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by

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On the Littlewood-Paley g function and the Lusin area function on real rank 1 semisimple Lie groups

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Abstract

Let G be a real rank one connected semisimple Lie group with finite center. Using the spherical Fourier transform and the classical one, we shall consider a pull back on G of $H^1(\mathbf{R})$ and introduce a real Hardy space $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ on G as a subspace of $L^1(G//K)$. We also define the Lusin area function $S_+(f)$ and the Littlewood-Paley g function $g(f)$ on G as analogues of the classical theory. We show that S_+ and g are bounded from $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$.

1. Notation. Let G be a real rank one connected semisimple Lie group with finite center and $G = KAN = K\overline{A^+}K$ respectively an Iwasawa and the Cartan decompositions of G. Let $\mathfrak{a}$ be the Lie algebra of A and $\mathcal{F} = \mathfrak{a}^*$ the dual space of $\mathfrak{a}$. Let γ be the positive simple root of (G, A) determined by N and H the unique element in $\mathfrak{a}$ satisfying $\gamma(H) = 1$. Let m_1 and m_2 denote the multiplicities of γ and 2γ respectively. We put

$$\alpha = \frac{m_1 + m_2 - 1}{2}, \quad \beta = \frac{m_2 - 1}{2}, \quad \rho = \alpha + \beta + 1.$$

We parameterize each element in A, $\mathfrak{a}$, and $\mathcal{F}$ as $a_x = \exp(xH)$, xH, and $x\gamma$ ($x \in \mathbf{R}$) respectively, and identify A, $\mathfrak{a}$, and $\mathcal{F}$ with $\mathbf{R}$. In this paper we shall treat only K-bi-invariant functions on G. Since $A^+ = \{a_x; x > 0\}$, all K-bi-invariant functions can be identified with even functions on $\mathbf{R}$.

Let $dg = e^{2\rho x} dk dx dn = \Delta(x) dk dx dk'$ denote the decompositions of a Haar measure dg on G respectively corresponding to the Iwasawa and Cartan

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decompositions of G, where dk, dx, dn denote Haar measures on K, A, N respectively, and $\Delta(x)$, $x \geq 0$, is explicitly given as

$$\Delta(x) = 2^{2\rho}(\mathrm{sh}x)^{2\alpha+1}(\mathrm{ch}x)^{2\beta+1}.$$

We extend this function on $\mathbf{R}_+$ as an even function on $\mathbf{R}$. Let $L^p(G//K)$ denote the space of K-bi-invariant functions on G with finite L^p-norm: $\|f\|_p = \left(\int_0^\infty |f(x)|^p \Delta(x)dx\right)^{1/p}$ and $L^1_{\mathrm{loc}}(G//K)$ the space of locally integrable, K-bi-invariant functions on G. Let $C_c^\infty(G//K)$ be the space of compactly supported C^∞, K-bi-invariant functions on G. We denote by $\hat{f}$ the spherical Fourier transform of f and by $f * h$ the convolution of f, h in $L^1(G//K)$ (cf. [**2**], [**9**, Chap.9]). Similarly, we denote by $\tilde{F}$ and $F \circledast H$ the Euclidean Fourier transform of F and the convolution of F, H in $L^1(\mathbf{R})$ respectively.

2. Real Hardy spaces. We shall introduce a real Hardy space on G by using a radial maximal function on G. Let ϕ be a positive compactly supported C^∞, K-bi-invariant function on G with $\|\phi\|_1 = 1$. We define the dilation ϕ_t, $t > 0$, of ϕ as

$$\phi_t(x) = \frac{1}{t}\frac{1}{\Delta(x)}\Delta\left(\frac{x}{t}\right)\phi\left(\frac{x}{t}\right). \tag{1}$$

Since this dilation has the same properties as in the Euclidean case; $\|\phi_t\|_1 = \|\phi\|_1$ and $\{\phi_t; t > 0\}$ approximates the identity in $L^p(G//K)$, $0 < p \leq \infty$, it is quite natural to introduce a radial maximal function $M_\phi f$ on G as

$$(M_\phi f)(g) = \sup_{0<t<\infty} |(f * \phi_t)(g)|, \quad g \in G.$$

As shown in [**3**, Theorem 3.4], this maximal operator M_ϕ satisfies the so-called maximal theorem: M_ϕ is bounded on $L^p(G//K)$ $(1 < p \leq \infty)$ and satisfies the weak type L^1 estimate. Analogously as the definition of the real Hardy space $H^1(\mathbf{R})$ on $\mathbf{R}$, we define the real Hardy space on G by

$$H^1(G//K) = \{f \in L^1_{\mathrm{loc}}(G//K)\ ;\ M_\phi f \in L^1(G//K)\}$$

and the norm by $\|f\|_{H^1(G)} = \|M_\phi f\|_1$. Then $H^1(G//K) \subset L^1(G//K)$ (see [**5**, §4]). For $f \in C_c^\infty(G//K)$, we define the Abel transform F_f^s, $s \in \mathbf{R}$, of f as

$$F_f^s(x) = e^{\rho(1+s)x}\int_N f(a_x n)dn.$$

For simplicity, we put $W_+(f) = F_f^1$ and we denote by W_- the inverse operator of W_+. As shown in [**6**, §3], $W_\pm$ are explicitly given by a composition of the generalized Weyl type fractional integral transforms. We recall (cf. [**6**, (3.7)]) that both $\hat{f}$ and $(F^s_f)^\sim$ are holomorphic functions on $\mathbf{C}$ of exponential type and they satisfy

$$\hat{f}(\lambda + is\rho) = (F^s_f)^\sim(\lambda), \quad \lambda \in \mathbf{C}. \tag{2}$$

Let $C(\lambda)$ denote Harish-Chandra's C-function (cf. [**6**, (2.6)]) and $\mathcal{M}_{C_\rho}$ the Euclidean Fourier multiplier corresponding to $C_\rho(\lambda) = C(-(\lambda + i\rho))$, that is, $\mathcal{M}_{C_\rho}(F)^\sim(\lambda) = C(-(\lambda + i\rho))\tilde{F}(\lambda)$. We define

$$W_-(H^1(\mathbf{R})) = \{f \in L^1_{\rm loc}(G//K)\ ;\ W_+(f) \in H^1(\mathbf{R})\}$$

and the norm by $\|f\|_{W_-} = \|W_+(f)\|_{H^1(\mathbf{R})}$. We also define $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ by replacing the condition that $W_+(f) \in H^1(\mathbf{R})$ in the above definition with $\mathcal{M}^{-1}_{C_\rho} \circ W_+(f) \in H^1(\mathbf{R})$ and the norm by $\|f\|_{W_-^{C_\rho}} = \|\mathcal{M}^{-1}_{C_\rho} \circ W_+(f)\|_{H^1(\mathbf{R})}$.

We note that $f * \phi_t = W_-(W_+(f * \phi_t)) = W_-(F \circledast W_+(\phi_t))$, where $F = W_+(f)$. Hence the H^1-norm $\|f\|_{H^1(G)}$ of f on G may be related to an L^1-norm of $F = W_+(f)$ on $\mathbf{R}$. Actually, let $\alpha - \beta = [\alpha - \beta] + \delta$ and $\beta + 1/2 = [\beta + 1/2] + \delta'$, where [ ] is the Gauss symbol, and set $\underline{n} = [\alpha - \beta] + [\beta - 1/2]$ and $\underline{D} = \{0, \delta, \delta', \delta + \delta'\}$. Then it follows from [**5**, Theorem 4.6] that

$$\|f\|_{H^1(G)} \sim \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \|M^{\mathbf{R}}_\phi \circ W^{\mathbf{R}}_{-(m+\xi)}(F)(x)(\mathrm{th}x)^{m+\xi}\|_{L^1(\mathbf{R})}, \tag{3}$$

where $W^{\mathbf{R}}_{-\gamma}$ is the Weyl type fractional integral transform on $\mathbf{R}$ and $M^{\mathbf{R}}_\phi$ is the maximal operator on $\mathbf{R}$ defined by

$$(M^{\mathbf{R}}_\phi F)(x) = \sup_{0<t<\infty} |(F \circledast W_+(\phi_t)(x)|, \quad x \in \mathbf{R}.$$

From the equivalence (3) it follows that

$$W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R}))) \subset H^1(G//K) \subset W_-(H^1(\mathbf{R}))$$

(see [**5**, Remark 4.7(1) and Corollary 4.3]).

3. Estimate of W_-. We retain the previous notations. In the process to deduce (3) we use a relation between the Weyl type fractional integral transforms W_- on G and $W^{\mathbf{R}}_{-\gamma}$ on $\mathbf{R}_+$. As shown in [**5**, Proposition 4.5, Lemma 4.4], if F is smooth, then $W_-(F)$ is estimated as follows. For $x > 0$,

$$\begin{aligned} |W_-(F)(x)| \quad &\leq \quad c\Delta(x)^{-1} \sum_{m=0}^{\underline{n}} \sum_{\xi \in \underline{D}} \Big(|W^{\mathbf{R}}_{-(m+\xi)}(F)(x)| \\ &\qquad + \int_x^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F)(s)| A_{m,\xi}(x,s) ds \Big), \end{aligned} \tag{4}$$

where $A_{m,0}(x,s) \equiv 0$, $A_{m,\xi}(x,s) \geq 0$ and there exists a constant c such that $\int_0^s A_{m,\xi}(x,s)dx \leq c$ for all $s \geq 0$. More precisely, $A_{m,\xi}(x,s)$ is dominated by $\chi_{[0,\infty)}(s-x)\chi_{[0,1]}(s)$ or $B_{m,\xi}(s-x)$, where $B_{m,\xi}(x)$ is integrable on $\mathbf{R}_+$.

Let $f \in W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ and put $F = W_+(f)$. By the definition, $\mathcal{M}_{C_\rho}^{-1}(F)$ belongs to $H^1(\mathbf{R})$. We note that $C(-(\lambda+i\rho)) \sim (1+|\lambda|)^{-(\alpha+1/2)}$ (cf. [**3**, Theorem 2]) and $(i\lambda)^\gamma/(\lambda+i\rho)^{\alpha+1/2}$, $0 \leq \gamma \leq \alpha+1/2$, satisfies the Hörmander condition (cf. [**8**, p.318]). Therefore, $W_{-\gamma}^{\mathbf{R}}(F)$ belongs to $H^1(\mathbf{R})$ (cf. [**8**, p.363]). Since $m+\xi \leq \underline{n}+\delta+\delta' = \alpha+1/2$, each $W_{-(m+\xi)}^{\mathbf{R}}(F)$ in (3) and (4) belongs to $H^1(\mathbf{R})$, that is,

$$\|W_{-(m+\xi)}^{\mathbf{R}}(F)\|_{H^1(\mathbf{R})} \leq c\|f\|_{W_-^{C_\rho}} \tag{5}$$

for all $0 \leq m \leq \underline{n}$ and $\xi \in \underline{D}$.

4. g **and area functions.** Let p_t denote the Poisson kernel on G, which is a K-bi-invariant function on G given by

$$\hat{p}_t(\lambda) = e^{-t\sqrt{\lambda^2+\rho^2}}.$$

We define the Littlewood-Paley g function $g(f)$ on G, $f \in C_c^\infty(G//K)$, as

$$g(f)(x) = \left(\int_0^\infty \left|t\frac{\partial}{\partial t} f * p_t(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

As shown in [**1**], [**7**] and [**8**], g satisfies the maximal theorem. We define the Lusin area function $S(f)$ on G as an analogue of the classical theory (cf. [**9**, p.314]). Let $B(t)$ denote the ball on G with radius t centered at the origin and $|B(t)|$ the volume of the ball. Let $\chi_{B(t)}$ denote the characteristic function of $B(t)$ and put

$$\chi_t(x) = \frac{1}{|B(t)|}\chi_{B(t)}(x).$$

We define the Lusin area function $S(f)$ on G as

$$S(f)(x) = \left(\int_0^\infty \chi_t * \left|t\frac{\partial}{\partial t} p_t * f(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

As shown in [**7**], S is bounded on $L^p(G//K)$, $0 < p < \infty$. We also define the modified one as

$$S_+(f)(x) = \left(\int\int_{\{\sigma(y)\geq\sigma(x)\}} \chi_t(xy^{-1}) \left|t\frac{\partial}{\partial t} f * p_t(y)\right|^2 dy\frac{dt}{t}\right)^{1/2},$$

where σ is the distance function on G (cf. [**10**, 8.1.2]). Our main theorem is the following.

Theorem. g and S_+ are bounded from $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$.

5. Sketch of the proof. We suppose that $f \in W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ and put $F = W_+(f)$. For simplicity we denote $K_t = t(\partial/\partial t)p_t$. Since $t(\partial/\partial t)f * p_t = f * K_t = W_-(W_+(f * K_t)) = W_-(F * W_+(K_t))$, it follows from (4) that

$$\|g(f)\|_1 \leq c \sum_{m=0}^{n} \sum_{\xi \in \underline{D}} \int_0^\infty \left(\int_0^\infty |W^{\mathbf{R}}_{-(m+\xi)}(F) \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \right)^{1/2} dx,$$

because

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty \left| \int_x^\infty H(s,t) A(x,s) ds \right|^2 \frac{dt}{t} \right)^{1/2} dx \\ \leq & \int_0^\infty \int_s^\infty \left(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \right)^{1/2} A(x,s) ds dx \\ = & \int_0^\infty \left(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \right)^{1/2} \int_0^s A(x,s) dx ds \\ \leq & \; c \int_0^\infty \left(\int_0^\infty |H(s,t)|^2 \frac{dt}{t} \right)^{1/2} ds. \end{aligned}$$

We here put

$$g_{\mathbf{R}}(H)(x) = \left(\int_0^\infty |H \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad H \in C_c^\infty(\mathbf{R}).$$

Then

$$\|g(f)\|_1 \leq c \sum_{m=0}^{n} \sum_{\xi \in \underline{D}} \int_0^\infty g_{\mathbf{R}}(W^{\mathbf{R}}_{-(m+\xi)}(F))(x) dx.$$

Since each $W^{\mathbf{R}}_{-(m+\xi)}(F)$ belongs to $H^1(\mathbf{R})$ (see (5)), it follows from the $(1,\infty,1)$-atomic decomposition of $H^1(\mathbf{R})$ that it is enough to show that there exists a constant C such that for all $(1,\infty,1)$-atoms A on $\mathbf{R}$,

$$\int_0^\infty g_{\mathbf{R}}(A)(x) dx \leq C. \tag{6}$$

Obviously, we may suppose that A is centered, that is, A is supported on $[-r, r]$, $\|A\|_\infty \leq (2r)^{-1}$ and $\int_{-\infty}^\infty A(x) x^k dx = 0$, $k = 0, 1$. First we shall prove that $g_{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$: For $H \in L^2(\mathbf{R})$,

$$\begin{aligned}
\|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} &= \int_0^\infty \|H \circledast W_+(K_t)\|^2_{L^2(\mathbf{R})} \frac{dt}{t} \\
&= \int_0^\infty \|\tilde{H} \cdot W_+(K_t)^\sim\|^2_{L^2(\mathbf{R})} \frac{dt}{t} \\
&= \int_0^\infty \|\tilde{H}(\lambda) \cdot t\sqrt{\lambda(\lambda+2i\rho)} e^{-t\sqrt{\lambda(\lambda+2i\rho)}}\|^2_{L^2(\mathbf{R})} \frac{dt}{t} \\
&= \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left(\int_0^\infty t|\lambda(\lambda+2i\rho)| e^{-2t\Re\sqrt{\lambda(\lambda+2i\rho)}} dt \right) d\lambda \\
&= \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 \left(\int_0^\infty tr e^{-2t\sqrt{r}\cos(\theta/2)} dt \right) d\lambda \\
&\leq c \int_{-\infty}^\infty |\tilde{H}(\lambda)|^2 d\lambda = c\|H\|^2_{L^2(\mathbf{R})},
\end{aligned}$$

where we set $\lambda(\lambda + 2i\rho) = re^{i\theta}$ and we used the fact that $\cos\theta \geq 0$ and $\cos(\theta/2) = \sqrt{(\cos\theta + 1)/2} \geq 1/\sqrt{2}$. Hence, by Schwarz' inequality, we have

$$\int_0^{2r} g_{\mathbf{R}}(A)(x)dx \leq c\|A\|_{L^2(\mathbf{R})}(2r)^{1/2} \leq C. \tag{7}$$

Next we suppose that $x \geq 2r$. We recall $W_+(K_t)(x) = te^{\rho x}(\partial/\partial t)F^0_{p_t}(x)$ and $F^0_{p_t}(x) = Ct(t^2+x^2)^{-1/2}\mathcal{K}_1(\rho(t^2+x^2)^{1/2})$, where $\mathcal{K}_\nu$ is the modified Bessel function (see [**1**], p. 289). Since $\mathcal{K}_\nu(x) = O(x^{-1/2}e^{-x})$ if $x \to \infty$, and $O(x^{-\nu})$ if $x \to 0$, and $x - y \geq x - r > r$ if $|y| \leq r$, it follows that

$$\begin{aligned}
&|A \circledast W_+(K_t)(x)| = \Big| \int_{-\infty}^\infty A(y)W_+(K_t)(x-y)dy \Big| \\
\leq\ & c\int_{-\infty}^\infty |A(y)|t^3(t^2+(x-y)^2)^{-3/4-1/2-\epsilon} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho(x-y)} dy \\
\leq\ & ct(t^2+(x-r)^2)^{-3/4} \leq ct(x-r)^{-3/2},
\end{aligned} \tag{8}$$

where $\epsilon = 0$ if $t^2 + (x-y)^2 \geq 1$, and $\epsilon = 1/4$ if $t^2 + (x-y)^2 < 1$. Actually, when $\epsilon = 0$, we used the fact that $t^{2\ell}e^{-\rho(t^2+(x-y)^2)^{1/2}}$, $\ell \in \mathbf{R}$, has the maximum $O((x-y)^\ell e^{-\rho(x-y)})$ at $t \sim (x-y)^{1/2}$. Thereby, letting $\ell = 1$, we see that $t^2(t^2+(x-y)^2)^{-1/2}e^{-\rho(t^2+(x-y)^2)^{1/2}} \leq t^2(x-y)^{-1}e^{-\rho(t^2+(x-y)^2)^{1/2}}$ $\leq ce^{-\rho(x-y)}$. When $\epsilon = 1/4$, we used the fact that $t^2(t^2+(x-y)^2)^{-3/4} \leq t^2(t^2+(x-y)^2)^{-1} \leq 1$. Next we note the moment condition of A, which implies that $B(x) = \int_{-\infty}^x \int_{-\infty}^u A(v)dvdu$ is supported on $[-r, r]$ and $\|B\|_\infty \leq 2r$.

Since $(d/dx)^k(K_\nu(x)e^x) = O(x^{-1/2-k})$ if $x \to \infty$, and $O(x^{-\nu-k})$ if $x \to 0$, integration by parts yields that

$$\begin{aligned}
&|A \circledast W_+(K_t)(x)| \\
\leq\ & c\int_{-\infty}^{\infty} |B(y)| t^3(t^2 + t(x-y) + (x-y)^2) \\
&\qquad \times (t^2 + (x-y)^2)^{-3/4-1/2-2-\epsilon} e^{-\rho(t^2+(x-y)^2)^{1/2}} e^{\rho(x-y)} dy \\
\leq\ & cr^2 t^{-2}(t^2 + (x-r)^2)^{-1} \leq cr^2 t^{-2}(x-r)^{-2}, \qquad (9)
\end{aligned}$$

where $\epsilon = 0$ if $t^2 + (x-y)^2 \geq 1$, and $\epsilon = 5/4$ if $t^2 + (x-y)^2 < 1$. Actually, when $\epsilon = 0$, letting $\ell = 5/2$, we see that $t^5(t^2 + (x-y)^2)^{-5/4} e^{-\rho(t^2+(x-y)^2)^{1/2}} \leq e^{-\rho(x-y)}$ and when $\epsilon = 5/4$, we have $t^5(t^2 + (x-y)^2)^{-5/2} \leq 1$. Hence, from (8) and (9) we see that

$$\begin{aligned}
&\int_0^\infty |A \circledast W_+(K_t)(x)|^2 \frac{dt}{t} \\
\leq\ & c(x-r)^{-3} \int_0^{\sqrt{r}} t\,dt + cr^4(x-r)^{-4} \int_{\sqrt{r}}^\infty t^{-5} dt \\
\leq\ & cr(x-r)^{-3} + cr^2(x-r)^{-4}
\end{aligned}$$

and thus

$$\int_{2r}^\infty g_{\mathbf{R}}(A)(x)dx \leq c\int_{2r}^\infty (r^{1/2}(x-r)^{-3/2} + r(x-r)^{-2})dx \leq C. \qquad (10)$$

Then (7) and (10) imply the desired estimate (6).

As for the area function $S_+(f)$, it follows from (4) that it is enough to show that for all centered $(1,\infty,1)$-atoms A on $\mathbf{R}$,

$$\int_0^\infty S^1_{\mathbf{R}}(A)(x)dx \leq C \quad \text{and} \quad \int_0^\infty S^2_{\mathbf{R}}(A)(x)dx \leq C, \qquad (11)$$

where

$$\begin{aligned}
S^1_{\mathbf{R}}(H)(x) \;=\; & \Big(\int_0^\infty \int_{\{y \geq |x|\}} \int_K \chi_t(a_x k a_y^{-1}) dk \\
& \qquad \times \Delta(y)^{-1} |H \circledast W_+(K_t)(y)|^2 dy \frac{dt}{t}\Big)^{1/2} \Delta(x).
\end{aligned}$$

and

$$\begin{aligned}
S^2_{\mathbf{R}}(H)(x) \;=\; & \Big(\int_0^\infty \int_{\{y \geq |x|\}} \int_K \chi_t(a_x k a_y^{-1}) dk \\
& \qquad \times \Delta(y)^{-1} \left|\int_y^\infty H \circledast W_+(K_t)(s) A(y,s) ds\right|^2 dy \frac{dt}{t}\Big)^{1/2} \Delta(x).
\end{aligned}$$

Here $A(y,s) \geq 0$ and $\int_0^s A(y,s)dy \leq c$ for all $s \geq 0$. More precisely, $A(y,s)$ is dominated by $\chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$ or $B(s-y)$, where $B(y)$ is integrable on $\mathbf{R}_+$. First we shall estimate $S^1_{\mathbf{R}}(A)$. Since $\Delta(y)^{-1} \leq \Delta(x)^{-1}$ if $y \geq |x|$ and $\int_0^\infty \int_K \chi_t(a_x k a_y^{-1})dk\Delta(x)dx = \int_G \chi_t(ga_y)dg = \|\chi_t\|_1 = 1$, it follows that for $H \in C_c^\infty(\mathbf{R})$,

$$\begin{aligned}
&\|S^1_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \\
\leq\ & \int_0^\infty \int_0^\infty \left(\int_0^\infty \int_K \chi_t(a_x k a_y^{-1})dk\Delta(x)dx\right) |H \circledast W_+(K_t)(y)|^2 dy\frac{dt}{t} \\
=\ & \|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})}
\end{aligned}$$

and thus, $S^1_{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$. Then $\int_0^{2r} S^1_{\mathbf{R}}(A)(x)dx \leq C$ as before. We suppose that $x \geq 2r$. We recall that, if $y \geq |x|$, then $\Delta(y)^{-1}\Delta(x)^2 \leq \Delta(y)$ and $|A \circledast W_+(K_t)(y)|$ is dominated by $t(x-r)^{-3/2}$ and $r^2t^{-2}(x-r)^{-2}$. Since $\|\chi_t\|_1 = 1$, as in the case of $g_{\mathbf{R}}$, $S^1_{\mathbf{R}}(A)(x)$ is estimated as $r^{1/2}(x-r)^{-3/2} + r(x-r)^{-2}$ and then $\int_{2r}^\infty S^1_{\mathbf{R}}(A)(x)dx \leq C$. Therefore, we can deduce (11) for $S^1_{\mathbf{R}}$. Next we shall estimate $S^2_{\mathbf{R}}$. As before, we have

$$\|S^2_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \leq \|\int_y^\infty g_{\mathbf{R}}(H)(s)A(y,s)ds\|^2_{L^2(\mathbf{R})}.$$

When $A(y,s)$ is dominated by $\chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$, we see that $0 \leq |x| \leq y \leq s \leq 1$ and thus,

$$\|S^2_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} \leq \Big(\int_0^1 g_{\mathbf{R}}(H)(s)\|A(\cdot,s)\|_{L^2(\mathbf{R})}ds\Big)^2 \leq c\|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})}$$

by Schwarz's inequality. When $A(y,s)$ is dominated by $B(s-y)$, we change the variable s to $s+y$ and thus,

$$\begin{aligned}
\|S^2_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})} &\leq \|\int_0^\infty g_{\mathbf{R}}(H)(s+y)B(s)ds\|^2_{L^2(\mathbf{R})} \\
&\leq c\|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})}\Big(\int_0^\infty B(s)ds\Big)^2 \leq c\|g_{\mathbf{R}}(H)\|^2_{L^2(\mathbf{R})}.
\end{aligned}$$

Hence $S^2_{\mathbf{R}}$ is bounded on $L^2(\mathbf{R})$ and $\int_0^{2r} S^2_{\mathbf{R}}(A)(x)dx \leq C$ as before. We suppose that $x \geq 2r$. When $A(y,s)$ is dominated by $\chi_{[0,\infty)}(s-y)\chi_{[0,1]}(s)$, we see that $2r \leq x \leq y \leq s \leq 1$ and thus, $A \circledast W_+(K_t)(s)$ is estimated as $t(x-r)^{-3/2}$ and $r^2t^{-2}(x-r)^{-2}$. Moreover, $\Delta(y)^{-1}\Delta(x)^2 \leq \Delta(y)$, $\|\chi_t\|_1 = 1$, and $\int_y^\infty A(y,s)ds \leq 1$. Therefore, $S^2_{\mathbf{R}}(A)(x)$ is dominated by $r^{1/2}(x-r)^{-3/2} + r(x-r)^{-2}$ and thereby, $\int_{2r}^\infty S^2_{\mathbf{R}}(A)(x)dx \leq C$ as in the case of $g_{\mathbf{R}}(A)$. When $A(y,s)$ is dominated by $B(s-y)$, we change the variable s to $s+(y-x)$.

We recall that, since $s+(y-x) \geq s \geq x > 2r$, $|A \circledast W_+(K_t)(s+(y-x))|$ is dominated by $H(t,s,r) = \min\{t(s-r)^{-3/2}, r^2t^{-2}(s-r)^{-2}\}$, which is independent of x, y. Therefore, noting $\Delta(y)^{-1}\Delta(x)^2 \leq \Delta(y)$, $\|\chi_t\|_1 = 1$ and $A(y, s+(y-x)) \leq B(s-x)$, we can deduce that

$$S^2_{\mathbf{R}}(A)(x) \leq \Big(\int_0^\infty \Big|\int_x^\infty H(t,s,r)B(s-x)ds\Big|^2 \frac{dt}{t}\Big)^{1/2}.$$

Since $\int_0^\infty |H(t,s,r)|^2 dt/t \leq r(s-r)^3 + r^2(s-r)^{-4}$ as before, it follows that

$$\begin{aligned} & \int_{2r}^\infty S^2_{\mathbf{R}}(A)(x)dx \\ \leq\ & \int_{2r}^\infty \int_x^\infty (r^{1/2}(s-r)^{3/2} + r(s-r)^{-2})B(s-x)dsdx \\ =\ & \int_{2r}^\infty (r^{1/2}(s-r)^{3/2} + r(s-r)^{-2})\left(\int_{2r}^s B(s-x)dx\right)ds \leq C. \end{aligned}$$

Therefore, we have (11) for $S^2_{\mathbf{R}}$. This completes the proof of the theorem.

Remark. We put $D_x = W_- \circ (d/dx) \circ W_+$. Then the operators g' and S'_+ defined by replaced $t(d/dt)p_t$ in the definitions of g and S_+ with tD_xp_t are also bounded from $W_-(\mathcal{M}_{C_\rho}(H^1(\mathbf{R})))$ to $L^1(G//K)$. Moreover, in the definition of S_+ we may replace χ_t with ϕ_t in (1).

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