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Partition Functions of Supersymmetric Gauge Theories in Noncommutative $\mathbb{R}^{2D}$ and their Unified Perspective

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Abstract

We investigate cohomological gauge theories in noncommutative $\mathbb{R}^{2D}$. We show that vacuum expectation values of the theories do not depend on noncommutative parameters, and the large noncommutative parameter limit is equivalent to the dimensional reduction. As a result of these fact, we show that a partition function of a cohomological theory defined in noncommutative $\mathbb{R}^{2D}$ and a partition function of a cohomological field theory in $\mathbb{R}^{2D+2}$ are equivalent if they are connected through dimensional reduction. Therefore, we find several partition functions of supersymmetric gauge theories in various dimensions are equivalent. Using this technique, we determine the partition function of the $\mathcal{N}=4$ U(1) gauge theory in noncommutative $\mathbb{R}^4$, where its action does not include a topological term. The result is common among (8-dim , $\mathcal{N}=2$), (6-dim , $\mathcal{N}=2$), (2-dim , $\mathcal{N}=8$) and the IKKT matrix model given by their dimensional reduction to 0-dim. We also discuss the partition function with the topological term.

1 Introduction

The first break through of the recent calculation technology for $\mathcal{N}=2$ supersymmetric Yang-Mills theories is brought by Nekrasov [1, 2]. After [1], many kinds of developments appear in $\mathcal{N}\geq 2$ supersymmetric Yang-Mills theories and string theories corresponding to them. From those analysis, it is found that different dimension theories are related each other [7, 3, 4, 5, 6]. There is more example that the different dimensional theories are connected to each other. For example, Dijkgraaf and Vafa show that some correlation functions in matrix theories and $\mathcal{N}=1$ Yang-Mills theories are equivalent [8]. It goes on and on. These facts imply the existence of some kind of unified perspectives. One of the ideas to explain the unification is the 'tHooft's large N gauge theory and string correspondence. Until now, many investigations from this point of view are reported. Meanwhile, the large N gauge theories are similar to noncommutative theories in the operator formalism in some infinite dimensional Hilbert space with discrete basis. In this article, we suggest a simple way to understand the reason why partition functions of various dimensional supersymmetric gauge theories are given as same form or have relations with each other. The basic idea of the way is given in [9, 10, 11]. Cohomological gauge theories in Euclidian spaces are invariant under the noncommutative parameter shifting, as we will see it in the next section. When we take the large noncommutative parameter limit, kinetic terms become irrelevant like dimensional reduction, then the partition function is essentially computable by using lower dimensional theories. From this fact, we will explain that partition functions in various dimensions are equivalent.

Here is the organization of this article. In section 2, invariance of cohomological field theories in noncommutative $\mathbb{R}^{2D}$ (N.C. $\mathbb{R}^{2D}$ for short) under deformation of noncommutative parameters will be proved formally. This invariance is not usual symmetry, because the action is deformed. Nevertheless, expectation values and partition functions are invariant. Particularly, we will treat the $\mathcal{N}=2$ and $\mathcal{N}=4$ Yang-Mills theories in N.C. $\mathbb{R}^4$ as examples. In section 3, universality of the partition functions will be investigated. By using the result of section 2, we will show that the several partition functions in different dimensions are equivalent. In section 4, by the technique of section 2 we will calculate the partition function of the $\mathcal{N}=4$ U(1) gauge theory in N.C. $\mathbb{R}^4$ without the terms proportional to the instanton number $\int F\wedge F$. This partition function is equal to partition functions of several dimensions. In section 5, the moduli space of $\mathcal{N}=4$ U(1) gauge theory in N.C. $\mathbb{R}^4$ will be discussed. The partition function of $\mathcal{N}=4$ U(1) theory with $\int F\wedge F$ will be investigated, too. In section 6, we will summarize this article.

2 N.C. Cohomological Yang-Mills Theory

In this section, we investigate some important properties of the cohomological Yang-Mills theories in N.C. $\mathbb{R}^{2D}$ whose noncommutativity is defined as

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \ , \tag{1}$$

where the $\theta^{\mu\nu}$ is an element of an antisymmetric matrix and called noncommutative parameters.

Since action functionals of cohomological field theories are defined by BRS-exact functionals like $\hat{\delta}\Psi[\phi_i]$, where $\hat{\delta}$ is a some BRS operator and $\{\phi_i\}$ represent all considered fields and Ψ is a some fermionic functional, the partition function of the cohomological field theory is invariant under any infinitesimal variation δ' which commutes (or anti-commutes) with the BRS transformation:

$$\begin{aligned} \hat{\delta}\delta' &= \pm\delta'\hat{\delta}, \\ \delta' \ Z_\theta &= \int \prod_i \mathcal{D}\phi_i \ \ \delta' \left(- \int dx^{2D} \hat{\delta}\Psi \right) \ \exp\left(-S_\theta\right) \\ &= \pm \int \prod_i \mathcal{D}\phi_i \ \ \hat{\delta} \left(- \int dx^{2D} \delta'\Psi \right) \ \exp\left(-S_\theta\right) = 0. \end{aligned} \tag{2}$$

Let δ_θ be the infinitesimal deformation operator of the noncommutative parameter θ which operates as

$$\delta_\theta \ \theta^{\mu\nu} = \delta\theta^{\mu\nu}, \tag{3}$$

where $\delta\theta^{\mu\nu}$ are some infinitesimal anti-symmetric two form elements. If δ_θ and BRS operator $\hat{\delta}$ commute each other, then the partition function is invariant. Indeed, there is some examples such that $\hat{\delta}\delta_\theta = \delta_\theta\hat{\delta}$, and partition functions are calculated by using this property [9, 10, 11].

In this article, cohomological Yang-Mills theories in noncommutative Euclidian spaces are discussed. If there is a gauge symmetry, the BRS-like transformation is slightly different from the one of non-gauge theory. The BRS-like symmetry is not nilpotent but

$$\hat{\delta}^2 = \delta_{g,\theta}, \tag{4}$$

where $\delta_{g,\theta}$ is a gauge transformation operator deformed by some noncommutative deformation method like the star product $*_\theta$. As occasion arises, the gauge transformation $\delta_{g,\theta}$ is defined as one including global symmetry transformations. The partition function of the noncommutative cohomological field theory is invariant under changing noncommutative parameters when the BRS transformation does not depend on the noncommutative parameters, because the BRS transformation $\hat{\delta}$ and the θ deformation δ_θ commute. Conversely, when the definition of the BRS-like operator (4) depends on the noncommutative parameter θ, then $\hat{\delta}$ and δ_θ do not commute :

$$\delta_\theta \hat{\delta} \neq \hat{\delta}\delta_\theta \Rightarrow \delta_\theta \hat{\delta} = \hat{\delta}' \delta_\theta, \tag{5}$$

where $\hat{\delta}'$ is a BRS-like operator that generates the same transformations as the original BRS-like operator $\hat{\delta}$, except for the square. The square of $\hat{\delta}$ is defined by

$$\hat{\delta}'^2 = \delta_{g,\theta+\delta\theta}. \tag{6}$$

Since the gauge symmetry is defined by using noncommutative parameter $\theta^{\mu\nu} + \delta\theta^{\mu\nu}$ after the δ_θ operation, this difference arises. This fact makes a little complex problem to prove the θ-shift invariance of noncommutative cohomological Yang-Mills theory in comparison with the case of non-gauge theory.

Note that the essential point of this problem is not nilpotent property changing, but θ dependence of the definition of the BRS operator. (In fact, we can construct the BRS operator for the cohomological Yang-Mills theory as a nilpotent operator [12].)

However, we can prove the invariance of the partition function of cohomological Yang-Mills theory in N.C. $\mathbb{R}^{2D}$ under the noncommutative parameter deformation. For simplicity, we take

$$(\theta^{\mu\nu}) = \bigoplus_i \epsilon^{2i-1,2i}\theta = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,$$

where $\epsilon^{2i-1,2i}$ is an antisymmetric tensor such that $\epsilon^{2i-1,2i} = -\epsilon^{2i,2i-1} = 1$, and we restrict the θ deformation to

$$\theta \to \theta + \delta\theta \quad .$$

In the following, we only use operator formalisms to describe the noncommutative field theory, therefore the fields are operators acting on the Hilbert space $\mathcal{H}$. Then differential operators ∂_μ are expressed by using commutation brackets $-i\theta^{-1}_{\mu\nu}[x^\nu, \ ] \equiv [\hat{\partial}_\mu, *]$ and $\int d^{2D}x$ is replaced with $det(\theta)^{1/2}Tr_{\mathcal{H}}$. Therefore the noncommutative parameter deformation is equivalent with replacing $-i\theta^{-1}_{\mu\nu}[x^\nu, \ ]$ and $det(\theta)^{1/2}Tr_{\mathcal{H}}$ by $-i(\theta+\delta\theta)^{-1}_{\mu\nu}[x^\nu, \ ]$ and $det(\theta+\delta\theta)^{1/2}Tr_{\mathcal{H}}$, respectively.

Let us consider Donaldson-Witten theory (topological twisted $\mathcal{N} = 2$ Yang-Mills theory) on N.C. $\mathbb{R}^4$ [13]. This theory is constructed by bosonic fields $(A_\mu, H^+_{\mu\nu}, \bar{\phi}, \phi)$ and fermionic fields $(\psi_\mu, \chi^+_{\mu\nu}, \eta)$, where $(A_\mu, H^+_{\mu\nu}, \bar{\phi})$ and $(\psi_\mu, \chi^+_{\mu\nu}, \eta)$ are supersymmetric (BRS) pairs.

$$\begin{aligned} \chi^+_{\mu\nu},\ H^+_{\mu\nu} &\in \Omega^{2,+}(\mathbb{R}^4, \mathrm{ad}P), \qquad \psi_\mu \in \Omega^1(\mathbb{R}^4, \mathrm{ad}P), \\ \eta,\ \bar{\phi},\ \phi &\in \Omega^0(\mathbb{R}^4, \mathrm{ad}P). \end{aligned} \tag{7}$$

Their ghost numbers are assigned as (A_μ,$\chi^+_{\mu\nu}$,$H^+_{\mu\nu}$,ψ_μ,η, $\bar{\phi}$,ϕ)=(0,−1,0,1,−1,−2,2). The BRS-like operator is defined by

$$\begin{aligned} \hat{\delta}A_\mu &= \psi_\mu, & \hat{\delta}\chi^+_{\mu\nu} &= H^+_{\mu\nu}, & & \\ \hat{\delta}\psi_\mu &= D_\mu\phi, & \hat{\delta}H^+_{\mu\nu} &= i[\chi^+_{\mu\nu},\phi], & & \\ \hat{\delta}\phi &= 0, & \hat{\delta}\bar{\phi} &= \eta, & \hat{\delta}\eta &= i[\bar{\phi},\phi], \end{aligned} \tag{8}$$

where the covariant derivative is defined by $D_\mu * := [\hat{\partial}_\mu + iA_\mu\ ,\ *\ ]$ with $\hat{\partial}_\mu := -i\theta^{-1}_{\mu\nu}x^\nu$. When we consider only the case of N.C. $\mathbb{R}^{2D}$, field theories are expressed by the Fock space formalism. (See appendix A.) In the Fock space representation, fields are expressed as $A_\mu = \sum A_{\mu}{}^{n_1 n_2}_{m_1 m_2}|n_1,n_2\rangle\langle m_1,m_2|$, $\psi_\mu = \sum \psi_{\mu}{}^{n_1 n_2}_{m_1 m_2}|n_1,n_2\rangle\langle m_1,m_2|$, etc. Therefore, the above BRS transformations are expressed as

$$\hat{\delta}A_{\mu}{}^{n_1 n_2}_{m_1 m_2} = \psi_{\mu}{}^{n_1 n_2}_{m_1 m_2} \quad ,\cdots \quad . \tag{9}$$

Let us define gauge fermions as

$$\begin{aligned} \Psi &= Tr_{\mathcal{H}}tr\left[2\chi^+_{\mu\nu}(-iF^{+\mu\nu} + \frac{1}{2}H^{+\mu\nu})\right], \\ \Psi_{\rm proj} &= -Tr_{\mathcal{H}}tr\left[\bar{\phi}D_\mu\psi^\mu\right], \end{aligned} \tag{10}$$

then the action functional is given by

$$\begin{aligned} S &= Tr_{\mathcal{H}}\ L(A_\mu,\ldots;\hat{\partial}_{z_i},\hat{\partial}_{\bar{z}_i}) \\ &= Tr_{\mathcal{H}}tr\hat{\delta}(\Psi + \Psi_{\rm proj}) \\ &= Tr_{\mathcal{H}}tr\Big(F^{+2} - 4i\chi^{+\mu\nu}D_\mu\psi_\nu - \eta D_\mu\psi^\mu + i\phi\{\chi^+_{\mu\nu},\chi^{+\mu\nu}\} \\ &\qquad -i\bar{\phi}\{\psi_\mu,\psi^\mu\} - \bar{\phi}D_\mu D^\mu\phi\Big)\quad , \end{aligned} \tag{11}$$

where tr is trace for gauge group. In this article, we omit to note $det(\theta)^{1/2}$ that is an overall factor, for economy of space. Let us change the dynamical variables as

$$\begin{aligned} &A_\mu \to \frac{1}{\sqrt{\theta}}\tilde{A}_\mu,\quad \psi_\mu \to \frac{1}{\sqrt{\theta}}\tilde{\psi}_\mu,\quad \bar{\phi} \to \frac{1}{\theta}\tilde{\bar{\phi}},\quad \eta \to \frac{1}{\theta}\tilde{\eta} \\ &\chi^+_{\mu\nu} \to \frac{1}{\theta}\tilde{\chi}^+_{\mu\nu},\quad H^+_{\mu\nu} \to \frac{1}{\theta}\tilde{H}^+_{\mu\nu},\quad \phi \to \tilde{\phi}\quad . \end{aligned} \tag{12}$$

Note that this changing does not cause nontrivial Jacobian from the path integral measure because of the BRS symmetry. Then, the action is rewritten as

$$S \to \frac{1}{\theta^2}\tilde{S} \quad ,\quad L(A_\mu,\ldots;\hat{\partial}_{z_i},\hat{\partial}_{\bar{z}_i}) \to \frac{1}{\theta^2}L(\tilde{A}_\mu,\ldots;-a_i^\dagger,a_i)\quad . \tag{13}$$

Here the action in the lefthand side depends on θ because the derivative is given by $\partial_{z_i} = -\sqrt{\theta^{-1}}[a_i^\dagger, \ ]$ and so on. In contrast, the action $\tilde{S}$ in the righthand side does not depend on θ because all θ parameters are factorized out. Using the BRS symmetry or the fact of eq.(2), it is proved that the partition function is invariant under the deformation of θ, because $\delta_\theta Z = -2(\delta\theta)\theta^{-3}\langle\tilde{S}\rangle = 0$. $Tr_{\mathcal{H}} tr(\phi F + \frac{1}{2}\psi\wedge\psi)$ and $tr\phi^n$ are known as observables of Donaldson-Witten theory. They are rewritten as $\frac{1}{\theta}Tr_{\mathcal{H}} tr(\tilde{\phi}\tilde{F} + \frac{1}{2}\tilde{\psi}\wedge\tilde{\psi})$ and $tr\tilde{\phi}^n$. Then $\delta_\theta\langle O\rangle = 0$ for the operators are proved in a similar way to the proof of $\delta_\theta Z = 0$. Therefore, invariance of Donaldson-Witten theory under $\theta \to \theta + \delta\theta$ is proved.

We can discuss the topological twisted $\mathcal{N} = 4$ Yang-Mills theory in noncommutative $\mathbb{R}^4$ similarly [1] [14]. There are additional fields $(B^+_{\mu\nu}, c, H_\mu)$ and $(\psi^+_{\mu\nu}, \bar{\eta}, \chi_\mu)$, where $(B^+_{\mu\nu}, c, H_\mu)$ are bosonic fields and $(\psi^+_{\mu\nu}, \bar{\eta}, \chi_\mu)$ are fermionic fields, where $B^+_{\mu\nu}, \psi^+_{\mu\nu} \in \Omega^{2,+}(\mathbb{R}^4, \mathrm{ad}P)$. They are supersymmetric partners, and the BRS multiples are expressed by the following diagram.

$$
\begin{array}{ccccc}
 & & \psi_\mu & & \\
 & \hat{\delta}_+ \nearrow & & \searrow \hat{\delta}_- & \\
A_\mu & & & & H_\mu \\
 & \searrow & & \nearrow & \\
 & & \chi_\mu & &
\end{array}
\quad , \quad
\begin{array}{ccccc}
 & & \psi^+_{\mu\nu} & & \\
 & \hat{\delta}_+ \nearrow & & \searrow \hat{\delta}_- & \\
B^+_{\mu\nu} & & & & H^+_{\mu\nu} \\
 & \searrow & & \nearrow & \\
 & & \chi^+_{\mu\nu} & &
\end{array}
$$

There are two BRS-like operators $\hat{\delta}_+$ and $\hat{\delta}_-$ because of the R-symmetry of the $\mathcal{N} = 4$. The $\hat{\delta}_+$ transformations are given by

$$\hat{\delta}_+ B^+_{\mu\nu} = \psi^+_{\mu\nu} \quad , \quad \psi^+_{\mu\nu} = i[B^+_{\mu\nu}, \phi] \tag{14}$$

$$\hat{\delta}_+ \chi_\mu = H_\mu \quad , \quad \hat{\delta}_+ H_\mu = i[\chi_\mu, \phi] \ , \ \hat{\delta}_+ c = \bar{\eta} \ , \ \hat{\delta}_+ \bar{\eta} = i[c, \phi], \tag{15}$$

and the same transformations as (8) for other fields. The action of the topological twisted $\mathcal{N} = 4$ Yang-Mills theory without the $\tau\int F\wedge F$ is

$$\begin{aligned}
S \ = \ & Tr_{\mathcal{H}} tr \ \ \hat{\delta}_+\{\chi^+_{\mu\nu}\left(H^{+\mu\nu} - i(F^{+\mu\nu} - i[B^+_{\mu\rho}, B^+_{\nu\sigma}]\delta^{\rho\sigma} - i[B^+_{\mu\nu}, c])\right)\} \\
& + Tr_{\mathcal{H}} tr \ \ \hat{\delta}_+\{\chi^\rho\left(H_\rho - i(-2D^\mu B^+_{\mu\rho} - D_\rho c)\right)\} \\
& + Tr_{\mathcal{H}} tr \ \ \hat{\delta}_+\{i[\phi, \bar{\phi}]\eta - i\bar{\eta}[c, \bar{\phi}] + i[B^{+\ \mu\nu}, \bar{\phi}]\psi^+_{\mu\nu} + (D_\mu\bar{\phi})\psi_\mu\} \ .
\end{aligned} \tag{16}$$

For this action, we change the variables as

$$B^+_{\mu\nu} \to \frac{1}{\sqrt{\theta}}\tilde{B}^+_{\mu\nu}, \quad \psi^+_{\mu\nu} \to \frac{1}{\sqrt{\theta}}\tilde{\psi}^+_{\mu\nu} \quad , \quad c \to \frac{1}{\sqrt{\theta}}\tilde{c}, \quad \bar{\eta} \to \frac{1}{\sqrt{\theta}}\tilde{\bar{\eta}} \quad ,$$
$$\chi_\mu \to \frac{1}{\theta}\tilde{\chi}_\mu, \quad H_\mu \to \frac{1}{\theta}\tilde{H}_\mu$$

[1]There are many kinds of topological twisted theories of $\mathcal{N} = 4$ Yang-Mills theory. We only consider Vafa-Witten type theory.

with (12), then $S \to \frac{1}{\theta^2}\tilde{S}$, and $\tilde{S}$ does not depend on θ. At last, invariance of the $\mathcal{N}=4$ topological twisted theory under $\theta \to \theta + \delta\theta$ is proved as same as Donaldson-Witten theory.

It is worth commenting on the topological term $\int F \wedge F$ that exists in usual Vafa-Witten theory but now is removed. This term is not written by a BRS exact term, so we can not adapt above discussion to the topological term. But, it is natural that we expect that $\int F \wedge F$ is invariant under the θ shift. Indeed, for instanton solutions constructed from noncommutative deformed ADHM data, we have proof of invariance of instanton number under θ shift [15, 16]. This is why, we expect that the partition functions or vacuum expectation values are still invariants even if the action of the cohomological Yang-Mills theories include $\int F \wedge F$. (See also section 5.)

By applying these facts for several physical models, some interesting information can be found. For example, as we will see soon, we can show that the partition function of the noncommutative cohomological gauge theory and the partition function of the IKKT matrix model have a correspondence. This correspondence is not only for certain classical background theory as we saw in [17]. The reason is as follows. The IKKT matrix model is constructed as dimensional reduction of the 10 dimensional super $U(N)$ Yang-Mills theory with large N limit [19, 18]. This dimensional reduction is regarded as the large noncommutative parameter limit ($\theta \to \infty$ in section 4). Taking the large N limit of the matrix model is similar to considering the Yang-Mills theories on noncommutative Moyal space, i.e. matrices are regarded as linear operators acting on the Hilbert space caused from noncommutativity. By the way, the noncommutative cohomological Yang-Mills model on Moyal space in the large θ limit is almost the same as the model of Moore, Nekrasov and Shatashvili (MNS) [20]. MNS show that the partition function is calculated by the cohomological matrix model in [20] and related works are seen in [24, 22, 23]. This cohomological matrix model is almost equivalent to the IKKT matrix model. That is why we can produce similar result by using N.C.cohomological Yang-Mills theories. To show these facts concretely, we will calculate the partition function of $\mathcal{N}=4$ d=4 U(1) theory on N.C.$\mathbb{R}^4$ in section 4 by using the facts given in this section.

3 Universality of Partition Functions

In this section, we show that the large θ limit is equivalent to dimensional reduction. From this fact, we find the universal perspective for the partition functions of supersymmetric Yang-Mills theories in N.C.$\mathbb{R}^{2D}$.

In the previous section, we consider the case of $\mathbb{R}^4$. There is two independent noncommutative parameters θ^1, θ^2 for the N.C.$\mathbb{R}^4$, that is to say, after choosing proper coordinate noncommutative parameters are expressed as

$$(\theta^{\mu\nu}) = \left(\begin{array}{cc|cc} 0 & \theta^1 & 0 & 0 \\ -\theta^1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \theta^2 \\ 0 & 0 & -\theta^2 & 0 \end{array}\right) \quad . \tag{17}$$

In the discussion of the previous section, we take noncommutative parameter shift coincidently, that is $\theta^1 = \theta^2 = \theta \to \theta + \delta\theta$. However, we can shift θ^1, θ^2 independently without changing partition functions and vacuum expectations. Further, this discussion is extended to other dimensional theories.

Let us consider more general cases than N.C.$\mathbb{R}^4$. Let noncommutative parameter matrix of N.C.$\mathbb{R}^{2D}$ be $(\theta^{\mu\nu}) = \oplus\theta^i\epsilon^{2i-1,\ 2i}$. In the large θ^i limit, terms with derivative operators $\partial_{x_{2i}} := (\theta^i)^{-1}[x_{2i-1}, *]$ and $-\partial_{x_{2i-1}} := (\theta^i)^{-1}[x_{2i}, *]$ vanish in lagrangians. In the complex coordinate expression, the terms including z_i and $\bar{z}_i$ derivatives are omitted. Meanwhile, an arbitrary operator is expressed as

$$\hat{\mathcal{O}} = \sum_{n_1,m_1} \cdots \sum_{n_D,m_D} \mathcal{O}^{n_1\cdots n_D}_{m_1\cdots m_D} \left|n_1, \cdots, n_D\right\rangle \left\langle m_1, \cdots, m_D\right| \ ,$$

by using fock space basis. (See appendix. A.) As a quantum theory of infinite dimensional matrix model, we cannot distinguish dynamical variables

$$\mathcal{O}^{n_1\cdots n_{i-1}n_{i+1}\cdots n_D}_{m_1\cdots m_{i-1}m_{i+1}\cdots m_D} \left|n_1,\cdots,n_{i-1},n_{i+1},\cdots,n_D\right\rangle \left\langle m_1,\cdots,m_{i-1},m_{i+1},\cdots,m_D\right| \tag{18}$$

from $\mathcal{O}^{n_1\cdots n_D}_{m_1\cdots m_D} \left|n_1, \cdots, n_D\right\rangle \left\langle m_1, \cdots, m_D\right|$ because both of them are infinite dimensional matrices. From the facts that there is no ∂_{z_i} or $\partial_{\bar{z}_i}$ and it is impossible to distinguish dynamical variables living in $\mathbb{R}^{2D}$ from variables in $\mathbb{R}^{2D-2}$, then we conclude that the large θ_i limit is equivalent to the dimensional reduction corresponding to x^{2i-1} and x^{2i} directions.

Note that naive path integrals contain zero mode integrals. To make story precise, let us define the zero mode here. Let $\{\phi_i\}$ be a set of fields and $S[\phi_i]$ be an action functional of a considered theory. Here, we define the zero mode ϕ_i^0 by $S[\phi_i^0] = 0$. To make the partition functions be well defined, we manage the zero modes, in general. But it is difficult that the dealing with the zero modes is discussed in general terms. To avoid this difficulty, the discussion of the zero mode integrals are taken up in the individual cases. In section 4, we will closely study the handling of the zero modes for the case of $\mathcal{N} = 4$ U(1) gauge theory in N.C.$\mathbb{R}^4$.

As a summary of these arguments, the following claim is obtained.

Claim
Let Z_{2D} and $\langle O\rangle_{2D}$ be a partition function and vacuum expectation value of O of a cohomological field theory in N.C.$\mathbb{R}^{2D}$ with $D \geq 1$ such that $\delta_\theta Z_{2D} = 0$ and $\delta_\theta \langle O\rangle_{2D} = 0$. Here, zero mode integrals are removed from the path integral of Z_{2D} and $\langle O\rangle_{2D}$. Let Z_{2D-2} and $\langle O\rangle_{2D-2}$ be the partition function and vacuum expectation value of O of a noncommutative cohomological field theory in N.C. $\mathbb{R}^{2D-2}$, where they are given by dimensional reduction of Z_{2D} and $\langle O\rangle_{2D}$. Then,

$$Z_{2D} = Z_{2D-2} \quad , \quad \langle O\rangle_{2D} = \langle O\rangle_{2D-2} \ , \tag{19}$$

i.e. the partition functions of such theories do not change under dimensional reduction from $2D$ to $2D-2$.

From this claim, we find that following partition functions of Super Yang-Mills theories on N.C. $\mathbb{R}^{2D}$ are equivalent:

$$Z^{8dim}_{\mathcal{N}=2} = Z^{6dim}_{\mathcal{N}=2} = Z^{4dim}_{\mathcal{N}=4} = Z^{2dim}_{\mathcal{N}=8} = Z^{0dim}_{***} \ , \tag{20}$$

where $Z^{Idim}_{\mathcal{N}=J}$ is a partition function of the $\mathcal{N} = J$ super Yang-Mills theory in noncommutative $\mathbb{R}^I$ with arbitrary gauge group. They are obtained by dimensional reduction of the 8 dimensional $\mathcal{N} = 2$ super Yang-Mills theory. Note that the topological terms in the actions of above theories should be removed because the topological terms is not universal between the different dimensional theories. The proof of (20) is as follows. In the $\mathbb{R}^{2D}$, a topological twist exists at any time for $\mathcal{N} \geq 2$. Using the topological twist, the partition functions are described as the one of cohomological field theories. Therefore, $Z^{8dim}_{\mathcal{N}=2}$ is invariant under θ-shift and satisfies the condition of the above claim. After all, (20) is obtained. We will calculate the partition functions concretely in the case of U(1) in the next section.

It is worth adding some comments about above models. We consider noncommutative Euclidean spaces. $\mathcal{N} = 4$ super Yang-Mills theory in N.C. $\mathbb{R}^4$ is given as follows. At first, we construct the 4-dimensional $\mathcal{N} = 4$ super Yang-Mills theory by dimensional reduction of the 10 dimensional $\mathcal{N} = 1$ super Yang-Mills defined on Minkowski space with SO(9,1) symmetry. In 4-dim, spinor in Euclidean space is defined as well as the spinor in Minkowski space. Therefore, we can construct the $\mathcal{N} = 4$ super Yang-Mills theory in $\mathbb{R}^4$ by formally replacing the metric, gamma matrices and so on. Since the θ-shift invariance of $Z^{4dim}_{\mathcal{N}=4}$ was shown explicitly in section 2, theories connected to the $\mathcal{N} = 4$ $d = 4$ super Yang-Mills theory through the dimensional reduction appear in (20).

This discussion is valid not only for the $\mathcal{N}$=4 case. For example, we saw that the θ-shift invariance of $Z^{4dim}_{\mathcal{N}=2}$ in section 2. Then, the similar relation should exist :

$$Z^{4dim}_{\mathcal{N}=2} = Z^{2dim}_{\mathcal{N}=4} = Z^{0dim}_{***}. \tag{21}$$

Let us summarize this section. Universality of partition functions and vacuum expectation values of observables of N.C.cohomological field theories are discussed. From the claim, we found that $\mathcal{N} \geq 2$ supersymmetric models or cohomological field theories in N.C. $\mathbb{R}^{2D}$ are invariant under dimensional reduction from $2D$ to $2D-2$.

4 $\mathcal{N} = 4$ $U(1)$ Gauge Theory in N.C. $\mathbb{R}^4$

In this section, we calculate the partition function of the topological twisted $\mathcal{N} = 4$ $U(1)$ gauge theory in N.C. $\mathbb{R}^4$, without the topological term $\int F \wedge F$ in its action.

We perform the calculation in the $\theta \to \infty$ limit. The reason why we take this limit is as follows. As explained in section 2, the partition function and other correlation functions of cohomological field theories on noncommutative spaces are invariant under the shift transformation of the noncommutative parameter θ. So we obtain the exact result by taking $\theta \to \infty$ limit. Also this limit makes the calculation executable.

In the operator formalism, field theories in N.C. $\mathbb{R}^4$ are expressed as infinite dimensional matrix models whose symmetry is $U(N)$ $(N \to \infty)$. The size of matrices appearing in this model is infinite. To perform the calculation, we introduce a cut off for the matrix size . In addition, this matrix model contains trace parts which correspond to zero modes in $\theta \to \infty$. Therefore we must carefully treat the trace parts to make the path integral well-defined.

In subsection 4.1, we give the action of the topological twisted $\mathcal{N} = 4$ $U(1)$ gauge theory in N.C. $\mathbb{R}^4$ in the operator formalism, i.e. in terms of infinite dimensional matrices. In subsection 4.2, we truncate the size of the matrices into finite size, a finite integer N. In subsection 4.3, we explain that the truncated $N \times N$ matrix model action obtained in the previous subsection is equivalent to the dimension reduction of the 10 dim. $\mathcal{N} = 1$ $U(N)$ super Yang-Mills action to 0 dim. This $U(N)$ matrix model contains traceless parts and trace parts. In subsection 4.4, we calculate the partition function of the traceless sector. The traceless sector is a $SU(N)$ matrix model. The partition function of this $SU(N)$ matrix model was obtained by MNS [20]. By modifying their arguments, we evaluate the $N \to \infty$ limit of the partition function of the traceless sector. In subsection 4.5, we introduce extra parts into the matrices to handle the trace parts which are zero modes. The extra parts and trace parts are the next leading terms in the $\theta \to \infty$ limit. In 4.6, the calculation of the trace sector is performed. Our result is presented at the end of this section.

4.1 Setting

In the Fock space formalism, i.e. in terms of (infinite dimensional) matrices, the action of the topological twisted $\mathcal{N}=4$ $U(1)$ gauge theory on N.C. $\mathbb{R}^4$ is expressed as

$$\begin{aligned} S^{4dim}_{\mathcal{N}=4} = Tr_{\mathcal{H}}\ \hat{\delta}_+\ \ [\ & +\chi^{+\ \mu\nu}\{H^+_{\mu\nu} - i(F^+_{\mu\nu} - i[B^+_{\mu\rho}, B^+_{\nu\sigma}]\delta^{\rho\sigma} - i[B^+_{\mu\nu}, c])\} \\ & +\chi^{\mu}\{H_\mu - i(-2D^\nu B^+_{\nu\mu} - D_\mu c)\} \\ & +i[\phi,\bar{\phi}]\eta - i\bar{\eta}[c,\bar{\phi}] + i[B^{+\ \mu\nu},\bar{\phi}]\psi^+_{\mu\nu} + (D_\mu\bar{\phi})\psi_\mu\ \ ]. \end{aligned} \tag{22}$$

After acting $\hat{\delta}_+$, (22) is rewritten as

$$\begin{aligned} S^{4dim}_{\mathcal{N}=4} \ =& \\ Tr_{\mathcal{H}}[\ \ & H^{+\ \mu\nu}\{H^+_{\mu\nu} - i(F^+_{\mu\nu} - i[B^+_{\mu\rho}, B^+_{\nu\sigma}]\delta^{\rho\sigma} - i[B^+_{\nu\sigma}, c])\} \\ + \ & \chi^{+\ \mu\nu}\{-i[\chi^+_{\mu\nu},\phi] + i(2D_\mu\phi_\nu - 2i[B^+_{\mu\rho},\psi^+_{\nu\sigma}]\delta^{\rho\sigma} - i[\psi^+_{\mu\nu}, c] - i[B^+_{\mu\nu},\bar{\eta}])\} \\ + \ & H^\mu\{H_\mu - i(-2D^\nu B^+_{\nu\mu} - D_\mu c)\} \\ + \ & \chi^\mu\{-i[\chi_\mu,\phi] - i(2D^\nu\psi^+_{\nu\mu} + 2i[\psi^\nu, B^+_{\nu\mu}] - D_\mu\bar{\eta} + i[\psi_\mu, c])\} \\ + \ & [\phi,\bar{\phi}]^2 + [c,\phi][c,\bar{\phi}] + [B^{+,\mu\nu},\bar{\phi}][B^+_{\mu\nu},\phi] + D^\mu\bar{\phi}D_\mu\phi \\ + \ & i[\phi,\eta]\eta + i\bar{\eta}[\bar{\eta},\bar{\phi}] + i\bar{\eta}[c,\eta] + i[\psi^{+\ \mu\nu},\bar{\phi}]\psi^+_{\mu\nu} + i[B^{+\ \mu\nu},\eta]\psi^+_{\mu\nu} \\ + \ & D^\mu\eta\psi_\mu + i[\psi^\mu,\bar{\phi}]\psi_\mu\ \ ]. \end{aligned} \tag{23}$$

From (22) or (23), we find the BPS equations. For example,

$$\begin{aligned} & F^+_{\mu\nu} - i[B^+_{\mu\rho}, B^+_{\nu\sigma}]\delta^{\rho\sigma} - i[B^+_{\mu\nu}, c] = 0, \\ & -2D^\nu B^+_{\nu\mu} - D_\mu c = 0. \end{aligned} \tag{24}$$

In the following, we calculate the partition function $Z^{4dim}_{\mathcal{N}=4}$ formally defined as

$$Z^{4dim}_{\mathcal{N}=4} = \int \mathcal{D}f e^{-S^{4dim}_{\mathcal{N}=4}[f]}, \tag{25}$$

where f means the all matrices $A_\mu, \psi_\mu, \cdots$. Also we use f_{boson} or $f_{fermion}$ to denotes bosonic matrices $A_\mu, H_\mu, \cdots$ or fermionic matrices $\psi_\mu, \chi_\mu \cdots$.

In usual commutative spaces, $U(1)$ gauge theories are free if all matters belong to the adjoint representation, because the gauge interactions between the fields belonging to the adjoint representation are described by commutators of matrices and all commutators vanish in the $U(1)$ case. However, in noncommutative spaces, the noncommutativity of the multiplication induces the $U(1)$ gauge theories to non-Abelian $U(N)$ $(N \to \infty)$ like gauge theories. This $U(N)$ $(N \to \infty)$ is identified with the unitary transformation group acting on state vectors of the Hilbert space $\mathcal{H}$ [2] .

[2] It is well known fact that the $U(\infty)$ is different from $\lim_{N\to\infty} U(N)$, in the meaning of the topology. In this article, we perform the all calculation by using $\lim_{N\to\infty} U(N)$, and there is denying that some extra collections appear from the difference. However, there is no doubt about validity of calculation of $U(N)$ $(N \to \infty)$ as a good approximation even in the case.

Let us consider to take the $\theta \to \infty$ limit in the calculation of the partition function $Z^{4dim}_{\mathcal{N}=4}$. We can evaluate the partition function exactly in this limit, as explained in section 2. In the $\theta \to \infty$ limit we naively expect that all differential terms in the action vanish and dimensional reduction occur as we saw in section 3 . Therefore, we can perform the calculation by using a matrix model in 0 dim. space whose symmetry is $U(N)$ $(N \to \infty)$. We define the action in 0 dim spacetime as

$$S^{\infty}_{MM} = S^{4dim}_{\mathcal{N}=4}|_{\theta\to\infty} \quad : \quad U(N)\ (N \to \infty) \text{ matrix model}, \tag{26}$$

then, we find $Z^{4dim}_{\mathcal{N}=4}$ is equal to the partition function of the matrix model (26)

$$Z^{4dim}_{\mathcal{N}=4} = \frac{1}{Vol.U(N)(N\to\infty)} \int \mathcal{D}f e^{-S^{\infty}_{MM}}. \tag{27}$$

To calculate the partition function of this infinite dimensional $U(N)$ $(N \to \infty)$ matrix model (26), we need to overcome the following problems.

(i)The size of the matrices is infinite. To perform the calculation, we truncate the size of the matrices into a finite integer N.

(ii)The matrices contain trace parts. These trace parts play a role of zero modes. To make the path integral well-defined, we must carefully treat the trace parts.

In the rest of this section, we solve these problems and obtain the partition function (27).

4.2 Cut off for matrix size

In this subsection, we truncate the size of the matrices, to calculate the partition function.

The Hilbert space of the $\mathcal{N} = 4$ $U(1)$ gauge theory on N.C.$\mathbb{R}^4$ is constructed by a Fock space

$$\mathcal{H} = \bigoplus_{n_1=0,n_2=0}^{n_1=\infty,n_2=\infty} \mathbb{C}\ |n_1, n_2\rangle. \tag{28}$$

We introduce a cut off, a finite integer N_c, and truncate the Hilbert space into a finite dimensional subspace $\mathcal{H}_N$ whose dimension is N. We can perform such truncation in several ways. For example, $\mathcal{H}_N$ is defined by

$$\mathcal{H}_N = \bigoplus_{n_1=0,n_2=0}^{n_1=N_c,n_2=N_c} \mathbb{C}\ |n_1, n_2\rangle. \tag{29}$$

For this case

$$\dim.\mathcal{H}_{\mathcal{N}} = N = (N_c + 1)^2, \tag{30}$$

and the unit matrix of $\mathcal{H}_N$ is given as

$$\mathbf{1}_N = \bigoplus_{n_1=0,n_2=0}^{n_1=N_c,n_2=N_c} |n_1, n_2\rangle\langle n_1, n_2|. \tag{31}$$

The results and calculations do not depend on the definition of the cut off in the following discussion. (See appendix A.) Therefore we do not use concrete expression of the example (29). By definition,

$$Tr_{\mathcal{H}}\mathbf{1}_N = \dim.\mathcal{H}_{\mathcal{N}} = N. \tag{32}$$

For later use, we define $\mathcal{I}$ as

$$\mathcal{I} = \frac{1}{\sqrt{N}}\mathbf{1}_N, \tag{33}$$

which satisfies

$$Tr_{\mathcal{H}}\ \mathcal{I}\mathcal{I} = 1. \tag{34}$$

We truncate the infinite dimensional matrices appearing in (26) into finite dimensional $N\times N$ matrices. We use the symbol f_N to denote the $N\times N$ truncation of f. For example of (29), if

$$f = \sum_{n_i=0}^{\infty}\sum_{m_i=0}^{\infty} f^{n_1 n_2}_{m_1 m_2}|n_1,n_2\rangle\langle m_1,m_2|,$$

then

$$f_N = \sum_{n_i=0}^{N_c}\sum_{m_i=0}^{N_c} f^{n_1 n_2}_{m_1 m_2}|n_1,n_2\rangle\langle m_1,m_2|\ .$$

Now we consider the finite dimensional $N\times N$ matrix model S^N_{MM} which is obtained by the truncation from (26)

$$S^N_{MM} = S^{\infty}_{MM}|_{N\times N\text{truncation}}\ . \tag{35}$$

The partition function of the truncated matrix model (35) is defined by

$$Z^{4dim}_{\mathcal{N}=4}|_N = \frac{1}{Vol.U(N)}\int \mathcal{D}f_N e^{-S^N_{MM}}. \tag{36}$$

$N\times N$ matrix f_N is decomposed into the traceless part and the trace part

$$f_N = f^{su} + f^{tr}, \tag{37}$$

where f^{su} is the traceless part and f^{tr} is the trace part. The traceless part f^{su} is expanded by the generators of the Lie algebra $su(N)$

$$f^{su} = \sum_{a=1}^{N^2-1} f_{(a)}\tau^a\ ,\ \tau^a\in su(N), \tag{38}$$

and f^{tr} is proportional to $\mathcal{I}$

$$f^{tr} = f_{(\mathbf{1})}\mathcal{I}. \tag{39}$$

The basis, τ^a and $\mathcal{I}$, satisfy the following orthonormal conditions

$$Tr_{\mathcal{H}}\ \tau^a\ \tau^b = \delta^{ab}\ \ ,\ \ Tr_{\mathcal{H}}\ \mathcal{I}\ \mathcal{I} = 1\ \ ,\ \ Tr_{\mathcal{H}}\ \tau^a\ \mathcal{I} = 0. \tag{40}$$

In the naive $\theta \to \infty$ limit (i.e. 0 dimension reduction), (35) contains no trace part f^{tr} [3]

$$Z^N_{MM} = Z^N_{MM}|_{\text{traceless}} \times \int \mathcal{D}f^{tr}, \tag{41}$$

where $Z^N_{MM}|_{\text{traceless}}$ is defined by

$$Z^N_{MM}|_{\text{traceless}} = \frac{1}{Vol.SU(N)} \int \mathcal{D}f^{su} e^{-S^N_{MM}[f^{su}]}. \tag{42}$$

So the trace part f^{tr} plays the role of the zero mode such that $S^N_{MM}[f^tr] = 0$. To make the path integral well-defined, we must carefully treat it. For other handling the zero modes, see for example [25]. However we postpone this task for the moment. First, we concentrate on the traceless sector. Before the calculation of the traceless sector, we explain the equivalence between (35) and the action cosidered in [20] in the next subsection.

4.3 Relation to the work of MNS and IKKT

To explain that the equivalence between (35) and the action considered in [20], we first recall the fact that the dimensional reduction model from the $D = 10$ $\mathcal{N} = 1$ super Yang-Mills theory to 0 dimension can be reformulated into a cohomological matrix model [22, 20]. The 0 dimension matrix model given by dimensional reduction from the $D = 10$ $\mathcal{N} = 1$ super Yang-Mills theory is expressed as

$$S^{10\to 0\ dim}_{\mathcal{N}=1} = tr\left(\frac{1}{4}[A_M, A_N]^2 + \frac{i}{2}\bar{\Psi}\Gamma^M[A_M, \Psi]\right), \tag{43}$$

where A_M is gauge vector fields and M, N takes $1 \cdots 10$ for the 10 dimension Euclid space, or $0, 1 \cdots 9$ for the 10 dimension Minkowski spacetime. Ψ is a Majorana-Weyl spinor of the 10 dimension spacetime. It contains real 16 components [4] .

In [22, 20], it is shown that (43) can be reformulated into a cohomological matrix model. The mapping rules between them are as follows [20]. A_M are arranged into

[3]Precisely speaking, the trace part of the auxiliary fields appear in (35). After integrating out the auxiliary fields, no trace part appears in (35).

[4]Note that there is no Majorana-Weyl spinor in 10 dim Euclidean space. So, if we consider 10 dim model, we should take Minkowski spacetime. To obtain low dimensional Euclidean model, we first perform dimensional reduction from 10 dim. to lower dim, and then carry out Wick rotation.

complex matrices ϕ and B_i $(i = 1, ..., 4)$,

$$\begin{aligned} B_i &= A_{2i-1} + iA_{2i} \quad (\text{for } i = 1,2,3), \\ B_4 &= A_9 + iA_8, \\ \phi &= A_7 + iA_{10}, \end{aligned} \tag{44}$$

and Ψ are arranged as

$$\Psi \to (\psi_i, \psi_i^\dagger) \oplus \vec{\chi} \oplus \eta, \tag{45}$$

where $\vec{\chi}$ belongs to the **7** representation of $Spin(7)$. Introducing the bosonic auxiliary matrices $\vec{H}$, we can rewrite (43) into a cohomological form

$$S_{MNS} = tr\ \hat{\delta} \left(\frac{1}{16}\eta[\phi, \bar{\phi}] - i\vec{\chi}\cdot\vec{\mathcal{E}} + \vec{\chi}\cdot\vec{H} + \frac{1}{4}\sum_{a=1}^{8}\Psi_a[A_a, \bar{\phi}] \right), \tag{46}$$

where $\vec{\mathcal{E}}$ is defined by

$$\vec{\mathcal{E}} = \left([B_i, B_j] + \frac{1}{2}\epsilon_{ijkl}[B_k^\dagger, B_l^\dagger]\ (i<j)\ ,\ \sum_i [B_i, B_i^\dagger] \right). \tag{47}$$

The BRS transformation rules are given as

$$\begin{aligned} \hat{\delta}A_a = \Psi_a \quad &, \quad \hat{\delta}\Psi_a = [\phi, A_a], \\ \hat{\delta}\vec{\chi} = \vec{H} \quad &, \quad \hat{\delta}\vec{H} = [\phi, \vec{\chi}], \\ \hat{\delta}\bar{\phi} = \eta \quad &, \quad \hat{\delta}\eta = [\phi, \bar{\phi}], \\ \hat{\delta}\phi = 0 \quad &. \end{aligned} \tag{48}$$

From (46) and (48), the following BPS equations are obtained

$$\vec{\mathcal{E}} = 0 \quad , \quad [\phi, \bar{\phi}] = 0 \quad , \quad [A_a, \phi] = 0. \tag{49}$$

One can show that (46) is equivalent to (26), by using the following correspondence rule [28]

$$\begin{aligned} (\phi\ ,\ c\ ,\ \bar{\phi}) \quad &\Longleftrightarrow \quad \left(\sqrt{2}\varphi_{34}\ ,\ i\frac{1}{\sqrt{2}}(\varphi_{14} - \varphi_{23})\ ,\ \sqrt{2}\varphi_{12} \right) \\ (B^+_{\mu\nu}\sigma^{\mu\nu}_{11}\ ,\ B^+_{\mu\nu}\sigma^{\mu\nu}_{12}\ ,\ B^+_{\mu\nu}\sigma^{\mu\nu}_{22}) \quad &\Longleftrightarrow \quad \left(\sqrt{2}\varphi_{13}\ ,\ -\frac{1}{\sqrt{2}}(\varphi_{14} + \varphi_{23})\ ,\ \varphi_{24} \right), \end{aligned} \tag{50}$$

where φ is defined by

$$\varphi_{k4} = -\varphi_{4k} = \frac{1}{\sqrt{2}}(A_{k+4} + iA_{k+7}) \quad , \quad \varphi_{ij} = (\epsilon^{ijk}\varphi_{k4})^* \quad , \ k = 1,2,3\ . \tag{51}$$

Remark that the equivalence among (35), (43) and (46) holds for both $U(N)$ group and $SU(N)$ group.

By choosing gauge group $SU(N)$ and setting N to be a finite integer, we obtain the equivalence between (35) and (46)

$$S_{MM}^{N}|_{traceless} = S_{MNS}|_{gauge\ group:SU(N)}^{N:finite}. \tag{52}$$

Therefore

$$Z_{MM}^{N}|_{traceless} = Z_{MNS}|_{gauge\ group:SU(N)}^{N:finite}, \tag{53}$$

where

$$\begin{aligned} & Z_{MNS}|_{gauge\ group:SU(N)}^{N:finite} \\ &= \frac{1}{Vol.SU(N)} \int \mathcal{D}f^{su} \exp\left\{-S_{MNS}[f^{su}]|_{gauge\ group:SU(N)}^{N:finite}\right\}. \end{aligned} \tag{54}$$

MNS obtained the partition function (54). It is equivalent to (42) [20] [5].

On the other hand, by choosing gauge group $U(N)$ and taking the $N \to \infty$ limit, the action (43) becomes the IKKT matrix model [18]

$$S_{IKKT} = \lim_{N\to\infty} S_{\mathcal{N}=1}^{10\to 0\ dim}|_{gauge\ group:U(N)}. \tag{55}$$

So, we obtain the equivalence between (26) and (55)

$$S_{MM}^{\infty} = S_{IKKT}. \tag{56}$$

4.4 Calculation of traceless sector

As explained in the previous subsection the partition function (42) is calculated in [20]. Their result tells us that

$$Z_{MM}^{N}|_{traceless} = \sum_{d|N} \frac{1}{d^2}, \tag{57}$$

where the summation is taken over all divisor d of the finite integer N.

Now we take the $N \to \infty$ limit

$$Z_{MM}^{\infty}|_{traceless} = \lim_{N\to\infty} Z_{MM}^{N}|_{traceless}, \tag{58}$$

to obtain the contribution of the traceless part f^{su} to (27). However we can not take the $N \to \infty$ limit in the righthand side of (57). The reason is as follows. We see that the righthand side of (57) is finite;

$$\sum_{d|N} \frac{1}{d^2} < \sum_{n=1}^{N} \frac{1}{n^2} < 1 + \int_1^\infty dx \frac{1}{x^2} = 2. \tag{59}$$

[5] See also [21] where the partition function of th D-instanton model was calculated.

But it is not monotonically increasing. So it does not converge. For example, if we constrain N to be prime numbers,

$$\lim_{N\to\infty}\sum_{d|N}\frac{1}{d^2}=\lim_{N\to\infty}(1+N^{-2})=1. \tag{60}$$

If we constrain $N=2^{N'}$,

$$\lim_{N\to\infty}\sum_{d|N}\frac{1}{d^2}=\lim_{N\to\infty}\sum_{n=0}^{N'}2^{-2n}=\frac{4}{3}. \tag{61}$$

To avoid this difficulty and to obtain a definite $N\to\infty$ limit, let us recall the argument of [20], where the result (57) is concluded for a finite N. The authors of [20] introduced a mass denoted by m, deformation into (46). This deformation corresponds to the supersymmetry breaking from $\mathcal{N}=4$ to $\mathcal{N}=1$ in the picture of 4 dimensional space. After this deformation, they obtained the following fixed point equations,

$$[B_i,B_j]=m\epsilon_{ijk4}B_k \quad , \quad [B_4,B_i]=0 \quad , \quad [B_4,\phi]=0 \quad , \quad i=1,2,3 \quad , \tag{62}$$

where B_i, B_4 and ϕ are all $N\times N$ matrices. The solution of (62) is given by

$$B_i=(L_i)_{a\times a}\otimes\mathbf{1}_{d\times d} \quad , \quad B_4=\mathbf{1}_{a\times a}\otimes(B_4)_{d\times d} \quad , \quad \phi=\mathbf{1}_{a\times a}\otimes\phi_{d\times d} \quad , \tag{63}$$

where a is a divisor of N and d is the quotient of N by a, and $(L_i)_{a\times a}$ denotes the generator of the $SU(2)$ group in the $a\times a$ representation. By integrating out B_i and corresponding fermionic partners and auxiliary ones, the partition function reduces to the sum of the contributions from the solution labeled by d, the divisors of N. The each contribution is given by $\frac{1}{d^2}$, which is another result obtained in the same paper [20], so they concluded (57).

Now let us take the $N\to\infty$ limit, in the equation (62). In this limit, we can construct solutions of (62) which are direct products of arbitrary dimensional matrices and the $SU(2)$ generators in infinite dimensional representations. Therefore in the $N\to\infty$ limit, d takes all natural numbers. For the $N\to\infty$ case, the partition function is still the sum of the contributions from the solutions labeled by d and each contribution is $\frac{1}{d^2}$. So we conclude that

$$Z_{MM}^{\infty}|_{\text{traceless}}=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}. \tag{64}$$

4.5 Introduction of extra terms

In this section, we deal with the zero mode problem. The origin of this problem is the fact that no trace part appears in (35). The reason why all trace parts vanish in (35) is that we drop all differential terms in the $\theta \to \infty$ limit. To solve the zero mode problem, we keep the next leading terms including the trace parts in the $\theta \to \infty$ limit.

Let us explain the outline of our calculation. To keep the next leading term, we bring back some extra part f^{ex} living in the out side of $\mathcal{H}_N$. The definition of f^{ex} is given later in this subsection. Roughly speaking, f^{ex} are matrices appearing in kinetic terms $\frac{1}{\theta} f^{ex} \Box f^{tr}$ in (23). By keeping f^{ex}, the part of (22) or (23) which includes the trace part f^{tr} does not vanish :

$$S_{tr\oplus ex}[f^{tr}, f^{ex}] \quad = \quad S^{4dim}_{\mathcal{N}=4}|_{\text{trace part}} - O(1/\theta^{1+\epsilon}) \quad \neq \quad 0, \tag{65}$$

where ϵ is an arbitrary positive real number. Then the partition function of (65) is well-defined

$$Z_{tr\oplus ex} \quad = \quad \int \mathcal{D} f^{tr} \mathcal{D} e^{ex} e^{-S_{tr\oplus ex}[f^{tr}, f^{ex}]} \quad : \quad \text{well-defined.} \tag{66}$$

We suppose f^{ex} has the following expansion form

$$f^{ex} = \sum_{\mu=1}^{4} f_{(\mu)} \mathcal{T}_\mu. \tag{67}$$

$\mathcal{T}_\mu$ is essentially defined by the commutator of $\hat{\partial}_\mu$ and $\mathbf{1}_N$. The precise definition of $\mathcal{T}_\mu$ is as follows. First of all, we define T_μ as the commutator of $\hat{\partial}_\mu$ and $\mathbf{1}_N$ i.e.

$$T_\mu = [\hat{\partial}_\mu, \mathbf{1}_N]. \tag{68}$$

In the Fock space formalism, $\hat{\partial}_\mu$ is given as

$$\begin{aligned} \hat{\partial}_1 &= \frac{1}{\sqrt{2\theta^1}}(a_1 - a_1^\dagger) \quad , \quad \hat{\partial}_2 = \frac{-i}{\sqrt{2\theta^1}}(a_1 + a_1^\dagger), \\ \hat{\partial}_3 &= \frac{1}{\sqrt{2\theta^2}}(a_2 - a_2^\dagger) \quad , \quad \hat{\partial}_4 = \frac{-i}{\sqrt{2\theta^2}}(a_2 + a_2^\dagger), \end{aligned} \tag{69}$$

where a_i is the annihilation operator and $a_i^\dagger$ is the creation operator. Given the definition of $\mathbf{1}_N$, for example (31), we obtain

$$\begin{aligned} T_1 &= \frac{\sqrt{N+1}}{\sqrt{2\theta^1}}\Big(-\sum_{n_2=0}^{N} |N, n_2\rangle\langle N+1, n_2| - \sum_{n_2=0}^{N} |N+1, n_2\rangle\langle N, n_2|\Big), \\ T_2 &= \frac{-i\sqrt{N+1}}{\sqrt{2\theta^1}}\Big(-\sum_{n_2=0}^{N} |N, n_2\rangle\langle N+1, n_2| + \sum_{n_2=0}^{N} |N+1, n_2\rangle\langle N, n_2|\Big), \\ T_3 &= \frac{\sqrt{N+1}}{\sqrt{2\theta^2}}\Big(-\sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N+1| - \sum_{n_1=0}^{N} |n_1, N+1\rangle\langle n_1, N|\Big), \\ T_4 &= \frac{-i\sqrt{N+1}}{\sqrt{2\theta^2}}\Big(-\sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N+1| + \sum_{n_1=0}^{N} |n_1, N+1\rangle\langle n_1, N|\Big). \end{aligned} \tag{70}$$

Using (70), we can show

$$Tr_{\mathcal{H}}\ T_\mu T_\nu = \frac{N}{\theta^{i(\mu)}}\delta_{\mu\nu}, \tag{71}$$

where $i(\mu) = [(\mu+1)/2]$ with the symbol [ ] indicating a Gaussian symbol. $\mathcal{T}_\mu$ is defined by

$$\mathcal{T}_\mu = \frac{\sqrt{\theta^{i(\mu)}}}{\sqrt{N}} T_\mu, \tag{72}$$

to satisfy

$$Tr_{\mathcal{H}}\ \mathcal{T}_\mu\ \mathcal{T}_\nu = \delta_{\mu\nu}. \tag{73}$$

Here we list some formulas about $\mathcal{I}$ and $\mathcal{T}_\mu$, which will be used in the calculation of the partition function. They are

$$Tr_{\mathcal{H}}\ \mathcal{I}\ \mathcal{I} = 1 \quad , \quad Tr_{\mathcal{H}}\ \mathcal{T}_\mu\ \mathcal{T}_\nu = \delta_{\mu\nu} \quad , \quad Tr_{\mathcal{H}}\ \mathcal{I}\ \mathcal{T}_\mu = 0 \quad , \tag{74}$$

$$\begin{aligned} Tr_{\mathcal{H}}\mathcal{I}[\hat{\partial}_\mu, \mathcal{I}] = 0 \quad &, \quad Tr_{\mathcal{H}}\mathcal{I}[\hat{\partial}_\mu, \mathcal{T}_\nu] = -\frac{1}{\sqrt{\theta^{i(\mu)}}}\delta_{\mu\nu}, \\ Tr_{\mathcal{H}}\mathcal{T}_\mu[\hat{\partial}_\nu, \mathcal{I}] = +\frac{1}{\sqrt{\theta^{i(\mu)}}}\delta_{\mu\nu} \quad &, \quad Tr_{\mathcal{H}}\mathcal{T}_\mu[\hat{\partial}_\nu, \mathcal{T}_\rho] = 0 \quad , \end{aligned} \tag{75}$$

and

$$\begin{aligned} Tr_{\mathcal{H}}\mathcal{I}[\mathcal{I}, \mathcal{I}] = 0 \quad &, \quad Tr_{\mathcal{H}}\mathcal{I}[\mathcal{I}, \mathcal{T}_\mu] = 0, \\ Tr_{\mathcal{H}}\mathcal{I}[\mathcal{T}_\mu, \mathcal{T}_\nu] = +\frac{i\theta^{i(\mu)}}{\sqrt{N}}\theta^{-1}_{\mu\nu} \quad &, \quad Tr_{\mathcal{H}}\mathcal{T}_\mu[\mathcal{T}_\nu, \mathcal{T}_\rho] = 0. \end{aligned} \tag{76}$$

For the proof of (74),(75) and (76), see the appendix A. Note that these formulas do not depend on the detail of the definition of the cut off or (31).

Remark that, in the $N \to \infty$ limit, $Tr_{\mathcal{H}}\mathcal{I}[\mathcal{T}_\mu, \mathcal{T}_\nu]$ vanishes,

$$\lim_{N\to\infty} Tr_{\mathcal{H}}\mathcal{I}[\mathcal{T}_\mu, \mathcal{T}_\nu] = 0. \tag{77}$$

We will use this $N \to \infty$ behavior to reduce the calculation of the partition function to the Gaussian integral.

4.6 Calculation of trace and extra sector

Now, let us calculate the partition function (66). First of all, we list the quantities appearing in the calculation. Because the model is constructed as a balanced topological field theory, it is natural to classify them into the BRS multiples. For $\{A_\mu, H_\mu, \psi_\mu, \chi_\mu, H_\mu\}$,

$$\begin{array}{ccccc} & & \psi_{\mu(\mathbf{1})}\ ,\ \phi_{\mu(\alpha)} & & \\ & \hat{\delta}_+ \nearrow & & \hat{\delta}_- \searrow & \\ A_{\mu(\mathbf{1})}\ ,\ A_{\mu(\alpha)} & & & & H_{\mu(\mathbf{1})}\ ,\ H_{\mu(\alpha)} \\ & \hat{\delta}_- \searrow & & \hat{\delta}_+ \nearrow & \\ & & \chi_{\mu(\mathbf{1})}\ ,\ \chi_{\mu(\alpha)} & & \end{array} \quad , \tag{78}$$

and for $\{B^+_{\mu\nu}, \psi^+_{\mu\nu}, \chi^+_{\mu\nu}, H^+_{\mu\nu}\}$,

$$\begin{array}{ccccc} & & \psi^+_{\mu\nu(\mathbf{1})}\ ,\ \psi^+_{\mu\nu(\alpha)} & & \\ & \hat{\delta}_+ \nearrow & & \hat{\delta}_- \searrow & \\ B^+_{\mu\nu(\mathbf{1})}\ ,\ B^+_{\mu\nu(\mu)} & & & & H^+_{\mu\nu(\mathbf{1})}\ ,\ H^+_{\mu\nu(\mu)} \\ & \hat{\delta}_- \searrow & & \hat{\delta}_+ \nearrow & \\ & & \chi^+_{\mu\nu(\mathbf{1})}\ ,\ \chi^+_{\mu\nu(\mu)} & & \end{array} \quad . \tag{79}$$

Note that $A_{\mu(\mathbf{1})}$ and $A_{\mu(\alpha)}$ are coefficients of $\mathcal{I}$ and $\mathcal{T}_\alpha$ i.e. $A_\mu = A_{\mu(\mathbf{1})}\mathcal{I} + \sum_{su(N)} A_{\mu a}\tau^a + \sum A_{\mu(\alpha)}\mathcal{T}_\alpha$, and other fields are noted by similar manner.

It is necessary to comment on the net components of $\{A_{\mu(\alpha)}, \psi_{\mu(\alpha)}, \chi_{\mu(\alpha)}, H_{\mu(\alpha)}\}$ in (78) and $\{B^+_{\mu\nu(\alpha)}, \psi^+_{\mu\nu(\alpha)}, \chi^+_{\mu\nu(\alpha)}, H^+_{\mu\nu(\alpha)}\}$ in (79). In the following, we use the term (μ,ν) selfdual which means that $A_{\mu(\nu)}$ satisfies that $A_{\mu(\nu)} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}A_{\rho(\sigma)}$.
(i) $\{A_{\mu(\alpha)}, \cdots\}$ have not sixteen but four components. Three of them satisfy the selfdual relation and the rest one is $A_{\mu(\mu)}$:

$$\{A_{\mu(\nu)} \mid A_{\mu(\nu)} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}A_{\rho(\sigma)} \ \ (\mu,\nu) \text{ selfdual } \} \text{ and } \{\sum_{\mu=1}^{4} A_{\mu(\mu)}\}. \tag{80}$$

(ii) $\{B^+_{\mu\nu(\mu)}, \cdots\}$ have four components corresponding to $B^+_{\mu\nu(\mu)}$

$$B^+_{\mu\nu(\mu)} = \sum_{\mu=1}^{4} B^+_{\mu\nu(\mu)}. \tag{81}$$

On the other hand, $\{\phi, c, \bar{\phi}, \bar{\eta}, \eta\}$ contain only trace parts

$$\begin{array}{ccc} \phi_{(\mathbf{1})} & & \\ & \hat{\delta}_- \searrow & \\ & & \bar{\eta}_{(\mathbf{1})} \\ & \hat{\delta}_+ \nearrow & \\ c_{(\mathbf{1})} & & \\ & \hat{\delta}_- \searrow & \\ & & \eta_{(\mathbf{1})} \\ & \hat{\delta}_+ \nearrow & \\ \bar{\phi}_{(\mathbf{1})} & & \end{array} \quad . \tag{82}$$

Later we obtain the Gaussian action (92-96,97-101,105,106). For example in (92) we find a term proportional to

$$\chi^{+}_{\mu\nu(\mathbf{1})}(A_{\nu(\mu)} - A_{\mu(\nu)}).$$

¿From this and other terms in (92-96,97-101,105,106), we see that the net components (80,81,82) should be taken to remove the zero modes.

Taking the net components (80,81,82) and using (74,75,76), we obtain

$$\begin{aligned} S_{tr\oplus ex} = Tr_{\mathcal{H}}\ \hat{\delta}_{+}\quad [\quad & +\chi^{+\ \mu\nu}_{(\rho)}\mathcal{T}^{\rho}\{H^{+}_{\mu\nu(\sigma)}\mathcal{T}^{\sigma} - ([\hat{\partial}_{\mu}, A_{\nu(\mathbf{1})}\mathcal{I}] - [\hat{\partial}_{\nu}, A_{\mu(\mathbf{1})}\mathcal{I}])\} \\ & +\chi^{+\ \mu\nu}_{(\mathbf{1})}\mathcal{I}\{H^{+}_{\mu\nu(\mathbf{1})}\mathcal{I} - ([\hat{\partial}_{\mu}, A_{\nu(\rho)}\mathcal{T}^{\rho}] - [\hat{\partial}_{\nu}, A_{\mu(\rho)}\mathcal{T}^{\rho}])\} \\ & +\chi^{\mu}_{(\rho)}\mathcal{T}^{\rho}\{H_{\mu(\sigma)}\mathcal{T}^{\sigma} - (-2[\hat{\partial}^{\nu}, B^{+}_{\nu\mu(\mathbf{1})}\mathcal{I}] - [\hat{\partial}_{\mu}, c_{(\mathbf{1})}\mathcal{I}])\} \\ & +\chi^{\mu}_{(\mathbf{1})}\mathcal{I}\{H_{\mu(\mathbf{1})}\mathcal{I} - (-2[\hat{\partial}^{\nu}, B^{+}_{\nu\mu(\rho)}\mathcal{T}^{\rho}] - [\hat{\partial}_{\mu}, c_{(\rho)}\mathcal{T}^{\rho}])\} \\ & -[\hat{\partial}^{\mu}, \bar{\phi}_{(\mathbf{1})}\mathcal{I}]\psi_{\mu(\nu)}\mathcal{T}^{\nu} \quad] \\ + \quad & \mathcal{O}(N^{-\frac{1}{2}}). \end{aligned} \tag{83}$$

Note that $B^{+}_{\nu\mu(\rho)}$ looks 12 components but only $B^{+}_{\nu\mu(\nu)}$ proportional terms survive in $Tr_{\mathcal{H}}\chi^{\mu}_{(\mathbf{1})}\mathcal{I}[\hat{\partial}^{\nu}, B^{+}_{\nu\mu(\rho)}\mathcal{T}^{\rho}]$. In the $N \to \infty$ limit, only quadratic terms survive [6]

$$S^{\infty}_{tr\oplus ex} \ = \ \lim_{N\to\infty} S_{tr\oplus ex} \quad : \text{quadratic action.} \tag{84}$$

The action (84) has the following gauge symmetry,

$$\delta_{gauge}A_{\mu(\nu)} = \frac{1}{\sqrt{\theta^{i(\mu)}}}\delta_{\mu\nu}\varphi_{(\mathbf{1})}. \tag{85}$$

Note that the gauge parameter φ contains only one component $\varphi_{(\mathbf{1})}$

$$\varphi = \varphi_{(\mathbf{1})}\mathcal{I}. \tag{86}$$

Now we give the BRS transformation rules for $f_{(\mathbf{1})}$ and $f_{(\mu)}$. Except for $A_{\mu(\nu)}, \psi_{\mu(\nu)}$ and $A_{\mu(\mathbf{1})}, \psi_{\mu(\mathbf{1})}$,

$$\begin{aligned} \hat{\delta}_{+}\mathcal{B}_{(\nu)} = \mathcal{F}_{(\nu)} \quad &, \quad \hat{\delta}_{+}\mathcal{F}_{(\nu)} = 0, \\ \hat{\delta}^{2}_{+}\mathcal{B}_{(\nu)} = 0 \quad &, \quad \hat{\delta}^{2}_{+}\mathcal{F}_{(\nu)} = 0, \end{aligned} \tag{87}$$

and

$$\begin{aligned} \hat{\delta}_{+}\mathcal{B}_{(\mathbf{1})} = \mathcal{F}_{(\mathbf{1})} \quad &, \quad \hat{\delta}_{+}\mathcal{F}_{(\mathbf{1})} = 0, \\ \hat{\delta}^{2}_{+}\mathcal{B}_{(\mathbf{1})} = 0 \quad &, \quad \hat{\delta}^{2}_{+}\mathcal{F}_{(\mathbf{1})} = 0 \end{aligned} \tag{88}$$

[6] Alternatively, we can take the weak coupling limit in the calculation. In general, partition functions of cohomological field theories are independent of coupling constants. So they can be evaluated exactly in the weak coupling limit.

where $\mathcal{B}$ denotes the bosonic matrix and $\mathcal{F}$ denotes the fermionic one.

For $A_{\mu(\nu)}, \psi_{\mu(\nu)}$ and $A_{\mu(\mathbf{1})}, \psi_{\mu(\mathbf{1})}$,

$$
\begin{aligned}
\hat{\delta}_+ A_{\mu(\nu)} = \psi_{\mu(\nu)} \quad &, \quad \hat{\delta}_+ \psi_{\mu(\nu)} = +\frac{1}{\sqrt{\theta^{i(\mu)}}}\delta_{\mu\nu}\phi_{\mathbf{1}}, \\
\hat{\delta}_+^2 A_{\mu(\nu)} = +\frac{1}{\sqrt{\theta^{i(\mu)}}}\delta_{\mu\nu}\phi_{\mathbf{1}} \quad &, \quad \hat{\delta}_+^2 \psi_{\mu(\nu)} = 0,
\end{aligned} \tag{89}
$$

and

$$
\begin{aligned}
\hat{\delta}_+ A_{\mu(\mathbf{1})} = \psi_{\mu\ (\mathbf{1})} \quad &, \quad \hat{\delta}_+ \psi_{\mu(\mathbf{1})} = 0, \\
\hat{\delta}_+^2 A_{\mu(\mathbf{1})} = 0 \quad &, \quad \hat{\delta}_+^2 \psi_{\mu(\mathbf{1})} = 0.
\end{aligned} \tag{90}
$$

For simplicity, in this section we set $\theta^1 = \theta^2 = \theta$ in the following. Using (74,75) and (89-90), (84) is shown to be

$$
S^{\infty}_{tr\oplus ex} = S^{\infty\ boson}_{tr\oplus ex} + S^{\infty\ fermion}_{tr\oplus ex}, \tag{91}
$$

where

$$
\begin{aligned}
S^{\infty\ boson}_{tr\oplus ex} \quad = \quad & +H^{+\ \mu\nu}_{(\mathbf{1})}\{H^{+}_{\mu\nu(\mathbf{1})} + \frac{i}{\sqrt{\theta}}(A_{\nu(\mu)} - A_{\mu(\nu)})\} & (92) \\
& +H^{+\ \mu\nu}_{(\alpha)}\{H^{+}_{\mu\nu(\alpha)} - \frac{i}{\sqrt{\theta}}(\delta^{\alpha}_{\mu}A_{\nu(\mathbf{1})} - \delta^{\alpha}_{\nu}A_{\mu(\mathbf{1})})\} & (93) \\
& +H^{\mu}_{(\mathbf{1})}\{H_{\mu(\mathbf{1})} + \frac{i}{\sqrt{\theta}}(-2B^{+}_{\alpha\mu(\alpha)})\} & (94) \\
& +H^{\mu}_{(\alpha)}\{H_{\mu(\alpha)} - \frac{i}{\sqrt{\theta}}(-2B^{+}_{\alpha\mu(\mathbf{1})} + \delta_{\mu\alpha}c_{(\mathbf{1})})\} & (95) \\
& +\frac{4}{\theta}\bar{\phi}_{(\mathbf{1})}\phi_{(\mathbf{1})}, & (96)
\end{aligned}
$$

and

$$
\begin{aligned}
S^{\infty\ fermion}_{tr\oplus ex} \quad = \quad & -\frac{i}{\sqrt{\theta}}\chi^{+\ \mu\nu}_{(\mathbf{1})}(\psi_{\nu(\mu)} - \psi_{\mu(\nu)}) & (97) \\
& -\frac{2i}{\sqrt{\theta}}\chi^{+\ \mu\alpha}_{(\alpha)}\psi_{\mu(\mathbf{1})} & (98) \\
& +\frac{i}{\sqrt{\theta}}\chi^{\mu}_{(\mathbf{1})}(2\psi^{+}_{\alpha\mu(\alpha)}) & (99) \\
& -\frac{i}{\sqrt{\theta}}\chi^{\mu}_{(\alpha)}(2\psi^{+}_{\alpha\mu(\mathbf{1})} + \delta_{\mu\alpha}\bar{\eta}_{(\mathbf{1})}) & (100) \\
& +\frac{i}{\sqrt{\theta}}\eta_{(\mathbf{1})}\psi^{\mu}_{(\mu)}. & (101)
\end{aligned}
$$

Now we fix the gauge symmetry (85). We introduce the ghost ρ, the anti-ghost $\bar{\rho}$ and the Nakanishi-Lautrup field b. Their ghost number are assigned as (+1,−1,0) for (ρ,$\bar{\rho}$,b), respectively. BRS transformations for $\{\bar{\rho}, b, \rho\}$ are defined as

$$\hat{\delta}_+ b = \rho \ , \ \ \hat{\delta}_+ \rho = 0 \ , \ \ \hat{\delta}_+ \bar{\rho} = 0. \tag{102}$$

Because the gauge symmetry is given by (85), $\{\bar{\rho}, b, \rho\}$ contain only the trace parts.

Let us introduce a gauge fixing action by

$$S_{g.f.} = Tr_{\mathcal{H}} \ \ \hat{\delta}_+ \ \ [\ \ \bar{\rho}_{(\mathbf{1})}\mathcal{I}(b_{(\mathbf{1})}\mathcal{I} + [\hat{\partial}^{\mu}, A_{\mu(\nu)}\mathcal{T}^{\nu}]) \ \ \ ]. \tag{103}$$

To get the BRS exact action including the gauge fixing action, let us deform the BRS transformation rules for $A_{\mu(\nu)}, \psi_{\mu(\nu)}$ (89) as

$$\begin{aligned} \hat{\delta}_+ A_{\mu(\nu)} &= \psi_{\mu(\nu)} + \frac{1}{\sqrt{\theta}} \delta_{\mu\nu} \rho_{(\mathbf{1})} \\ \hat{\delta}_+ \psi_{\mu(\nu)} &= + \frac{1}{\sqrt{\theta}} \delta_{\mu\nu} \phi_{(\mathbf{1})}. \end{aligned} \tag{104}$$

(103) is rewritten into

$$S_{g.f} = +b_{(\mathbf{1})}(b_{(\mathbf{1})} - \frac{1}{\sqrt{\theta}} A_{\mu,(\mu)}) \tag{105}$$

$$+\frac{4}{\theta}\bar{\rho}_{(\mathbf{1})}\rho_{(\mathbf{1})} + \frac{1}{\sqrt{\theta}}\bar{\rho}_{(\mathbf{1})}\psi_{\mu(\mu)}. \tag{106}$$

We list degrees of the Gaussian integral in (92-96),(97-101) and (105,106).

from bosons (107)

degree		
$3+3$	$H^{+}_{\mu\nu(\mathbf{1})}$, $A_{\mu(\nu)}$ (μ,ν) selfdual	in (92)
$4+4$	$H^{+\ \mu\nu}_{(\nu)}$, $A_{\mu(\mathbf{1})}$	in (93)
$4+4$	$H^{\mu}_{(\mathbf{1})}$, $B^{+\ \alpha\mu}_{(\alpha)}$	in (94)
$3+1+3+1$	$H_{\mu(\alpha)}$ (μ,α) selfdual , $H_{\mu(\mu)}$, $B^{+}_{\alpha\mu(\mathbf{1})}$, $c_{(\mathbf{1})}$	in (95)
$1+1$	$\phi_{(\mathbf{1})}$, $\bar{\phi}_{(\mathbf{1})}$	in (96)
$1+1$	$b_{(\mathbf{1})}$, $A_{\mu(\mu)}$	in (105)

from fermions (108)

degree		
$3+3$	$\chi^{+}_{\mu\nu(\mathbf{1})}$, $\psi_{\nu(\mu)}$ (μ,ν) selfdual	in (97)
$4+4$	$\chi^{+\ \mu\alpha}_{(\alpha)}$, $\psi_{\mu(\mathbf{1})}$	in (98)
$4+4$	$\chi^{\mu}_{(\mathbf{1})}$, $\psi^{+\ \alpha\mu}_{(\alpha)}$	in (99)
$3+1+3+1$	$\chi_{\mu(\nu)}$ (μ,ν) selfdual , $\chi^{\mu}_{(\mu)}$, $\psi^{+}_{\alpha\mu(\mathbf{1})}$, $\bar{\eta}_{(\mathbf{1})}$	in (100)
$1+1$	$\eta_{(\mathbf{1})}$, $\psi^{\mu}_{(\mu)}$	in (101)
$1+1$	$\rho_{(\mathbf{1})}$, $\bar{\rho}_{(\mathbf{1})}$	in (106)

From (107) and (108), we see that the path integral contains no zero mode, so we obtain a definite partition function. Also, the noncommutative parameter θ does not appear in the partition function, which is expected feature of our model.

We adopt the following path integral measure $\mathcal{D}f^{tr}\mathcal{D}f^{ex}$,

$$\mathcal{D}f^{tr}\mathcal{D}f^{ex} = \prod \frac{df_{boson}}{\sqrt{\pi}} \prod df_{fermion}, \tag{109}$$

where f_{boson} denotes a bosonic field and $f_{fermion}$ denotes a fermionic field. Then we obtain

$$Z^{\infty}_{tr\oplus ex} = \int \mathcal{D}f^{tr}\mathcal{D}f^{ex} e^{-(S^{\infty}_{tr\oplus ex}+S_{g.f.})} = 1. \tag{110}$$

From (64) and (110), we conclude that the partition function of the $\mathcal{N}=4$ $U(1)$ gauge theory on N.C. $\mathbb{R}^4$ is given by

$$Z^{4dim}_{\mathcal{N}=4} = Z^{\infty}_{MM} = Z^{\infty}_{MM}|_{\text{trace}} \times Z^{\infty}_{tr\oplus ex} = \frac{\pi^2}{6}. \tag{111}$$

5 Moduli Space and Instanton Number

In this section, we concentrate on the relation between the moduli space of the Monads and the partition function of the $\mathcal{N}=4$ supersymmetric Yang-Mills theory. The partition function of Vafa-Witten theory is given by the generating function of the Euler number of the some vector bundle over the moduli space :

$$Z = \sum_{k=1} \chi(\mathcal{M}_k) q^k \tag{112}$$

$$q^k = e^{2\pi i k\tau}. \tag{113}$$

Here τ is the complex coupling constant and $\mathcal{M}_k$ is the moduli space defined by

$$\left\{A, B, c | F^{+\mu\nu} - i[B^+_{\mu\rho}, B^{+\rho}_{\nu}] - i[B^+_{\mu\nu}, c] = 0 \ , 2D^{\mu}B_{+\mu\rho} + D_{\rho}c = 0\right\}/\mathcal{G}, \tag{114}$$

where $\mathcal{G}$ is the gauge transformation group. In addition, if $\chi^{\mu\nu}, \chi^{\mu}$ zero-modes are sections of the cotangent bundle of $\mathcal{M}_k$, then $\chi(\mathcal{M}_k)$ is the Euler number of $\mathcal{M}_k$. Particularly, the base 4-fold satisfies the vanishing theorem in [14], then the moduli space is identified with the instanton moduli space with its instanton number k. Therefore, it is important to investigate the $\mathcal{M}_k$.

It is natural to assume that the topology of the moduli space does not change under the θ-shift. After dimensional reduction (large θ limit) , let us replace variables as (44), (50) and (51). As operators, fields are infinite dimensional matrices. If matrix size of these B_i is cut off at N , BPS eqs. (24) are replaced by hyperkähler momentum maps

$$\mu_{\mathbb{C}} := [B_i, B_j] + \frac{1}{2}\epsilon_{ijkl}[B_k^\dagger, B_l^\dagger] = 0,$$
$$\mu_{\mathbb{R}} := \sum_{i=1}^{4}[B_i, B_i^\dagger] = 0, \tag{115}$$

then the moduli space is determined by

$$\mathcal{M}_N = (\mu_{\mathbb{C}}^{-1}(0) \cap \mu_{\mathbb{R}}^{-1}(0))/U(N). \tag{116}$$

It is known that the solutions of eqs.(115) include the solutions of simultaneous ADHM eqs. [29]. θ deformation realizes the continuous connection between (114) and (116). This is a direct correspondence between BPS equations of noncommutative field theory and Monads by means of changing the noncommutative parameter.

Turning now to next issue, let us study the partition function whose action functional includes the topological term. In section 4, we perform the calculation with the action functional which does not include the term of $\tau \int F \wedge F$ (or $\tau\ Tr_{\mathcal{H}} F \wedge F$). In the MNS calculation, they use the mass deformation to decompose the theory to more simple ones whose partition function is given by $1/d^2$ in (57). (See section 7 in [20] and section 4.4 in this article.) This mass deformation causes supersymmetry breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 1$. B_1, B_2, B_3 become massive, and $B_4, B_4^\dagger$ and $\phi, \bar{\phi}$ are left for massless fields. If we consider this mass deformation in the finite θ theory, we find that gauge fields are given from $B_4, B_4^\dagger$ and $\phi, \bar{\phi}$ as 4-dim theory, because the massless fields correspond to the unbroken gauge fields. In the reduced theory after integrating out B_1, B_2 and B_3, fixed point loci are defined by

$$[B_4, B_4^\dagger] = 0\ ,\ [\phi, \bar{\phi}] = 0\ ,\ [B_4, \phi] = 0, \tag{117}$$

where $B_4, B_4^\dagger$ and $\phi, \bar{\phi}$ are $d \times d$ matrices where d is a divisor of N and is appearing in the argument of (57). Furthermore, contributions for the partition function are given by isolated fixed points, as MNS mentioned in the end of section 5 in [20]. At least one of B_4 and ϕ is the rank d , when B_4 and ϕ are solutions of the fixed point equations and the fixed points contribute to the path integral. Because if rank $< d$ then there are zero modes of the equations

$$[\delta B_4, B_4^\dagger] + [B_4, \delta B_4^\dagger] = 0\ ,\ [\delta\phi, \bar{\phi}] + [\phi, \delta\bar{\phi}] = 0\ ,\ [\delta B_4, \phi] + [B_4, \delta\phi] = 0, \tag{118}$$

where these equations are given by variation of (117). These zero modes mean that the fixed point loci are non-zero dimension and path integrals vanish by the fermionic zero modes. With attention to these points, if we specify the instanton numbers corresponding to solutions of (117) labeled by d, then we determine the partition function whose action functional includes the topological term.

A hint to speculate the instanton number is ADHM correspondence. The solutions of (117) is included in the set of solutions of noncommutative deformed ADHM eqs. corresponding to d instanton, i.e.

$$[B_4, B_4^\dagger] + [\phi, \bar{\phi}] + II^\dagger - J^\dagger J = 0 \ , \ [B_4, \phi] + IJ = 0, \tag{119}$$

where I and $J^\dagger$ are d-dim. vectors. This is ADHM equations of noncommutative U(1) theories under the condition of noncommutativity $\theta^1 = -\theta^2$ [30]. Here we have to fix I and $J^\dagger$ to compare (119) with (117) as

$$I = 0_d \quad , \quad J^\dagger = 0_d, \tag{120}$$

where 0_d is 0 vector of d-dim. Then, the solutions of (119) are given by the solutions of (117). From this observation, we find that the moduli space of $B_4, B_4^\dagger$ and $\phi, \bar{\phi}$, which are gauge fields in this case, is the submanifold in instanton moduli space of instanton number d.

Therefore, someone might think it is not so strange to expect that the instanton number is given as $-\frac{det(\theta)^{\frac{1}{2}}}{16\pi^2} Tr_{\mathcal{H}} F \wedge F = d$, where the gauge fields correspond to $B_4, B_4^\dagger$ and $\phi, \bar{\phi}$, and we conjecture that the partition function of the $\mathcal{N} = 4$ U(1) gauge theory in noncommutative $\mathbb{R}^4$ with the topological term $\tau \int F \wedge F$ is given by $\tilde{Z}^{4dim}_{\mathcal{N}=4,\tau} = \sum_{d=0}^{\infty} \frac{1}{d^2} e^{2\pi i \tau d}$. However, It would still be unwise to conclude $\tilde{Z}^{4dim}_{\mathcal{N}=4,\tau} = \sum_{d=0}^{\infty} \frac{1}{d^2} e^{2\pi i \tau d}$, because the direct corresponding with the instanton number and $B_4, B_4^\dagger$ and $\phi, \bar{\phi}$ fixed point locus labeled by d is unknown. Meanwhile, it might be possible to investigate this conjecture from Montonen and Olive duality [31, 14] if such a duality of noncommutative version exists. (See also [32].) For example, if we assume that the partition function takes the form as

$$\tilde{Z}^{4dim}_{\mathcal{N}=4,\tau} = \sum_{d=1}^{\infty} \frac{1}{d^2} e^{2\pi i \tau k(d)} \ , \tag{121}$$

where $k(d)$ is a instanton number depending on d, restriction to the modular like form

$$\tilde{Z}^{4dim}_{\mathcal{N}=4,1/\tau} = \pm \left(\frac{\tau}{i}\right)^n \tilde{Z}^{4dim}_{\mathcal{N}=4,\tau} \tag{122}$$

might determine $k(d)$, where n is a some number. Unfortunately, we do not know how to chose a suitable modular like form , and above naive conjecture $\tilde{Z}^{4dim}_{\mathcal{N}=4,\tau} = \sum_{d=0}^{\infty} \frac{1}{d^2} e^{2\pi i \tau d}$ does not satisfy this condition. Anyway, further investigations are necessary to determine the contribution of the topological term.

6 Conclusions and Discussions

We investigated cohomological gauge theories in N.C. $\mathbb{R}^{2D}$. We saw that vacuum expectation values of the theories do not depend on noncommutative parameters, and the large noncommutative parameter limit is equivalent to the dimensional reduction. As a result of these facts, we showed that two types of cohomological theories defined in N.C. $\mathbb{R}^{2D}$ and N.C. $\mathbb{R}^{2D+2}$ are equivalent, if they are connected through dimensional reduction. Therefore, we found several partition functions of noncommutative supersymmetric Yang-Mills theories in various dimensions are equivalent, when they are connected by dimensional reduction from $2+2D$ to $2D$. Using this technique, we determine the partition function of the $\mathcal{N}=4$ U(1) gauge theory in N.C. $\mathbb{R}^4$, where the action does not include the topological term $\tau \int F \wedge F$, and the result is equivalent to the partition function of (8dim , $\mathcal{N}=2$) , (6dim , $\mathcal{N}=2$) , (2dim , $\mathcal{N}=8$) and the IKKT matrix model given by their dimensional reduction to 0 dim. The case including the topological term was discussed, too.

Let us list some left problems below. In this article, concrete partition functions are given for the $\mathcal{N}=4$ U(1) gauge theory in N.C. $\mathbb{R}^4$ and the series connecting to it by dimensional reduction. So, we are interested in N.C.non-abelian cases. To calculate them, we have to find some new formulation like MNS, because we know the partition function concerning $su(N)$ but we need it for $su(N) \times su(M)$ for $U(M)$ theory.

Next, we had qualitative observation of $\mathcal{N}=2$ 4-dim case but we do not do quantitative approach. So, we have to do the more detail analysis for the $\mathcal{N}=2$ super Yang-Mills cases. We saw in section 5, after taking large θ limit, moduli space is described by Monads in $\mathcal{N}=4$ 4-dim case. From the analogy with $\mathcal{N}=4$ 4-dim case, direct and smooth connections between noncommutative instanton moduli spaces and ADHM spaces might be given in $\mathcal{N}=2$ 4-dim case.

Other important problems are applications to the various fuzzy spaces, T^d_θ , $\mathbb{C}P^d_N$, and so on. Since these noncommutative spaces are expressed by finite dimensional Hilbert spaces, the dimensional reduction will not occur at the large θ limit despite omitting kinetic terms.

Wide spread applications of the technology of this article are going to happen in many cases other than above subjects. All of them are left for future works.

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A Fock Space

Let us consider N.C. $\mathbb{R}^{2D}$. First of all, we introduce following operators,

$$
\begin{aligned}
a_i &\equiv \frac{z_i}{\sqrt{\theta^{2i-1,2i}}} \quad , \quad z_i \equiv \frac{1}{\sqrt{2}}(x^{2i-1} + ix^{2i}), \\
a_i^\dagger &\equiv \frac{\bar{z}_i}{\sqrt{\theta^{2i-1,2i}}} \quad , \quad \bar{z}_i \equiv \frac{1}{\sqrt{2}}(x^{2i-1} - ix^{2i}),
\end{aligned} \tag{123}
$$

where i runs from 1 to D, and a_i and $a_i^\dagger$ satisfy

$$
[a_i, a_j^\dagger] = \delta_{ij}. \tag{124}
$$

We often use the symbol θ^i defined as

$$
\theta^i = +\theta^{2i-1,2i} = -\theta^{2i,2i-1}. \tag{125}
$$

The Hilbert space is constructed as the Fock space,

$$
\begin{aligned}
\mathcal{H} &= \bigoplus \mathbb{C} \left| n_1, \cdots, n_D \right\rangle , \\
\left| n_1, \cdots, n_D \right\rangle &\equiv \frac{(a_1^\dagger)^{n_1} \cdots (a_D^\dagger)^{n_D}}{\sqrt{n_1! \cdots n_D!}} \left| 0, \cdots, 0 \right\rangle .
\end{aligned} \tag{126}
$$

a_i and $a_i^\dagger$ operate on $\left| n_1, \cdots, n_D \right\rangle$ as follows

$$
\begin{aligned}
a_i \left| n_1, \cdots, n_D \right\rangle &= \sqrt{n_i} \left| n_1, \cdots, n_i - 1, \cdots, n_D \right\rangle , \\
a_i^\dagger \left| n_1, \cdots, n_D \right\rangle &= \sqrt{n_i + 1} \left| n_1, \cdots, n_i + 1, \cdots, n_D \right\rangle .
\end{aligned} \tag{127}
$$

$\left| n_1, \cdots, n_D \right\rangle$ are the eigenstates of the number operator $\hat{n}_i \equiv a_i^\dagger a_i$,

$$
\hat{n}_i \left| n_1, \cdots, n_D \right\rangle = n_i \left| n_1, \cdots, n_D \right\rangle . \tag{128}
$$

Arbitrary operator has following expression;

$$
\hat{\mathcal{O}} = \sum_{n_1, m_1} \cdots \sum_{n_D, m_D} \mathcal{O}^{n_1 \cdots n_D}_{m_1 \cdots m_D} \left| n_1, \cdots, n_D \right\rangle \left\langle m_1, \cdots, m_D \right| .
$$

Let us consider $2D = 4$ case. The Hilbert space $\mathcal{H}$ is expanded by the Fock basis $|n_1, n_2\rangle$,

$$
\begin{aligned}
\mathcal{H} &= \bigoplus \mathbb{C} |n_1, n_2\rangle, \\
|n_1, n_2\rangle &= \frac{(a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2}}{\sqrt{n_1! n_2!}} |0, 0\rangle.
\end{aligned} \tag{129}
$$

$a_i^\dagger$ and a_i are expressed as

$$a_1^\dagger = \sum_{n_1=0}^{\infty} \sqrt{n_1+1}|n_1+1, n_2\rangle\langle n_1, n_2| \quad , \quad a_2^\dagger = \sum_{n_2=0}^{\infty} \sqrt{n_2+1}|n_1, n_2+1\rangle\langle n_1, n_2|,$$
$$a_1 = \sum_{n_1=0}^{\infty} \sqrt{n_1+1}|n_1, n_2\rangle\langle n_1+1, n_2| \quad , \quad a_2 = \sum_{n_2=0}^{\infty} \sqrt{n_2+1}|n_1, n_2\rangle\langle n_1, n_2+1|. \tag{130}$$

The finite dimensional truncation $\mathcal{H}_N$ can be defined by several ways. One definition of $\mathcal{H}_N$ is given by

$$\mathcal{H}_N = \bigoplus_{n_1=0,n_2=0}^{n_1=N_c,n_2=N_c} \mathbb{C}|n_1, n_2\rangle, \tag{131}$$

where N_c is a finite integer number. By the definition, we obtain

$$\text{dim. of } \mathcal{H}_N = (N_c+1)^2 = N, \tag{132}$$

and

$$\mathbf{1}_N = \sum_{n_1=0,n_2=0}^{n_1=N_c,n_2=N_c} |n_1, n_2\rangle\langle n_1, n_2|. \tag{133}$$

Another definition of $\mathcal{H}_N$ is given by

$$\mathcal{H}_N = \bigoplus_{n_1=0,n_2=0}^{n_1+n_2=N_c} \mathbb{C}|n_1, n_2\rangle. \tag{134}$$

In this case,

$$\text{dim. of } \mathcal{H}_N = \frac{(N_c+1)(N_c+2)}{2} = N, \tag{135}$$

and

$$\mathbf{1}_N = \sum_{n_1=0,n_2=0}^{n_1+n_2=N_c} |n_1, n_2\rangle\langle n_1, n_2|. \tag{136}$$

By using the definition of $\mathbf{1}_N$, (133) or (136), and the following expressions of the differential operators $\hat{\partial}_\mu$ in terms of $a_i^\dagger$ and a_i,

$$\hat{\partial}_1 = \frac{1}{\sqrt{2\theta^1}}(a_1 - a_1^\dagger) \quad , \quad \hat{\partial}_2 = \frac{-i}{\sqrt{2\theta^1}}(a_1 + a_1^\dagger),$$
$$\hat{\partial}_3 = \frac{1}{\sqrt{2\theta^2}}(a_2 - a_2^\dagger) \quad , \quad \hat{\partial}_4 = \frac{-i}{\sqrt{2\theta^2}}(a_2 + a_2^\dagger), \tag{137}$$

Given the definition of $\mathcal{H}_N$, for example by (31), we obtain

$$
\begin{aligned}
[a_1, \mathbf{1}_N] &= -\sqrt{N+1} \sum_{n_2=0}^{N} |N, n_2\rangle\langle N+1, n_2|, \\
[a_1^\dagger, \mathbf{1}_N] &= +\sqrt{N+1} \sum_{n_2=0}^{N} |N+1, n_2\rangle\langle N, n_2|, \\
[a_2, \mathbf{1}_N] &= -\sqrt{N+1} \sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N+1|, \\
[a_2^\dagger, \mathbf{1}_N] &= +\sqrt{N+1} \sum_{n_1=0}^{N} |n_1, N+1\rangle\langle n_1, N|.
\end{aligned} \tag{138}
$$

¿From (138) and (137), we obtain

$$
\begin{aligned}
T_1 = \frac{1}{\sqrt{2\theta^1}} \quad (\quad & -\sqrt{N+1} \sum_{n_2=0}^{N} |N, n_2\rangle\langle N+1, n_2| \\
& -\sqrt{N+1} \sum_{n_2=0}^{N} |N+1, n_2\rangle\langle N, n_2| \qquad), \\
T_2 = \frac{-i}{\sqrt{2\theta^1}} \quad (\quad & -\sqrt{N+1} \sum_{n_2=0}^{N} |N, n_2\rangle\langle N+1, n_2| \\
& +\sqrt{N+1} \sum_{n_2=0}^{N} |N+1, n_2\rangle\langle N, n_2| \qquad), \\
T_3 = \frac{1}{\sqrt{2\theta^2}} \quad (\quad & -\sqrt{N+1} \sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N+1| \\
& -\sqrt{N+1} \sum_{n_1=0}^{N} |n_1, N+1\rangle\langle n_1, N| \qquad), \\
T_4 = \frac{-i}{\sqrt{2\theta^2}} \quad (\quad & -\sqrt{N+1} \sum_{n_1=0}^{N} |n_1, N\rangle\langle n_1, N+1| \\
& +\sqrt{N+1} \sum_{n_1=0}^{N} |n_1, N+1\rangle\langle n_1, N| \qquad).
\end{aligned} \tag{139}
$$

Using (133) and (139), we can show

$$
Tr_{\mathcal{H}}\ \mathbf{1}_N\ \mathbf{1}_N = N \quad , \quad Tr_{\mathcal{H}}\ T_\mu\ T_\nu = +\frac{1}{\theta^i} N \delta_{\mu\nu} \quad , \quad Tr_{\mathcal{H}} \mathbf{1}_N\ T_\mu = 0. \tag{140}
$$

Also, we can obtain

$$Tr_{\mathcal{H}}\mathbf{1}_N[\hat{\partial}_\mu, \mathbf{1}_N] = 0 \quad , \quad Tr_{\mathcal{H}}\mathbf{1}_N[\hat{\partial}_\mu, T_\nu] = -\frac{N}{\theta^i}\delta_{\mu\nu},$$
$$Tr_{\mathcal{H}}T_\mu[\hat{\partial}_\nu, \mathbf{1}_N] = +\frac{N}{\theta^i}\delta_{\mu\nu} \quad , \quad Tr_{\mathcal{H}}T_\mu[\hat{\partial}_\nu, T_\rho] = 0 \quad , \tag{141}$$

and

$$Tr_{\mathcal{H}}\mathbf{1}_N[\mathbf{1}_N, \mathbf{1}_N] = 0 \quad , \quad Tr_{\mathcal{H}}\mathbf{1}_N[\mathbf{1}_N, T_\mu] = 0,$$
$$Tr_{\mathcal{H}}\mathbf{1}_N[T_\mu, T_\nu] = +iN\theta^{-1}_{\mu\nu} \quad , \quad Tr_{\mathcal{H}}T_\mu[T_\nu, T_\rho] = 0. \tag{142}$$

Let us define $\mathcal{I}$ and $\mathcal{T}_\mu$ as,

$$\mathcal{I} = \frac{1}{\sqrt{N}}\mathbf{1}_N, \tag{143}$$

and

$$\mathcal{T}_\mu = \frac{\sqrt{\theta_i}}{\sqrt{N}}T_\mu. \tag{144}$$

By definition,

$$\mathcal{T}_\mu = \sqrt{\theta_i}[\hat{\partial}_\mu, \mathcal{I}]. \tag{145}$$

Using $\mathcal{I}$ and $\mathcal{T}_\mu$, (140),(141) and (142) are rewritten into

$$Tr_{\mathcal{H}}\ \mathcal{I}\ \mathcal{I} = 1 \quad , \quad Tr_{\mathcal{H}}\ \mathcal{T}_\mu\ \mathcal{T}_\nu = \delta_{\mu\nu} \quad , \quad Tr_{\mathcal{H}}\ \mathcal{I}\ \mathcal{T}_\mu = 0, \tag{146}$$

$$Tr_{\mathcal{H}}\mathcal{I}[\hat{\partial}_\mu, \mathcal{I}] = 0 \quad , \quad Tr_{\mathcal{H}}\mathcal{I}[\hat{\partial}_\mu, \mathcal{T}_\nu] = -\frac{1}{\sqrt{\theta^i}}\delta_{\mu\nu},$$
$$Tr_{\mathcal{H}}\mathcal{T}_\mu[\hat{\partial}_\nu, \mathcal{I}] = +\frac{1}{\sqrt{\theta^i}}\delta_{\mu\nu} \quad , \quad Tr_{\mathcal{H}}\mathcal{T}_\mu[\hat{\partial}_\nu, \mathcal{T}_\rho] = 0 \quad , \tag{147}$$

and

$$Tr_{\mathcal{H}}\mathcal{I}[\mathcal{I}, \mathcal{I}] = 0 \quad , \quad Tr_{\mathcal{H}}\mathcal{I}[\mathcal{I}, \mathcal{T}_\mu] = 0,$$
$$Tr_{\mathcal{H}}\mathcal{I}[\mathcal{T}_\mu, \mathcal{T}_\nu] = +\frac{i\theta^i}{\sqrt{N}}\theta^{-1}_{\mu\nu} \quad , \quad Tr_{\mathcal{H}}\mathcal{T}_\mu[\mathcal{T}_\nu, \mathcal{T}_\rho] = 0. \tag{148}$$

The same formulae as (146), (147) and (148) hold for the case of (134). The difference between the definitions of $\mathcal{H}_N$'s, (131) and (134), are absorbed in dim. of $\mathcal{H}_N$.

It is worthwhile to notice that the independence of the precise definitions of $\mathcal{H}_N$ holds generally. The proof is done by using the discrete version of Stokes's theorem for the boundary of the finite truncated Fock space [15, 16].

References

[1] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv.Theor.Math.Phys. 7 (2004) 831-864, hep-th/0206161.

[2] N. Nekrasov and A. Okounkov, *Seiberg-Witten theory and random partitions*, hep-th/0306238.

[3] T. Maeda, T. Nakatsu, K. Takasaki, T. Tamakoshi, *Five-Dimensional Supersymmetric Yang-Mills Theories and Random Plane Partitions*, hep-th/0412327.

[4] T. Matsuo, S. Matsuura, K. Ohta, *Large N limit of 2D Yang-Mills Theory and Instanton Counting*, hep-th/0406191.

[5] Y. Tachikawa, *Five-dimensional Chern-Simons terms and Nekrasov's instanton counting*, JHEP 0402 (2004) 050 , hep-th/0401184 .

[6] M. Wijnholt, *Five-Dimensional Gauge Theories and Unitary Matrix Models*, hep-th/0401025.

[7] T. Eguchi, H Kanno, *Topological Strings and Nekrasov's formulas*, JHEP 0312 (2003) 006 , hep-th/0310235.

[8] R. Dijkgraaf, C. Vafa, *Matrix Models, Topological Strings, and Supersymmetric Gauge Theories* , Nucl.Phys. B644 (2002) 3-20 , hep-th/0206255 R. Dijkgraaf, C. Vafa, *On Geometry and Matrix Models* , Nucl.Phys. B644 (2002) 21-39, hep-th/0207106 R. Dijkgraaf, C. Vafa, *A Perturbative Window into Non-Perturbative Physics*, hep-th/0208048 .

[9] A. Sako, S-I. Kuroki and T. Ishikawa, *Noncommutative Cohomological Field Theory and GMS soliton*, J.Math.Phys.43(2002)872-896, hep-th/0107033.

[10] A. Sako, S-I. Kuroki and T. Ishikawa, *Noncommutative-shift invariant field theory*, proceeding of 10th Tohwa International Symposium on String Theory, (AIP conference proceedings 607, 340).

[11] A.Sako, *Noncommutative Cohomological Field Theories and Topological Aspects of Matrix models*, hep-th/0312120.

[12] D.Birmingham, N.Blau, M.Rakowski, G.Thomson, *Topological Field Theory* Phys.Rep.209(1991)129.

[13] E. Witten, *Topological quantum field theory*, Commun.Math.Phys.117(1988)353; E. Witten, *Introduction to cohomological field theories*, Int.J.Mod.Phys.A.6(1991)2273.

[14] C. Vafa and E. Witten, *A Strong coupling test of S-duality*, Nucl.Phys.B431(1994)3-77, hep-th/9408074.

[15] T. Ishikawa, S-I. Kuroki, A. Sako, *Instanton number calculas on Noncommutative R4* , JHEP 08(2002)028 hep-th/0201196.

[16] A. Sako *Instanton number of Noncommutative U(n) gauge theory*, JHEP 04(2003)023, hep-th/0209139.

[17] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa, T. Tada, *Noncommutative Yang-Mills in IIB Matrix Model* , Nucl.Phys. B565 (2000) 176-192, hep-th/9908141 .

[18] N.Ishibashi, H. Kawai, Y. Kitazawa, A. Tsuchiya , *A Large N Reduction Model as Superstring*, Nucl.Phys.B498(1997)467-491, hep-th/9612115 .

[19] H. Aoki, S. Iso, H. Kawai, Y. Kitazawa, A. Tsuchiya , T. Tada, *IIB Matrix Model*, Prog.Theor.Phys.Suppl.134(1999)47-83, hep-th/9908038 .

[20] G.W. Moore, N. Nekrasov, S. Shatashvili, *D-particle bound states and generalized instantons*, Commun.Math.Phys.209(2000)77-95, hep-th/9803265 .

[21] N. Dorey, T. J. Hollowood, V. V. Khoze *The D-Instanton Partition Function* , JHEP 0103 (2001) 040 , hep-th/0011247.

[22] S. Hirano, M. Kato, *Topological Matrix Model*, Prog.Theor.Phys.98(1997)1371-1384, hep-th/9708039 .

[23] F.Sugino,*Cohomological Field Theory Approach To Matrix Strings*, Int.J.Mod.Phys.A14(1999)3979-4002, hep-th/9904122 .

[24] U. Bruzzo, F. Fucito, J. F. Morales, A. Tanzini, *Multi-Instanton Calculus and Equivalent Cohomology*, JHEP 0305(2003)054, hep-th/0211108 .

[25] T. Suyama, A. Tsuchiya, *Exact Result in* $N_C = 2$ *IIB Matrix Model*, Prog.Theor.Phys.99(1998)321-325, hep-th/9711073.

[26] N. Nekrasov and A. Schwarz, *Instantons on noncommutative* R^4 *and (2,0) superconformal six dimensional theory*, Commun.Math. Phys,198(1998)689, hep-th/9802068.

[27] N.A. Nekrasov, *Trieste lectures on solitons in noncommutative gauge theories*, hep-th/0011095.

[28] J.M.F. Labastida and C. Lozano, *Mathai-Quillen Formulation of Twisted N=4 Supersymmetric Gauge Theories in Four Dimensions* Nucl.Phys. B502 (1997) 741-790, hep-th/9702106.

[29] J-S. Park, *Monads and D-instantons*, Nucl.Phys. B493 (1997) 198-230, hep-th/9612096.

[30] N. Nekrasov and A. Schwarz, *Instantons on noncommutative* R^4 *and (2,0) superconformal six dimensional theory*, Commun.Math.Phys. 198 (1998) 689-703, hep-th/9802068.

[31] C.Montonen and D.Olive, *Magnetic Monopoles as Gauge Particles?* , Phys.Lett.B72(1977)117;
P.Goddard,J.Nuyts and D.Olive , *Gauge Theories and Magnetic Charge* , Nucl.Phys.B125(1977)1.

[32] F. Fucito, J. F. Morales, R. Poghossian, *Multi instanton calculus on ALE spaces* , Nucl.Phys. B703 (2004) 518-536, hep-th/0406243.