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by

Taka-aki Tanaka

Taka-aki Tanaka
Department of Mathematics
Keio University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

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Abstract

In this paper we establish, using Mahler's method, the algebraic independence of reciprocal sums of products of Fibonacci numbers including slowly increasing factors in their numerators (see Theorems 1, 5, and 6 below). Theorems 1 and 4 are proved by using Theorems 2 and 3 stating key formulas of this paper, which are deduced from the crucial Lemma 2. Theorems 5 and 6 are proved by using different technique. From Theorems 2 and 5 we deduce Corollary 2, the algebraic independence of the sum of a certain series and that of its subseries obtained by taking subscripts in a geometric progression.

1 Introduction

Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0). \tag{1}$$

Brousseau [2] proved that for every $k \in \mathbb{N}$

$$\sigma_k = \sum_{n=1}^{\infty} \frac{(-1)^n}{F_n F_{n+k}} = \frac{1}{F_k}\left(\frac{k(1-\sqrt{5})}{2} + \sum_{n=1}^{k} \frac{F_{n-1}}{F_n}\right).$$

Rabinowitz [8] proved that for every $k \in \mathbb{N}$

$$\sigma_k^* = \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2k}} = \frac{1}{F_{2k}} \sum_{n=1}^{k} \frac{1}{F_{2n-1}F_{2n}}.$$

In this paper we consider the arithmetic nature of the sums of similarly constructed series such as

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

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and

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}),$$

where $[x]$ denotes the largest integer not exceeding the real number x. These sums are not only transcendental but also algebraically independent in contrast with the sums σ_k and σ_k^* which are algebraic numbers.

In what follows, let $\{R_n\}_{n\geq 0}$ be the binary linear recurrence defined by

$$R_{n+2} = A_1 R_{n+1} + A_2 R_n \quad (n \geq 0), \tag{2}$$

where A_1, A_2 are nonzero integers with $\Delta = A_1^2 + 4A_2 > 0$ and R_0, R_1 are integers with $R_0 R_2 \neq R_1^2$ and $A_1 R_0 (A_1 R_0 - 2R_1) \leq 0$. We can express $\{R_n\}_{n\geq 0}$ as follows:

$$R_n = a\alpha^n + b\beta^n \quad (n \geq 0),$$

where $\alpha,\ \beta$ ($|\alpha| \geq |\beta|$) are the roots of $\Phi(X) = X^2 - A_1 X - A_2$ and $a, b \in \mathbb{Q}(\sqrt{\Delta})$. It is easily seen that $|\alpha| > |\beta| > 0$. Since $R_0 R_2 - R_1^2 = ab\Delta$ and $A_1 R_0 (A_1 R_0 - 2R_1) = (\alpha^2 - \beta^2)(b^2 - a^2)$, we see that $|a| \geq |b| > 0$. Therefore $\{R_n\}_{n\geq 0}$ is not a geometric progression and $R_n \neq 0$ for any $n \geq 1$.

Theorem 1. *The numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{A_2^n [\log_d n]}{R_n R_{n+2k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}).$$

EXAMPLE 1. Let $\{F_n\}_{n\geq 0}$ be the sequence of the Fibonacci numbers defined by (1). Then the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{F_n F_{n+2k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}).$$

EXAMPLE 2. Let $\{L_n\}_{n\geq 0}$ be the sequence of Lucas numbers defined by

$$L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geq 0). \tag{3}$$

Then the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{L_n L_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{[\log_d n]}{L_n L_{n+2k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}).$$

Theorem 1 is deduced from Theorems 2 and 3 below. The proof will be given in Section 3.

Let $f(x)$ be a real-valued function on $x \geq 0$ such that $f'(x) > 0$ for any $x > 0$ and $f(\mathbb{N}) \subset \mathbb{N}$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. For any $k \in \mathbb{N}$ we put

$$S_k = \sum_{n=f(1)}^{\infty} \frac{(-A_2)^n [f^{-1}(n)]}{R_n R_{n+k}}, \quad S_k^* = \sum_{n=f(1)}^{\infty} \frac{A_2^n [f^{-1}(n)]}{R_n R_{n+k}},$$

$$T_k = \sum_{n=f(1)}^{\infty} \frac{(-A_2)^n [f^{-1}(n)]}{R_{n+k-1} R_{n+k}},$$

and

$$U_k = \sum_{n=1}^{\infty} \frac{(-A_2)^{f(n)}}{R_{f(n)} R_{f(n)+k}}.$$

Let $\{F_n^*\}_{n\geq 0}$ be the Fibonacci type sequence defined by

$$F_0^* = 0, \quad F_1^* = 1, \quad F_{n+2}^* = A_1 F_{n+1}^* + A_2 F_n^* \quad (n \geq 0).$$

Theorem 2. *For any* $k \in \mathbb{N}$

$$S_k = \frac{1}{F_k^*} \sum_{l=1}^{k} (-A_2)^{l-1} T_l$$

and

$$U_k = \frac{1}{F_k^*} \left(T_1 - (-A_2)^k T_{k+1} \right).$$

Hence the sets of the numbers $\{S_1, \ldots, S_{k+1}\}$, $\{T_1, \ldots, T_{k+1}\}$, *and* $\{S_1 (= T_1), U_1, \ldots, U_k\}$ *generate the same vector space over* $\mathbb{Q}$.

Theorem 3. *If $f(n) \equiv f(1) \pmod 2$ for any $n \geq 1$, then*

$$S_{2k}^{*} = \frac{(-1)^{f(1)}}{F_{2k}^{*}} \sum_{l=1}^{2k} A_2^{l-1} T_l$$

for any $k \in \mathbb{N}$. Hence the numbers $\{S_{2l} \mid 1 \leq l \leq k\}$ are expressed as linearly independent linear combinations over $\mathbb{Q}$ of the numbers $\{T_l \mid 1 \leq l \leq 2k\}$.

Using Theorem 2, we prove also the following:

Theorem 4. *The numbers*

$$\sum_{n=1}^{\infty} \frac{A_2^{d^n}}{R_{d^n} R_{d^n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

are algebraically independent.

Example 3. The numbers

$$\sum_{n=1}^{\infty} \frac{1}{F_{d^n} F_{d^n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

are algebraically independent and so are the numbers

$$\sum_{n=1}^{\infty} \frac{1}{L_{d^n} L_{d^n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}).$$

Using different technique to that used in the proof of Theorem 4, we prove the following:

Theorem 5. *Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{n^l \xi^n (-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N}) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} \tag{4}$$

are algebraically independent.

As a special case of Theorem 5 we have the following:

Corollary 1. *Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}}, \quad \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (k \in \mathbb{N}), \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}}$$

are algebraically independent.

Combining Corollary 1 and Theorem 2 with $f(x) = d^x$, we immediately have the following:

Corollary 2. *Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+k}}, \quad \sum_{n=1}^{\infty} \frac{n(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

It is interesting that the second series of Corollary 2 is regarded as a subseries of the first one obtained by replacing n by d^n. It seems difficult to find in literature the results which assert the algebraic independence of the sum of a certain series and that of its subseries with subscripts taken in a geometric progression. For example, the algebraic independency of the numbers $\sum_{n=1}^{\infty} 1/F_n$ and $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \geq 3)$ is open. On the other hand, Lucas [3] showed that $\sum_{n=1}^{\infty} 1/F_{2^n} = (5-\sqrt{5})/2$. André-Jeannin [1] proved the irrationality of $\sum_{n=1}^{\infty} 1/F_n$, while its transcendency is open. Nishioka, Tanaka, and Toshimitsu [7] proved that the numbers $\sum_{n=1}^{\infty} 1/F_{d^n}$ $(d \geq 3)$ are algebraically independent.

Example 4. Let $\{F_n\}_{n \geq 0}$ be the sequence of the Fibonacci numbers defined by (1) and d an integer greater than 1. Then the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{F_n F_{n+k}}, \quad \sum_{n=1}^{\infty} \frac{n}{F_{d^n} F_{d^n+k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

Example 5. Let $\{L_n\}_{n \geq 0}$ be the sequence of Lucas numbers defined by (3) and d an integer greater than 1. Then the numbers

$$\sum_{n=1}^{\infty} \frac{(-1)^n [\log_d n]}{L_n L_{n+k}}, \quad \sum_{n=1}^{\infty} \frac{n}{L_{d^n} L_{d^n+k}} \quad (k \in \mathbb{N})$$

are algebraically independent.

If Δ is not a perfect square, we can prove the algebraic independence of the sums of the series (4) of Theorem 5 without the factor $(-A_2)^{d^n}$ in their numerators as follows:

Theorem 6. *Assume in addition that Δ is not a perfect square. Let d be an integer greater than 1. Then the numbers*

$$\sum_{n=1}^{\infty} \frac{n^l \xi^n}{R_{d^n} R_{d^n+k}} \quad (\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N}) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} \tag{5}$$

are algebraically independent.

2 Lemmas

The following lemma will be used in the proof of Theorems 1 and 4.

Lemma 1 (Tanaka [9])**.** *Let $\{R_n\}_{n\geq 0}$ be as in Section 1. Then the numbers*

$$\sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+k-1} R_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$$

are algebraically independent.

The following lemma plays an essential role in the proof of Theorems 2 and 3.

Lemma 2. *Let $f(x)$ be a real-valued function on $x \geq 0$ such that $f'(x) > 0$ for any $x > 0$ and $f(\mathbb{N}) \subset \mathbb{N}$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Let K be any field of characteristic 0 endowed with an absolute value $|\ \ |_v$. Let $\{a_n\}_{n\geq 1}$ be a sequence in K with $|a_n|_v = o(1/f^{-1}(n))$. Suppose the sum $\sum_{n=1}^{\infty} |a_n|_v$ converges in $\mathbb{R}$. Then in the completion K_v of K we have*

$$\sum_{n=f(1)}^{\infty} [f^{-1}(n)](a_n - a_{n+1}) = \sum_{h=1}^{\infty} a_{f(h)}. \tag{6}$$

Proof. Let $h \in \mathbb{N}$ and $n \in \mathbb{N}$. Since $f'(x) > 0$ for any $x > 0$, $(f^{-1}(x))' > 0$ for any $x \geq f(1)$. Hence, if $f(h) \leq n < f(h+1)$, then $h \leq f^{-1}(n) < h+1$ and so $[f^{-1}(n)] = h$. Therefore, letting

$$\chi(n) = \begin{cases} 1 & (n = f(h)) \\ 0 & (\text{otherwise}) \end{cases} \qquad \text{and} \qquad s_n = \sum_{k=1}^{n} \chi(k),$$

we see that $s_n = [f^{-1}(n)]$ for $n \geq f(1)$. Then, letting $H \in \mathbb{N}$ and $N = f(H)$, we have

$$\sum_{h=1}^{H} a_{f(h)} \quad = \quad \sum_{n=f(1)}^{N} \chi(n) a_n$$

$$= \sum_{n=f(1)}^{N-1} s_n(a_n - a_{n+1}) + s_N a_N$$

$$= \sum_{n=f(1)}^{N-1} [f^{-1}(n)](a_n - a_{n+1}) + [f^{-1}(N)]a_N. \tag{7}$$

Since $|a_n|_v = o(1/f^{-1}(n))$, $[f^{-1}(N)]a_N$ tends to 0 as $N \to \infty$. Since $\sum_{n=1}^{\infty} |a_n|_v$ converges in $\mathbb{R}$, the sum of the subseries $\sum_{h=1}^{\infty} a_{f(h)}$ also converges in K_v. Letting $H \to \infty$ in (7), we have (6). This completes the proof of the lemma.

Remark 1. The condition $|a_n|_v = o(1/f^{-1}(n))$ of Lemma 2 is satisfied if

$$|a_n|_v = o(n^{-1}), \tag{8}$$

since we have $[f^{-1}(n)] = s_n \le n$. We shall use the condition (8) instead in the proof of Theorems 2 and 3.

The following lemma is a special case of Theorem 3.3.2 in Nishioka [5], since its assumption is satisfied by Masser's vanishing theorem [4].

Lemma 3. *Let K be an algebraic number field and d an integer greater than* 1*. Suppose that $f_{ij}(z_1, z_2) \in K[[z_1, z_2]]$ $(i = 1, \ldots, m,\ j = 1, \ldots, n(i))$ are algebraically independent over $K(z_1, z_2)$ and convergent in a polydisc $U \subset \mathbb{C}^2$ around the origin. Assume that, for every i, $f_{i1}(z_1, z_2), \ldots, f_{in(i)}(z_1, z_2)$ satisfy the system of functional equations*

$$\begin{pmatrix} f_{i1}(z_1, z_2) \\ \vdots \\ \vdots \\ f_{in(i)}(z_1, z_2) \end{pmatrix} = \begin{pmatrix} a_i & 0 & \cdots & 0 \\ a_{21}^{(i)} & a_i & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ a_{n(i)\,1}^{(i)} & \cdots & a_{n(i)\,n(i)-1}^{(i)} & a_i \end{pmatrix} \begin{pmatrix} f_{i1}(z_1^d, z_2^d) \\ \vdots \\ \vdots \\ f_{in(i)}(z_1^d, z_2^d) \end{pmatrix} + \begin{pmatrix} b_{i1}(z_1, z_2) \\ \vdots \\ \vdots \\ b_{in(i)}(z_1, z_2) \end{pmatrix}, \tag{9}$$

where $a_i, a_{st}^{(i)} \in K$ and $b_{ij}(z_1, z_2) \in K(z_1, z_2)$. If $(\alpha_1, \alpha_2) \in U$ is an algebraic point with $0 < |\alpha_1|, |\alpha_2| < 1$ such that α_1, α_2 are multiplicatively independent, then the values $f_{ij}(\alpha_1, \alpha_2)$ $(i = 1, \ldots, m,\ j = 1, \ldots, n(i))$ are algebraically independent.

Remark 2. It is not necessary in Lemma 3 to assume that $b_{ij}(\alpha_1^{d^k}, \alpha_2^{d^k})$ $(i = 1, \ldots, m,\ j = 1, \ldots, n(i))$ are defined for all $k \ge 0$, which is satisfied by (9) and the fact that $f_{ij}(\alpha_1^{d^k}, \alpha_2^{d^k})$ $(i = 1, \ldots, m,\ j = 1, \ldots, n(i))$ are defined for all $k \geq 0$ since $(\alpha_1^{d^k}, \alpha_2^{d^k}) \in U$.

Lemma 4 (Theorem 3.2.1 in Nishioka [5]). *Let C be a field of characteristic 0. Suppose that $f_{ij}(z_1, z_2) \in C[[z_1, z_2]]$ $(i = 1, \ldots, m,\ j = 1. \ldots, n(i))$ satisfy the functional equations of the form (9) with $a_i, a_{st}^{(i)} \in C, a_i \neq 0, a_{s\,s-1}^{(i)} \neq 0$ $(2 \leq s \leq n(i))$, and $b_{ij}(z_1, z_2) \in C(z_1, z_2)$. If $f_{ij}(z_1, z_2)$ $(i = 1, \ldots, m,\ j = 1. \ldots, n(i))$ are algebraically dependent over $C(z_1, z_2)$, then there exists a non-empty subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, m\}$ with $a_{i_1} = \cdots = a_{i_r}$ such that $f_{i_1 1}, \ldots, f_{i_r 1}$ are linearly dependent over C modulo $C(z_1, z_2)$, that is, there exist $c_1, \ldots, c_r \in C$, not all zero, such that*

$$c_1 f_{i_1 1} + \cdots + c_r f_{i_r 1} \in C(z_1, z_2).$$

Lemma 5 (Nishioka [6, Lemmas 2, 3, and 6]). *Let ξ be a nonzero complex number and $a_1, \ldots, a_n$ nonzero complex numbers satisfying $|a_i| \neq 1,\ |a_i| \neq |a_j|$ $(i \neq j)$. Let $f_i(z) \in \mathbb{C}[[z]]$ $(0 \leq i \leq n)$ satisfy the functional equations*

$$\begin{aligned} f_0(z) &= \xi f_0(z^d) + \frac{z^r}{1 + \varepsilon z^r}, \\ f_i(z) &= \xi f_i(z^d) + \frac{z^r}{1 + a_i z^r} \quad (1 \leq i \leq n), \end{aligned}$$

where $r \in \mathbb{N}$ and $\varepsilon = \pm 1$. If $d = \xi = 2$ and $\varepsilon = 1$, then $f_i(z)$ $(1 \leq i \leq n)$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$, otherwise so are $f_i(z)$ $(0 \leq i \leq n)$.

Remark 3. If $d = \xi = 2$ and $\varepsilon = 1$, then

$$f_0(z) = \sum_{h=0}^{\infty} \frac{2^h z^{r2^h}}{1 + z^{r2^h}} = \frac{z^r}{1 - z^r} \in \mathbb{C}(z).$$

Lemma 6 (A special case of Theorem 3.3.10 in Nishioka [5]). *Let C be a field and F a subfield of C. If*

$$f(z_1, z_2) \in C[[z_1, z_2]] \cap F(z_1, z_2),$$

then there exist $A(z_1, z_2), B(z_1, z_2) \in F[z_1, z_2]$ such that

$$f(z_1, z_2) = \frac{A(z_1, z_2)}{B(z_1, z_2)}, \quad B(0, 0) \neq 0.$$

3 Proof of Theorems 1, 2, 3, and 4

Proof of Theorem 1. Let

$$S_{d,k} = \sum_{n=1}^{\infty} \frac{(-A_2)^n[\log_d n]}{R_n R_{n+k}} = \sum_{n=d}^{\infty} \frac{(-A_2)^n[\log_d n]}{R_n R_{n+k}},$$

$$S^*_{d,k} = \sum_{n=1}^{\infty} \frac{A_2^n[\log_d n]}{R_n R_{n+k}} = \sum_{n=d}^{\infty} \frac{A_2^n[\log_d n]}{R_n R_{n+k}},$$

and

$$T_{d,k} = \sum_{n=1}^{\infty} \frac{(-A_2)^n[\log_d n]}{R_{n+k-1} R_{n+k}} = \sum_{n=d}^{\infty} \frac{(-A_2)^n[\log_d n]}{R_{n+k-1} R_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}).$$

Letting $f(x) = d^x$ in Theorem 2, we see that for any fixed d

$$S_{d,k} = \frac{1}{F^*_k} \sum_{l=1}^{k} (-A_2)^{l-1} T_{d,l} \quad (k \in \mathbb{N}).$$

Hence the sets of the numbers $\{S_{d,l} \mid 2 \le d \le m,\ 1 \le l \le k\}$ and $\{T_{d,l} \mid 2 \le d \le m,\ 1 \le l \le k\}$ generate the same vector space over $\mathbb{Q}$ for any fixed $m \in \mathbb{N} \setminus \{1\}$ and for any fixed $k \in \mathbb{N}$. Since the numbers $T_{d,k}$ $(d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$ are algebraically independent by Lemma 1, the numbers $S_{d,k}$ $(d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$ are algebraically independent.

Again letting $f(x) = d^x$ and noting that $f(n) \equiv f(1) \pmod 2$ for any $n \in \mathbb{N}$, we see by Theorem 3 that for any fixed d

$$S^*_{d,2k} = \frac{(-1)^{f(1)}}{F^*_{2k}} \sum_{l=1}^{2k} A_2^{l-1} T_{d,l} \quad (k \in \mathbb{N}).$$

Hence the numbers $\{S^*_{d,2l} \mid 2 \le d \le m,\ 1 \le l \le k\}$ are expressed as linearly independent linear combinations over $\mathbb{Q}$ of the numbers $\{T_{d,l} \mid 2 \le d \le m,\ 1 \le l \le 2k\}$ for any $m \in \mathbb{N} \setminus \{1\}$ and for any $k \in \mathbb{N}$. Since the numbers $T_{d,k}$ $(d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$ are algebraically independent by Lemma 1, the numbers $S^*_{d,2k}$ $(d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N})$ are algebraically independent, which completes the proof of the theorem.

Before stating the proof of Theorems 2 and 3, we recall that $\{R_n\}_{n \ge 0}$ is expressed as

$$R_n = a\alpha^n + b\beta^n \quad (n \ge 0),$$

where $\alpha,\ \beta$ are the roots of $\Phi(X) = X^2 - A_1X - A_2$ such that $|\alpha| > |\beta| > 0$ and $a, b \in \mathbb{Q}(\sqrt{\Delta})$ satisfy $|a| \geq |b| > 0$. Using the same α and β, we can express the sequence $\{F_n^*\}_{n\geq 0}$ defined before Theorem 2 by

$$F_n^* = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \geq 0).$$

PROOF OF THEOREM 2. Since $R_n = a\alpha^n + b\beta^n$ $(n \geq 0)$ and $-A_2 = \alpha\beta$, we have

$$\begin{aligned} \frac{(-A_2)^n}{R_nR_{n+k}} &= \frac{1}{a(\alpha^k - \beta^k)}\left(\frac{\beta^n}{a\alpha^n + b\beta^n} - \frac{\beta^{n+k}}{a\alpha^{n+k} + b\beta^{n+k}}\right) \\ &= \frac{1}{a(\alpha^k - \beta^k)}\left(\frac{\beta^n}{R_n} - \frac{\beta^{n+k}}{R_{n+k}}\right). \end{aligned} \tag{10}$$

Hence, noting that $n|\beta^n/R_n| \to 0$ as $n \to \infty$, we have by Lemma 2 with Remark 1

$$\begin{aligned} S_k &= \frac{1}{a(\alpha^k - \beta^k)}\sum_{n=f(1)}^{\infty}[f^{-1}(n)]\left(\sum_{l=0}^{k-1}\frac{\beta^{n+l}}{R_{n+l}} - \sum_{l=0}^{k-1}\frac{\beta^{n+l+1}}{R_{n+l+1}}\right) \\ &= \frac{1}{a(\alpha^k - \beta^k)}\sum_{h=1}^{\infty}\sum_{l=0}^{k-1}\frac{\beta^{f(h)+l}}{R_{f(h)+l}}. \end{aligned} \tag{11}$$

Letting $k = 1$ and replacing n by $n + l - 1$ in (10), we have

$$\frac{(-A_2)^{n+l-1}}{R_{n+l-1}R_{n+l}} = \frac{1}{a(\alpha - \beta)}\left(\frac{\beta^{n+l-1}}{R_{n+l-1}} - \frac{\beta^{n+l}}{R_{n+l}}\right).$$

Hence by Lemma 2

$$\begin{aligned} T_l &= \frac{(-A_2)^{1-l}}{a(\alpha - \beta)}\sum_{n=f(1)}^{\infty}[f^{-1}(n)]\left(\frac{\beta^{n+l-1}}{R_{n+l-1}} - \frac{\beta^{n+l}}{R_{n+l}}\right) \\ &= \frac{(-A_2)^{1-l}}{a(\alpha - \beta)}\sum_{h=1}^{\infty}\frac{\beta^{f(h)+l-1}}{R_{f(h)+l-1}}. \end{aligned} \tag{12}$$

Therefore we have

$$S_k = \frac{1}{F_k^*}\sum_{l=1}^{k}(-A_2)^{l-1}T_l.$$

Replacing n by $f(h)$ in (10), we have

$$\frac{(-A_2)^{f(h)}}{R_{f(h)}R_{f(h)+k}} = \frac{1}{a(\alpha^k - \beta^k)}\left(\frac{\beta^{f(h)}}{R_{f(h)}} - \frac{\beta^{f(h)+k}}{R_{f(h)+k}}\right). \tag{13}$$

Hence

$$U_k = \frac{1}{a(\alpha^k - \beta^k)} \sum_{h=1}^{\infty} \left(\frac{\beta^{f(h)}}{R_{f(h)}} - \frac{\beta^{f(h)+k}}{R_{f(h)+k}} \right)$$

and so

$$U_k = \frac{1}{F_k^*} \left(T_1 - (-A_2)^k T_{k+1} \right),$$

which completes the proof of the theorem.

Proof of Theorem 3. Replacing k by $2k$ in (10) and multiplying its both sides by $(-1)^n$, we have

$$\begin{aligned} \frac{A_2^n}{R_n R_{n+2k}} &= \frac{1}{a(\alpha^{2k} - \beta^{2k})} \left(\frac{(-\beta)^n}{R_n} - \frac{(-\beta)^{n+2k}}{R_{n+2k}} \right) \\ &= \frac{1}{a(\alpha^{2k} - \beta^{2k})} \left(\sum_{l=0}^{2k-1} \frac{(-\beta)^{n+l}}{R_{n+l}} - \sum_{l=0}^{2k-1} \frac{(-\beta)^{n+l+1}}{R_{n+l+1}} \right). \end{aligned}$$

Hence, noting that $n|\beta^n/R_n| \to 0$ as $n \to \infty$, we have by Lemma 2 with Remark 1

$$\begin{aligned} S_{2k}^* &= \frac{1}{a(\alpha^{2k} - \beta^{2k})} \sum_{n=f(1)}^{\infty} [f^{-1}(n)] \left(\sum_{l=0}^{2k-1} \frac{(-\beta)^{n+l}}{R_{n+l}} - \sum_{l=0}^{2k-1} \frac{(-\beta)^{n+l+1}}{R_{n+l+1}} \right) \\ &= \frac{1}{a(\alpha^{2k} - \beta^{2k})} \sum_{h=1}^{\infty} \sum_{l=0}^{2k-1} \frac{(-\beta)^{f(h)+l}}{R_{f(h)+l}} \\ &= \frac{1}{a(\alpha^{2k} - \beta^{2k})} \sum_{l=0}^{2k-1} (-1)^{l+f(1)} \sum_{h=1}^{\infty} \frac{\beta^{f(h)+l}}{R_{f(h)+l}}, \end{aligned}$$

since $f(h) \equiv f(1) \pmod 2$ for any $h \geq 1$. Therefore we have by (12)

$$S_{2k}^* = \frac{(-1)^{f(1)}}{F_{2k}^*} \sum_{l=1}^{2k} A_2^{l-1} T_l,$$

which completes the proof of the theorem.

Proof of Theorem 4. Let

$$U_{d,k} = \sum_{n=1}^{\infty} \frac{A_2^{d^n}}{R_{d^n} R_{d^n+k}}$$

and

$$T_{d,k} = \sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+k-1} R_{n+k}} = \sum_{n=d}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_{n+k-1} R_{n+k}} \quad (d \in \mathbb{N} \setminus \{1\},\ k \in \mathbb{N}).$$

Letting $f(x) = d^x$ in Theorem 2 and noting that $(-1)^{d^n} = (-1)^d$ $(n \geq 1)$, we see that for any fixed d

$$(-1)^d U_{d,k} = \sum_{n=1}^{\infty} \frac{(-A_2)^{d^n}}{R_{d^n} R_{d^n+k}} = \frac{1}{F_k^*} \left(T_{d,1} - (-A_2)^k T_{d,k+1} \right) \quad (k \in \mathbb{N}).$$

Hence the numbers $\{U_{d,l} \mid 2 \leq d \leq m,\ 1 \leq l \leq k\}$ are expressed as linearly independent linear combinations over $\mathbb{Q}$ of the numbers $\{T_{d,l} \mid 2 \leq d \leq m,\ 1 \leq l \leq k+1\}$ for any $m \in \mathbb{N}\backslash\{1\}$ and for any $k \in \mathbb{N}$. Since the numbers $T_{d,k}$ $(d \in \mathbb{N}\backslash\{1\},\ k \in \mathbb{N})$ are algebraically independent by Lemma 1, the numbers $U_{d,k}$ $(d \in \mathbb{N}\backslash\{1\},\ k \in \mathbb{N})$ are algebraically independent, which completes the proof of the theorem.

4 Proof of Theorems 5 and 6

Remark 4. For $Q(z_1, z_2) \in \mathbb{C}(z_1, z_2)$ with $Q(0, 0) = 0$, we define

$$f(x, z_1, z_2) = \sum_{n=1}^{\infty} x^n Q(z_1^{d^n}, z_2^{d^n}),$$

where x is a variable and d is an integer greater than 1. Letting $D = x\partial/\partial x$, we see that

$$f_l(x, z_1, z_2) := D^l f(x, z_1, z_2) = \sum_{n=1}^{\infty} n^l x^n Q(z_1^{d^n}, z_2^{d^n}) \qquad (l \geq 0)$$

satisfy

$$\begin{aligned}
f_0(x, z_1, z_2) &= x f_0(x, z_1^d, z_2^d) + x Q(z_1^d, z_2^d), \\
f_1(x, z_1, z_2) &= x f_1(x, z_1^d, z_2^d) + x f_0(x, z_1^d, z_2^d) + x Q(z_1^d, z_2^d), \\
&\ \vdots \\
f_m(x, z_1, z_2) &= \sum_{l=0}^{m} \binom{m}{l} x f_l(x, z_1^d, z_2^d) + x Q(z_1^d, z_2^d).
\end{aligned}$$

Hence for a complex number x, the functions $f_0(x, z_1, z_2), \ldots, f_m(x, z_1, z_2)$ satisfy a system of functional equations of the form (9).

Proof of Theorem 5. Let $c = a^{-1}b,\ \gamma = \alpha^{-1}\beta$, and

$$f_{\xi l k}(z) = \sum_{n=1}^{\infty} n^l \xi^n \left(\frac{z^{d^n}}{1 + c z^{d^n}} - \frac{\gamma^k z^{d^n}}{1 + c\gamma^k z^{d^n}} \right) \quad (\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N}).$$

Then

$$f_{\xi lk}(\gamma) = a^2(\alpha^k - \beta^k) \sum_{n=1}^{\infty} \frac{n^l \xi^n (-A_2)^{d^n}}{R_{d^n} R_{d^n+k}}. \tag{14}$$

Using (11) in the proof of Theorem 2 and letting $k = 1$, $f(x) = d^x$, and $g(z) = \sum_{n=1}^{\infty} z^{d^n}/(1 + cz^{d^n})$, we have

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} = \frac{1}{a(\alpha - \beta)} \sum_{n=1}^{\infty} \frac{\beta^{d^n}}{R_{d^n}} = \frac{g(\gamma)}{a^2(\alpha - \beta)}. \tag{15}$$

Therefore it is enough by (14) and (15) to prove the algebraic independence of the values $f_{\xi lk}(\gamma)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(\gamma)$. We see that each $f_{\xi 0k}(z)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ k \in \mathbb{N})$ satisfies the functional equation

$$f_{\xi 0k}(z) = \xi f_{\xi 0k}(z^d) + \xi \left(\frac{z^d}{1 + cz^d} - \frac{\gamma^k z^d}{1 + c\gamma^k z^d} \right)$$

and $f_{\xi lk}(z)$ $(l \geq 0)$ satisfy a system of functional equations of the form (9) for every fixed ξ and k by Remark 4. We see also that $g(z)$ satisfies the functional equation

$$g(z) = g(z^d) + \frac{z^d}{1 + cz^d}.$$

Hence by Lemma 3 the values $f_{\xi lk}(\gamma)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(\gamma)$ are algebraically independent if the functions $f_{\xi lk}(z)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z)$ are algebraically independent over $\mathbb{C}(z)$.

We assert that for every fixed $\xi \neq 1$ the functions $f_{\xi 0k}(z)$ $(k \in \mathbb{N})$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ and so are the functions $f_{10k}(z)$ $(k \in \mathbb{N})$ with $g(z)$, which implies by Lemma 4 that the functions $f_{\xi lk}(z)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z)$ are algebraically independent over $\mathbb{C}(z)$. Let

$$h_{\xi k}(z) = \sum_{n=1}^{\infty} \frac{\gamma^k \xi^n z^{d^n}}{1 + c\gamma^k z^{d^n}} \quad (\xi \in \overline{\mathbb{Q}}^{\times},\ k \geq 0).$$

Then

$$f_{\xi 0k}(z) = h_{\xi 0}(z) - h_{\xi k}(z)$$

for every fixed $\xi \in \overline{\mathbb{Q}}^{\times}$ and $k \in \mathbb{N}$ and each $h_{\xi k}(z)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ k \geq 0)$ satisfies the functional equation

$$h_{\xi k}(z) = \xi h_{\xi k}(z^d) + \frac{\xi \gamma^k z^d}{1 + c\gamma^k z^d}.$$

Suppose there exists a $\xi \neq 1$ such that $f_{\xi 01}(z), \ldots, f_{\xi 0k}(z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ for some k. If $d = \xi = 2$ and $c = 1$, we see by Remark 3 that $h_{20}(z) = 2z^2/(1-z^2) \in \mathbb{C}(z)$ and so $h_{21}(z), \ldots, h_{2k}(z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$; otherwise, so are $h_{\xi 0}(z), h_{\xi 1}(z), \ldots, h_{\xi k}(z)$, which contradicts Lemma 5, since $H_{\xi k}(z) := \xi^{-1}\gamma^{-k} h_{\xi k}(z)$ satisfies the functional equation

$$H_{\xi k}(z) = \xi H_{\xi k}(z^d) + \frac{z^d}{1 + c\gamma^k z^d}.$$

Therefore, if $f_{\xi l k}(z)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z) = h_{10}(z)$ are algebraically dependent over $\mathbb{C}(z)$, then $h_{10}(z), f_{101}(z), \ldots, f_{10k}(z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$ for some k, and hence so are $h_{10}(z), h_{11}(z), \ldots, h_{1k}(z)$, which contradicts Lemma 5. Therefore the functions $f_{\xi l k}(z)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z)$ are algebraically independent over $\mathbb{C}(z)$ and so the values $f_{\xi l k}(\gamma)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(\gamma)$ are algebraically independent, which completes the proof of the theorem.

Proof of Theorem 6. First we consider the case where α, β are multiplicatively dependent. Then there exist integers m, n, not both zero, with $\alpha^m \beta^n = 1$. Since α and β are field conjugates in the quadratic number field $\mathbb{Q}(\sqrt{\Delta})$, $\beta^m \alpha^n = 1$ must also hold. This implies

$$(\alpha\beta)^{m+n} = (\alpha/\beta)^{m-n} = 1.$$

Since $|\alpha/\beta| > 1$, we have $m = n \neq 0$, and hence $\alpha\beta$ must be a real root of unity, i.e., $-A_2 = \alpha\beta = \pm 1$. Therefore this case is proved by Theorem 5 since $(-A_2)^{d^n} = (-A_2)^d$ $(n \geq 1)$.

Secondly we consider the case where α, β are multiplicatively independent. Define

$$f_{\xi l k}(z_1, z_2) = \sum_{n=1}^{\infty} n^l \xi^n \left(\frac{z_1^{d^n}}{1 + c z_2^{d^n}} - \frac{\gamma^k z_1^{d^n}}{1 + c\gamma^k z_2^{d^n}} \right) \quad (\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N}),$$

where $c = a^{-1}b$ and $\gamma = \alpha^{-1}\beta$. Then

$$f_{\xi l k}(\alpha^{-2}, \gamma) = a^2(\alpha^k - \beta^k) \sum_{n=1}^{\infty} \frac{n^l \xi^n}{R_{d^n} R_{d^n + k}}.$$

Using (11) in the proof of Theorem 2 and letting $k = 1$, $f(x) = d^x$, and $g(z_1, z_2) = \sum_{n=1}^{\infty} z_2^{d^n}/(1 + c z_2^{d^n})$, we have

$$\sum_{n=1}^{\infty} \frac{(-A_2)^n [\log_d n]}{R_n R_{n+1}} = \frac{1}{a(\alpha - \beta)} \sum_{n=1}^{\infty} \frac{\beta^{d^n}}{R_{d^n}} = \frac{g(\alpha^{-2}, \gamma)}{a^2(\alpha - \beta)}.$$

Therefore it is enough to prove the algebraic independence of the values $f_{\xi lk}(\alpha^{-2},\gamma)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(\alpha^{-2},\gamma)$. We see that each $f_{\xi 0k}(z_1,z_2)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ k \in \mathbb{N})$ satisfies the functional equation

$$f_{\xi 0k}(z_1,z_2) = \xi f_{\xi 0k}(z_1^d,z_2^d) + \xi\left(\frac{z_1^d}{1+cz_2^d} - \frac{\gamma^k z_1^d}{1+c\gamma^k z_2^d}\right)$$

and $f_{\xi lk}(z_1,z_2)$ $(l \geq 0)$ satisfy a system of functional equations of the form (9) for every fixed ξ and k by Remark 4. We see also that $g(z_1,z_2)$ satisfies the functional equation

$$g(z_1,z_2) = g(z_1^d,z_2^d) + \frac{z_2^d}{1+cz_2^d}.$$

Hence, noting that α^{-2},γ are multiplicatively independent, we see by Lemma 3 that the values $f_{\xi lk}(\alpha^{-2},\gamma)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(\alpha^{-2},\gamma)$ are algebraically independent if the functions $f_{\xi lk}(z_1,z_2)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z_1,z_2)$ are algebraically independent over $\mathbb{C}(z_1,z_2)$. We assert that for every fixed $\xi \neq 1$ the functions $f_{\xi 0k}(z_1,z_2)$ $(k \in \mathbb{N})$ are linearly independent over $\mathbb{C}$ modulo $\mathbb{C}(z_1,z_2)$ and so are the functions $f_{10k}(z_1,z_2)$ $(k \in \mathbb{N})$ with $g(z_1,z_2)$, which implies by Lemma 4 that the functions $f_{\xi lk}(z_1,z_2)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z_1,z_2)$ are algebraically independent over $\mathbb{C}(z_1,z_2)$.

Suppose there exists a $\xi \neq 1$ such that $f_{\xi 01}(z_1,z_2),\dots,f_{\xi 0k}(z_1,z_2)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z_1,z_2)$ for some k. Thus there are complex numbers $c_1,\dots,c_k$, not all zero, such that

$$c_1 f_{\xi 01}(z_1,z_2) + \cdots + c_k f_{\xi 0k}(z_1,z_2) \in \mathbb{C}(z_1,z_2).$$

Since $f_{\xi 01}(z_1,z_2),\dots,f_{\xi 0k}(z_1,z_2) \in \mathbb{C}[[z_1,z_2]]$, by Lemma 6 there exist $A(z_1,z_2), B(z_1,z_2) \in \mathbb{C}[z_1,z_2]$ such that

$$c_1 f_{\xi 01}(z_1,z_2) + \cdots + c_k f_{\xi 0k}(z_1,z_2) = \frac{A(z_1,z_2)}{B(z_1,z_2)}, \quad B(0,0) \neq 0.$$

Letting $z_1 = z_2 = z$, we have

$$c_1 f_{\xi 01}(z,z) + \cdots + c_k f_{\xi 0k}(z,z) \in \mathbb{C}(z),$$

which contradicts Lemma 5 by the same way as in the proof of Theorem 5. Therefore, if $f_{\xi lk}(z_1,z_2)$ $(\xi \in \overline{\mathbb{Q}}^{\times},\ l \geq 0,\ k \in \mathbb{N})$ and $g(z_1,z_2)$ are algebraically dependent over $\mathbb{C}(z_1,z_2)$, then $g(z_1,z_2), f_{101}(z_1,z_2),\dots,f_{10k}(z_1,z_2)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z_1,z_2)$ for some k. By the same way as above

$g(z,z), f_{101}(z,z), \ldots, f_{10k}(z,z)$ are linearly dependent over $\mathbb{C}$ modulo $\mathbb{C}(z)$, which again contradicts Lemma 5. Therefore the functions $f_{\xi lk}(z_1,z_2)$ ($\xi \in \overline{\mathbb{Q}}^{\times}$, $l \geq 0$, $k \in \mathbb{N}$) and $g(z_1,z_2)$ are algebraically independent over $\mathbb{C}(z_1,z_2)$ and so the values $f_{\xi lk}(\alpha^{-2},\gamma)$ ($\xi \in \overline{\mathbb{Q}}^{\times}$, $l \geq 0$, $k \in \mathbb{N}$) and $g(\alpha^{-2},\gamma)$ are algebraically independent, which completes the proof of the theorem.

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Taka-aki Tanaka
Department of Mathematics
Keio University
Hiyoshi 3–14–1, Kohoku-ku
Yokohama 223–8522
Japan
E-mail: takaaki@math.keio.ac.jp