.1, it is natural to regard $\{\exp_{*_\kappa} a\zeta\}_{\kappa\in\mathbb{C}}$ as the star exponential function $\exp_* a\zeta$, which may be called a q-number exponential function.

From the star exponential functions $\exp_{*_\kappa} a\zeta$, we construct a type of *delta function* via the star Fourier transform: Namely, we call

$$\delta_{*_\kappa}(\zeta) = \int_{-\infty}^{\infty} \exp_{*_\kappa} it\zeta\, dt \tag{10}$$

the $*_\kappa$-delta function. Using (9), we have

Lemma 2.8. *The* $*_\kappa$*-delta function* $\delta_{*_\kappa}(\zeta)$ *is well defined as an element of* $\mathcal{E}_{2+}(\mathbb{C})$ *for every* $\kappa{\in}\mathbb{C}$ *such that* $\mathrm{Re}(\kappa) > 0$.

Using integration by parts, we easily see that

$$e^{i\theta}\int_{-\infty}^{\infty} \exp_{*_\kappa} e^{i\theta} it\zeta\, dt, \qquad \mathrm{Re}\, e^{2i\theta}\kappa{>}0$$

does not depend on θ whenever $\mathrm{Re}(e^{2i\theta}\kappa) > 0$. This allows us to define $\delta_{*_\kappa}(\zeta){\in}\mathcal{E}_{2+}(\mathbb{C})$ for $\kappa{\in}\mathbb{C}{-}\{0\}$.

Lemma 2.9. *The mapping* $\delta_* : \mathbb{C}{-}\{0\} \to \mathcal{E}_{2+}(\mathbb{C})$ *defined by* $\kappa{\to}\delta_{*_\kappa}$ *is double-valued.*

Proof. We set

$$\delta(\zeta; e^{i\theta}, \kappa) = e^{i\theta}\int_{-\infty}^{\infty} \exp(ie^{i\theta}t\zeta - \frac{\kappa}{4}e^{2i\theta}t^2)dt. \tag{11}$$

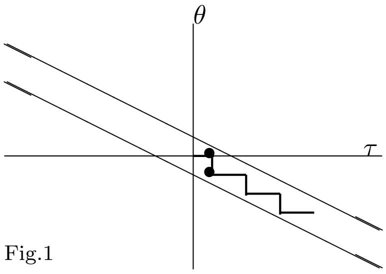

Fig.1

Setting $\kappa{=}e^{i\tau}$ gives that $\delta(\zeta; e^{i\theta}, e^{i\tau})$ is well defined on the strip bounded by $\theta = -\frac{\pi}{2} \pm \frac{\pi}{4}$ given in Fig.1. Note that $\delta(\zeta; e^{i\theta}, \kappa)$ depends only on τ in this strip $-\pi/2{<}\tau{+}2\theta{<}\pi/2$ and $\delta(\zeta; e^{i\theta}, \kappa)$ is a parallel section with respect to κ. By varying θ, we may move t from 0 to 2π such that (t, θ) is contained in the strip as indicated in the figure. Moving along such a path from $t = 0$ to $t = 2\pi$ gives

$$\delta(\zeta; 1, c){=}\int_{-\infty}^{\infty} \exp(it\zeta{-}\frac{1}{4}ct^2)dt{=}-\int_{-\infty}^{\infty} \exp(-it\zeta{-}\frac{1}{4}ct^2)dt{=}{-}\delta(\zeta; 1, c).$$

□

Let us consider the trivial vector bundle over $\mathbb{C}-\{0\}$. Lemma 2.9 tells us that $\delta_*(\zeta)$ can be viewed as a double-valued holomorphic parallel section over $\mathbb{C}-\{0\}$. Note that $\delta(\zeta;1,c)=\frac{2\sqrt{\pi}}{\sqrt{c}}e^{-\frac{1}{c}\zeta^2}$, and $\lim_{c\to 0}\delta(\zeta;1,c)$ gives us the ordinary delta function.

As seen in the construction of the star delta functions, the notion of *densely defined multi-valued parallel sections* arises naturally, which could be handled as leaves of a foliation. However, as mentioned in §2.1, we prefer to interpret this object as an alternative geometric notion.

2.3 Star exponential functions of quadratic functions

We set

$$P^{(2)}(\mathbb{C})=\{f(\zeta)=a\zeta^2+b\,|\,a,b\in\mathbb{C}\},\quad \mathbb{C}^{(2)}_*[\mathbb{C}]=\{f_*(\zeta)=a\zeta{*}\zeta\in\mathbb{C}_*[\zeta]\,|\,a\in\mathbb{C}\}$$

Thus, we view $a\zeta{*}\zeta$ as the section $\gamma(\kappa)=a\zeta^2+\frac{a}{2}\kappa\in\Gamma(\mathbb{C}\times P^{(2)}(\mathbb{C}))$, where $\pi:\mathbb{C}\times P^{(2)}(\mathbb{C})\to\mathbb{C}$ is the trivial bundle over $\mathbb{C}$ with fiber $P^{(2)}(\mathbb{C})$. We now attempt to give a meaning to the star exponential function $\exp_* a\zeta{*}\zeta$, $a\in\mathbb{C}$ along the argument in §2.1.

We consider a quadratic element $f_*\in\mathbb{C}^{(2)}_*[\zeta]$. Then the corresponding polynomial f_κ is given by

$$f_\kappa=\zeta{*_\kappa}\zeta=\zeta^2+\frac{\kappa}{2}. \tag{12}$$

As in §2.1, we view $\{f_\kappa\}_{\kappa\in\mathbb{C}}$ as a parallel section of $\mathbb{C}\times P^{(2)}(\mathbb{C})$. We consider the following evolution equation.

$$\partial_t F_\kappa(t)=f_\kappa(\zeta)*_\kappa F_\kappa(t),\quad F_\kappa(0)=g_\kappa, \tag{13}$$

where f_κ can be given by (12). (13) is rewritten as

$$\partial_t F_\kappa(t)=(\zeta^2+\frac{\kappa}{2})F_\kappa+\kappa\zeta\partial_\zeta F_\kappa+\frac{\kappa^2}{4}\partial_\zeta^2 F_\kappa,\quad F_\kappa(0)=g_\kappa. \tag{14}$$

We assume that the initial condition g_κ is given by the form $g_\kappa=\rho_{\kappa,0}\exp a_{\kappa,0}\zeta^2$, where $\rho_{\kappa,0}\in\mathbb{C}_\times=\mathbb{C}-\{0\}$ and $a_{\kappa,0}\in\mathbb{C}$. Putting g_κ=1 gives the star exponential function $\exp_{*_\kappa} f_\kappa(\zeta)$. To solve (14) explicitly, we assume that F_κ is of the following form:

$$F_\kappa(t)=\rho_\kappa(t)\exp a_\kappa(t)\zeta^2. \tag{15}$$

Plugging (15) into (14), we have

$$\left\{\begin{array}{l}\partial_t a_\kappa=1+2a_\kappa\kappa+a_\kappa^2\kappa^2,\\ \partial_t\rho_\kappa=\frac{\kappa}{2}\big(1+\kappa a_\kappa\big)\rho,\\ a_\kappa(0)=a_{\kappa,0},\quad \rho_\kappa(0)=\rho_{\kappa,0}.\end{array}\right. \tag{16}$$

Proposition 2.10. *The solution of* (16) *is given by*

$$a_\kappa(t)=\frac{a_{\kappa,0}+t(1+\kappa a_{\kappa,0})}{1-\kappa t(1+\kappa a_{\kappa,0})},\quad \rho_\kappa(t)=\frac{\rho_{\kappa,0}}{\sqrt{1-\kappa t(1+\kappa a_{\kappa,0})}},\tag{17}$$

where we note the ambiguity in choosing the sign of the square root in (17). We define a subset $\mathcal{E}^{(2)}(\mathbb{C})$ of $\mathcal{E}(\mathbb{C})$ by

$$\mathcal{E}^{(2)}(\mathbb{C})=\{f=\rho\exp a\zeta^2 \mid \rho\in\mathbb{C}_\times,\, a\in\mathbb{C}\}.$$

Identifying $f{=}\rho\exp a\zeta^2\in\mathcal{E}^{(2)}(\mathbb{C})$ with (ρ,a) gives $\mathcal{E}^{(2)}(\mathbb{C})\cong\mathbb{C}_\times{\times}\mathbb{C}$. Note that $\mathcal{E}^{(2)}(\mathbb{C})$ is not contained in $\mathcal{E}_2(\mathbb{C})$ but in $\mathcal{E}_{2+}(\mathbb{C})$, on which the product $*_\kappa$ may give rise to strange phenomena (cf. [12]).

Consider the trivial bundle $\pi:\mathbb{C}{\times}\mathcal{E}^{(2)}(\mathbb{C})$ over $\mathbb{C}$ with fiber $\mathcal{E}^{(2)}(\mathbb{C})$. In particular, putting $a_{\kappa,0}{=}0$, $\rho_{\kappa,0}{=}1$ and $t{=}a$ in Proposition 2.10, we see that

$$\exp_{*_\kappa} a\zeta{*_\kappa}\zeta=\frac{1}{\sqrt{1-a\kappa}}\exp\frac{a}{1-a\kappa}\zeta^2\tag{18}$$

where the right hand side of (18) still has an ambiguous choice for the the sign of the square root.

Keeping this ambiguity in mind, we have a kind of fuzzy one parameter group property for the exponential function of (18). Namely, for $g_\kappa=\exp_{*_\kappa} b\zeta{*_\kappa}\zeta$, where $b\in\mathbb{C}$, the solutions of (14) yield the exponential law:

$$\exp_{*_\kappa} a\zeta{*_\kappa}\zeta *_\kappa \exp_{*_\kappa} b\zeta{*_\kappa}\zeta=\frac{1}{\sqrt{1-(a{+}b)\kappa}}e^{\frac{a+b}{1-(a+b)\kappa}\zeta^2}=\exp_{*_\kappa}(a{+}b)\zeta{*_\kappa}\zeta,\tag{19}$$

where (19) still contains an ambiguity in the sign of the square root.

Recall the connection ∇ on the trivial bundle $\pi:\mathbb{C}{\times}\mathcal{E}(\mathbb{C}){\to}\mathbb{C}$. It is easily seen that the connection ∇ gives a specific trivialization of the bundle $\pi:\mathbb{C}{\times}\mathcal{E}^{(2)}(\mathbb{C}){\to}\mathbb{C}$. According to the identification $\mathcal{E}^{(2)}(\mathbb{C})\cong\mathbb{C}_\times{\times}\mathbb{C}$, we write $\gamma(\kappa)=\rho(\kappa)\exp a(\kappa)\zeta^2$ as $(\rho(\kappa),a(\kappa))$. Then the equation $\nabla_{\partial_t}\gamma=0$ gives

$$\begin{cases}\partial_t a(t)=a(t)^2,\\ \partial_t\rho(t)=\frac{1}{2}\rho(t)a(t).\end{cases}\tag{20}$$

We easily see that (18) gives a densely defined parallel section. As seen in [12], it should also be considered as a densely defined multi-valued section of this bundle. Thus, we may view the star exponential function $\exp_* a\zeta{*}\zeta$ as a family

$$\big\{F_\kappa(\zeta)=\frac{1}{\sqrt{1-a\kappa}}\exp\frac{a}{1-a\kappa}\zeta^2\big\}_{\kappa\in\mathbb{C}}.$$

This realization of $\exp_* a\zeta{*}\zeta$ is a densely defined and multi-valued parallel section $\gamma(\kappa)=\rho(\kappa)\exp a(\kappa)\zeta^2$ of the bundle $\pi:\mathbb{C}{\times}\mathcal{E}^{(2)}(\mathbb{C}){\to}\mathbb{C}$. In the next section, we investigate the solution of (20) more closely.

3 Bundle gerbes as a non-cohomological notion

The bundle $\pi : \mathbb{C}\times\mathcal{E}(\mathbb{C})\to\mathbb{C}$ with the flat connection ∇ gave us the notion of parallel sections, where we extended this notion to be densely defined and multi-valued sections. This is in fact the notion of leaves of the foliation given by the flat connection ∇. We now analyze the moduli space of densely defined multi-valued parallel sections of the bundle $\pi : \mathbb{C}\times\mathcal{E}^{(2)}(\mathbb{C})\to\mathbb{C}$ with respect to the connection ∇. The moduli space has an unusual bundle structure, which we would call a *pile*. We analyse the evolution equation (20) for parallel sections as a toy model of the phenomena of movable branch singularities.

3.1 Non-linear connections

First, consider a non-linear connection on the trivial bundle $\coprod_{\kappa\in\mathbb{C}}\mathbb{C} = \mathbb{C}\times\mathbb{C}$ over $\mathbb{C}$ given by a holomorphic horizontal distribution

$$(21) \qquad H(\kappa; y) = \{(t; y^2t); t \in \mathbb{C}\} \quad \text{(independent of } \kappa.)$$

.

The first equation of parallel translation (20) is given by $\frac{dy}{d\kappa} = y^2$. Hence, parallel sections are given in general by

$$(22) \qquad (\kappa; y(\kappa)){=}(\kappa; \frac{1}{c{-}\kappa}){=}(\kappa; \frac{c^{-1}}{1{-}c^{-1}\kappa}).$$

There is also the singular solution $(\kappa; 0)$, corresponding to $c^{-1} = 0$. Note that $(\kappa, -\frac{1}{\kappa})$ is not a singular solution. For consistency, we think that the singular point of the $(\kappa, 0)$ section as at ∞.

Let $\mathcal{A}$ be the set of parallel sections including the singular solution $(\kappa, 0)$. Every $f{\in}\mathcal{A}$ has one singular point at a point $c \in S^2{=}\mathbb{C} \cup \{\infty\}$. The assignment of $f \in \mathcal{A}$ to its singular point $\sigma(f) = c$ gives a bijection $\sigma : \mathcal{A}{\to}S^2{=}\mathbb{C} \cup \{\infty\}$. Namely, $\mathcal{A}$ is parameterized by S^2 by

$$(23) \qquad \sigma(f) = c \Leftrightarrow f = (\kappa, \frac{1}{c{-}\kappa}), \quad \sigma(f) = \infty \Leftrightarrow f = (\kappa, 0) \in \mathcal{A}.$$

In this way, we give a topology on $\mathcal{A}$.

Let $T_\kappa^{\kappa'}(y)$ be the parallel translation of $(\kappa; y)$ along a curve from κ to κ'. Since (21) is independent of the base point κ, $T_\kappa^{\kappa'}(y)$ is given by

$$T_\kappa^{\kappa'}(y) = \frac{y}{1{-}y(\kappa'{-}\kappa)}, \quad T_\kappa^{\kappa'}(\infty) = \frac{1}{\kappa{-}\kappa'}.$$

We easily see that $T_\kappa^{\kappa''} = T_{\kappa'}^{\kappa''}T_\kappa^{\kappa'}$, $T_\kappa^\kappa = I$. Every $f{\in}\mathcal{A}$ satisfies $T_\kappa^{\kappa'}f(\kappa) = f(\kappa')$ where they are defined.

3.1.1 Extension of the non-linear connection

We now extend the non-linear connection H defined by (21) to the space $\mathbb{C}\times\mathbb{C}^2$ by giving the holomorphic horizontal distributions

$$\tilde{H}(\kappa; y, z) = \{(t; y^2 t, -yt); t \in \mathbb{C}\} \quad \text{(independent of } \kappa, z). \tag{24}$$

Parallel translation with respect to (24) is given by the following equations:

$$\frac{dy}{d\kappa} = y^2, \quad \frac{dz}{d\kappa} = -y. \tag{25}$$

For the equation (25), multi-valued parallel sections are given by both ways

$$(\kappa, \frac{a}{1-a\kappa}, z+\log(1-a\kappa)), \quad (\kappa, \frac{1}{b-\kappa}, w+\log(\kappa-b)), \quad (a, b \in \mathbb{C}) \tag{26}$$

although they are infinitely valued. The singular solution $(\kappa; 0, z)$ occurs in the first expression. The set-to-set correspondence

$$(a, z+2\pi i\mathbb{Z}) \overset{\iota}{\Longleftrightarrow} (b, w+2\pi i\mathbb{Z}) = (a^{-1}, z+\log a+\pi i+2\pi i\mathbb{Z}) \tag{27}$$

identifies these two sets of parallel sections, which gives multi-valued parallel sections. However, because of the ambiguity of $\log a$, we can not make this correspondence a univalent correspondence (cf. Proposition 3.1).

Denote by $\tilde{\mathcal{A}}$ the set of all parallel sections written in the form (26). Denote by $\pi_3 : \tilde{\mathcal{A}}\to\mathcal{A}$ be the mapping which forgets the last component. This is surjective. For every $v\in\mathcal{A}$ such that $\sigma(v)=b=a^{-1}\in S^2$, we see

$$\pi_3^{-1}(v)=\{(\kappa, \frac{1}{b-\kappa}, w+\log(\kappa-b)); w\in\mathbb{C}\}=\{(\kappa, \frac{a}{1-a\kappa}, z+\log(1-a\kappa)); z\in\mathbb{C}\}.$$

Since there is one dimensional freedom of moving, $\pi_3^{-1}(v)$ should be parameterized by $\mathbb{C}$. However, there is no natural parameterization and there are many technical choices.

3.1.2 Tangent spaces of $\tilde{\mathcal{A}}$

For an element $f=(\kappa, \frac{a}{1-a\kappa}, z+\log(1-a\kappa))=(\kappa, \frac{1}{b-\kappa}, w+\log(\kappa-b))$, the *tangent space* $T_f\tilde{\mathcal{A}}$ of $\tilde{\mathcal{A}}$ at f is

$$\begin{aligned} T_f\tilde{\mathcal{A}} &=\Big\{\frac{d}{ds}\Big|_{s=0}(\frac{a(s)}{1-a(s)\kappa}, z(s)+\log(1-a(s)\kappa)); (a(0), z(0))=(a, z)\Big\} \\ &=\left\{(\frac{\dot{a}}{(1-a\kappa)^2}, \dot{z}-\frac{\dot{a}\kappa}{1-a\kappa}); \dot{a}, \dot{z}\in\mathbb{C}\right\} = \left\{(\frac{-\dot{b}}{(b-\kappa)^2}, \dot{w}-\frac{\dot{b}}{\kappa-b}); \dot{b}, \dot{w}\in\mathbb{C}\right\}. \end{aligned}$$

Hence

$$\begin{bmatrix}\dot{b}\\ \dot{w}\end{bmatrix} = \begin{bmatrix}-a^{-2} & 0\\ a^{-1} & 1\end{bmatrix}\begin{bmatrix}\dot{a}\\ \dot{z}\end{bmatrix} =(d\iota)_{(a,z)}\begin{bmatrix}\dot{a}\\ \dot{z}\end{bmatrix}, \quad \text{and} \quad T_f\tilde{\mathcal{A}} \cong \mathbb{C}^2.$$

Consider now a subspace H_f of $T_f\tilde{\mathcal{A}}$ obtained by setting $\dot{z}=0$ in the definition of $T_f\tilde{\mathcal{A}}$. Then, $\{H_f; f\in\tilde{\mathcal{A}}\}$ is defined without ambiguity $2\pi i\mathbb{Z}$, and obviously $H_f\cong\mathbb{C}$. We regard $\{H_f; f\in\tilde{\mathcal{A}}\}$ an unambiguously defined horizontal distribution on $\pi_3:\tilde{\mathcal{A}}\to\mathcal{A}$.

The invariance in the vertical direction gives that $\{H_f; f\in\tilde{\mathcal{A}}\}$ may be viewed as an *infinitesimal trivialization* of $\pi_3:\tilde{\mathcal{A}}\to\mathcal{A}$.

Parallel translation $I_\kappa^{\kappa'}$ for (25) is given by

$$(28)\qquad \begin{aligned} I_\kappa^{\kappa'}(y,z)&=(\frac{y}{1-y(\kappa'-\kappa)}, z+\log(1-y(\kappa'-\kappa))),\\ &=(\frac{1}{y^{-1}-\kappa'+\kappa}, z+\log y+\log(y^{-1}-\kappa'+\kappa)),\end{aligned}$$

which is obtained by solving for (25) under the initial data (κ, y, z).

By definition we see $I_\kappa^\kappa=I$, and $I_\kappa^{\kappa''}=I_{\kappa'}^{\kappa''}I_\kappa^{\kappa'}$, as a set-to-set mapping Every $f\in\tilde{\mathcal{A}}$ satisfies $I_\kappa^{\kappa'}f(\kappa)=f(\kappa')$ where they are defined.

Proposition 3.1. *Parallel translation via the horizontal distribution* $\{H_f : f\in\tilde{\mathcal{A}}\}$ *does not give a local trivialization of* $\pi_3:\widetilde{\mathcal{A}}\to\mathcal{A}$.

Proof. For a point $g=(\kappa,\frac{a}{1-a\kappa})$ of $\mathcal{A}$, and a small neighborhood V_a of a, $\tilde{V}_a=\{\frac{a'}{1-a'\kappa}; a'\in V_a\}$ is a neighborhood of f in $\mathcal{A}$. Consider the set

$$\pi_3^{-1}(\tilde{V}_a)=\{(\kappa,\frac{a'}{1-a'\kappa}, z+\log(1-a'\kappa)); a'\in V_a,\ z\in\mathbb{C}\}.$$

The horizontal lift of the curve $\frac{a'(s)}{1-a'(s)\kappa}$, $a'(s)=a+s(a'-a)$ along the infinitesimal trivialization is given by solving the equation

$$\frac{d}{ds}z(s)=-\frac{(a'-a)\kappa}{1-a'(s)\kappa},\quad z(0)\in\log(1-a\kappa).$$

Hence $z(s)=\log(1-(a+s(a'-a))\kappa)$, and $z(1)=\log(1-a'\kappa)$. Thus it is impossible to eliminate the ambiguity of $\log(1-a'\kappa)$ on V_a, no matter how small the neighborhood V_a is. □

Proposition3.1 shows that $\pi_3:\widetilde{\mathcal{A}}\to\mathcal{A}$ is not an affine bundle. In spite of this, one may say that the curvature of its connection vanishes.

3.1.3 Affine bundle gerbes

Although $\pi_3:\widetilde{\mathcal{A}}\to\mathcal{A}$ does not have a bundle structure, we can consider *local trivializations* by restricting the domain of κ.

(a) Let $V_\infty=\{b; |b|>3\}\subset S^2$ be a neighborhood of ∞. First, we define a fiber preserving mapping $p_{\infty,D}$ from the trivial bundle $\pi: V_\infty\times\mathbb{C}\to V_\infty$ into $\pi_3:\widetilde{\mathcal{A}}\to\mathcal{A}$ such that $\pi_3 p_{\infty,D}=\sigma^{-1}\pi$ by restricting the domain of κ in a unit disk D: Consider $(\kappa,\frac{a}{1-a\kappa}, z+\log(1-a\kappa))$ for $(\kappa, a^{-1})\in D\times V_\infty$. Since $|a\kappa|<1/3$, $\log(1-a\kappa)$

is defined as a univalent function $\log(1-a\kappa) = \log|1-a\kappa| + i\theta$, $-\pi < \theta < \pi$ on this domain by setting $1-a\kappa = |1-a\kappa|e^{i\theta}$, which will be denoted by $\log(1-a\kappa)_{D\times V_\infty}$. We define

$$p_{\infty,D}(b,z) = (\kappa, \frac{a}{1-a\kappa}, z+\log(1-a\kappa)), \quad a^{-1}=b\in V_\infty, \quad z\in\mathbb{C} \tag{29}$$

where $\log(1-a\kappa)$ in the right hand side is the analytic continuation of $\log(1-a\kappa) = \log(1-a\kappa)_{D\times V_\infty}$.

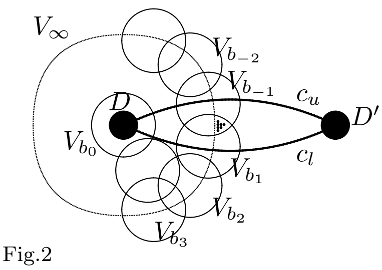

Fig.2

(b) We take a simple covering of the domain $|z| \le 3$ by unit disks $V_{b_{-k}}, \ldots, V_{b_{-1}}$, V_{b_0}, V_{b_1}, $\ldots$, V_{b_ℓ} as in Fig.2, and fix a unit disk D' apart from all V_{b_i}. We define a fiber preserving mapping $p_{V_{b_i},D'}$ from the trivial bundle $\pi : V_{b_i}\times\mathbb{C} \to V_{b_i}$ to the bundle $\pi_3 : \widetilde{\mathcal{A}} \to \mathcal{A}$ such that $\pi_3 p_{V_{b_i},D'} = \sigma^{-1}\pi$ by restricting the domain of κ in a unit disk D'.

We see that setting $\kappa-b = |\kappa-b|e^{i\theta}$, $\log(\kappa-b)$ is defined as a univalent function on the domain $D'\times V_{b_i}$ as $\log|\kappa-b| + i\theta$, $-\pi<\theta<\pi$, which is denoted by $\log(\kappa-b)_{D'\times V_{b_i}}$.

Consider $(\kappa, \frac{1}{b-\kappa}, w+\log(\kappa-b))$ for $(\kappa,b)\in D'\times V_{b_i}$. We define

$$p_{V_{b_i},D'}(b',w) = (\kappa, \frac{1}{b'-\kappa}, w+\log(\kappa-b')), \quad (b',w)\in V_b\times\mathbb{C} \tag{30}$$

where $\log(\kappa-b)$ on the r.h.s. is the analytic continuation of $\log(\kappa-b)_{V_{b_i}\times D'}$.

(c) Suppose $c\in V_{b_i}\cap V_{b_j}$ and $p_{V_{b_i},D'}(c,w) = p_{V_{b_j},D'}(c,w')$. Then we see that there exists a unique $n(i,j)\in\mathbb{Z}$ such that $w' = w+2\pi i n(i,j)$. For the above covering, we see $n(i,j)=0$ for every pair (i,j).

Let $c\in V_{b_i}\cap V_\infty$ and $p_{V_{b_i},D'}(c,w) = p_{\infty,D}(c,z)$. To fix the coordinate transformation, we have to choose the identification of two sets of values $\log(\kappa-b)_{D'\times V_{b_i}}$ and $\log(1-a\kappa)_{D\times V_\infty}$. For b_i, except b_1, we identify these through the analytic continuation along the (lower) curve c_l joining D and D', but for b_1, we identify $\log(\kappa-b)_{D'\times V_{b_1}}$ and $\log(1-a\kappa)_{D\times V_\infty}$ through the analytic continuation along the (upper) curve c_u joining D and D'.

Therefore there is a positive integer $n(i,\infty)$ such that $w' = z+2\pi i n(i,\infty)$ by the same argument. For the above covering we see in fact that if $n(1,\infty)=m$ for $i=1$, then $n(i,\infty)=m+1$ for every $i\neq 1$.

These give coordinate transformations. However, the collection of these local trivializations do not glue together, for these do not satisfy the cocycle condition on the triple intersection marked with the triangle in Fig. 2.

We denote by $\coprod_{b\in S^2} \mathbb{C}_b$ the collection of these local trivializations. Thus we have a commutative diagram

$$\begin{array}{ccc} \coprod_{b\in S^2} \mathbb{C}_b & \xrightarrow{p(\bullet)} & \tilde{\mathcal{A}} \\ \pi\downarrow & & \pi_3\downarrow \\ S^2 & \xleftarrow{\sigma} & \mathcal{A} \end{array} \tag{31}$$

One can consider various local trivializations of the bundle-like object of the left hand side. $\coprod_{b\in S^2} \mathbb{C}_b$ is not an affine bundle, but an "affine bundle gerbe" with a holomorphic flat connection (cf.[8]). However, the geometric realization of a holomorphic parallel section is nothing but an element of $\widetilde{\mathcal{A}}$ given by (26).

3.2 Geometric notions on $\tilde{\mathcal{A}}$

Recall that the discordance (the Japanese word *sogo* in the term used in [10]) of patching of three local coordinate neighbourhood occurs only on the small dotted triangle in Fig.2.

In this section, we construct two examples which give almost the same phenomena as in the previous section for gluing local bundles.

3.2.1 Geometric quantization for a non-integral 2-form

Consider the standard volume form dV on S^2 with total volume 4π. Let Ω be a non-integral, closed smooth 2-form (current) on S^2 such that $\int_{S^2}\Omega = 4\pi\lambda$, and with the support of Ω concentrated on a small disk neighborhood of the north pole N. For $\{U_i\}_{i\in I}$ a simple cover of S^2, on each U_i, Ω is of the form $\Omega = d\omega_i$, and hence $\omega_{ij} = \omega_i - \omega_j$ on $U_{ij} = U_i \cap U_j$ is a closed 1-form (current), and is written by $\omega_{ij} = df_{ij}$ on U_{ij} for a smooth 0-form (current) f_{ij}.

Now we want to make a $U(1)$-vector bundle using $e^{\sqrt{-1}f_{ij}}$ as transition functions. However, since on $U_{ijk} = U_i \cap U_j \cap U_k$ we only have

$$e^{\sqrt{-1}f_{ij}} e^{\sqrt{-1}f_{jk}} e^{\sqrt{-1}f_{ki}} = e^{\sqrt{-1}(f_{ij}+f_{jk}+f_{ki})},$$

$e^{\sqrt{-1}f_{ij}}$ cannot be used as patching diffeomorphisms. In spite of these difficulties, we see that the horizontal distributions defined by ω_i glue together.

Thus, we can define a linear connection on such *broken* vector bundle, which is precisely the notion of *bundle gerbes.* Since $\Omega = d\omega_i$, the curvature form of this connection is given by Ω. Note that we can make a parallel translation along any smooth curve $c(t)$ in S^2.

Since the support of Ω is concentrated in a small neighborhood V_N of the north pole N. Therefore any closed curve in $S^2 - V_N$ can be shrunk to a point in $S^2 - V_N$. In spite of this, the homotopy lifting of parallel translation does not succeed, because of the discordance (sogo) of the patching diffeomorphisms.

If U_i does not intersect V_N, then we have a product bundle $U_i \times \mathbb{C}$ with the trivial flat connection. Since $\omega_i = d\log e^{h_i}$, the integral submanifold of the horizontal distribution of ω_i is given by $\log e^{h_i}$. This looks like a *pile.* Thus, even if the object is restricted to $S^2 - V_N$, we have a *non-trivial* bundle gerbe which is apparently not classified by a cohomology class.

We note that this gives also a concrete example of the local line bundles over a manifold treated by [7].

3.2.2 A simple example

The simplest example of objects we propose in this paper is given by the Hopf-fibering $S^3 \xrightarrow{S^1} S^2$. Viewing $S^3 = \coprod_{q \in S^2} S^1_q$ (disjoint union), we consider the ℓ-covering $\tilde{S}^1_q$ of each fiber S^1_q, and denote by $\tilde{S}^3$ the disjoint union $\coprod_{q \in S^2} \tilde{S}^1_q$. We are able to define local trivializations of $\tilde{S}^3|_{U_i} \cong U_i \times \tilde{S}^1$ naturally through the trivializations $S^3|_{U_i}$ given on a simple open covering $\{U_i\}_{i \in \Gamma}$ of S^2. This structure permits us to treat $\tilde{S}^3$ as a local Lie group, and hence it looks like a topological space. On the other hand, we have a projection

$$\pi \ : \ \tilde{S}^3 = \coprod_{q \in S^2} \tilde{S}^1_q \to S^3 = \coprod_{q \in S^2} S^1_q$$

as the union of fiberwise projections, as if it were a non-trivial ℓ- covering. However $\tilde{S}^3$ cannot be a manifold, since S^3 is simply connected. In particular, the *points* of $\tilde{S}^3$ should be regarded as *ℓ-valued elements.*

We now consider a 1-parameter subgroup S^1 of S^3 and the inverse image $\pi^{-1}(S^1)$. Since all points of $\tilde{S}^3$ are "ℓ-valued", this simply looks like a combined object of $S^1 \times \mathbb{Z}_\ell$ and the ℓ covering group, i.e. in some restricted region, this object can be regarded as a point set by several ways. In such a region, the ambiguity is caused simply by the reason that two pictures of point sets are mixed up.

3.2.3 Conceptual difficulties beyond ordinary mathematics

Let P_c be the parallel translation along a closed curve. Let $c_s(t)$ be a family of closed curves. Suppose $c_s(0) = c_s(1) = p$ and $c_1(t) = p$. We see that there is $(p; v)$ such that $P_{c_s}(p; v) \neq v$. Therefore there must be somewhere a singular point for the homotopy chasing, caused by the discordance. However the position of singular point can not be specified.

Even though the parallel translation is defined for every fixed curve, these parallel translations are in general set-to-set mappings when one parameter family of closed curves are considered.

Thus, we have some conceptual difficulty that may be explained as follows: a parallel translation along a curve has a definite meaning, but when we think this in a family of curves, then we have to think *suddenly* this a set-to-set mapping. Recall here the "Schrödinger's cat".

Such a strange phenomena caused in $\tilde{\mathcal{A}}$ by movable branch singularities. In §3.1, we considered a non-linear connection on the trivial bundle $S^2 \times \mathbb{C}$, and an extended connection to treat the amplitude of the star exponential functions of the quadratic form.

4 Broken associative products and extensions

In this section we give an example where such fuzzy phenomena play a crucial role in defining a concrete algebraic structure. We consider the product bundle $\coprod_{\kappa\in\mathbb{C}}\mathbb{C}$, and we define in each fiber an associative product which is *broken* in the sense that each product is not necessarily defined for all pairs (a,b).

4.1 Associative products combined with the Cayley transform

First of all, we give such a product on the fiber at κ=0. Let S^2 be the 2-sphere identified with $\mathbb{C}\cup\{\infty\}$. Consider the Cayley transform $C_0 : S^2{\to}S^2$, $C_0(X)=\frac{1-X}{1+X}$, and define the product by

$$a{\bullet_0}b=\frac{a{+}b}{1{+}ab}\sim C_0^{-1}(C_0(a)C_0(b)). \tag{32}$$

Here $\sim$ means algebraic equality where defined: an algebraic procedure through the calculations such as follows:

$$\frac{1-\frac{1-a}{1+a}\cdot\frac{1-b}{1+b}}{1+\frac{1-a}{1+a}\cdot\frac{1-b}{1+b}}\sim\frac{(1{+}a)(1{+}b)-(1{-}a)(1{-}b)}{(1{+}a)(1{+}b)+(1{-}a)(1{-}b)}=\frac{a+b}{1+ab}.$$

The product is defined for every pair (a,b) such that $ab\neq -1$, and is commutative and associative whenever they are defined. Note also that

$$a{\bullet_0}b=\frac{a{+}b}{1{+}ab}\sim\frac{a^{-1}{+}b^{-1}}{1{+}(ab)^{-1}}=a^{-1}{\bullet_0}b^{-1}. \tag{33}$$

Hence we set $\infty{\bullet_0}b=b^{-1}$, $\infty{\bullet_0}\infty$=0, in particular.

One can extend this broken product to pairs $(a:g){\in}\mathbb{C}{\times}\mathbb{C}$ as follows:

$$(a:g){\bullet_0}(b:g')=(a{\bullet_0}b:gg'(1{+}ab)).$$

This is an associative product, which follows from (32).

$$(1{+}bc)(1{+}a\frac{b{+}c}{1{+}bc})=(1{+}\frac{a{+}b}{1{+}ab}c)(1{+}ab).$$

It is worthwhile to write this identity in the logarithmic form

$$\log(1{+}bc){+}\log(1{+}a\frac{b{+}c}{1{+}bc})=\log(1{+}\frac{a{+}b}{1{+}ab}c){+}\log(1{+}ab),\quad \bmod 2\pi i\mathbb{Z} \tag{34}$$

although the logarithmic form uses infinitely valued functions. If one sets $C(a,b)=\log(1{+}ab)$, then (34) is a the Hochschild 2-cocycle condition:

$$C(b,c){-}C(a{\bullet_0}b,c){+}C(a,b{\bullet_0}c){-}C(a,b)=0,\quad \bmod 2\pi i\mathbb{Z}.$$

We extend the product as follows:

$$(a:g){\bullet}_{ln}(b:g') = (a{\bullet}_0 b : g{+}g'{+}\log(1{+}ab)). \tag{35}$$

This is associative as a set-to-set mapping. By using (27), (35) is rewritten as

$$(a^{-1}:g){\bullet}_{ln}(b^{-1}:g') = (a^{-1}{\bullet}_0 b^{-1} : g{+}g'{+}\log(1{+}a^{-1}b^{-1})).$$

Next we define a family of products defined on each fiber at κ. To define such a product, we use the twisted Cayley transform defined by $C_\kappa \sim C_0 T^0_\kappa$, where T^0_κ is given in the equality (1). The result is

$$C_\kappa(y) = \frac{1{-}y(1{-}\kappa)}{1{+}y(1{+}\kappa)}, \tag{36}$$

and we define

$$a{\bullet}_\kappa b = \frac{a{+}b{+}2ab\kappa}{1{+}ab(1{-}\kappa^2)} \sim C_\kappa^{-1}(C_\kappa(a)C_\kappa(b)). \tag{37}$$

The point is that the singular set of the product depends on κ. $a{\bullet}_\kappa b$ is defined for every pair (a,b) such that $ab(1{-}\kappa^2){\neq}{-}1$. In other word, for an arbitrary pair $(a,b)\in\mathbb{C}^2$, the product $a{\bullet}_\kappa b$ is defined for some κ in an open dense domain.

For the parallel sections given in (22), we see that

$$\frac{a}{1{-}a\kappa}{\bullet}_\kappa\frac{b}{1{-}b\kappa} = \frac{a{+}b}{1{-}(a{+}b)\kappa{+}ab}. \tag{38}$$

In particular,

$$-\kappa^{-1}{\bullet}_\kappa{-}\kappa^{-1} = 0, \quad -\kappa^{-1}{\bullet}_\kappa\frac{1}{b^{-1}{-}\kappa} = \frac{1}{b{-}\kappa}.$$

For simplicity, we denote by $f(\kappa)$ the section f of the bundle $\pi: S^2\times\mathbb{C}\to S^2$.

Proposition 4.1. *For parallel sections $f(\kappa), g(\kappa)$ defined on open subsets, the product $f(\kappa){\bullet}_\kappa g(\kappa)$ is also a parallel section where defined.*

4.1.1 Extension of the product

Using (35), one can extend the product $a{\bullet}_\kappa b$ by the formula

$$(a;g){\bullet}_\kappa(b;g') \sim I^\kappa_0\big((I^0_\kappa(a;g)){\bullet}_{ln}(I^0_\kappa(a;g'))\big).$$

Indeed, we see how the algebraic trick works:

$$\begin{aligned}(a:g){\bullet}_\kappa(b:g') &= (a{\bullet}_\kappa b : g{+}g'{+}\log(1{+}ab(1{-}\kappa^2)) \\ &= \Big(\frac{a{+}b{+}2ab\kappa}{1{+}ab(1{-}\kappa^2)} : g{+}g'{+}\log(1{+}ab(1{-}\kappa^2))\Big).\end{aligned} \tag{39}$$

Proposition 4.2. *The extended product $(a:g)\bullet_\kappa(b:g')$ is defined with a $2\pi i\mathbb{Z}$ ambiguity. However, the $\bullet_\kappa$ product is associative where defined.*

The point of such a fiberwise product is the following:

Proposition 4.3. *For parallel sections $f(\kappa), g(\kappa)$ defined on open subsets, the product $f(\kappa)\bullet_\kappa g(\kappa)$ is also a parallel section where defined.*

Proof. We have only to prove $I_\kappa^{\kappa'}(f\bullet_\kappa h) = I_\kappa^{\kappa'}(f)\bullet_{\kappa'} I_\kappa^{\kappa'}(h)$.

For $f = (\frac{a}{1-a\kappa}, \log(1-a\kappa))$, $h = (\frac{b}{1-b\kappa}, \log(1-b\kappa))$, we see that

$$
\begin{aligned}
f\bullet_\kappa h =&\Big(\frac{a+b}{1-(a+b)\kappa+ab}, \log\big((1-a\kappa)(1-b\kappa)(1+\frac{a}{1-a\kappa}\frac{b}{1-b\kappa}(1-\kappa^2))\big)\Big)\\
=&\Big(\frac{a+b}{1-(a+b)\kappa+ab}, \log(1-(a+b)\kappa+ab)\Big).
\end{aligned}
$$

It is easily seen that $I_\kappa^{\kappa'}(f\bullet_\kappa h) = \Big(\frac{a+b}{1-(a+b)\kappa'+ab}, \log(1-(a+b)\kappa'+ab)\Big)$. $\square$

5 The notion of q-number functions

Using Propositions 4.1, 4.3, we define a multiplicative structure on the sets $\mathcal{A}$ and $\tilde{\mathcal{A}}$ of parallel sections. A notion of q-number functions which describe quantum observables was introduced in [1], and our notion of parallel sections is stimulated by this idea. From this point of view, we may employ the notation $:f:_\kappa$ for a section f of the bundle $\pi : \coprod_{\kappa\in\mathbb{C}} \mathbb{C} \to \mathbb{C}$.

For $f{\in}\mathcal{A}$, we view κ as an indeterminate. For every $f, g{\in}\mathcal{A}$, excluding the pair $(f,g) = (\frac{1}{1-\kappa}, \frac{-1}{1+\kappa})$, we define an element $f\bullet g{\in}\mathcal{A}$ by

$$
:f\bullet g:_\kappa = f(\kappa)\bullet_\kappa g(\kappa). \tag{40}
$$

Some product formulas on $\mathcal{A}$ are given as follows:

$$
0\bullet f = f, \quad \frac{-1}{\kappa}\bullet\frac{-1}{\kappa} = 0, \quad \frac{1}{1-\kappa}\bullet f = \frac{1}{1-\kappa}, \quad \frac{-1}{1+\kappa}\bullet f = \frac{-1}{1+\kappa},
$$

where 0 stands for the singular solution $(\kappa, 0)$. These formulas say that $\frac{\pm 1}{1\mp\kappa}$ acts like 0 or ∞. Hence $\mathcal{A}$ is viewed naturally as the Riemann sphere with standard multiplicative structure such that $a0 = 0$, $a\infty = \infty$, but 0∞ is not defined. By the definition of $\bullet_\kappa$, we have $C_\kappa(f\bullet_\kappa g) = C_\kappa(f)C_\kappa(g)$.

Here the correspondence is given by the family of twisted Cayley transforms $\coprod_{\kappa\in\mathbb{C}} C_\kappa : \mathcal{A}{\to}\mathbb{C}\cup\{\infty\}$. We view $\mathcal{A}$ as a topological space through the identification $\coprod_{\kappa\in\mathbb{C}} C_\kappa$.

The table of correspondence is as follows:

$\mathcal{A}$	0	$\frac{-1}{\kappa}$	$\frac{1}{1-\kappa}$	$\frac{-1}{1+\kappa}$	$\frac{a}{1-a\kappa}$	$\frac{1-a}{1-\kappa+a(1+\kappa)}$	$f(\kappa)$
Image C_κ	1	-1	0	∞	$\frac{1-a}{1+a}$	a	$\frac{1-f(\kappa)(1-\kappa)}{1+f(\kappa)(1+\kappa)}$
singular point	∞	0	1	-1	$\frac{1}{a}$	$\frac{1+a}{1-a}$	$-$

Note that

$$C_\kappa^{-1}(a) = \frac{1-a}{1-\kappa+a(1+\kappa)} \sim \frac{\frac{1-a}{1+a}}{1-\frac{1-a}{1+a}\kappa} \sim T_0^\kappa C_0^{-1}(a)$$

is a parallel section, and $\frac{1-f(\kappa)(1-\kappa)}{1+f(\kappa)(1+\kappa)}$ is independent of κ for every parallel section f.

5.1 A product on $\tilde{\mathcal{A}}$

Let $\tilde{\mathcal{A}}$ be the space of all parallel sections given in (26), and consider the product $\bullet$ on $\tilde{\mathcal{A}}$ is given by the product formula (39). For $f, f' \in \tilde{\mathcal{A}}$, we set $f = (\kappa, y(\kappa), z(\kappa))$, $f' = (\kappa, y'(\kappa), z'(\kappa))$. $f{\bullet}g$ is defined as a parallel section defined on the open dense domain where $y(\kappa), y'(\kappa) \neq \infty$.

Note that

$$(\kappa, \frac{a}{1-a\kappa}, \log(1-a\kappa)){\bullet}(\kappa, \frac{-1}{1+\kappa}, \log(1+\kappa)) = (\kappa, \frac{-1}{1+\kappa}, \log(1-a)+\log(1+\kappa)),$$

$$(\kappa, \frac{a}{1-a\kappa}, \log(1-a\kappa)){\bullet}(\kappa, \frac{1}{1-\kappa}, \log(1-\kappa)) = (\kappa, \frac{1}{1-\kappa}, \log(1+a)+\log(1-\kappa)).$$

Although $\frac{\pm 1}{1\mp\kappa}$ plays the role of 0 or ∞, the third component depends on a.

For simplicity, we denote in particular

$$(41) \qquad \varpi_c = (\kappa; \frac{1}{1-\kappa}, c+\log\frac{1}{2}(1-\kappa)), \quad \bar{\varpi}_c = (\kappa; \frac{-1}{1+\kappa}, c+\log\frac{1}{2}(1+\kappa)).$$

It is easy to see that

$$\varpi_c{\bullet}\varpi_{c'} = \varpi_{c+c'}, \quad \bar{\varpi}_c{\bullet}\bar{\varpi}_{c'} = \bar{\varpi}_{c+c'},$$

but $\varpi_c{\bullet}\bar{\varpi}_{c'}$ diverges.

Let $\tilde{\mathcal{A}}_\times$ be the subset of $\tilde{\mathcal{A}}$ excluding the parallel sections $(\kappa; \frac{\pm 1}{1\mp\kappa}, c+\log(1\mp\kappa))$. We also set

$$\tilde{\mathcal{A}}_0 = \tilde{\mathcal{A}}_\times \cup \{\varpi_c\}, \quad \tilde{\mathcal{A}}_\infty = \tilde{\mathcal{A}}_\times \cup \{\bar{\varpi}_c\}.$$

Proposition 5.1. *$\widetilde{\mathcal{A}}$ is closed under the extended product $\bullet_\kappa$, where defined. In particular, $\widetilde{\mathcal{A}}_\times$, $\widetilde{\mathcal{A}}_0$ $\widetilde{\mathcal{A}}_\infty$ are each closed respectively under the $\bullet$-product.*

5.2 The infinitesimal left action

Note that the singular solution $\mathbf{1}=(\kappa, 0, 0) \in \tilde{\mathcal{A}}$ is the multiplicative identity. A neighborhood of $\mathbf{1}$ is given by $(\kappa, \frac{a}{1-a\kappa}, g+\log(1-a\kappa))$ by taking (a, g) in a small neighborhood of 0. For $g = 0$, we set $f_a = (\kappa, \frac{a}{1-a\kappa}, \log(1-a\kappa))$. For a parallel section $h = (\kappa, y(\kappa), z(\kappa)) \in \tilde{\mathcal{A}}$, the product $f_a{\bullet}h$ is given by

$$f_a{\bullet}h = \left(\frac{a + y(\kappa) + ay(\kappa)\kappa}{1 - a\kappa + ay(\kappa)(1-\kappa^2)}, z+\log(1-a\kappa+ay(\kappa)(1-\kappa^2))\right),$$

Consider the infinitesimal action

$$\frac{d}{ds}\Big|_{s=0} f_{as}\bullet_{\kappa}(y,z) = (a(1{+}2y\kappa{-}y^2(1{-}\kappa^2)), a(-\kappa + y(1{-}\kappa^2)).$$

Define for every fixed κ the invariant distribution

$$\tilde{L}_{\kappa}(y,z) = \{(a((1{+}y\kappa)^2{-}y^2), a(-\kappa + y(1{-}\kappa^2))); a \in \mathbb{C}\}.$$

By Proposition5.1, we have $dI_0^{\kappa}\tilde{L}_0 I_{\kappa}^0(y,z) = \tilde{L}_{\kappa}$.

5.3 The exponential mapping

The equation for the integral curves of the invariant distribution $\tilde{L}_{\kappa}$ through the identity $(0,0)$ is

$$\frac{d}{dt}(y(t),z(t)) = (a((1{+}y(t)\kappa)^2{-}y(t)^2), ay(t)(1{-}\kappa^2)), \quad (y(0),z(0)) = (0,0).$$

For the case $\kappa = 0$, $a = 1$, we have $(y(t),z(t)) = (\tanh t, \log\cosh t)$.

We define $\mathrm{Exp}_{\bullet} : \mathbb{C} \to \mathcal{A}_{\times}$ by the family of Exp_{κ}:

$$\mathrm{Exp}_{\bullet_{\kappa}} t = T_0^{\kappa}(\tanh t) = \frac{\sinh t}{\cosh t{-}(\sinh t)\kappa},$$

(42) $$\mathrm{Exp}_{\bullet} t = (\kappa; T_0^{\kappa}(\tanh t)) = \Big(\kappa; \frac{\sinh t}{\cosh t{-}(\sinh t)\kappa}\Big).$$

For a fixed t, $\mathrm{Exp}_{\bullet} t$ is a parallel section with the exponential law

$$\mathrm{Exp}_{\bullet} s \bullet \mathrm{Exp}_{\bullet} t = \mathrm{Exp}_{\bullet}(s{+}t), \quad \text{and} \quad \mathrm{Exp}_{\bullet}(s{+}2\pi i) = \mathrm{Exp}_{\bullet} s.$$

For the extended product, let $\widetilde{\mathrm{Exp}}_0\, t = (\tanh t; \log\cosh t)$, and let

$$\widetilde{\mathrm{Exp}}_{\kappa}\, t = I_0^{\kappa}\widetilde{\mathrm{Exp}}_0\, t = \Big(\frac{\sinh t}{\cosh t{-}(\sinh t)\kappa}, \log(\cosh t{-}(\sinh t)\kappa\Big)).$$

Although $\widetilde{\mathrm{Exp}}_{\kappa}$ is not defined for all $t \in \mathbb{C}$, viewing κ as an indeterminate permits us to define the exponential mapping $\widetilde{\mathrm{Exp}}_{\bullet} : \mathbb{C} \to \widetilde{\mathcal{A}}_{\times}$ by

(43) $$\widetilde{\mathrm{Exp}}_{\bullet} t = (\kappa; \frac{\sinh t}{\cosh t{-}(\sinh t)\kappa}, \log(\cosh t{-}(\sinh t)\kappa)).$$

This is a parallel section with the exponential law

$$\widetilde{\mathrm{Exp}}_{\bullet} s \bullet \widetilde{\mathrm{Exp}}_{\bullet}\, t = \widetilde{\mathrm{Exp}}_{\bullet}(s{+}t).$$

We remark here that $\widetilde{\mathrm{Exp}}_{\bullet} s$ is infinitely many valued. We have $\widetilde{\mathrm{Exp}}_{\bullet}(s{+}2\pi i) = \widetilde{\mathrm{Exp}}_{\bullet} s$, but these equalities hold as a set-to-set correspondence.

Consider the formula

$$\log(\cosh t - (\sinh t)\kappa) = \int_0^t \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds. \tag{44}$$

The multi-valuedness of the left hand side is caused by dependence of the integral on the homology class of the route of integration. Thus, for a fixed κ, the left hand side must be considered as a univalent function on the homological universal covering space of $\mathbb{C} - \{\cosh s - (\sinh s)\kappa = 0\}$ (the path space factored by the group of all homologically trivial loops).

Keeping this in mind, we see the following:

Proposition 5.2. *If κ is treated as an indeterminate, then $2\pi i$ periodicity does not appear in the integral*

$$\int_0^t \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds.$$

Proof If the periodicity appears, then $\int_a^{a+2\pi i} \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds$ must be of the form $2\pi n$. Hence the derivative by κ must vanish. Consider now the following quantity:

$$\frac{d}{d\kappa} \int_a^{a+2\pi i} \frac{\sinh s - (\cosh s)\kappa}{\cosh s - (\sinh s)\kappa} ds = - \int_a^{a+2\pi i} \frac{1}{(\cosh s - (\sinh s)\kappa)^2} ds.$$

This does not vanish if we regard κ as an indeterminate. □

We see also that for every $\alpha \in \mathbb{C}$

$$\widetilde{\mathrm{Exp}}_\bullet^{(\alpha)} s = \Big(\kappa; \frac{\sinh s}{\cosh t - (\sinh s)\kappa}, \log e^{\alpha s}(\cosh s - (\sinh s)\kappa)\Big)$$

satisfies the exponential law. Using this formula, it is easily seen that for $t \in \mathbb{R}$,

$$:\varpi_0:_\kappa = \lim_{t\to\infty} \Big(\frac{\sinh t}{\cosh t - (\sinh t)\kappa}, \log e^{-t}(\cosh t - (\sinh t)\kappa)\Big),$$

$$:\bar{\varpi}_0:_\kappa = \lim_{t\to -\infty} \Big(\frac{\sinh t}{\cosh t - (\sinh t)\kappa}, \log e^{t}(\cosh t - (\sinh t)\kappa)\Big).$$

We end by noting that $\tilde{\mathcal{A}}_\times$ is a strange object, which one cannot treat as an usual manifold. $\tilde{\mathcal{A}}_\times$ is a group-like object and the mapping which forget the last component for the map $\tilde{\mathcal{A}}_\times \to \mathcal{A}_\times$ is a homomorphism onto $\mathcal{A}_\times \cong \mathbb{C}_\times$. The mapping $\widetilde{\mathrm{Exp}}_\bullet : \mathbb{C} \to \tilde{\mathcal{A}}_\times$ may be viewed as an injective homomorphism.

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Department of Mathematics
Faculty of Science and Technology
Keio University

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