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From Stokes' formula to cyclic Hochschild cocycles on classical symbols

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Abstract

We first show that the cut-off integral on non integer order classical symbols extends to symbol valued forms and obeys Stokes' property on non integer order classical symbol valued forms. Similarly, the Wodzicki residue extends to classical symbol valued forms and we show it relates to the complex residue of cut-off integrals of holomorphic symbol valued forms. The extended Wodzicki residue yields a cycle on classical symbol valued forms, the residue cycle.

Secondly, given a deformed algebra $(A[[\nu]], \star)$, where $(A, \cdot)$ is a unital commutative complex algebra, we investigate antisymmetrized $2k$-cochains (which we refer to as trace forms)

$$\mathrm{Alt}\,\psi_{2k}(a_0, \cdots, a_{2k}) := \frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T_k(a_0 \star a_{\sigma(1)} \star \cdots \star a_{\sigma(2k)}),$$

where $T(\sum_{k=0}^{\infty} a_k\, \nu^k) = \sum_{k=0}^{\infty} T_k(a)\, \nu^k$ is a $\mathbb{C}[[\nu]]$ extension to $\mathcal{N} := N[[\nu]]$ of some linear form τ_0 defined on some vector space $N \subset A$. We give a local description of these cochains when $A = C^\infty(W)$ where W is a Poisson manifold, equipped with a star product compatible with the Poisson structure. Whenever $\tau_k : \alpha \mapsto \tau_0(\langle \Lambda^{\wedge k}, \alpha \rangle)$, where Λ is the Poisson tensor, satisfies a Stokes' property, we show that $\mathrm{Alt}\psi_{2k}$ is a closed b-cocycle.

We finally combine cut-off integrals with the Moyal, resp. the left product on the algebra $CS_{c.c}(\mathbb{R}^n)$ of classical symbols on $\mathbb{R}^n$ with constant coefficients to build meromorphic families of trace forms, the residue of which yields a cyclic b-cocycle. We show that the $n+1$-trace form built this way is proportional to the character associated with the residue cycle on classical symbol valued forms on $\mathbb{R}^n$ with constant coefficients.

Introduction

The paper is organised in three parts. In Part 1, we prove Stokes' formula for cut-off integrals on non integer order classical symbol valued forms. In Part 2, we give the abstract setting to build cyclic Hochschild cocycles associated with star products using a Stokes' type formula. In Part 3,using the results of Part 1, we apply the general construction described in Part 2 to build closed Hochschild cocycles on classical symbols.

The first part of the paper is devoted to Stokes' property on symbols. We first define cut-off integrals to non integer order classical symbol valued forms on M and show that the cut-off integral extended to forms satisfies Stokes' property (see Theorem 3). Similarly, we extend the Wodzicki residue to all classical symbol valued forms. We show that the relation between complex residues and the Wodzicki residue extends to symbols valued forms (see Theorem 1):

$$\mathrm{Res}_{z=z_0} -\!\!\!\!\!\!\int \omega(z) = -\frac{1}{\alpha'(z_0)} \mathrm{res}(\omega(z_0)),$$

where $\omega(z)$ is a holomorphic family of classical symbol valued forms of order $\alpha(z)$ and $\mathrm{res}(\omega(z_0))$ the Wodzicki residue of $\omega(z_0)$.
The extended Wodzicki residue also satisfies Stokes' property and therefore yields a closed linear form on classical symbol valued forms on a closed manifold M thereby giving rise to a cycle $(\Omega\, CS(M), d, \mathrm{res})$ (see Theorem 2) which we refer to as the residue cycle.

Let us now describe the contents of the second part of the paper. Given an algebra $\mathcal{A}$ over some ring R, equipped with an associative product $\star$ and some R-linear form $T : \mathcal{A} \to R$, we consider *trace forms*

$$\Psi_n(a_0, \cdots, a_n) := T\left(a_0 \star a_1 \star a_2 \star \cdots \star a_{n-1} \star a_n\right) \tag{1}$$

which provide R-valued $n+1$- multilinear maps on $\mathcal{A}$. We consider the corresponding *antisymmetrized trace forms*:

$$\mathrm{Alt}\, \Psi_n(a_0, \cdots, a_n) := \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \epsilon(\sigma) \Psi_n\left(a_0, a_{\sigma(1)}, \cdots, a_{\sigma(n)}\right).$$

We check (see Proposition 6) that the traciality of T w.r. to $\star$, i.e. the fact that T vanishes on $\star$-brackets:

$$T\left([a, b]_*\right) = 0 \quad \forall a, b \in \mathcal{A}$$

is equivalent on one hand to the cyclicity of $Alt\, \Psi_{2k}$ and on the other hand to the vanishing of the trace forms investigated by Helton and Howe [HH]:

$$\mathcal{A}lt\, \Psi_{2k}(a_1, \cdots, a_{2k}) := T\left([a_1, \cdots, a_{2k}]_\star\right)$$

where we have set

$$[a_1, \cdots, a_{2k}]_\star := \frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) a_{\sigma(1)} \star \cdots \star a_{\sigma(2k)}$$

for all $k \in \mathbb{N}$. If $\mathcal{A}$ has unit 1, Helton and Howe's trace forms correspond in this context to $\mathrm{Alt}\, \Psi_{2k}(1, a_1, \cdots, a_n)$.
Specialising to a deformation $(\mathcal{A} = A[[\nu]], \star)$ with unit $1 \in A$ of a commutative algebra $(A, \cdot)$, we can either see $\mathcal{A}$ as a complex algebra taking $R = \mathbb{C}$ or see $\mathcal{A}$ as a $\mathbb{C}[[\nu]]$ -algebra taking $R = \mathbb{C}[[\nu]]$. Writing $T(\sum_{k=0}^\infty a_k\, \nu^k) = \sum_{k=0}^\infty T_k(a)\, \nu^k$, in the first case the tracial property of $T : \mathcal{A} \to \mathbb{C}[[\nu]]$ translates to the *strong closedness* of $\star$ w.r. to T. In the second case, the tracial property of $T_k : \mathcal{A} \to$

$\mathbb{C}$ translates to the closedness of $\star$ w.r. to T_k [F]. In this setting, we show (see Proposition 9) that antisymmetrised trace forms $\mathrm{Alt}\,\Psi_{2k}$ considered here coincide with the antisymmetrised cochains $\mathrm{Alt}\,\Phi_{2k}$ built from the cochains considered in [CFS], [H]:

$$\Phi_{2k}(a_0, \cdots, a_{2k}) := T\left(a_0 \star \theta(a_1, a_2) \star \cdots \star \theta(a_{2k-1}, a_{2k})\right). \tag{2}$$

From this it follows that

$$\mathrm{Alt}\,\psi_{2k}(a_0, \cdots, a_{2k}) := \mathrm{Alt}\,T_k\left(a_0 \star a_1 \star a_2 \star \cdots \star a_{2k-1} \star a_k\right)$$

and

$$\mathrm{Alt}\,\phi_{2k}(a_0, \cdots, a_{2k}) := \mathrm{Alt}\,T_k\left(a_0 \star \theta(a_1, a_2) \star \cdots \star \theta(a_{2k-1}, a_{2k})\right)$$

coincide. When T is a $\mathbb{C}[[\nu]]$-linear extension of a $\mathbb{C}$-linear map τ_0 then (see Proposition 10)

$$\mathrm{Alt}\,\psi_{2k}(a_0, \cdots, a_{2k}) = 2^{-k}\,Alt\,\tau_0\left(a_0 \cdot \{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right),$$

where $\{a, b\}$ is the Poisson bracket associated with $\star$.
We further specialise to a star product associated with a Poisson structure on a Poisson manifold W, and show (Theorem 4) that if T is the $\mathbb{C}[[\nu]]$-linear extension of a $\mathbb{C}$-linear form τ_0 defined on a subspace of A then

$$\mathrm{Alt}\,\psi_{2k}(f_0, f_1, \cdots, f_{2k}) = \tau_0\left(\langle \Lambda^{\wedge k},\, f_0\, df_1 \wedge \cdots \wedge\, df_{2k}\rangle\right) \tag{3}$$

where f_i are smooth functions on W for which the expression on the r.h.s. makes sense, i.e. whenever $\langle \Lambda^{\wedge k},\, f_0\, df_1 \wedge \cdots \wedge\, df_{2k}\rangle$ lies in N. Here Λ stands for the Poisson bracket associated with the Poisson structure. Taking $f_0 = 1$ yields back a formula similar to results by Helton and Howe obtained in a different context [HH]:

$$T_k\left([f_1, \cdots, f_{2k}]_\star\right) = \tau_0\left(\langle \Lambda^{\wedge k},\, df_1 \wedge \cdots \wedge\, df_{2k}\rangle\right).$$

It follows from there that if

$$\tau_k(\alpha) = \tau_0\left(\langle \Lambda^{\wedge k}, \alpha\rangle\right)$$

satisfies Stokes' property:

$$\tau_k(d\beta) = 0$$

for any $(2k-1)$-form β on W such that $\langle \Lambda^{\wedge k}, d\beta\rangle \in N$, then $\mathrm{Alt}\,\psi_{2k} = \mathrm{Alt}\,\phi_{2k}$ is cyclic and closed for the Hochschild coboundary operator b induced by the commutative product on functions. In Appendix B we discuss why one should not expect ϕ_{2k} to be closed and hence to be able to derive (using the identification $Alt\,\psi_{2k} = \mathrm{Alt}\,\psi_{2k}$), b-closedness of $\mathrm{Alt}\,\psi_{2k}$ from b-closedness of $\mathrm{Alt}\,\phi_{2k}$ as a consequence of the b- closedess of ϕ_{2k} since the latter does not hold in general.
Specialising down to the case of a $2l$-dimensional symplectic manifold (W, ω), formula (3) reads:

$$\mathrm{Alt}\,\psi_{2k}(f_0, f_1, \cdots, f_{2k}) = l^k\,\tau_l\left(f_0\, df_1 \wedge \cdots \wedge\, df_{2k} \wedge \omega^{l-k}\right)$$

and provided τ_l satisfies Stokes' property then $\mathrm{Alt}\,\psi_{2k}$ is cyclic and closed for the Hochschild coboundary operator b.

In particular, taking $k=l$, when $A=C_0^\infty(W)$ and $T_l(f)=\int_W f\omega^l$, we recover a local formula which can be derived from a more general formula proven in [CFS] [H]:

$$\mathrm{Alt}\,\psi_{2l}(f_0,f_1,\cdots,f_{2l})=\int_W f_0\,df_1\wedge\cdots\wedge\,df_{2l}\quad\forall f_i\in C_0^\infty(W).$$

In Part 3, we apply equation (3) to the class $CS_{c.c}^{\notin\mathbb{Z}}(\mathbb{R}^{2l})$ of non integer order classical symbols on $\mathbb{R}^{2l}$ with constant coefficients, setting τ_0 to be a cut-off integral in order to build cut-off antisymmetrised trace forms $\mathrm{Alt}\,\psi_{2k}^{cut-off}$ and to show (see Theorem 5) that they are cyclic and b-closed.
Approximating a general formal classical symbol by a holomorphic family of non integer order formal symbols, we then construct (see Theorem 6) meromorphic families of antisymmetrised trace forms on $CS_{c.c}(\mathbb{R}^{2l})$ with simple poles, the complex residue of which yields the character of the residue cycle. Specialising to the left product on formal symbols, we also build meromorphic families of closed $2k$-cocycles for a perturbed product $f\cdot_z g=f\cdot|\xi|^{-z}\cdot g$:

$$\mathrm{Alt}\,\psi_{2k}(z)(\sigma_0,\cdots,\sigma_{2k})=l^k\,-\!\!\!\!\!\!\int_{T^*U}\sigma_0(z)\wedge d\sigma_1(z)\wedge\cdots\wedge d\sigma_{2k}(z)\wedge\omega^{l-k}$$

where we have set $\sigma(z):=\sigma\cdot|\xi|^{-z}$ for any $\sigma\in CS_{c.c}(\mathbb{R}^{2l})$. When $k=l$, its complex residue at $z=0$ yields back the character associated to the residue cycle $(\Omega\,CS_{c.c}(\mathbb{R}^l),d,\mathrm{res})$).
The paper is organised as follows:

- Part 1.
  1. Cut-off integrals on non integer order classical symbols
  2. Cut-off integrals extended to classical symbol valued forms
  3. The Wodzicki residue extended to classical symbol valued forms
  4. Integrals of holomorphic families of symbols valued forms
  5. Stokes' property for cut-off integrals and the Wodzicki residue
- Part 2.
  1. Trace forms on a unital algebra $(\mathcal{A},\star)$
  2. Trace forms on a deformed algebra
  3. The case of a Poisson manifold
  4. The symplectic case
- Part 3.
  1. A cyclic cocycle for non integer order classical symbols on $\mathbb{R}^{2l}$
  2. Meromorphic families of trace forms and the residue cocycle on classical symbols on $\mathbb{R}^{2l}$
  3. Perturbed star products
  4. Meromorphic cocycles associated with the left product on classical symbols on $\mathbb{R}^{2l}$
- Appendix A
- Appendix B

1 Stokes' formula for regularised integrals on symbols valued forms

1.1 Cut-off integrals on non integer order symbols and the Wodzicki residue

This section is a review of known results which we need in the subsequent section. Let U be an open subset of $\mathbb{R}^n$ and x a point in U. Let $\mathcal{S}^m(U) \subset C^\infty(T^*U)$ denote the set of scalar valued symbols on U of order $m \in \mathbb{R}$, $\mathcal{S}(U) := \bigcup_{m \in \mathbb{R}} \mathcal{S}^m(U) \subset C^\infty(T^*U)$ the algebra of all scalar valued symbols on U, $\mathcal{S}^{-\infty}(U) := \bigcap_{m \in \mathbb{R}} \mathcal{S}^m(U)$ the algebra of scalar smoothing symbols.

Definition 1 *$\sigma \in \mathcal{S}^m(U)$ is a classical symbol if for any positive integer N there is an integer K_N such that*

$$\sigma = \sum_{i=0}^{K_N} \chi\, \sigma_i + \sigma_{(N)}$$

where σ_i is positively homogeneous of order $m-i$ (i.e. $\sigma_i(x, t\xi) = t^{m-i}\sigma_i(x,\xi)\, \forall t > 0$, $\forall (x,\xi) \in T_x^*U$), $\sigma_{(N)}$ is of order $-N$ and where χ is a smooth function T_x^*U which is constant and equal to 1 outside the open unit ball B_x^*U in the cotangent space T_x^*U at point x, which vanishes around 0. We write for short

$$\sigma \sim \sum_{i=0}^{\infty} \chi\, \sigma_i.$$

Let $CS^m(U)$ denote the class of scalar classical symbols of order m and $CS(U) = \bigcup_{m \in \mathbb{R}} CS^m(U)$ the algebra of scalar classical symbols.

Using a partition of unity on the closed manifold M, we can extend these definitions from an open subset U to the manifold M.

Definition 2 *Let $CS^m(M)$ denote the set of functions $\sigma \in C^\infty(T^*M)$ such that given a partition of unity subordinated to a trivialising atlas (U_i, ψ_i), $\sigma(x,\xi) = \sum_i \psi_i(x)\, \sigma_i(x,\xi)$ with $\sigma_i \in CS^m(U_i)$.*
$CS(M) = \bigcup_{m \in \mathbb{R}} CS^m(M)$ denotes the algebra of all classical symbols on M.

This definition makes sense since σ would have the same form (in particular, the order m would not change) for another partition of unity subordinated to another coordinate chart.
When $M = \mathbb{R}^n$ it makes sense to consider symbols with constant coefficients, i.e. symbols which are independent of $x \in M$.

Definition 3 *Let $CS^m_{c.c.}(\mathbb{R}^n)$, resp. $CS_{c.c.}(\mathbb{R}^n)$ denote the set of classical symbols of order m with constant coefficients on $\mathbb{R}^n$, resp. the algebra of classical symbols with constant coefficients.*

Remark 1 *Changing the cut-off function amounts to changing $\sigma_{(N)}$ by a smoothing symbol.*

Let us recall the notion of Wodzicki residue on classical symbols.[W], [K]

Definition 4 *Let U be an open subset in $\mathbb{R}^n$ and x a point in U. The (local) Wodzicki residue of a classical symbol $\sigma \in CS(U)$ at point x is given by*

$$\mathrm{res}_x(\sigma) = \int_{|\xi|=1} \sigma_{-n}(x,\xi)\, d_S\xi,$$

where $d_S\xi = \sum_{i=1}^n (-1)^{i+1}\xi_i\, d\xi_1 \wedge \cdots \wedge d\hat{\xi}_i \wedge \cdots \wedge d\xi_n$ and $|\xi|^2 = \sum_{i=1}^n \xi_i^2$ is the canonical norm in $\mathbb{R}^n$.

Remark 2 *For any $t > 0$ we have $d_S(t\xi) = t^n d_S\xi$ and $\sigma_{-n}(x,t\xi) = t^n\sigma_{-n}(x,\xi)$ so that the form $\sigma_{-n}(x,\xi)\, d_S\xi$ is positively homogeneous of degree 0.*

Proposition 1 *[W],[K] It extends to $\sigma \in CS(M)$ by linearity using a partition of unity and $x \mapsto \mathrm{res}_x(\sigma) = \int_{|\xi|=1}\sigma_{-n}(x,\xi)\, d_S\xi$ defines a global density so that we can set:*

$$\mathrm{res}(\sigma) = \int_M dx \int_{|\xi|=1} \sigma_{-n}(x,\xi)\, d_S\xi,$$

which is independent of the chosen partition of unity and coordinate chart.

Remark 3 *The Wodzicki residue does not depend on the choice of cut-off function χ chosen to write $\sigma \sim \sum_{i=0}^\infty \chi\, \sigma_{m-i}$ where m is the order of σ, since χ only modifies the smoothing part and hence not the homogeneous part of order $-n$ involved in the definition of the residue.*

We now extract finite parts from otherwise divergent integrals [H], [G], [W], [KV]:

Proposition 2 *Let U be an open subset of $\mathbb{R}^n$ and let $x \in U$. Given $\sigma \sim \sum_{i=0}^\infty \chi\,\sigma_i \in CS^m(U)$, the expression $\int_{B_x^*(0,R)} \sigma(x,\xi)\, d\xi$ has a formal asymptotic expansion*

$$\int_{B_xj(0,R)} \sigma(x,\xi)\, d\xi = c(x) + \sum_{i=0,\, m-i+n\neq 0}^{\infty} a_i(x)\, \frac{R^{m-i+n}}{m-i+n} + b(x)\log R$$

where $c(x), a_i(x), b(x) \in \mathbb{C}$. The constant $b(x)$ coincides with the local Wodzicki residue density $\mathrm{res}_x(\sigma)$. Here $B_x^*(0,R)$ is the ball of radius R in T_x^*U centered at 0. Whenever $\mathrm{res}_x(\sigma) = 0$, the constant term $c(x) = \mathrm{fp}_{R\to\infty}\int_{B_x^*(0,R)}\sigma(x,\xi)\, d\xi$ is independent of the rescaling $R \to \lambda R$ so that the finite part

$$\begin{aligned} -\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)\, d\xi &= \sum_{i=0}^\infty \int_{B_x^*(0,1)} \chi(\xi)\, \sigma_i(x,\xi)\, d\xi \\ &+ \int_{T_x^*U} \sigma_{(N)}(x,\xi)\, d\xi \\ &- \sum_{i=0,\, m-i+n\neq 0}^{K_N} \frac{1}{m-i+n} \int_{|\xi|=1} \sigma_k(x,\xi)\, d_S\xi \end{aligned} \tag{4}$$

is well defined. It is also called the cut-off integral of $\sigma(x,\cdot)$.

Remark 4 *This cut-off integral extends the ordinary integral in the following sense; if σ has order smaller than $-n$ then $\int_{B_x^*(0,R)}\sigma(x,\xi)\, d\xi$ converges when $R \to \infty$ and $-\!\!\!\!\!\int_{T_x^*U}\sigma(x,\xi)\, dx = \int_{T_x^*U}\sigma(x,\xi)\, d\xi$.*

Proof: We drop the explicit mention of x in the proof and identifying T_x^*U with $\mathbb{R}^n$. We also write $B(0,R)$ instead of $B_x^*(0,R)$ and $S(0,1)$ instead of S_x^*U. Using the splitting $\sigma = \sum_{i=0}^{K_N} \chi\,\sigma_i + \sigma_{(N)}$, we first split the expression

$$\int_{B(0,R)} \sigma(\xi)\,d\xi = \sum_{i=0}^{\infty} \int_{B(0,1)} \chi(\xi)\,\sigma_i(\xi)\,d\xi + \sum_{i=0}^{K_N} \int_{D(1,R)} \sigma_i(\xi)\,d\xi + \int_{B(0,R)} \sigma_{(N)}(\xi)\,d\xi.$$

The integrals on $B(0,1)$ converge and we want to investigate the integral on $D(1,R)$. By linearity, we can restrict ourselves to a homogeneous component σ_i. We have

$$\int_{D(1,R)} \sigma_i(\xi)\,d\xi = \int_1^R r^{m-i+n-1} dr \int_{|\xi|=1} \sigma_i(\xi)\,d_S\xi$$

where $D(r,R) = B(0,R) - B(0,r)$ for $r < R$. We distinguish two cases:

- $m-i+n=0$ in which case we get $\int_{D(1,R)} \sigma_i(\xi)\,d\xi = \int_{D(1,R)} \sigma_i(\xi)\,d\xi = \log R \int_{S(0,1)} \sigma_i(\xi)\,d\xi$.
- $m-i+n \neq 0$ in which case we get $\int_{D(1,R)} \sigma_i(\xi)\,d\xi = \int_{D(1,R)} \sigma_i(\xi)\,d\xi = \left(\frac{R^{m-i+n}}{m-i+n} - \frac{1}{m-i+n}\right) \int_{|\xi|=1} \sigma_i(\xi)\,d_S\xi$.

Combining these different cases we get:

$$\begin{aligned}\int_{B(0,R)} \sigma(\xi)\,d\xi &= \int_{B(0,1)} \chi(\xi)\,\sigma(\xi)\,d\xi + \int_{\mathbb{R}^n} \sigma_{(N)}(\xi)\,d\xi \\ &+ \sum_{i=0,\,m-i+n\neq 0}^{K_N} \left(\frac{R^{m-i+n}}{m-i+n} - \frac{1}{m-i+n}\right) \int_{|\xi|=1} \sigma_i(\xi)\,d_S\xi + \\ &+ \log R \int_{|\xi|=1} \sigma_{-n}(\xi)\,d_S\xi.\end{aligned}$$

Extracting the finite part yields

$$\begin{aligned}\mathrm{fp}_{R\to\infty} \int_{B(0,R)} \sigma(\xi)\,d\xi &= \int_{B(0,1)} \chi(\xi)\,\sigma(\xi)\,d\xi + \int_{\mathbb{R}^n} \sigma_{(N)}(\xi)\,d\xi \\ &- \sum_{i=0,\,m-i+n\neq 0}^{K_N} \frac{1}{m-i+n} \int_{|\xi|=1} \sigma_i(\xi)\,d_S\xi.\end{aligned}$$

Changing R to λR introduces an extra finite part $\int_{S(0,1)} \sigma_{-n}(\xi)\,d\xi$, which vanishes whenever $\mathrm{res}(\sigma) = \int_{|\xi|=1} \sigma_{-n}(\xi)\,d_S\xi = 0$.

The following covariance property will be useful to extend cut-off integrals to symbols in $CS(M)$.

Lemma 1 *[L] Let U be an open subset of $\mathbb{R}^n$ and let $x \in U$. Given $\sigma \in CS(U)$, whenever $\sigma_{-n} = 0$ then for any $A \in GL_n(\mathbb{R})$,*

$$|\mathrm{det} A| ⨍_{T_x^*U} \sigma(x, A\,\xi) d\xi = ⨍_{T_x^*U} \sigma(x,\xi) d\xi.$$

Definition 5

$CS^{\notin \mathbb{Z}}(U) := \{\sigma \in CS^m(U), \quad m \notin \mathbb{Z}\}, \quad CS^{\notin \mathbb{Z}}_{c.c.}(\mathbb{R}^n) = CS^{\notin \mathbb{Z}}(\mathbb{R}^n) \cap CS_{c.c}(\mathbb{R}^n).$

Similarly, we define

$$CS^{\notin \mathbb{Z}}(M) := \{\sigma \in CS^m(M), \quad m \notin \mathbb{Z}\}).$$

Proposition 3 *The cut-off integral $-\!\!\!\!\!\int_{T_x^*U} d\xi$, $x \in U$ is well defined on $CS^{\notin \mathbb{Z}}(U)$ and the cut-off integral $-\!\!\!\!\!\int_{T^*M} d\xi\, dx$ is well defined on $CS^{\notin \mathbb{Z}}(M)$ by*

$$-\!\!\!\!\!\!\int_{T^*M} \sigma(x,\xi)\, dx\, d\xi := \sum_i \int_M dx -\!\!\!\!\!\!\int_{T_x^*U_i} \psi_i(\xi)\sigma_i(x,\xi)\, d\xi$$

for any $\sigma = \sum_i \psi_i \sigma_i \in CS(M)$ where $\{\psi_i, i \in I\}$ is a partition of unity subordinated to an atlas $(U_i, i \in I)$ and $\sigma_i \in CS(U_i)$.

Proof: Since the positively homogeneous components σ_i of any $\sigma \in CS^{\notin \mathbb{Z}}(U)$ have non integer order, there is no positively homogeneous component of order $-n$ and the Wodzicki residue vanishes. It then follows from Proposition 2 that the cut-off integral $\int_{T_x^*U} \sigma(x,\xi)\, d\xi$ is well defined for any $x \in U$.
Let $\sigma = \sum_i \psi_i \sigma_i \in CS(M)$ where $\sigma_i \in CS(U_i)$ and $\{\psi_i, i \in I\}$ is a partition of unity subordinated to an atlas $(U_i, i \in I)$. If $\sigma \in CS(M)^{\notin \mathbb{Z}}$, then $\sigma_i \in CS(U_i)^{\notin \mathbb{Z}}$ so that the component σ_{-n} vanishes in any local coordinate chart. By lemma 1, we know that in that case, the cut-off integrals $-\!\!\!\!\!\int_{T_x^*U_i} \psi_i(\xi)\sigma_i(x,\xi)\, d\xi$ transform covariantly under a change of coordinate, so that they patch up to an integral

$$-\!\!\!\!\!\!\int_{T^*M} \sigma(x,\xi)\, dx\, d\xi := \sum_i \int_M dx -\!\!\!\!\!\!\int_{T_x^*U_i} \psi_i(\xi)\sigma_i(x,\xi)\, d\xi.$$

1.2 Classical symbol valued forms

Let us first define symbol valued forms on T^*U where U is an open subset of $\mathbb{R}^n$.

Definition 6 *Let k be a non negative integer, m a real number. We set*

$$\begin{aligned} \Omega^k\, CS^m(U) = \{\alpha \in \Omega^k(T^*U), & \\ \alpha \quad &= \sum_{I\subset\{1,\cdots,n\}, J\subset\{1,\cdots,n\}, |I|+|J|=k} \alpha_{I,J}(x,\xi)\, dx_I \wedge d\xi_J \\ \text{with} \quad & \alpha_{I,J} \in CS^{m-|J|}(U)\}, \end{aligned}$$

and

$$\Omega^k\, CS(U) := \bigcup_{m \in \mathbb{R}} \Omega^k\, CS^m(U).$$

The order of $\alpha \in \Omega^k CS^m(U)$ is given by m. Furthermore, let

$$\Omega^k\, CS^{\notin \mathbb{Z}}(U) = \bigcup_{m \notin \mathbb{Z}} \Omega\, CS^m(U).$$

Here, $(x_1, \cdots, x_n, \xi_1, \cdots, \xi_n)$ is some coordinate system on U and we have set $dx_I = d_{x_{i_1}} \wedge \cdots \wedge dx_{i_l}$, $d\xi_J = d\xi_{j_1} \wedge \cdots d\xi_{j_m}$ when $I = \{i_1, i_2, \cdots, i_l\}$, $J = \{j_1, \cdots, j_m\}$.

Remark 5 • *With these conventions,* $d\xi_j$ *is of order* 1*. Also, a* k*-form of order* 0 *reads* $\alpha = \sum_{|I|+|J|=k} \alpha_{I,J}(x,\xi)\, dx_I \wedge d\xi_J$ *with* $\alpha_{I,J}$ *of order* $-|J|$*.*

- *These definitions are independent of the coordinate system since a coordinate change does not modify the behaviour in* ξ *of the* $\alpha_{I,J}$*; in particular it leaves the order unchanged.*

- *The order of a zero degree symbol valued form* $\sigma \in \Omega^0\, CS^m(U)$ *coincides with the order of the corresponding classical symbol* σ*.*

Lemma 2 *A classical symbol valued form* $\alpha \in \Omega^k\, CS^m(U)$ *of order* m *has an asymptotic expansion of the following form. For any non negative integer* N*, there is a symbol valued form* $\alpha_{(N)}$ *of order* $-N$ *such that*

$$\alpha = \sum_{i=0}^{N} \alpha_{m-i} + \alpha_{(N)}$$

with $\alpha_{m-i} := \sum_{|I|+|J|=k} \alpha_{I,J,m-|J|-i}\, dx_I \wedge d\xi_J$ *is positively homogeneous of order* $m-i$*, with* $\alpha_{I,J,m-|J|-i}$ *positively homogeneous of order* $m-|J|-i$*. Moreover for any integer* $j \leq m$*, we have*

$$(d\alpha)_j = d\,\alpha_j.$$

Proof: The first part of the statement follows trivially from the description of α combined with the properties of ordinary classical symbols. Indeed,

$$\alpha = \sum_{I\subset\{1,\cdots,n\},J\subset\{1,\cdots,n\},|I|+|J|=k} \alpha_{I,J}(x,\xi)\, dx_I \wedge d\xi_J$$

with $\alpha_{I,J} \in CS^{m-|J|}(U)$, and for each multi index I, J, there is some $\alpha_{I,J,(N)}$ of order $m-|J|-N$ such that

$$\alpha_{I,J} = \sum_{i=0}^{N} \alpha_{I,J,m-i} + \alpha_{I,J,(N)}$$

with $\alpha_{I,J,m-i}$ positively homogeneous of order $m-i$. Adding the latter we get

$$\begin{aligned}\alpha &= \sum_{i=0}^{N}\sum_{I,J} \alpha_{I,J,m-i}\, dx_I \wedge d\xi_J + \sum_{I,J} \alpha_{I,J,(N)}\, dx_I \wedge d\xi_J \\ &= \sum_{i=0}^{N} \alpha_{m-i} + \alpha_{(N)}\end{aligned}$$

where we have set $\alpha_{m-i} := \sum_{I,J} \alpha_{I,J,m-i}\, dx_I \wedge d\xi_J$ which is positively homogeneous of order $m-i$ and $\alpha_{(N)} := \sum_{I,J} \alpha_{I,J,(N)}$ which is of order $m-N-|J|-N$.
As for the second part of the statement we have

$$(d\,\alpha)_j = \left(d \sum_{|I|+|J|=k} \alpha_{I,J}\, dx_I \wedge d\xi_J \right)_j$$

$$
\begin{aligned}
&= \left(\sum_{l=1}^{n} \sum_{|I|+|J|=k} \frac{\partial}{\partial x_l} \alpha_{I,J}\, dx_l \wedge dx_I \wedge d\xi_J \right)_j \\
&+ \left(\sum_{l=1}^{n} \sum_{|I|+|J|=k} \frac{\partial}{\partial \xi_m} \alpha_{I,J}\, d\xi_m \wedge dx_I \wedge d\xi_J \right)_j \\
&= \sum_{l=1}^{n} \sum_{|I|+|J|=k} \left(\frac{\partial}{\partial x_l} \alpha_{I,J} \right)_{j-|J|} dx_l \wedge dx_I \wedge d\xi_J \\
&+ \sum_{l=1}^{n} \sum_{|I|+|J|=k} \left(\frac{\partial}{\partial \xi_m} \alpha_{I,J} \right)_{j-|J|-1} d\xi_m \wedge dx_I \wedge d\xi_J \\
&= \sum_{l=1}^{n} \sum_{|I|+|J|=k} \frac{\partial}{\partial x_l} \alpha_{I,J,j-|J|}\, dx_l \wedge dx_I \wedge d\xi_J \\
&+ \sum_{m=1}^{n} \sum_{|I|+|J|=k} \frac{\partial}{\partial \xi_m} \alpha_{I,J,j-|J|} dx_l \wedge dx_I \wedge d\xi_J \\
&= d \sum_{I,J} \alpha_{I,J,j-|J|}\, dx_I \wedge d\xi_J \\
&= d\,\alpha_j.
\end{aligned}
$$

The last line follows from our conventions setting $j = m - i$.

Remark 6 *In particular, for $\alpha \in \Omega\, CS(U)$ we have:*

$$(d\alpha)_{-n} = d\,\alpha_{-n}.$$

1.3 Cut-off integrals extended to non integer order classical symbol valued forms

Let U be an open subset in $\mathbb{R}^n$. Just as the ordinary Lebesgue integral extends to forms, the cut-off integral on T_x^*U extends to any $\alpha \in \Omega^k\, CS^{\notin \mathbb{Z}}(U)$ by:

$$-\!\!\!\!\!\!\int_{T_x^*U} \alpha := \sum_{|I|+|J|=k} d\,x_I\, -\!\!\!\!\!\!\int_{T_x^*U} \alpha_{IJ}\, d\xi_J.$$

Since computing the cut-off integral $-\!\!\!\!\!\int_{T_x^*U}$ boils down to taking the finite part when $R \to \infty$ of some ordinary integral on a ball $B(0, R)$, it vanishes on terms $\alpha_{IJ} dx_I \wedge d\xi_J$ whenever $|J| < n$ and we have:

$$-\!\!\!\!\!\!\int_{T_x^*U} \alpha = \sum_{|I|=k-n} d\,x_I \sum_{|J|=n} -\!\!\!\!\!\!\int_{T_x^*U} \alpha_{IJ}\, d\xi_J,$$

which lies in $\Omega^{k-n}(U)$. Since the α_{IJ}'s lie in $CS^{\notin \mathbb{Z}}(U)$, the cut-off integrals $-\!\!\!\!\!\int_{T_x^*U} \alpha_{IJ}\, d\xi_J$ are defined without ambiguity.
If T_x^*U is equipped with the volume form $d\mathrm{vol}_x(\xi) = d\xi_1 \wedge \cdots \wedge d\xi_n$, as in the case of ordinary integrals, we recover the cut-off integral on symbol valued functions

$\sigma \in CS^{\notin \mathbb{Z}}(U)$ via the integral on forms by integrating the top form $\sigma(x,\xi)\, d\mathrm{vol}_x(\xi)$ setting:

$$\,-\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi) := \,-\!\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)\, d\mathrm{vol}_x(\xi).$$

Definition 7 *Let for any non negative integer k, $\Omega^k CS(M)$ denote the space of forms $\alpha \in \Omega^k(T^*M)$ such that given a partition of unity subordinated to a trivialising atlas (U_i,ψ_i), $\alpha(x,\xi) = \sum_i \psi_i(x)\, \alpha_i(x,\xi)$ with $\alpha_i \in \Omega^k CS(U_i)$.*
Similarly, let $\Omega^k CS^{\notin \mathbb{Z}}(M)$ denote the space of forms $\alpha \in \Omega^k(T^*M)$ such that given a partition of unity subordinated to a trivialising atlas (U_i,ψ_i), $\alpha(x,\xi) = \sum_i \psi_i(x)\, \alpha_i(x,\xi)$ with $\alpha_i \in \Omega^k CS^{\notin \mathbb{Z}}(U_i)$.

Remark 7 *This definition makes sense since α has the same expression for another partition of unity subordinated to another coordinate chart.*
When $M = \mathbb{R}^n$ it makes sense to consider constant coefficients symbols valued forms.

Definition 8 *Let for any non negative integer k, $\Omega^k CS_{c.c.}(\mathbb{R}^n)$ denote the the space of forms $\alpha \in \Omega^k(\mathbb{R}^{2n})$ such that $\alpha = \sum_{|I|+|J|=k} \alpha_{I,J}\, dx_I \wedge d\xi_J$ with $\alpha_{I,J} \in CS_{c.c}(\mathbb{R}^n)$.*

The cut-off integral can be extended to non integer order symbol valued forms on M.

Definition 9 *Given $\alpha(x,\xi) = \sum_i \psi_i(x)\, \alpha_i(x,\xi) \in \Omega^k CS^{\notin \mathbb{Z}}(M)$ –where as before, (U_i,ψ_i) is a partition of unity subordinated to a trivialising atlas $(U_i, i \in I)$ of M,– we set for any $x \in M$:*

$$\,-\!\!\!\!\!\!\int_{T_x^*M} \alpha(x,\xi) = \sum_i \psi_i(x) \,-\!\!\!\!\!\!\int_{T_x^*U_i} \alpha_i(x,\xi)$$

and

$$\,-\!\!\!\!\!\!\int_{T^*M} \alpha(x,\xi) = \int_M \sum_i \psi_i(x) \,-\!\!\!\!\!\!\int_{T_x^*U_i} \alpha(x,\xi).$$

Remark 8 *This definition makes sense in as far as it is independent of the choice of coordinate chart and partition of unity. Indeed, on each trivialising chart U_i, when $|J| = n$, the cut-off integral $-\!\!\!\!\!\int_{T_x^*U_i} (\alpha_i)_{IJ}\, d\xi_J$ is proportional to $-\!\!\!\!\!\int_{T_x^*U_i} (\alpha_i)_{IJ}\, d\xi_1 \wedge \cdots \wedge d\xi_n$ which is a cut-off integral of an ordinary non integer order symbol. Since cut-off integrals of non integer order classical symbols patch up correctly to build a well-defined cut-off integral on M, the same holds for non integer order classical symbol valued forms.*

1.4 The Wodzicki residue extended to classical symbol valued forms

Let us now extend the Wodzicki residue density from $CS(U)$ to $\Omega CS(U)$. The definition is based on the following straight forward lemma.

Lemma 3 *Let $\rho : T^*U \to S^*U$ denote the radial projection $\rho(x,\xi) = (x, \frac{\xi}{|\xi|})$. A form $\alpha \in \Omega CS(U)$ has order zero if and only if it can be written:*

$$\alpha(x, r\cdot\omega) := \rho^*\bar{\alpha}(x, r\cdot\omega) \wedge dr$$

for some $\bar{\alpha}(x,\omega) \in \Omega(S^*U)$.

In particular, given any $\sigma \in CS(U)$, the top form

$$\alpha_\sigma(x, r\cdot\omega) := \sigma_{-n}(x, r\cdot\omega)\,\rho^* d_S\omega \wedge dr$$

where $d_S\omega$ is the volume form on S^*U, provides an example of zero order symbol valued n-form. More generally, any form $\alpha = \sum_{|I|+|J|=k} \alpha_{I,J} d\,x_I \wedge d\,\xi_J \in \Omega^k\, CS(U)$ with $\alpha_{I,J}$ of order $-|J|$ is a zero order symbol valued k- form and therefore induces a form $\bar\alpha$ on S^*U.

Definition 10 *Let U be an open subset of $\mathbb{R}^n$. Given a zero order symbol valued form $\alpha \in \Omega\, CS(U)$ we set for any $x \in U$:*

$$\mathrm{res}_x(\alpha) := \int_{S_x^*U} \bar\alpha(x, \omega).$$

It extends by 0 to all of $\Omega\, CS(U)$.

Remark 9

- *The Wodzicki residue vanishes on $\Omega^k\, CS(U)$ for $k < n$ since $\alpha \in \Omega^k\, CS(U) \Rightarrow \bar\alpha \in \Omega^{k-1}(S^*U)$ and $\bar\alpha$ has degree $k-1 < n-1$ so that the integral of $\bar\alpha$ on S_x^*U vanishes.*
- *The Wodzicki residue on forms relates to the Wodzicki residue defined on ordinary symbols as integrals on forms relate to integrals on functions. Indeed, given any ordinary classical symbol $\sigma \in CS(U)$, the Wodzicki residue density of the associated symbol valued form α_σ defined above reads*

$$\mathrm{res}_x(\alpha_\sigma) = \int_{S_x^*U} \bar\alpha_\sigma(x, \xi) = \int_{S_x^*U} \sigma_{-n}(x, \omega) d_S\omega = \mathrm{res}_x(\sigma),$$

where $\mathrm{res}_x(\sigma)$ is the ordinary Wodzicki residue density of σ.

Proposition 4 *Let U be an open subset of $\mathbb{R}^n$ and k be a non negative integer. Let $\alpha = \sum_{I,J} \alpha_{IJ}\, dx_I \wedge d\xi_J \in \Omega^k\, CS(U)$. For any $x \in U$ we have:*

$$\mathrm{res}_x(\alpha) = \sum_{|I|=k-n, |J|=n} \mathrm{res}_x(\alpha_{IJ})\, dx_I,$$

where $\mathrm{res}_x(\alpha_{IJ})$ is the ordinary Wodzicki residue density of $\alpha_{IJ} \in CS(U)$.

Proof: It follows from the second part of the above remark, that $\mathrm{res}_x(\sigma d\xi_1 \wedge \wedge \cdots \wedge d\xi_n) = \mathrm{res}_x(\sigma)$ for any $\sigma \in CS(U)$. Applying this to $\sigma = \epsilon_J\, \alpha_{IJ}$ ($|J| = n$) where ϵ_J is the signature of the permutation $i \mapsto j_i$ with $J = \{j_1, \cdots, j_n\}$ yields the result.

In order to generalise this extended Wodzicki residue from an open subset in $\mathbb{R}^n$ to the manifold M, it is useful to caracterise zero order symbol valued forms on M using the Liouville field on T^*M. The group $\mathbb{R}^+$ of positive real numbers acts on T^*M by:

$$\begin{array}{rcl} \mathbb{R}^+ \times T^*M & \to & T^*M \\ (t, (x, \xi)) & \to & f_t(x, \xi) := (x, t\,\xi). \end{array}$$

In local coordinates, the Liouville field:

$$X(x,\xi) := \frac{d}{dt}_{|t=0} f_t(x,\xi)$$

reads

$$X(x,\xi) = \sum_{i=1}^n \xi_i \frac{\partial}{\partial \xi_i}.$$

Lemma 4 *A form $\alpha \in \Omega\, CS(M)$ has zero order whenever*

$$\mathcal{L}_X(\alpha) = 0$$

where $\mathcal{L}_X$ is the Lie derivative in direction X.

Proof: α has zero order whenever

$$\alpha(x, e^t\,\xi) = \alpha(x,\xi) \quad \forall t > 0.$$

Differentiating on either side at $t = 0$ yields the result since

$$L_X \alpha = \frac{d}{dt}_{|t=0} f_t^* \alpha = \frac{d}{dt}_{|t=0} \alpha(x, e^t\,\xi).$$

Remark 10 *This confirms the fact that the requirement that the symbol valued form have vanishing order is covariant, which was to be expected since the order is a covariant concept.*

Definition 11 *Let $\alpha \in \Omega\, CS(M)$ be of zero order or equivalently let it satisfy $\mathcal{L}_X(\alpha) = 0$, where X is the Liouville field defined above. Then, the restriction α_U to a coordinate chart U reads*

$$\alpha_U(x, r\cdot\omega) := \rho^* \bar\alpha(x, r\cdot\omega) \wedge dr$$

for some $\bar\alpha \in \Omega(S^*U)$ where $\bar\alpha(x,\omega)$ is a uniquely defined form on S^*M.
The Wodzicki residue of α_U is defined at a point $x \in U$ by:

$$\mathrm{res}_x(\alpha_U) := \int_{S^*_x M} \bar\alpha_U(x,\omega).$$

Given an atlas on M and a partition of unity $(U_i, \psi_i, i \in I)$ subordinated to it, if $\alpha = \sum_i \psi_i\, \alpha_i$ then

$$\mathrm{res}(\alpha) = \sum_i \int_M \psi_i(x)\, \mathrm{res}_x(\alpha_i),$$

where $\mathrm{res}_x(\alpha_i)$ is the Wodzicki density of the restriction α_i to U_i.

1.5 Integrals of holomorphic families of symbol valued forms

Recall that given an open subset $U \subset \mathbb{R}^n$ (resp. an n-dimensional manifold M), for any real number m the class $CS_0^m(U)$ ($CS^m(M)$) of classical symbols of order m with compact support on U (resp. of classical symbols of order m) can be equipped with a natural Fréchet topology so that $CS(U) = \bigcup_{m\in \mathbb{R}} CS^m(U)$ (and hence $CS(M) = \bigcup_{m\in \mathbb{R}} CS^m(U)$) comes equipped with an inductive limit Fréchet topology. We first recall the notion of holomorphic regularisation (see e.g. [P] for a review of various regularisations):

Definition 12 *A holomorphic regularisation procedure on* $CS(U)$ *is a map*

$$\begin{array}{rcl}\mathcal{R}: CS(U) & \to & \text{Hol}\,(CS(U)) \\ \sigma & \mapsto & \sigma(z)\end{array}$$

where $\text{Hol}\,(CS(U))$ *is the algebra of holomorphic maps with values in* $CS(U)$*, such that*

1. $\sigma(0)=\sigma$,
2. $\sigma(z)$ *has holomorphic order* $\alpha(z)$ *(in particular,* $\alpha(0)$ *is equal to the order of* σ*) such that* $\alpha'(0)\neq 0$.
   Taking M *instead of* U *leads to a holomorphic regularisation on* $CS(M)$.

Examples of holomorphic regularisations are the well known Riesz regularisation $\sigma \mapsto \sigma(z)(x,\xi) := \sigma(x,\xi)\cdot|\xi|^{-z}$ and generalisations of the type $\sigma \mapsto \sigma(z)(x,\xi) := H(z)\cdot\sigma(x,\xi)\cdot|\xi|^{-z}$ where H is a holomorphic function such that $H(0)=1$. The latter include dimensional regularisation (see [P]).

Proposition 5 *[G], [KV], [L] Given a holomorphic regularisation procedure* $\mathcal{R}: \sigma\mapsto\sigma(z)$ *on* $CS(U)$ *and any symbol* $\sigma\in CS(U)$ *(resp.* $CS(M)$*), the map* $z\mapsto ⨍_{T_x^*U} d\xi\,\sigma(z)$ *(resp.* $z\mapsto ⨍_{T^*M} dx\,d\xi\,\sigma(z)$*) is meromorphic with simple poles at points in* $\alpha^{-1}([-n,+\infty[\cap\mathbb{Z})$ *where* α *is the order of* $\sigma(z)$*. Moreover for any* $x\in U$

$$\text{Res}_{z=0} ⨍_{T_x^*U}\sigma(z)(x,\xi)\,d\xi = -\frac{1}{\alpha'(0)}\text{res}_x(\sigma(0)),$$

respectively

$$\text{Res}_{z=0} ⨍_{T^*M}\sigma(z)(x,\xi)\,d\xi = -\frac{1}{\alpha'(0)}\text{res}(\sigma(0)).$$

The finite part when $z\to 0$ *is defined by:*

$$\begin{aligned}\int_{T_x^*U}^{\mathcal{R}}\sigma(x,\xi)\,d\xi &:= \text{fp}_{z=0} ⨍_{T_x^*U}\sigma(z)(x,\xi)\,d\xi \\ &:= \lim_{z\to 0}\left(⨍_{T_x^*U} d\xi\,\sigma(z)(x,\xi) - \frac{1}{z}\text{Res}_{z=0} ⨍_{T_x^*U} d\xi\,\sigma(z)(x,\xi)\right),\end{aligned}$$

respectively

$$\begin{aligned}\int_{T^*M}^{\mathcal{R}}\sigma(x,\xi)\,dx\,d\xi &:= \text{fp}_{z=0}\int_M dx ⨍_{T_x^*M}\sigma(z)(x,\xi)\,d\xi \\ &:= \lim_{z\to 0}\int_M dx\left(⨍_{T_x^*M} d\xi\sigma(z)(x,\xi) - \frac{1}{z}\text{Res}_{z=0} ⨍_{T_x^*M} d\xi\,\sigma(z)(x,\xi)\right).\end{aligned}$$

Proof: By linearity and using a partition of unity, we can restrict ourselves to the case $\sigma\in CS(U)$. We identify T_x^*U with $\mathbb{R}^n$ using a coordinate chart. From equation (4) we have

$$⨍_{\mathbb{R}^n}\sigma(z)(x,\xi)\,d\xi = \int_{B(0,1)}\sigma(z)(x,\xi)\,d\xi$$

$$
\begin{aligned}
& - \sum_{i=0,\, \alpha(z)-i+n\neq 0}^{K_N} \frac{1}{\alpha(z)-i+n} \int_{S(0,1)} \sigma_i(z)(x,\xi)\, d\xi \\
& + \int_{\mathbb{R}^n} \sigma_{(N)}(z)(\xi)\, d\xi \\
& = \int_{B(0,1)} \sigma(z)(x,\xi)\, d\xi \\
& - \sum_{i=0,\, \alpha(z)-i+n\neq 0}^{K_N} \frac{1}{\alpha(0)-i+n+\alpha'(0)\, z+o(z)} \int_{S(0,1)} \sigma_i(z)(x,\xi)\, d\xi \\
& + \int_{\mathbb{R}^n} \sigma_{(N)}(z)(x,\xi)\, d\xi,
\end{aligned}
$$

where we have written $\alpha(z) = \alpha(0) + \alpha'(0) + o(z)$. As a consequence, we have that:

$$
\begin{aligned}
\mathrm{Res}_{z=0} {-\!\!\!\!\!\!\int}_{\mathbb{R}^n} \sigma(z)(x,\xi)\, d\xi &= \mathrm{Res}_{z=0} \int_{B(0,1)} \sigma(z)(x,\xi)\, d\xi \\
&- \mathrm{Res}_{z=0} \sum_{i=0}^{K_N} \frac{1}{\alpha(0)-i+n+\alpha'(0)\, z+o(z)} \int_{S(0,1)} \sigma_i(z)(x,\xi)\, d\xi \\
&+ \mathrm{Res}_{z=0} \int_{\mathbb{R}^n} \sigma_{(N)}(z)(x,\xi)\, d\xi \\
&= -\frac{1}{\alpha'(0)} \int_{S(0,1)} \sigma_{-n}(0)(x,\xi)\, d\xi \\
&= -\frac{1}{\alpha'(0)} \mathrm{res}_x(\sigma(0)).
\end{aligned}
$$

This result extends to classical symbol valued forms.

Definition 13 *A holomorphic regularisation procedure on $\Omega\, CS(U)$ is a map*

$$
\begin{aligned}
\mathcal{R} : \Omega\, CS(U) &\to \Omega\, \mathrm{Hol}\, (CS(U)) \\
\omega &\mapsto \omega(z)
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega\, \mathrm{Hol}\, (CS(U)) := & \{ z \mapsto \omega(z) = \sum_{I,J} \omega_{IJ} dx_I \wedge d\xi_J \in \Omega\, CS(U), \\
& z \mapsto \omega_{IJ}(z) \quad \text{lies} \quad \text{in} \quad \mathrm{Hol}\, CS(U) \\
& \text{for} \quad \text{all} \quad \text{multi}-\text{indices} \quad I, J \}
\end{aligned}
$$

and

1. *$\omega(0) = \omega$,*
2. *$\omega(z)$ has holomorphic order $\alpha(z)$ (in particular, $\alpha(0)$ is equal to the order of ω) such that $\alpha'(0) \neq 0$.*
   Replacing U by M defines a holomorphic regularisation on $\Omega\, CS(M)$.

Remark 11 *Clearly, any holomorphic regularisation* $\mathcal{R}$ *on* $CS(U)$ *induces one on* $\Omega\, CS(U)$ *setting:*

$$\mathcal{R}(\omega) = \sum_{I,J} \mathcal{R}(\omega_{IJ}) dx_I \wedge d\xi_J.$$

Theorem 1 *Given a holomorphic regularisation procedure* $\mathcal{R} : \omega \mapsto \omega(z)$ *on* $\Omega\, CS(U)$ *(resp. on* $\Omega\, CS(M)$*) induced by a regularisation* $\mathcal{R} : \sigma \mapsto \sigma(z)$ *on* $CS(U)$ *(resp. on* $CS(M)$*) and any symbol valued form* $\omega \in \Omega\, CS(U)$ *(resp.* $\Omega\, CS(M)$*), the map* $z \mapsto ⨍_{T_x^*U} \omega(z)$ *(resp.* $z \mapsto ⨍_{T^*M} \omega(z)$*) is meromorphic with simple poles at points in* $\alpha^{-1}([-n, +\infty[\cap \mathbb{Z})$ *where* α *is the order of* $\omega(z)$*. Moreover for any* $x \in U$

$$\mathrm{Res}_{z=0} ⨍_{T_x^*U} \omega(z)(x,\xi) = -\frac{1}{\alpha'(0)} \mathrm{res}_x(\omega(0)),$$

respectively

$$\mathrm{Res}_{z=0} ⨍_{T^*M} \omega(z)(x,\xi) = -\frac{1}{\alpha'(0)} \mathrm{res}(\omega(0)).$$

The finite part when $z \to 0$ *is defined by:*

$$\begin{aligned}
\int_{T_x^*U}^{\mathcal{R}} \omega(x,\xi) &:= \mathrm{fp}_{z=0} ⨍_{T_x^*U} \omega(z)(x,\xi) \\
&:= \lim_{z\to 0} \left(⨍_{T_x^*U} \omega(z)(x,\xi) - \frac{1}{z} \mathrm{Res}_{z=0} ⨍_{T_x^*U} \omega(z)(x,\xi) \right),
\end{aligned}$$

respectively

$$\begin{aligned}
\int_{T^*M}^{\mathcal{R}} \omega(x,\xi) &:= \mathrm{fp}_{z=0} ⨍_{T^*M} \omega(z)(x,\xi) \\
&:= \lim_{z\to 0} \left(⨍_{T^*M} \omega(z)(x,\xi) - \frac{1}{z} \mathrm{Res}_{z=0} ⨍_{T^*M} \omega(z)(x,\xi) \right),
\end{aligned}$$

Proof: The result follows from applying Proposition 5 to each component $\omega_{IJ}(z)$ of the form $\omega(z) = \sum_{IJ} \omega_{IJ}(z) dx_I \wedge d\xi_J$. The symbol valued form $\omega_{IJ}(z)$ has order $\alpha_{IJ}(z) = \alpha(z) - |J|$ so that $\alpha'_{IJ}(0) = \alpha'(0)$. Since $z \mapsto ⨍_{T_x^*U} \omega_{IJ}(z)$ is meromorphic with simple poles so is $z \mapsto ⨍_{T_x^*U} \omega(z)$ and we have

$$\begin{aligned}
\mathrm{Res}_{z=0} ⨍_{T_x^*U} \omega(z)(x,\xi) &= \sum_{IJ} \mathrm{Res}_{z=0} ⨍_{T_x^*U} \omega_{IJ}(z)(x,\xi)\, dx_I \wedge d\xi_J \\
&= -\sum_{IJ} \frac{1}{\alpha'_{IJ}(0)} \mathrm{res}_x(\omega_{IJ}(0))\, dx_I \wedge d\xi_J \\
&\qquad \text{by Proposition 5} \\
&= -\frac{1}{\alpha'(0)} \sum_{IJ} \mathrm{res}_x(\omega_{IJ}(0))\, dx_I \wedge d\xi_J \\
&= -\frac{1}{\alpha'(0)} \mathrm{res}_x(\omega(0)),
\end{aligned}$$

where we have used Proposition 4 in the last equality.

1.6 Stokes' property for cut-off integrals and the Wodzicki residue

$CS(U)$ is equipped with the left product on symbols:

$$\sigma \cdot_L \sigma' := \sum_{|\alpha|} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma \cdot \partial_x^\alpha \sigma'$$

which lies in $CS^{m+m'}(U)$ if $\sigma \in CS^m(U)$ and $\sigma' \in CS^{m'}(U)$. $(CS(U), \cdot_L)$ is an algebra [1] on which the Wodzicki residue defines a trace [W], [K]:

$$\mathrm{res}(\sigma \cdot_L \sigma') = \mathrm{res}(\sigma' \cdot_L \sigma) \quad \forall \sigma, \sigma' \in CS(U).$$

The left product $\cdot_L$ on symbols extends to forms. Indeed for two classical symbol valued forms $\alpha = \sum_{i\in I} \psi_i \alpha_i = \sum_{i\in I} \psi_i \sum_I \alpha_{I,J,i} d\xi_I^i \wedge d\xi_J^i$ and $\beta = \sum_{i\in I} \psi_i \beta_i = \sum_{i\in I} \psi_i \sum_{K,L} \alpha_{K,L,i} d\xi_K^i \wedge d\xi_L^i$, the product

$$\alpha \wedge_L \beta := \sum_{i\in I} \psi_i \sum_{I,J} \sum_{K,L} (\alpha_{I,J,i} \cdot_L \beta_{K,L,i})\, dx_I^i \wedge d\xi_J^i \wedge dx_K^i \wedge d\xi_L^i$$

is independent of the choice of atlas and subordinated partition of unity $(U_i, \psi_i, i \in I)$. $(\Omega\, CS(M), \wedge_L)$is an algebra. [2]

Theorem 2 *For any $\beta \in \Omega\, CS(M)$ with order 0 (or equivalently such that $\mathcal{L}_X(\beta) = 0$ where X is the Liouville field), then*

$$\mathrm{res}\,(d\beta) = 0.$$

The triple $(\Omega\, CS(M), d, \mathrm{res})$ yields a cycle which we refer to as the Wodzicki residue cycle. The associated n-character (see Appendix A) reads:

$$\chi_n^{res}(\sigma_0, \cdots, \sigma_n) = \mathrm{res}\,(\sigma_0 \cdot_L d\,\sigma_1 \wedge_L \cdots \wedge_L \sigma_n), \quad \forall \sigma_0, \cdots, \sigma_n \in CS(M).$$

Proof: The first part of the theorem follows from Stokes' property of ordinary integrals. First observe that if β is of order 0 then so is $d\,\beta$ since $(d\beta)_j = d\,\beta_j$ (this can also be seen from the fact that $\mathcal{L}_X(d\beta) = d\mathcal{L}_X(\beta) = 0$) so that $\mathrm{res}(d\beta)$ is defined by integration on S^*M. But,

$$\mathrm{res}(d\,\beta) = \int_{S^*M} d\,\beta = 0$$

since S^*M is boundaryless. This says that the linear form res on $\Omega\, CS(M)$ is closed.

The fact that the ordinary Wodzicki residue defines a trace on $CS(M)$ then implies that $\mathrm{res}\,(\alpha_{I,J,i} \cdot_L (\psi_i \cdot \beta_{K,L,i})) = \mathrm{res}\,((\psi_i \cdot \beta_{K,L,i}) \cdot_L \alpha_{I,J,i})$ from which it follows that:

$$\mathrm{res}(\alpha \wedge_L \beta) = (-1)^{|\alpha|\cdot|\beta|} \mathrm{res}(\beta \wedge_L \alpha)$$

so that $(\Omega\, CS(M), d, \mathrm{res})$ defines a cycle.

Similarly to ordinary integrals, cut-off integrals on forms satisfy Stokes' property and cut-off integrals on symbols satisfy an integration by parts property.

[1] Strictly speaking, only the integer order classical symbols form an algebra, by $CS(U)$ we actually mean (as it is commonly done in the literature) the algebra generated by symbols in $CS(U)$

[2] Here again, strictly speaking, only the integer order symbol valued forms form an algebra, but abusing notations, we write $\Omega\, CS(M)$ for the algebra generated by $CS(M)$-valued forms.

Theorem 3 *Let U be an open subset of $\mathbb{R}^n$ and let $\beta \in \Omega^{n-1}\, CS^{\notin \mathbb{Z}}(U)$ be a symbol valued form with compact support in U. Then*

$$-\!\!\!\!\!\!\int_{T^*U} d\beta = 0.$$

In particular, let $\sigma \in CS^{\notin \mathbb{Z}}(U)$ be a symbol valued form with compact support in U. Then

$$-\!\!\!\!\!\!\int_{T^*U} \frac{\partial}{\partial \xi_i}\sigma(x,\xi)\, dx\, d\xi = 0 \quad \forall i \in \{1, \cdots, n\}.$$

Similarly, for a closed manifold M, given any $\beta \in \Omega^{n-1}\, CS^{\notin \mathbb{Z}}(M)$ we have,

$$-\!\!\!\!\!\!\int_{T^*M} d\beta = 0.$$

Remark 12

- *Integration by parts formula in ξ as stated in this theorem combined with the usual integration by parts formula in x yields the cyclicity of the canonical trace TR on non integer order symbols introduced by Kontsevich and Vishik in[KV] defined by*

$$\mathrm{TR}(A) := \frac{1}{(2\pi)^n} -\!\!\!\!\!\!\int_{T^*M} \sigma_A(x,\xi)\, dx\, d\xi,$$

where σ_A is the (local) symbol of A. Indeed, taking the product of two operators boils down to taking the left product of their local symbols, which involves partial differentiation both in x and in ξ so that the proof of the cyclicity of TR *boils down to integrating by parts in both x and ξ.*

- *The integration by parts formula also yields translation invariance of cut-off integrals on non integer order symbols. Indeed, using a Taylor expansion $\eta \mapsto \sigma(\xi+\eta)$ in η at 0 yields, for any $x \in U$, the existence of some $\theta \in]0,1[$ such that:*

$$\begin{aligned} -\!\!\!\!\!\!\int_{T^*_xU} \sigma(x,\xi+\eta)\, d\xi &= \sum_{|\alpha|\leq K} -\!\!\!\!\!\!\int_{T^*_xU} d\xi\, \frac{D^\alpha_\xi \sigma(x,\xi)}{\alpha!}\eta^\alpha + \sum_{|\alpha|=K} -\!\!\!\!\!\!\int_{T^*_xU} d\xi\, \frac{D^\alpha_\xi \sigma(x,\xi+\theta\eta)}{\alpha!}\eta^\alpha \\ &= -\!\!\!\!\!\!\int_{T^*_xU} d\xi\, \sigma(x,\xi), \end{aligned}$$

where we have used that if σ has non integer symbol then so has $D^\alpha\sigma$ so that all the terms corresponding to $|\alpha| \neq 0$ vanish by the integration by parts formula as a result of which we are left with the $|\alpha| = 0$ term.

Proof: It is sufficient to prove the statement for an open subset U. Indeed, using a partition of unity on M, by linearity of the cut-off integral, we can restrict ourselves from $\beta \in \Omega\, CS(M)$ to $\beta \in \Omega\, CS(U)$ with compact support in U. Moreover, the integration by parts formula on T^*U easily follows from Stokes' formula on T^*U as follows. For any $x \in U$ we have:

$$\begin{aligned} -\!\!\!\!\!\!\int_{T^*_xU} \frac{\partial}{\partial \xi_i}\sigma(x,\xi)\, d\xi &= (-1)^{i-1} -\!\!\!\!\!\!\int_{T^*_xU} d\left(\sigma(x,\xi)\, d\xi_1 \wedge \cdots \wedge \hat{d\xi_i} \wedge \cdots \wedge d\xi_n\right) \\ &= 0 \end{aligned}$$

by Stokes' applied to $\beta := \sigma(x,\xi)\, d\xi_1 \wedge \cdots \wedge \hat{d\xi_i} \wedge \cdots \wedge d\xi_n$ where we have left out $d\xi_i$ in the wedge product. Integrating on U then yields the result.
We are therefore left with the proof of $-\!\!\!\!\!\int_{T^*U} d\beta = 0$. In local coordinates on T^*U, the $n-1$ form reads $\beta(x,\xi) = \sum_{I,J\subset\{1,\cdots,n\},|I|+|J|=2n-1} \beta_{I,J}(x,\xi)\, dx_I \wedge d\xi_J$ with $\beta_{I,J} \in CS^{\notin \mathbb{Z}}(U)$ so that, letting $B^*(0,R)$, resp. $S^*(0,R)$ be respectively the ball in the cotangent bundle of radius R centered at the origin, and the sphere in the cotangent bundle of radius R centered at the origin, we have

$$\begin{aligned}
-\!\!\!\!\!\!\int_{T^*U} d\beta(x,\xi) &= \sum_{I,J} -\!\!\!\!\!\!\int_{T^*U} d\,\left(\beta_{I,J}(x,\xi)\, dx_I \wedge d\xi_J\right) \\
&= \sum_{I,J} \mathrm{fp}_{R\to\infty} \int_{B^*(0,R)} d\,\left(\beta_{I,J}(x,\xi)\, dx_I \wedge d\xi_J\right) \\
&= \sum_{I,J} \mathrm{fp}_{R\to\infty} \int_{S^*(0,R)} \beta_{I,J}(x,\xi)\, dx_I \wedge d\xi_J \\
&\quad \text{using Stokes}' \text{ property for ordinary integrals} \\
&= \sum_{I,J} \sum_{j_{I,J}=0}^{K_{N_{I,J}}} \mathrm{fp}_{R\to\infty} \int_{S^*(0,R)} \chi_{I,J}(\xi)\, \beta_{I,J,j_{I,J}}(x,\xi)\, dx_I \wedge d\xi_J \\
&+ \lim_{R\to\infty} \int_{S^*(0,R)} \beta_{I,J\,(N_{I,J})} \\
&\quad \text{where} \quad \beta_{I,J} = \sum_{j_{I,J}=0}^{K_{N_{I,J}}} \chi_{I,J}\, \beta_{I,J,j_{I,J}} + \beta_{I,J\,(N_{I,J})} \\
&= \sum_{I,J} \sum_{j_{I,J}=0}^{K_{N_{I,J}}} \mathrm{fp}_{R\to\infty} \int_{S^*(0,R)} \chi_{I,J}(\xi)\, \beta_{I,J\,j_{I,J}}(x,\xi)\, dx_I \wedge d\xi_J \\
&\quad \text{since} \quad \lim_{|\xi|\to\infty} \beta_{I,J\,(N_{I,J})}(x,\xi) = 0 \\
&= \sum_{I,J} \sum_{j_{I,J}=0}^{K_{N_{I,J}}} \mathrm{fp}_{R\to\infty} \int_{S^*(0,R)} \beta_{I,J\,j_{I,J}}(x,\xi)\, dx_I \wedge d\xi_J \\
&\quad \text{since} \quad \chi_{I,J} \quad \text{equals} \quad 1 \quad \text{outside} \quad B^*(0,1) \\
&= \sum_{I,J} \sum_{j_{I,J}=0}^{K_{N_{I,J}}} \mathrm{fp}_{R\to\infty} R^{m_{I,J}-j_{I,J}+n-1} \int_{S^*(0,1)} \beta_{I,J\,j_{I,J}}(x,\xi)\, dx_I \wedge d\xi_J \\
&= 0 \quad \text{since} \quad m_{I,J} - j_{I,J} + n - 1 \neq 0,
\end{aligned}$$

where $m_{I,J} \notin \mathbb{Z}$ is the order of $\beta_{I,J}$.

Corollary 1 *let U be an open subset of $\mathbb{R}^n$. Given any holomorphic regularisation $\mathcal{R} : \omega \mapsto \omega(z)$ on $\Omega\, CS(U)$ (resp. $\Omega\, CS(M)$) induced by a holomorphic regularisation on $CS(U)$ (resp. $CS(M)$) and given any $\beta \in \Omega^{n-1}\, CS(U)$ with compact support (resp. $\beta \in \Omega^{n-1}\, CS(M)$), then*

$$\mathrm{fp}_{z=0} -\!\!\!\!\!\!\int_{T^*U} d\,(\beta(z)) = 0,$$

respectively

$$\mathrm{fp}_{z=0} -\!\!\!\!\!\!\int_{T^*M} d\,(\beta(z)) = 0.$$

In particular, this yields an integration by parts formula. For any $\sigma \in CS(U)$ with compact support,

$$\mathrm{fp}_{z=0} -\!\!\!\!\!\!\int_{T^*U} \frac{\partial}{\partial\,\xi_i}\,(\sigma(z))\; dx\, d\xi = 0 \quad \forall i \in \{1, \cdots, n\}.$$

Similarly,

$$\mathrm{Res}_{z=0} -\!\!\!\!\!\!\int_{T^*U} d\,(\beta(z)) = -\frac{1}{b'(0)}\mathrm{res}(d\beta) = 0,$$

respectively

$$\mathrm{Res}_{z=0} -\!\!\!\!\!\!\int_{T^*M} d\,(\beta(z)) = -\frac{1}{b'(0)}\mathrm{res}(d\beta) = 0$$

where $b(z)$ is the order of $\beta(z)$.

Remark 13

- *The first part of this corollary can be summarised by the more compact statements*

$$-\!\!\!\!\!\!\int_{T^*U}^{\mathcal{R}} d\beta = 0, \quad -\!\!\!\!\!\!\int_{T^*U}^{\mathcal{R}} \frac{\partial}{\partial\,\xi_i}\sigma(x,\xi)dx\, d\xi = 0$$

  by which it is understood that the regularisation procedure $\mathcal{R}$ applies to β and σ before differentiation and not to the readily differentiated symbols $d\,\beta$ or to $\frac{\partial}{\partial\,\xi_i}\sigma$.

- *It follows from translation invariance for cut-off integrals that $-\!\!\!\!\!\int_{T_x^*U} \sigma(x, \xi + \eta)(z)\, d\xi = -\!\!\!\!\!\int_{T_x^*U} \sigma(x,\xi)(z)\, d\xi$ outside the set of points z_0 for which $\sigma(z)$ has integer order. Taking finite parts yields translation invariance for regularised integrals:*

$$-\!\!\!\!\!\!\int_{T_x^*U}^{\mathcal{R}} \sigma(x, \xi + \eta)\, d\xi = -\!\!\!\!\!\!\int_{T_x^*U}^{\mathcal{R}} \sigma(x,\xi)\, d\xi.$$

  By this we mean that the regularised symbol $\mathcal{R}(\sigma)$ is translated before taking finite parts.

- *When $\mathcal{R}$ corresponds to dimensional regularisation $\sigma(x,\xi) \mapsto H(z)\sigma(x,\xi)\cdot|\xi|^{-z}$, these two remarks justify the use of integration by parts and translation invarinace in computations involving dimensional regularisation in physics. Namely, one can apply integration by parts or translation invariance after having "complexified the dimension $d \mapsto d(z) := d - z$" (or after having "regularised" $\sigma \mapsto \sigma(z)$ in our terminology), and only then can one take finite parts letting the dimensional parameter $d(z)$ tend to d. However, integration by parts or translation invariance cannot directly be applied to finite parts or in integer dimensions.*

- *The second part of this corollary yields back Stokes' property for the Wodzicki residue extended to forms.*

Proof: The proof easily follows from applying Stokes' formula to $\beta(z)$ or integration by parts formula to $\frac{\partial}{\partial\,\xi_i}\sigma(z)$ outside the discrete set of points z_0 for which $b(z)$, resp. the order of $\sigma(z)$ is an integer. Taking either finite parts or complex residues then yields the result.

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2 From Stokes' formula to cyclic Hochschild cocycles associated with star products

2.1 Trace forms on a unital algebra $(\mathcal{A}, \star)$

Let $(\mathcal{A}, \star)$ be an associative algebra over some ring R with unit 1 and let $\mathcal{N}$ be an R-submodule of $\mathcal{A}$. Let $C^n(\mathcal{N}, R)$ denote the space of R-multilinear valued forms on $\mathcal{N}^{\otimes n+1}$. When $\mathcal{N} = \mathcal{A}$, $C^n(\mathcal{N}, R)$ corresponds to the space of n-cochains on $\mathcal{A}$.

Definition 14 *We introduce the antisymmetrization maps on $C^\bullet(\mathcal{N})$ defined by:*

$$\begin{array}{rcl} \mathrm{Alt}: C^n(\mathcal{N}, \mathbb{C}[[\nu]]) & \to & C^n(\mathcal{N}, \mathbb{C}[[\nu]]) \\ \chi & \mapsto & \mathrm{Alt}\,\chi(a_0, \cdots, a_n) := \frac{1}{n!}\sum_{\sigma\in\Sigma_n} \epsilon(\sigma)\chi\left(a_0, a_{\sigma(1)}, \cdots, a_{\sigma(n)}\right) \end{array}$$

which corresponds to antisymmetrization on all but the first variable. [3]

To a linear map $T : \mathcal{N} \to R$ we associate *trace forms*

$$\Psi_n(a_0, \cdots, a_n) := T\left(a_0 \star a_1 \star a_2 \star \cdots \star a_{n-1} \star a_n\right) \tag{5}$$

which are well-defined provided $a_0 \star a_1 \star a_2 \star \cdots \star a_{n-1} \star a_n$ lies in $\mathcal{N}$. When $\mathcal{N} = \mathcal{A}$ this yields a n-cochain on $\mathcal{A}$. We focus on antisymmetrized trace forms:

$$\mathrm{Alt}\,\Psi_{2k}(a_0, \cdots, a_{2k}) = T\left(a_0 \star [a_1, \cdots, a_{2k}]_\star\right)$$

Since 1 is a unit element in $\mathcal{A}$ we have:

$$\mathrm{Alt}\,\Psi_{2k}(1, a_0, \cdots, a_{2k}) = \mathcal{A}lt\,\Psi_{2k-1}(a_1, \cdots, a_{2k}) = T\left([a_1, \cdots, a_{2k}]_\star\right)$$

where, following the notations of [HH],we have set:

$$[a_1, \cdots, a_n]_\star := \frac{1}{n!}\sum_{\sigma\in\Sigma_n} \epsilon(\sigma) a_{\sigma(1)} \star \cdots \star a_{\sigma(n)}.$$

The following proposition relates the B-boundary of the antisymmetrized trace forms with Helton and Howe's fundamental trace forms [HH] which correspond to $\mathrm{Alt}\,\Psi_{2k}(1, a_1, \cdots, a_{2k}) = T([a_1, \cdots, a_{2k}]_\star)$ in our setting.

Proposition 6 *For any $a_1, \cdots, a_{2k} \in \mathcal{A}$*

$$(B_0\mathrm{Alt}\,\Psi_{2k})(a_1, \cdots, a_{2k}) = T\left([a_1, \cdots, a_{2k}]_\star\right)$$

and

$$(B\mathrm{Alt}\,\Psi_{2k})(a_1, \cdots, a_{2k}) = 2k \cdot T\left([a_1, \cdots, a_{2k}]_\star\right)$$

[3] In the course of the paper we shall also use the following "total antisymmetrisation" denoted by a calligraphic $\mathcal{A}lt$:

$$\begin{array}{rcl} \mathcal{A}lt: C^n(\mathcal{N}, \mathbb{C}[[\nu]]) & \to & C^n(\mathcal{N}, \mathbb{C}[[\nu]]) \\ \chi & \mapsto & \mathcal{A}lt\,\chi(a_0, \cdots, a_n) := \frac{1}{(n+1)!}\sum_{\sigma\in\Sigma_{n+1}} \epsilon(\sigma)\chi\left(a_{\sigma(0)}, \cdots, a_{\sigma(n)}\right). \end{array}$$

Here Σ_n denotes the permutation group on n elements and $\epsilon(\sigma)$ the signature of a permutation σ.

are proportional to Helton and Howe's trace forms. Moreover,

$$T\,\text{tracial} \Leftrightarrow \mathcal{A}lt\,T(a_1,\cdots,a_{2k}) = 0\,\forall k \in \mathbb{N} \Leftrightarrow B\,\text{Alt}\,\Psi_{2k} = 0 \quad \forall k \in N.$$

Here B_0 and B are operators on cochains the definitions of which are recalled in Appendix A.

Proof: The first part of the proposition follows from the fact that 1 is a unit for the product $\star$:

$$\begin{aligned}(B_0\,\text{Alt}\,\Psi_{2k})(a_1,\cdots,a_{2k}) &= \text{Alt}\,\Psi_{2k}(1,a_1,\cdots,a_{2k}) + \text{Alt}\,\Psi(a_1,\cdots,a_{2k},1)\\ &= T(1\star[a_1,\cdots,a_{2k}]_\star) + T(a_1,[a_2,\cdots,a_{2k},1]_\star)\\ &= T([a_1,\cdots,a_{2k}]_\star) + \frac{1}{2^k}T\left(a_1,[[a_2,a_3]_\star\cdots,[a_{2k},1]_\star]_\star\right)\\ &= T([a_1,\cdots,a_{2k}]_\star).\end{aligned}$$

As for second part of the proposition, all we need to check is the equivalence

$$T\,\text{tracial} \Leftrightarrow \mathcal{A}lt\,T(a_1,\cdots,a_{2k}) = 0 \quad \forall k \in \mathbb{N}.$$

The implication from right to left is easily obtained choosing $k = 1$. The other implication follows from an observation made by Helton and Howe [HH], namely that:

$$[a_1,\cdots,a_{2k}]_\star = \frac{1}{2}\left([a_1,[a_2,\cdots,a_{2k}]_\star]_\star - [a_2,[a_1,\cdots,a_{2k}]_\star]_\star - \cdots - [a_{2k},[a_1,\cdots,a_{2k-1}]_\star]_\star\right).$$

Applying T then yields the result since T vanishes on $\star$-brackets.

Proposition 7

$$T\,\text{tracial} \Leftrightarrow b_\star\Psi_{2k} = 0 \quad \forall k \in \mathbb{N}\cup\{0\}$$

where $b_\star$ is the Hochschild coboundary operator associated with the product $\star$ as defined in Appendix A.

Proof: $b_\star\Psi_0 = 0 \Rightarrow T([a,b]_\star) = 0$ which yields the implication from right to left. The implication form left to right also follows from a straightforward computation using the cyclicity of T w.r. to $\star$:

$$\begin{aligned}&b_\star\Psi_{2k}(a_0,a_1,\ldots,a_{2k+1}) = T((a_0\star a_1)\star\cdots a_{2k+1}) - T(a_0\star(a_1\star a_2)\star\cdots a_{2k+1})\\ \cdots\ &+T(a_0\star\cdots\star(a_{2i-2}\star a_{2i-1})\star\cdots\star a_{2k+1}) - T(a_0\star\cdots\star(a_{2i-1}\star a_{2i})\star\cdots\star a_{2k+1}) +\\ \cdots\ &+T(a_0\star a_1\star\cdots\star(a_{2k}\star a_{2k+1})) - T((a_{2k+1}\star a_0)\star a_1\cdots\star a_{2k})\\ =\ &0.\end{aligned}$$

Remark 14 *It follows from the above that when T is tracial, then* $\text{Alt}\,\Psi_{2k}$ *is cyclic and* Ψ_{2k} *is $b_\star$-closed. But one does not expect* $\text{Alt}\,\Psi_{2k}$ *to be a cyclic $b_\star$-cocycle in general.*

2.2 Trace forms on a deformed algebra

Let now $(A,\cdot)$ be an algebra over $\mathbb{C}$ and $\mathcal{A} := A[[\nu]]$ so that elements of $\mathcal{A}$ are formal power series in ν, $a = \sum_{k=0}^{\infty} a_k\nu^k$, $a_k \in A$. $\mathcal{A}$ is equipped with a star-product $\star$ which makes $(\mathcal{A},\star)$ an associative algebra.

With the notations of the previous section, we can set

1. $R = \mathbb{C}[[\nu]]$ and see $\mathcal{A}$ as a $\mathbb{C}[[\nu]]$-algebra,
2. $R = \mathbb{C}$ in which case $\mathcal{A}$ is seen as a complex algebra.

If $N \subset A$ is a complex subspace of A then $\mathcal{N} = N[[\nu]] \subset \mathcal{A} = A[[\nu]]$ can be seen in the first case as a $\mathbb{C}[[\nu]]$-submodule of $\mathcal{A}$ and in the second case as a complex submodule of $\mathcal{A}$. We introduce the following terminology:

Definition 15 *Let $T : \mathcal{N} \to \mathcal{R}$ be an $\mathcal{R}$ linear map. The traciality of T w.r. to $\star$ i.e.*

$$T([a,b]_\star) = 0 \quad \forall a, b \in \mathcal{N} \quad \text{s.t.} \quad [a,b]_\star := a \star b - b \star a \in \mathcal{N} \tag{6}$$

is refered to as

1. *strong closedness of the star product w.r. to T if $\mathcal{R} = \mathbb{C}[[\nu]]$,*
2. *closedness of the star product w.r. to T if $\mathcal{R} = \mathbb{C}$.*

When $\mathcal{R} = \mathbb{C}[[\nu]]$, the map T can be written:

$$\begin{aligned} T : \mathcal{N} &\to \mathbb{C}[[\nu]] \\ \sum_{k=0}^{\infty} a_k \nu^k &\mapsto \sum_{k=0}^{\infty} T_k(a) \nu^k, \end{aligned}$$

where T_k is a $\mathbb{C}$-linear map on $\mathcal{N}$ for each non negative integer k. Clearly, if $\star$ is strongly closed w.r. to T then it is closed w.r. to any of the projections T_k, so that we recove the usual concept of strong closedness. Taking $T = T_l$ for a given l coincides with the usual concept of closedness of a star product associated with a symplectic form on a $2l$ dimensional manifold.

Proposition 8 *The following statements are equivalent:*

1. *$\star$ is strongly closed w.r. to T*
2. *$b_\star \Psi_{2k} = 0 \, \forall \, k \in \mathbb{N} \cup \{0\}$*
3. *$B \, \mathrm{Alt} \, \Psi_{2k} = 0 \, \forall \, k \in \mathbb{N}$.*

Remark 15 *One does not expect $\mathrm{Alt}\,\Psi_{2k}$ to be $b_\star$- closed. For $k = 1$ for example, we have $b_\star \mathrm{Alt}\Psi_2(a_0, a_1 a_2, a_3) = T([a_0, a_2]_\star [a_1, a_3]_\star)$. Taking the cooefficient of ν^2 in the formal power series expansion yields $b_\star \mathrm{Alt}\Psi_2(a_0, a_1 a_2, a_3)_{[2]} = T(\{a_0, a_2\} \cdot \{a_1, a_3\})$. Setting $a_0 = a_1, a_2 = a_3$, $b_\star$- closedness of $\mathrm{Alt}\,\Psi_2$ would lead to $T(\{a, b\}^2) = 0$ for all $a, b \in A$ which one cannot expected to hold in general.*

Proof: This follows from Propositions 6 and 7 of the previous section since $\star$ is strongly closed if and only if T is tracial.

Given a deformation $(A[[\nu]], \star)$ of aa commutative algebra $(A, \cdot)$, we set

$$\theta(a,b) := a \star b - b \cdot a \quad \forall a, b \in A. \tag{7}$$

The corresponding Poisson bracket is given by

$$\{a, b\} := (\theta(a,b) - \theta(b,a))_{[1]} \, .$$

Connes, Flato and Sternheimer introduced in [CFS] a cochain further investigated by Halbout [H] in view of an index type theorem for closed star products. It is defined as:

$$\Phi_{2k}(a_0, \cdots, a_{2k}) := T\left(a_0 \star \theta(a_1, a_2) \star \cdots \star \theta(a_{2k-1}, a_{2k})\right) \tag{8}$$

for any $a_0, \cdots, a_{2k} \in N$ such that $a_0 \star \theta(a_1, a_2) \star \cdots \star \theta(a_{2k-1}, a_{2k}) \in \mathcal{N}$. We shall refer to this $2k$-cochain as the CFS-H cochain. The next proposition compares $\mathrm{Alt}\,\Psi_{2k}$ with the antisymmetrized form $\mathrm{Alt}\,\Phi_{2k}$ of the CFS-H cochain:

Proposition 9 *For any non negative integer k,*

$$\mathrm{Alt}\,\Psi_{2k} = \mathrm{Alt}\,\Phi_{2k}.$$

Proof: First observe that since $(A, \cdot)$ is commutative,

$$\mathcal{A}lt\,\theta(a,b) = a \star b - b \star a := [a,b]_\star$$

where $\mathcal{A}lt$ stands for antisymmetrization in all variables. Let τ_{ij} be the transposition that exchanges i and j.

$$\begin{aligned}
& \mathrm{Alt}\,\Phi_{2k}\,(a_0, \cdots, a_{2k}) \\
= \;& \frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \theta(a_{\sigma(1)}, a_{\sigma(2)}) \star \cdots \star \theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})\right) \\
= \;& \frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma\tau_{12}) T\left(a_0 \star \theta(a_{\sigma(\tau_{12}(1))}, a_{\sigma(\tau_{12}(2))}) \star \cdots \star \theta(a_{\sigma(\tau_{12}(2k-1))}, a_{\sigma(\tau_{12}(2k))})\right) \\
= \;& -\frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \theta(a_{\sigma(2)}, a_{\sigma(1)}) \star \theta(a_{\sigma(3)}, a_{\sigma(4)}) \star \cdots \star \theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})\right) \\
= \;& \frac{1}{2}\frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \mathcal{A}lt\,\theta(a_{\sigma(1)}, a_{\sigma(2)}) \star \theta(a_{\sigma(3)}, a_{\sigma(4)}) \star \cdots \star \theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})\right) \\
= \;& \frac{1}{2}\frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \left[a_{\sigma(1)}, a_{\sigma(2)}\right]_\star \star \theta(a_{\sigma(3)}, a_{\sigma(4)}) \star \cdots \star \theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})\right) \\
= \;& -\frac{1}{2}\frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \left[a_{\sigma(\tau_{34}(1))}, a_{\sigma(\tau_{34}(2))}\right]_\star \star \theta(a_{\sigma(\tau_{34}(3)}, a_{\sigma(\tau_{34}(4))}) \cdots\right) \\
= \;& \frac{1}{2^2}\frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \left[a_{\sigma(1)}, a_{\sigma(2)}\right]_\star \star \left[a_{\sigma(3)}, a_{\sigma(4)}\right]_\star \star \theta(a_{\sigma(5)}, a_{\sigma(6)}) \star \cdots\right) \\
= \;& \cdots \\
= \;& \frac{1}{2^k}\frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star \left[a_{\sigma(1)}, a_{\sigma(2)}\right]_\star \star \cdots \star \left[a_{\sigma(2k-1)}, a_{\sigma(2k)}\right]_\star\right) \\
= \;& \frac{1}{2k!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) T\left(a_0 \star a_{\sigma(1)} \star a_{\sigma(2)} \star \cdots \star a_{\sigma(2k-1)} \star a_{\sigma(2k)}\right) \\
= \;& \mathrm{Alt}\,\Psi_{2k}\,(a_0, \cdots, a_{2k})\,.
\end{aligned}$$

A straightforward consequence of this identification is:

Corollary 2 *$\star$ is strongly closed w.r. to T if and only if $B\,\mathrm{Alt}\,\Phi_{2k} = 0$.*

Proof: This follows from the fact that if $\star$ is strongly closed w.r to T if and only if $B\,\mathrm{Alt}\,\Psi_{2k} = 0$.

We now consider a projected trace form.

Definition 16 *Given a $\mathbb{C}[[\nu]]$-linear map T on A we set*

$$\psi_k(a_0, \cdots, a_{2k}) := T_k\left(a_0 \star a_1 \star \cdots, a_{2k}\star\right) = \left(\Psi_{2k}(a_0, \cdots, a_{2k})\right)_{[k]},$$

and accordingly

$$\phi_k(a_0, \cdots, a_{2k}) := T_k\left(a_0 \star [a_1, \cdots, a_{2k}]_\star\right) = \left(\Phi_{2k}(a_0, \cdots, a_{2k})\right)_{[k]}.$$

The next proposition shows that $\mathrm{Alt}\,\psi_k$, $\mathrm{Alt}\,\phi_k$ can be written in terms of the Poisson brackets.

Proposition 10 *Let T be a $\mathbb{C}[[\nu]]$-linear extension of a linear map τ_0 on N. Given any elements $a_0, \cdots, a_{2k} \in A$ such that*

$$\mathrm{Alt}\,\left(a_0\{a_1, a_2\} \cdots \{a_{2k-1}, a_{2k}\}\right) := \frac{1}{(2k)!} \sum_{\sigma\in\Sigma_{2k}} \epsilon(\sigma) a_0 \cdot \{a_{\sigma(1)}, a_{\sigma(2)}\} \cdots \{a_{\sigma(2k-1)}, a_{\sigma(2k)}\} \in N,$$

then

$$\begin{aligned}
\mathrm{Alt}\,\psi_{2k}(a_0, \cdots, a_{2k}) &= \mathrm{Alt}\,\phi_{2k}(a_0, \cdots, a_{2k}) \\
&= 2^{-k}\,\tau_0\left(\mathrm{Alt}\,a_0 \cdot \{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right) \\
&:= 2^{-k}\,\mathrm{Alt}\,\tau_0\left(a_0 \cdot \{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right).
\end{aligned}$$

Remark 16 *In particular Helton and Howe's trace forms in this context read*

$$\begin{aligned}
T_k\left([a_1, \cdots, a_{2k}]_\star\right) &= 2^{-k}\,\tau_0\left(\mathrm{Alt}\,\{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right) \\
&= 2^{-k}\,\mathcal{A}lt\,\tau_0\left(\{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right)
\end{aligned}$$

where the antisymmetrization $\mathcal{A}lt$ applies to all variables.

Proof: It follows from Proposition 9 that $\mathrm{Alt}\,\psi_{2k}(a_0, \cdots, a_{2k}) = \mathrm{Alt}\,\phi_{2k}(a_0, \cdots, a_{2k})$. From the computations of the previous section we further derive that

$$\begin{aligned}
&\mathrm{Alt}\,\Psi_{2k}(a_0, \cdots, a_{2k})_{[k]} = \mathrm{Alt}\,\psi_{2k}(a_0, \cdots, a_{2k})_{[k]} \\
= &\frac{1}{2^k}\frac{1}{(2k)!} \sum_{\sigma\in\Sigma_{2k}} \epsilon(\sigma) \left[T\left(a_0 \star \mathcal{A}lt\,\theta(a_{\sigma(1)}, a_{\sigma(2)}) \star \cdots \star \mathcal{A}lt\,\theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})\right)\right]_{[k]} \\
= &\frac{1}{2^k}\frac{1}{(2k)!} \sum_{\sigma\in\Sigma_{2k}} \epsilon(\sigma) T_k\left(a_0 \star \mathcal{A}lt\,\theta(a_{\sigma(1)}, a_{\sigma(2)}) \star \cdots \star \mathcal{A}lt\,\theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})\right) \\
= &\frac{1}{2^k}\frac{1}{(2k)!} \sum_{\sigma\in\Sigma_{2k}} \epsilon(\sigma) \tau_0\left(a_0 \star \mathcal{A}lt\,\theta(a_{\sigma(1)}, a_{\sigma(2)})_{[1]} \cdot \cdots \cdot \mathcal{A}lt\,\theta(a_{\sigma(2k-1)}, a_{\sigma(2k)})_{[1]}\right) \\
= &\frac{1}{2^k(2k)!}\tau_0\left(\sum_{\sigma\in\Sigma_{2k}} \epsilon(\sigma) a_0 \cdot \{a_{\sigma(1)}, a_{\sigma(2)}\} \cdot \cdots \cdot \{a_{\sigma(2k-1)}, a_{\sigma(2k)}\}\right) \\
= &2^{-k}\tau_0\left(\mathrm{Alt}\,a_0 \cdot \{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right) \\
= &2^{-k}\mathrm{Alt}\,\tau_0\left(a_0 \cdot \{a_1, a_2\} \cdot \cdots \cdot \{a_{2k-1}, a_{2k}\}\right).
\end{aligned}$$

2.3 The case of a Poisson manifold

We now specialise to a star product associated with a Poisson manifold. Let W be an n- dimensional Poisson manifold and let Λ be the associated Poisson tensor. The Poisson bracket is defined by:

$$\{f, g\} = \langle \Lambda, df \wedge dg \rangle$$

for any smooth functions f and g on W.
Given a subspace $N \subset C^\infty(W)$, let

$$N^{2k} := \{2k - \text{forms} \quad \alpha \quad \text{on} \quad W, \langle \Lambda^{\wedge k}, \alpha \rangle \in N\}.$$

Any linear form $\tau_0 : N \subset C^\infty(W) \to \mathbb{C}$ induces a map $\tau_k : N^{2k} \to \mathbb{C}$ defined by:

$$\tau_k(\alpha) := \tau_0 \left(\langle \Lambda^{\wedge k}, \alpha \rangle \right).$$

Lemma 5 *Let $k \leq l$ and let $f_0, \cdots, f_{2k} \in N$. Then for any $f_0, \cdots, f_{2k} \in N$ such that*

$$\text{Alt}\, f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\} := \frac{1}{(2k)!} \sum_{\sigma \in \Sigma_{2k}} \epsilon(\sigma) f_0 \{f_{\sigma(1)}, f_{\sigma(2)}\} \cdots \{f_{\sigma(2k-1)}, f_{\sigma(2k)}\}$$

lies in N, then

- $\langle \Lambda^{\wedge k}, f_0 df_1 \wedge df_2 \cdots \wedge df_{2k-1} \wedge df_{2k} \rangle = 2^{-k}\, \text{Alt}\, \left(f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\} \right)$ *also lies in N,*
- $\tau_k(f_0 df_1 \wedge \cdots \wedge df_{2k}) = 2^{-k}\, \text{Alt}\, \tau_0 \left(f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\} \right),$

where as before, Alt *denotes cyclic antisymetrization on all but the first variable.*

Proof: We first observe that

$$\begin{aligned}
\text{Alt}\, \left(f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\} \right) &= \text{Alt}\, \left(f_0 \langle \Lambda, df_1 \wedge df_2 \rangle \cdots \langle \Lambda, df_{2k-1} \wedge df_{2k} \rangle \right) \\
&= 2^k \text{Alt}\, \left(f_0 \langle \Lambda, df_1 \otimes df_2 \rangle \cdots \langle \Lambda, df_{2k-1} \otimes df_{2k} \rangle \right) \\
&= 2^k \text{Alt}\, \left(f_0 \langle \Lambda^{\otimes k}, df_1 \otimes df_2 \otimes \cdots \otimes df_{2k-1} \otimes df_{2k} \rangle \right) \\
&= 2^k f_0 \langle \Lambda^{\wedge k}, df_1 \wedge df_2 \wedge \cdots \wedge df_{2k-1} \wedge df_{2k} \rangle \\
&= 2^k \langle \Lambda^{\wedge k}, f_0 df_1 \wedge df_2 \cdots \wedge df_{2k-1} \wedge df_{2k} \rangle.
\end{aligned}$$

It follows from there that $\langle \Lambda^{\wedge k}, f_0 df_1 \wedge df_2 \cdots \wedge df_{2k-1} \wedge df_{2k} \rangle \in N^{2k}$ since by assumption Alt $\left(f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\} \right)$ lies in N. Applying τ_0 then yields the result.

Definition 17 *The linear map $\tau_k : N^{2k} \to \mathbb{C}$ is said to satisfy Stokes' property if*

$$\tau_k(d\beta) = 0 \quad , \forall\, (k-1) - \text{form} \quad \beta \quad \text{on} \quad W \quad \text{s.t.} \quad d\beta \in N^{2k}.$$

Theorem 4 *Let $\star$ be a star product associated with the Poisson structure on a Poisson manifold W, and let $T(\sum_{k=0}^\infty a_k \nu^k) = \sum_{k=0}^\infty T_k(a) \nu^k$ be a $\mathbb{C}[[\nu]]$-linear extension to an N-module $\mathcal{N} = N[[\nu]]$ of a linear form $\tau_0 : N \subset C^\infty(W) \to \mathbb{C}$. For any $f_0, \cdots, f_{2k} \in C^\infty(W)$ such that* $\text{Alt}\, f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\}$ *lies in N we have*

$$\begin{aligned}
\text{Alt}\, \psi_{2k}(f_0, \cdots, f_{2k}) &= \text{Alt}\, \phi_{2k}(f_0, \cdots, f_{2k}) \\
&= \tau_k(f_0\, df_1 \wedge \cdots \wedge df_{2k}).
\end{aligned}$$

If τ_k satisfies Stokes' property then $\mathrm{Alt}\,\psi_{2k} = \mathrm{Alt}\,\phi_{2k}$ *satisfies the following conditions:*

1. *Whenever* $\mathrm{Alt}\,(f_0\{f_1, f_2\}\cdots\{f_{2k-1}, f_{2k}\}) \in N$ *and* $\mathrm{Alt}\,(f_1\{f_2, f_3\}\cdots\{f_{2k}, f_0\}) \in N$ *then*

$$\mathrm{Alt}\,\psi_{2k}(f_0, \cdots, f_{2k}) = \mathrm{Alt}\,\psi_{2k}(f_1, \cdots, f_{2k}, f_0) \quad \text{(cyclicity)}$$

2. *Whenever* $\mathcal{A}lt\,(\{f_1, f_2\}\cdots\{f_{2k-1}, f_{2k}\}) = \mathrm{Alt}\,(1\,\{f_1, f_2\}\cdots\{f_{2k-1}, f_{2k}\}) \in N$ *then* $\mathrm{Alt}\,\psi_{2k}(1, f_1, \cdots, f_{2k}) = 0$.

3. $b\,\mathrm{Alt}\,\psi_{2k}(f_0, \cdots, f_{2k+1}) = 0$ *whenever the l.h.s makes sense, where b is the Hochschild coboundary associated with the commutative product $\cdot$ on $\mathcal{A}$.*

If $N = A$ then $\mathrm{Alt}\,\psi_{2k}$ *is the $2k$-character associated with the $2k$-dimensional cycle $(\Omega(\mathcal{A}), d, \tau_k)$ where $\Omega(\mathcal{A})$ is equipped with the ordinary exterior product built from the commutative product $\cdot$ on A.*

Remark 17 *Since* $\mathrm{Alt}\,\psi_{2k} = \mathrm{Alt}\,\phi_{2k}$ *one could expect that the b-closedness of* $\mathrm{Alt}\,\psi_{2k}$ *might follow from that of ϕ_{2k} (which would imply the b-closedness of* $\mathrm{Alt}\,\phi_{2k}$ *since the product is commutative). We show in Appendix B that there is no reason to expect ϕ_{2k} to be b-closed.*

Proof: Combining Proposition 10 applied to $T_k = \tau_0$ combined with Lemma 5 yields that

$$\begin{aligned} \mathrm{Alt}\,\psi_{2k}(f_0, \cdots, f_{2k}) &= \mathrm{Alt}\,\phi_{2k}(f_0, \cdots, f_{2k}) \\ &= \tau_k(f_0\, df_1 \wedge \cdots \wedge df_{2k}). \end{aligned}$$

Conditions 1,2 then follow from Stokes' property of τ_k. Indeed,

1. We first observe that whenever $\mathrm{Alt}\, f_0\{f_1, f_2\}\cdots\{f_{2k-1}, f_{2k}\}$ and $\mathrm{Alt}\, f_1\{f_2, f_3\}\cdots\{f_{2k}, f_0\}$ lie in N, then $d\,(f_0\, f_1\, df_2 \wedge \cdots \wedge df_{2k})$ lies in $\mathcal{N}_{2k}$ since

$$\begin{aligned} & \langle \Lambda^{\wedge k}, d\,(f_1\, f_0\, df_2 \wedge \cdots \wedge df_{2k})\rangle \\ = \; & \langle \Lambda^{\wedge k}, f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_{2k}\rangle + \langle \Lambda^{\wedge k}, f_1\, df_0 \wedge df_2 \wedge \cdots \wedge df_{2k}\rangle \\ = \; & \langle \Lambda^{\wedge k}, f_0\, df_1 \wedge df_2 \wedge \cdots \wedge df_{2k}\rangle - \langle \Lambda^{\wedge k}, f_1\, df_2 \wedge \cdots \wedge df_{2k} \wedge df_0\rangle \\ = \; & 2^{-k}\mathrm{Alt}\;(\,f_0\{f_1, f_2\}\cdots\{f_{2k-1}, f_{2k}\}) - 2^{-k}\mathrm{Alt}\;(\,f_1\{f_2, f_3\}\cdots\{f_{2k}, f_0\}) \end{aligned}$$

lies in N. Applying Stokes' property to $\beta := f_0\, f_1\, df_2 \wedge \cdots \wedge df_{2k}$ we get

$$\begin{aligned} \mathrm{Alt}\,\psi_{2k}(f_0, \cdots, f_{2k}) &= \tau_k(f_0\, df_1 \wedge \cdots \wedge df_{2k}) \\ &= \tau_k\,(d\,(f_1\, f_0\, df_2 \wedge \cdots \wedge df_{2k})) - \tau_k\,(f_1\, df_0 \wedge \cdots \wedge df_{2k}) \\ &= -\tau_k\,(f_1\, df_0 \wedge df_2\, \cdots \wedge df_{2k}) \\ &= \tau_k\,(f_1\, df_2 \wedge \cdots \wedge df_{2k} \wedge df_0) \\ &= \mathrm{Alt}\,\psi_{2k}(f_1, \cdots, f_{2k}, f_0). \end{aligned}$$

2. Applying this to $f_0 = 1$ yields 2.

3. The last condition follows from the Leibniz rule $d(fg) = df\, g + f\, dg$.

$$\begin{aligned} b\,\mathrm{Alt}\,\psi_{2k}(f_0,\cdots,f_{2k+1}) &= b\,\tau_k(f_0\,df_1\wedge\cdots\wedge df_{2k+1})\\ &= \tau_k(f_0\,f_1\,df_2\wedge\cdots\wedge df_{2k+1}) - \tau_k(f_0\,d(f_1\,f_2)\wedge\cdots\wedge df_{2k+1})\cdots\\ \cdots\ & +\tau_k(f_0\,df_1\wedge\cdots\wedge d(f_{2k}f_{2k+1})) - \tau_k(f_{2k+1}f_0\,df_1\wedge\cdots\wedge df_{2k})\\ &= \tau_k(f_0\,f_1\,df_2\wedge\cdots\wedge df_{2k+1}) - \tau_k(f_0\,f_1\,df_2\wedge\cdots\wedge df_{2k+1})\\ -\ & \tau_k(f_0\,df_1\,f_2\,df_3\wedge\cdots\wedge df_{2k+1}) + \tau_k(f_0\,df_1\wedge f_2\,df_3\wedge\cdots\wedge df_{2k+1})\cdots\\ \cdots\ & +\tau_k(f_0\,df_1\wedge\cdots\wedge df_{2k}\,f_{2k+1}) - \tau_k(f_{2k+1}f_0\,df_1\wedge\cdots\wedge df_{2k})\\ &= 0 \end{aligned}$$

where we have used the commutativity of the product $\cdot$ in the last identity.

2.4 The symplectic case

We now further specialize to a $2l$-dimensional symplectic manifold W equipped with the symplectic form ω. The non degenerate 2-form ω induces an isomorphism of vector bundles:

$$\begin{aligned} \omega^\flat : TM &\to T^*M\\ v &\mapsto v^\flat(w) := \omega(v,w) \end{aligned}$$

with inverse:

$$\begin{aligned} \omega^\sharp : T^*M &\to TM\\ \alpha &\mapsto \alpha^\sharp, \quad \omega(v,\alpha^\sharp) = \alpha(v). \end{aligned}$$

The Hamiltonian vector field associated to a smooth function f on W is defined by:

$$H_f(x) := \omega_x^\sharp(df)$$

and the Poisson bracket by:

$$\{f,g\} = \langle \omega, H_f \wedge H_g\rangle = \langle \Lambda, df\wedge dg\rangle$$

for any smooth functions f and g on W. From this it follows that:

$$\Lambda = \left(\wedge^2\omega^\sharp\right)(\omega).$$

Lemma 6

$$\langle\omega,\Lambda\rangle = l, \quad \langle\omega^{\wedge k},\Lambda^{\wedge k}\rangle = l^k.$$

Proof: The second property easily follows from the first one using the usual normalisation conventions for inner products of forms. Let us check the first property. Using Darboux coordinates, we have:

$$\langle\Lambda,\omega\rangle = \langle\Lambda,\sum_{i=1}^l dp_i\wedge dq_i\rangle = \sum_{i=1}^l\{p_i,q_i\} = l$$

which ends the proof.

Given a subspace $N \subset C^\infty(W)$, we set as before $N^{2k} := \{2k-\text{forms} \quad \alpha \quad \text{on} \quad W, \langle \Lambda^{\wedge k}, \alpha\rangle \in N\}$. and let $\tau_k : N^{2k} \to \mathbb{C}$ defined by:

$$\tau_k(\alpha) := \tau_0\left(\langle \Lambda^{\wedge k}, \alpha\rangle\right)$$

be the map induced by some linear form $\tau_0 : N \subset C^\infty(W) \to \mathbb{C}$.

Lemma 7

$$\tau_k(\alpha) = l^k\, \tau_l(\alpha \wedge \omega^{l-k}) \quad \forall \alpha \in N^{2k}.$$

Proof: On a Darboux coordinate chart with local coordinates $(x_1, \cdots, x_{2l})$ we can write

$$\alpha = \sum_{\{i_1,\cdots,i_{2k}\}\subset\{1,\cdots,2l\}} \alpha_{i_1\cdots i_{2k}} dx_{i_1} \wedge \cdots \wedge dx_{i_{2k}}$$

so that the proof boils down to showing the property for some form $\alpha = f\, dx_{i_1} \wedge \cdots \wedge dx_{i_{2k}}$ in which case we have:

$$\begin{aligned}
\tau_k(\alpha) &= \tau_k(f\, dx_{i_1} \wedge \cdots \wedge dx_{i_{2k}}) \\
&= \tau_0\left(\langle \Lambda^{\wedge k}, f\, dx_{i_1} \wedge \cdots \wedge dx_{i_{2k}}\rangle \omega^l\right) \\
&= \epsilon(\sigma) l^k \tau_l\left(f\omega^l\right) \\
&= l^k\, \tau_l\left(f\, dx_{i_1} \wedge \cdots dx_{i_{2k}} \wedge \omega^{l-k}\right) \\
&= l^k \tau_l(\alpha \wedge \omega^{l-k})
\end{aligned}$$

where $\epsilon(\sigma)$ is the signature of the permutation $\sigma : j \to i_j$. In the third identity, we use Lemma 6 setting $p_i = x_{2i+1}$, $q_i = x_{2i}$.

Proposition 11 *Let as before W be a $2l$-dimensional symplectic manifold. If τ_l satisfies Stokes' property: $\tau_l(d\beta) = 0 \quad \forall (2l-1) - \text{form} \quad \beta \quad \text{on} \quad W, \text{s.t.} \quad d\beta \in N^{2l}$, then so does τ_k verify Stokes' property for any integer $0 \leq k \leq l$, i.e. $\tau_k(d\beta) = 0 \quad \forall (2k-1) - \text{form} \quad \beta \quad \text{on} \quad W, \text{s.t.} \quad d\beta \in N^{2k}$.*

Proof: Let ω denote the symplectic form on W. The statement follows from Lemma 7 using the closedness of ω since $\tau_k(d\,\beta) = l^k\, \tau_l(d\beta \wedge \omega^{l-k}) = l^k\, \tau_l\left(d(\beta \wedge \omega^{l-k})\right)$.

The following Corollary is then a direct consequence of Theorem 4.

Corollary 3 *Let $\star$ be a star product associated with a symplectic manifold (W, ω), and let $T(\sum_{k=0}^\infty a_k \nu^k) = \sum_{k=0}^\infty \tau_0(a_k)\nu^k$ be a $\mathbb{C}[[\nu]]$-linear extension to a submodule $\mathcal{N} \subset N[[\nu]]$ of a linear form $\tau_0 : N \subset C^\infty(W) \to \mathbb{C}$. For any non negative integer k and any $f_0, \cdots, f_{2k} \in C^\infty(W)$ such that $\text{Alt}\, f_0\{f_1, f_2\} \cdots \{f_{2k-1}, f_{2k}\}$ lies in N we have*

$$\begin{aligned}
\text{Alt}\, \psi_{2k}(f_0, \cdots, f_{2k}) &= \text{Alt}\, \phi_{2k}(f_0, \cdots, f_{2k}) \\
&= l^k\, \tau_l\left(f_0\, df_1 \wedge \cdots \wedge df_{2k} \wedge \omega^{l-k}\right).
\end{aligned}$$

If τ_l satisfies Stokes' property, then conditions 1,2, 3 of Theorem 4 are fulfilled.
Whenever $N = A$, then for any non negative integer k, $\text{Alt}\, \psi_{2k}$ is the character (see Appendix A) associated with the $2k$-dimensional cycle $(\Omega(\mathcal{A}), d, \tau_k)$ where $\Omega(\mathcal{A})$ is equipped with the ordinary exterior product built from the commutative product $\cdot$ on A.

When $A = C_0^\infty(W)$ *and* $T_k(f) = \int_W f\omega^l$ *is the ordinary integral then*

$$\mathrm{Alt}\,\phi_{2k}(f_0, \cdots, f_{2k}) = l^k \int_W f_0\, df_1 \wedge \cdots \wedge df_{2k} \wedge \omega^{l-k}.$$

Proof: From Proposition 11 we know that τ_k satisfies Stokes' property for any k since by assumption τ_l does. Applying Theorem 4 then yields the result.

Remark 18 *When* $k = l$ *this last corollary yields back:*

$$\mathrm{Alt}\,\phi_{2l}(f_0, \cdots, f_{2k}) = l^l \int_W f_0\, df_1 \wedge \cdots \wedge df_{2l},$$

which corresponds to a particular case of a more general formula shown in [CFS] and [H].

3 Cyclic Hochschild cocycles on classical symbols

3.1 A cyclic cocycle for non integer order classical symbols on $\mathbb{R}^{2l}$

Following the notations of the first part of the paper, we set

- $n = 2l$, $W = \mathbb{R}^{2l}$ equipped with the canonical symplectic form ω,
- $N := CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l})$, $\mathcal{N} = N[[\nu]] = CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l})[[\nu]]$,
- $\tau_0^{cut-off}(\sigma) := -\!\!\!\!\!\!\int_{\mathbb{R}^{2l}} \sigma\,\omega^l$ which defines a linear form on N,
- for $\sigma = \sum_{k=0}^\infty \sigma_k \nu^k$, $T\left(\sum_{k=0}^\infty \sigma_k\, \nu^k\right) = \sum_{k=0}^\infty T_k(\sigma)\, \nu^k$, where $T_k(\sigma) = \tau_0^{cut-off}(\sigma_k)$,
- $N^{2k} := \{\alpha \quad k-\text{form} \quad \text{on } \mathbb{R}^{2l} \quad \text{s.t.} \quad \langle \Lambda^{2k}, \alpha\rangle \in N\}$.

A form $\alpha \in \Omega^{2k}\, CS(\mathbb{R}^{2l})$, locally reads $\alpha = \sum \alpha_{|I|+|J|=k} \alpha_{I,J}\, dx_I \wedge d\xi_J$ with $\alpha_{I,J} \in CS(U)$. Hence

$$\langle \Lambda^{\wedge k}, \alpha\rangle = \sum_\alpha |I| + |J| = k\alpha_{I,J}\, \langle \Lambda^{\wedge k}, dx_I \wedge d\xi_J\rangle$$

so that if α has non integer order,$\langle \Lambda^{\wedge k}, \alpha\rangle$ also has non integer order. As a consequence, $\Omega^{2k}\, CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l}) \subset N^{2k}$ so that

$$\begin{aligned} \tau_k^{cut-off} : \Omega^{2k}\, CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l}) &\to \mathbb{C}[[\nu]] \\ \alpha &\mapsto -\!\!\!\!\!\!\int \langle \Lambda^{\wedge k}, \alpha\rangle\, \omega^l \end{aligned}$$

is well defined.

Lemma 8 *The linear form* τ_l *coincides on* $\Omega^{2l}\, CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l})$ *with the cut-off integral extended to top forms:*

$$\tau_l(\alpha) = -\!\!\!\!\!\!\int_{\mathbb{R}^{2l}} \alpha \quad \forall \alpha \in \Omega^{2l}\, CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l}).$$

Proof: A straightforward computation shows that $\langle \Lambda^{2l}, \alpha\rangle\, \omega^l = \alpha$ from which the result then follows.
We equip $CS_{c.c}(\mathbb{R}^{2l})[[\nu]]$ with the Moyal star product. Given two symbols $\sigma \in CS^m_{c.c}(\mathbb{R}^{2l})$, $\sigma' \in CS^{m'}_{c.c}(\mathbb{R}^{2l})$, their Moyal product:

$$\sigma \star \sigma' = \sum_{k=0}^{\infty} C_k(\sigma, \sigma')\nu^k$$

where

$$C_k(\sigma, \sigma') := \sum_{k=0}^{\infty} \sum_{i_1,\cdots,i_k} \sum_{j_1,\cdots,j_k} \omega^{i_1 j_1} \cdots \omega^{i_k j_k} \partial^{i_1\cdots i_k}\sigma \partial^{i_1\cdots i_k}\sigma'$$

lies in $CS^{m+m'}_{c.c}(\mathbb{R}^{2l})[[\nu]]$ since $C_k(\sigma, \sigma')$ has order $m + m' - 2k$. In particular, $C_1(\sigma, \sigma') = \frac{1}{2}\{\sigma, \sigma'\}$. We set

$$\theta(\sigma, \sigma') := \sum_{k=1}^{\infty} C_k(\sigma, \sigma')\, \nu^k,$$

which is of order $m + m' - 2$.
Let $\sigma_0, \cdots, \sigma_{2k} \in N$ of orders $m_0, m_1, \cdots, m_{2k}$ such that $\sum_{i=0}^{2k} m_i \notin \mathbb{Z}$. Then $\sigma_0 \star \sigma_1 \star \cdots \star \sigma_{2k}$ and $\sigma_0 \star \theta(\sigma_1, \sigma_2) \star \cdots \star \theta(\sigma_{2k-1}, \sigma_{2k})$ also lie in N.

Definition 18 *Let $\sigma_0, \cdots, \sigma_{2k} \in N$ of orders $m_0, m_1, \cdots, m_{2k}$ such that $\sum_{i=0}^{2k} m_i \notin \mathbb{Z}$. We set*

$$\Psi^{cut-off}_{2k}(\sigma_0, \sigma_1, \cdots, \sigma_{2k}) := T^{cut-off}\left(\sigma_0 \star \sigma_1 \star \cdots \star \sigma_{2k}\right),$$

$$\psi^{cut-off}_{2k}(\sigma_0, \sigma_1, \cdots, \sigma_{2k}) := T^{cut-off}_k\left(\sigma_0 \star \sigma_1 \star \cdots \star \sigma_{2k}\right).$$

We define the cut-off CFS-H cocycle:

$$\Phi^{cut-off}_{2k}(\sigma_0, \sigma_1, \cdots, \sigma_{2k}) := T^{cut-off}\left(\sigma_0 \star \theta(\sigma_1, \sigma_2) \star \cdots \star \theta(\sigma_{2k-1}, \sigma_{2k})\right)$$

and its projection

$$\phi^{cut-off}_{2k}(\sigma_0, \sigma_1, \cdots, \sigma_{2k}) := T^{cut-off}_k\left(\sigma_0 \star \theta(\sigma_1, \sigma_2) \star \cdots \star \theta(\sigma_{2k-1}, \sigma_{2k})\right).$$

As a consequence of Corollary 3, $\mathrm{Alt}\, \psi^{cut-off}_{2k} = \mathrm{Alt}\, \phi^{cut-off}_{2k}$ is a cyclic "b-cocycle" (the terminology is improperly used here since non integer order symbols do not build an algebra) on non integer order symbols on $\mathbb{R}^{2l}$ with constant coefficients:

Theorem 5 $\mathrm{Alt}\, \psi^{cut-off}_{2k} = \mathrm{Alt}\, \phi^{cut-off}_{2k}$ *reads:*

$$\mathrm{Alt}\, \psi^{cut-off}_{2k}(\sigma_0, \cdots, \sigma_{2k}) = l^k \,⨍_{\mathbb{R}^{2l}} \sigma_0\, d\sigma_1 \wedge \cdots \wedge d\sigma_{2k} \wedge \omega^{l-k}$$

and satisfies the following conditions. For any symbols $\sigma_0, \cdots, \sigma_{2k+1} \in CS_{c.c}(\mathbb{R}^{2l})$ with orders $m_i, i = 0, \cdots, 2k+1$ such that

1. *$\sum_{i=0}^{2k} m_i \notin \mathbb{Z}$, we have (cyclicity)*

$$\mathrm{Alt}\, \psi^{cut-off}_{2k}(\sigma_0, \cdots, \sigma_{2k}) = \mathrm{Alt}\, \psi^{cut-off}_{2k}(\sigma_1, \cdots, \sigma_{2k}, \sigma_0)$$

2. $\sum_{i=1}^{2k} m_i \notin \mathbb{Z}$, *we have*

$$\mathrm{Alt}\,\psi_{2k}^{cut-off}(1, \sigma_1, \cdots, \sigma_{2k}) = 0$$

3. $\sum_{i=0}^{2k+1} m_i \notin \mathbb{Z}$, *we have*

$$b\,\mathrm{Alt}\,\psi_{2k}^{cut-off}(\sigma_0, \cdots, \sigma_{2k+1}) = 0$$

where b *is the Hochschild coboundary associated with the commutative product* $\cdot$ *on* $\mathcal{A}$.

Proof: This follows from Corollary 3 applied to $\tau_0^{cut-off} = -\!\!\!\!\!\int$ and $N = CS_{\notin\mathbb{Z}}(\mathbb{R}^{2l})$, which satisfies Stokes' property by Theorem 3. Indeed the fact that $\sum_{i=0}^{2k} m_i \notin \mathbb{Z}$ implies that $\mathrm{Alt}\,\sigma_0\,\{\sigma_1, \sigma_2\} \cdots \{\sigma_{2k-1}, \sigma_{2k}\} \in N$ and $\mathrm{Alt}\,\sigma_1\,\{\sigma_2, \sigma_3\} \cdots \{\sigma_{2k}, \sigma_0\} \in N$ so that their cut-off integrals are well-defined and condition 1 makes sense. Similarly, assumption $\sum_{i=1}^{2k} m_i \notin \mathbb{Z}$ ensures that condition 2 makes sense and $\sum_{i=1}^{2k+1} m_i \notin \mathbb{Z}$ that condition 3 makes sense.

Setting $k = l$ and $\sigma_0 = 1$ we get back a formula similar to formulae proven in [HH]:

Corollary 4 *For any classical symbols* $\sigma_1, \cdots, \sigma_{2l} \in CS_{c.c}(\mathbb{R}^{2l})$ *with orders* $m_i, i = 1, \cdots, 2k$ *such that* $\sum_{i=1}^{2l} m_i \notin \mathbb{Z}$,

$$T_l^{cut-off}\,[\sigma_1, \cdots, \sigma_{2l}]_\star = l^l -\!\!\!\!\!\!\int_{\mathbb{R}^{2l}} d\sigma_1 \wedge \cdots \wedge d\sigma_{2l}.$$

3.2 Meromorphic families of trace forms and the residue cocycle on classical symbols on $\mathbb{R}^{2l}$

Recall from Theorem 2 the character $(\sigma_0, \sigma_1, \cdots, \sigma_p) \mapsto \mathrm{res}(\sigma_0 d\sigma_1 \wedge_L \cdots \wedge_L d\sigma_p)$ associated with the cycle $(\Omega\, CS(M), d, \mathrm{res})$ where $CS(M)$ was equipped with the left product on symbols. On $CS_{c.c}(\mathbb{R}^{2l})$, the left product boils down to an ordinary product so that the associated $2l$-residue character reads:

$$(\sigma_0, \cdots, \sigma_{2l}) \mapsto \mathrm{res}\,(\sigma_0\, d\sigma_1 \wedge \cdots \wedge d\sigma_{2l}).$$

Theorem 6 *Let* $CS_{c.c}(\mathbb{R}^{2l})[[\nu]]$ *be equipped with the Moyal* $\star$*-product. Given a regularisation procedure* $\mathcal{R} : \sigma \mapsto \sigma(z)$ *on* $CS_{c.c}(\mathbb{R}^{2l})$, *for any* $\sigma_0, \cdots, \sigma_{2k} \in CS_{c.c}(\mathbb{R}^{2l})$, *the map* $z \mapsto \mathrm{Alt}\,\psi_{2k}^{cut-off}\,(\sigma_0(z), \sigma_1(z), \cdots, \sigma_{2k}(z))$ *is meromorphic with simple poles and its complex residue at zero*

$$\mathrm{Alt}\,\psi_{2k}^{res}(\sigma_0, \sigma_1, \cdots, \sigma_{2k}) := -\alpha'(0) \cdot \mathrm{Res}_{z=0} \mathrm{Alt}\,\psi_{2k}^{cut-off}\,(\sigma_0(z), \sigma_1(z), \cdots, \sigma_{2k}(z))$$

where $\alpha(z)$ *is the order of the symbol valued form* $\sigma_0\, d\sigma_1 \wedge \cdots \wedge d\sigma_{2k} \wedge \omega^{l-k}$, *reads:*

$$\mathrm{Alt}\,\psi_{2k}^{res}(\sigma_0, \cdots, \sigma_{2k}) := l^k\, \mathrm{res}\left(\sigma_0\, d\sigma_1 \wedge \cdots \wedge d\sigma_{2k} \wedge \omega^{l-k}\right). \tag{9}$$

$\mathrm{Alt}\,\psi_{2l}$ *is proportional to the* $2l$*-character associated with the cycle*

$$\left(\Omega\left(CS_{c.c}(\mathbb{R}^{2l})\right),\, d,\, \mathrm{res}\right).$$

It vanishes on symbols $\sigma_0, \sigma_1, \cdots, \sigma_i, \cdots$ *with orders* $m_0, m_1, m_2, \cdots$ *such that their sum is non integer.*

Proof: Let $m_i(z)$ denote the order of $\sigma_i(z)$, then $\sigma_0(z) \star \sigma_1(z) \star \cdots \star \sigma_{2k}(z)$ lies in $CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l})[[\nu]]$ provided $\sum_{i=1}^{2k} m_i(z) \notin \mathbb{Z}[[\nu]]$ so that it lies in $CS_{c.c}^{\notin \mathbb{Z}}(\mathbb{R}^{2l})$ outside a discrete number of points. Applying Theorem 5 to the $\sigma_i(z)$ outside this set yields that

$$\begin{aligned} & \mathrm{Alt}\, \psi_{2k}^{cut-off}(\sigma_0(z), \cdots, \sigma_{2k}(z)) \\ = \;& l^k ⨍ \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k}. \end{aligned}$$

Applying Theorem 1 to $\omega(z) := \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k}$, yields:

$$\begin{aligned} \mathrm{Alt}\, \psi_{2k}^{res}(\sigma_0, \cdots, \sigma_{2k}) &= -\alpha'(0) \cdot l^k \cdot \mathrm{Res}_{z=0} \mathrm{Alt}\, \psi_{2k}^{cut-off}(\sigma_0(z), \cdots, \sigma_{2k}(z)) \\ &= -\alpha'(0) \cdot l^k \cdot \mathrm{Res}_{z=0} ⨍ \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k} \\ &= l^k \mathrm{res}\left(\sigma_0\, d\sigma_1 \wedge d\sigma_2 \wedge \cdots \wedge d\sigma_{2k} \wedge \omega^{l-k}\right). \end{aligned}$$

Hence $\mathrm{Alt}\, \psi_{2k}^{res}(\sigma_0, \sigma_1, \cdots, \sigma_{2l})$ is proportional to the $2l$-character associated with the cycle $\left(\Omega\left(CS_{c.c}(\mathbb{R}^{2l})\right), d, \mathrm{res}\right)$. It clearly vanishes on symbols $\sigma_0, \sigma_1, \sigma_2, \cdots$ with orders $m_0, m_1, m_2, \cdots$ such that their sum is non integer since in that case the form $\sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k}$ also has non integer order.

3.3 Perturbed star products

Going back to a general associative algebra $(\mathcal{A}, \star)$ equipped with a linear map T, given a fixed element $\kappa \in \mathcal{A}$, we can *perturb the product* $\star$ and the corresponding linear map T as follows:

$$a \star_\kappa b := a \star \kappa \star b; \quad T_\kappa(a) := T(a \star \kappa) = T(\kappa \star a) \quad \forall a, b \in \mathcal{A}.$$

This gives rise to a new associative algebra $(\mathcal{A}, \star_\kappa)$ equipped with a new linear form $T_\kappa(\sum_{k=0}^\infty a_k \nu^k) = \sum_{k=0}^\infty T_{k,\kappa}(a_k)\nu^k$ and to corresponding perturbed trace forms:

$$\Psi_{2k}^\kappa(a_0, \cdots, a_n) = T_\kappa\left(a_0 \star_\kappa a_1 \cdots \star_\kappa a_n\right) \quad \forall a_0, \cdots, a_n \in \mathcal{A}$$

and

$$\psi_{2k}^\kappa(a_0, \cdots, a_n) = T_{k,\kappa}\left(a_0 \star_\kappa a_1 \cdots \star_\kappa a_n\right) \quad \forall a_0, \cdots, a_n \in \mathcal{A}.$$

A straighforward computation shows that

Lemma 9
- *If $\star$ is (resp. strongly) closed w.r. to T_k (resp. T) then $\star_\kappa$ is (strongly) closed w.r. to $T_{k,\kappa}$ (resp. T_κ).*
- *Set $\mathcal{R}_\kappa(a) := a \star \kappa$ with $\kappa \in N$. Then*

$$a_0 \star_\kappa a_1 \star_\kappa \cdots \star_\kappa a_{n-1} \star_\kappa a_n = \mathcal{R}_\kappa(a_0) \star \mathcal{R}_\kappa(a_1) \star \cdots \star \mathcal{R}_\kappa(\sigma_{n-1}) \star \mathcal{R}_\kappa(a_n)$$

and

$$T_\kappa\left(\sigma_0 \star_\kappa a_1 \star_\kappa \cdots \star_\kappa a_{n-1} \star_\kappa a_n\right) = T\left(\mathcal{R}_\kappa(a_0) \star \cdots \star a_n\right).$$

3.4 Meromorphic cocycles associated with the left product on classical symbols on $\mathbb{R}^n$

Let us now specialise to $CS(U)$ where U is an open subset of $\mathbb{R}^n$. We equip $CS(U)$ with a left star product on symbols.

Definition 19 *Given two symbols $\sigma \in CS^m(U)$, $\sigma' \in CS^{m'}(U)[[\nu]]$, their (left) star product is defined by:*

$$\sigma \star_L \sigma' = \sum_{k=0}^{\infty} \tilde{C}_k(\sigma, \sigma')\nu^k$$

with

$$\tilde{C}_k(\sigma, \sigma') := \sum_{|\alpha|=k} \frac{1}{\alpha!}\partial_\xi^\alpha \sigma \cdot \partial_x^\alpha \sigma'.$$

It lies in $CS^{m+m'}(U)$ since $\tilde{C}_k(\sigma, \sigma')$ has order $m + m' - k$.

Note that

$$\tilde{C}_1(\sigma, \sigma') - \tilde{C}_1(\sigma', \sigma) = \{\sigma, \sigma'\}$$

so that the left star product $\star_L$, like the Moyal product $\star$, is associated with the canonical symplectic form ω on $W = T^*U$.
It is useful to note that if σ' is independent of x then

$$\sigma \star_L \sigma' = \sigma \cdot_L \sigma' = \sigma \cdot \sigma'.$$

We are now ready to perturb the left product on formal symbols. Let U be an open subset of $\mathbb{R}^n$. A holomorphic family

$$\kappa \in \mathrm{Hol}\,(CS(U))$$

such that

$$\kappa(0) = Id, \text{ and } \quad \mathrm{ord}\kappa(z) = \alpha(z) \,\text{with}\, \alpha'(0) \neq 0,$$

provides on one hand a map:

$$\mathcal{R}_\kappa : \sigma \to \sigma(z) = \sigma \star_L \kappa(z)$$

and on the other hand a holomorphic family of perturbed star products

$$\sigma \star_{L,z} \sigma' := \sigma \star_L \kappa(z) \star_L \sigma'$$

with corresponding perturbed linear forms:

$$T(z) := T_{\kappa(z)}, \quad T_k(z) := T_{k,\kappa(z)},\, k \in \mathbb{N}.$$

Take $\kappa(z)$ independent of x. Then

$$\sigma \star_L \kappa(z) = \sigma \cdot \kappa(z)$$

so that

$$\mathcal{R}_\kappa : \sigma \mapsto \sigma(z) = \sigma \cdot \kappa(z)$$

provides a regularisation procedure on $CS(U)$.

In particular, choosing $\kappa(z) = |\xi|^{-z}$ yields back the well known Riesz regularisation:

$$\sigma \mapsto \sigma(z) := \sigma \star_L \mid xi|^{-z} = \sigma \cdot |\xi|^{-z}.$$

Remark 19 *In order to generalise this construction from an open subset U of $\mathbb{R}^n$ to a manifold M, one would need to weaken the requirement $\sigma \star_L \kappa(z) = \sigma \cdot \kappa(z)$, requiring instead that at each point $x \in M$,*

$$\sigma \star_L \kappa(z) = \sigma \cdot \kappa(z) + z\tilde{\sigma}_x(z)$$

for some classical symbol $\tilde{\sigma}_x(z)$ which might depend on the point $x \in M$. Riesz regularisation on a Riemannian manifold M, for which $\kappa(z) = |\xi|_x^{-z}$ where $|\cdot|_x$ is the norm on T_x^*U induced by the metric on M, satisfies such a requirement.

Theorem 7 *Let $CS_{c.c}(\mathbb{R}^{2l})[[\nu]]$ be equipped with the left star product $\star_L$. Let $\mathcal{R} : \sigma \mapsto \sigma(z) = \sigma \cdot \kappa(z)$ on $CS_{c.c}(\mathbb{R}^n)$ be a regularisation procedure induced by a map $\kappa \in \mathrm{Hol}\left(CS_{c.c}(\mathbb{R}^{2l})\right)$ (which is therefore independent of x). For $\sigma_0, \cdots, \sigma_{2k} \in CS_{c.c}(\mathbb{R}^{2l})$, set:*

$$\begin{aligned} \psi_{\kappa,2k}^{cut-off}(z)(\sigma_0, \cdots, \sigma_{2k}) &:= T_p^{cut-off}(\sigma_0(z) \star_L \sigma_1(z) \star_L \cdots \star_L \sigma_{2k}(z)) \\ &= T_{2k}^{cut-off}(z)(\sigma_0 \star_{L,z} \sigma_1 \star_{L,z} \cdots \star_{L,z} \sigma_{2k}). \end{aligned}$$

Then

$$z \mapsto \mathrm{Alt}\psi_{\kappa,2k}^{cut-off}(z)(\sigma_0, \cdots, \sigma_{2k}) = l^k ⨍_{\mathbb{R}^n} \sigma_0(z) d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k}$$

yields a meromorphic family of cyclic b_z-cocycles on $CS_{c.c}(\mathbb{R}^{2l})$ with simple poles. Here b_z is the Hochschild coboundary associated with the perturbed product $f \cdot_z g := f \cdot \kappa(z) \cdot g$.
When $k = l$

$$\mathrm{Alt}\psi_{\kappa,2k}^{res}(\sigma_0, \cdots, \sigma_{2k}) := -\alpha'(0)\, \mathrm{Res}_{z=0} \mathrm{Alt}\psi_{\kappa,2k}^{cut-off}(z)(\sigma_0, \cdots, \sigma_{2k})$$

is proportional (independently of the choice of κ) to the n- character associated with the n-cycle $(\Omega(CS_{c.c}(\mathbb{R}^n)), d, \mathrm{res})$ where res *is the Wodzicki residue extended to forms:*

$$\begin{aligned} \mathrm{Alt}\psi_{\kappa,2k}^{res}(\sigma_0, \cdots, \sigma_{2k}) &= l^k \cdot \mathrm{res}\,(\sigma_0 d\sigma_1 \wedge \cdots \wedge d\sigma_n) \\ &= l^k \cdot \mathrm{res}\,(\sigma_0 d\sigma_1 \wedge_L \cdots \wedge_L d\sigma_n)\,. \end{aligned}$$

Remark 20
- *The fact that starting from the left star product, we recover the residue character we had come across in the previous section starting from the Moyal star product is due to the fact that they are associated with the same Poisson brackets.*
- *On a closed Riemannian manifold M, one can take $\kappa(z)(x, \xi) := |\xi|_x$ defined using the metric $g(x)$ on T_xM at point x. Then,as was pointed out above, $\sigma(z) := \sigma \star_L \kappa(z) = \sigma(z) \cdot |\xi|^{-z} + z\tilde{\sigma}_x(z)$ for some classical symbol $\sigma_x(z)$. The perturbation by $z\sigma_x(z) = O_x(z)$ does not affect the complex residue so that the here again, the complex residue at zero of $\mathrm{Alt}\psi_{\kappa,p}(z)$ reads:*

$$\alpha'(0)\, \mathrm{Res}_{z=0} \mathrm{Alt}\psi_{\kappa,2k}^{cut-off}(z)(\sigma_0, \cdots, \sigma_{2k}) = l^k \mathrm{res}\,(\sigma_0 d\sigma_1 \wedge \cdots \wedge d\sigma_{2k})\,.$$

But when $k = l$ this expression is no longer proportional to the $2l$- charachter

$$\mathrm{res}\,(\sigma_0 d\sigma_1 \wedge_L \cdots \wedge_L d\sigma_{2l})$$

associated with the $2l$-cycle $(\Omega(CS(M)), d, \mathrm{res})$ which involves the left product on symbols as well as the wedge product on forms unlike in the case of symbols with constant coefficients for which the left product reduced to the ordinary product.

Proof: Applying Theorem 6 to $\sigma_i(z) = \sigma \cdot \kappa(z)$ yields the meromorphicity with simple poles together with the fact that the complex residue

$$\mathrm{Res}_{z=0}\,\mathrm{Alt}\psi_{\kappa,2k}(z)(\sigma_0, \cdots, \sigma_{2k}) = -\frac{l^k}{\alpha'(0)}\,\mathrm{res}\left(\sigma_0\, d\sigma_1 \wedge \cdots \wedge d\sigma_{2k} \wedge \omega^{l-k}\right)$$

is proportional to the $2k$-character associated with the cycle $(\Omega(CS(U)), d, \mathrm{res})$. This holds independently of the choice of κ.
On the other hand, by Theorem 5, we have

$$\mathrm{Alt}\psi_{\kappa,2k}(z)(\sigma_0, \cdots, \sigma_{2k}) = l^k \,⨍\, \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k}.$$

A straightforward computation then yields that it is $b(z)$-closed:

$$\begin{aligned}
b(z)\,\mathrm{Alt}\psi_{\kappa,2k}(z)(\sigma_0, \cdots, \sigma_{2k+1}) &= l^k\, b(z) \,⨍\, \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k+1}(z) \wedge \omega^{l-k} \\
&= l^k \,⨍\, (\sigma_0 \cdot_\kappa \sigma_1)(z)\, d\sigma_2(z) \wedge \cdots \wedge d\sigma_{2k+1}(z) \wedge \omega^{l-k} \\
&- l^k \,⨍\, \sigma_0(z)\, d\,(\sigma_1 \cdot_\kappa \sigma_2)(z) \wedge d\sigma_3(z) \wedge \cdots \wedge d\sigma_{2k+1}(z) \wedge \omega^{l-k} \\
&\quad \cdots + l^k \,⨍\, \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\,(\sigma_{2k} \cdot_\kappa \sigma_{2k+1})(z) \wedge \omega^{l-k} \\
&- l^k \,⨍\, (\sigma_{2k+1} \cdot_\kappa \sigma_0)(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k} \\
&= l^k \,⨍\, \sigma_0(z) \cdot \sigma_1(z)\, d\sigma_2(z) \wedge \cdots \wedge d\sigma_{2k+1}(z) \wedge \omega^{l-k} \\
&- l^k \,⨍\, \sigma_0(z)\, d\,(\sigma_1(z) \cdot \sigma_2(z)) \wedge d\sigma_3(z) \wedge \cdots \wedge d\sigma_{2k+1}(z) \wedge \omega^{l-k} \\
&\quad \cdots + l^k \,⨍\, \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\,(\sigma_{2k}(z) \cdot \sigma_{2k+1}(z)) \wedge \omega^{l-k} \\
&- l^k \,⨍\, (\sigma_{2k}(z) \cdot \sigma_0(z))\; d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k} \\
&= 0
\end{aligned}$$

since by Theorem 5, $\mathrm{Alt}\psi_{2k}^{cut-off}(\sigma_0(z), \cdots, \sigma_{2k}(z)) = l^k ⨍ \sigma_0(z)\, d\sigma_1(z) \wedge \cdots \wedge d\sigma_{2k}(z) \wedge \omega^{l-k}$ is a b-cocycle.

Appendix A

We recall here a few definitions borrowed from non commutative geometry see e.g.[C], [GVF]. Let $(\mathcal{A}, \star)$ be an associative algebra over some ring R with unit 1. The space

$C^n(\mathcal{A}, R)$ of R-valued $n+1$-linear forms on $\mathcal{A}$ corresponds to the space of n-cochains on $\mathcal{A}$. Equivalently, these spaces can be seen as spaces of R-multilinear n-forms on $\mathcal{A}$ with values in the R-algebraic dual $\mathcal{A}^*$, seen as an $\mathcal{A}$-bimodule, where for $\chi \in \mathcal{A}^*$ we put $a'\chi(a)a'' = \chi(a''aa')$.
Following [C] we define the operators B_0 and B acting on cochains:

Definition 20 *Let*

$$\begin{array}{rcl} C^n(\mathcal{A}) & \to & C^{n-1}(\mathcal{A}) \\ \chi & \mapsto & B_0\chi(a_0, \cdots, a_{n-1}) := \chi(1, a_0, \cdots, a_{n-1}) - (-1)^n \chi(a_0, \cdots, a_{n-1}, 1). \end{array}$$

Let $B := \mathcal{A}\, B_0$ where $\mathcal{A}$ denotes cyclic antisymmetrisation in all variables so that

$$B\chi(a_0, \cdots, a_{n-1}) = \sum_{i=0}^{n-1}(-1)^i \chi(1, a_i, a_{i+1}, \cdots) - (-1)^n \sum_{i=0}^{n-1}(-1)^i \chi(a_i, a_{i+1}, \cdots, a_{i-1}, 1)$$

One can check that $B^2 = 0$ so that B defines a homology on $C^\bullet(\mathcal{A})$ [C].

Definition 21 *The Hochschild coboundary for the product $\star$ of an n-cochain χ is defined by:*

$$b_\star\chi(a_0, \cdots, a_{n+1}) = \sum_{j=0}^{n}(-1)^j \chi(a_0, \cdots, a_j \star a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1}\chi(a_{n+1}\star a_0, \cdots, a_n).$$

It satisfies the condition $b_\star^2 = 0$ and hence defines a cohomology called the *Hochschild cohomology* of $(\mathcal{A}, \star)$.

Definition 22 *An n-dimensional cycle is given by a triple $\left(\Omega, d, \int\right)$ where Ω is a graded differential algebra on $\mathbb{C}$ equipped with the differential d such that $d^2 = 0$ and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace i.e. $\int$ is a linear map which, when extended to Ω by 0, satisfies*

$$\int \alpha \wedge \beta = (-1)^{|\alpha|\cdot|\beta|} \cdot \int \beta \wedge \alpha, \qquad \int d\beta = 0 \quad \forall\, \beta \in \Omega^{n-1}(\mathcal{A}).$$

An n-cycle on an algebra $\mathcal{A}$ on $\mathbb{C}$ is a cycle $(\Omega, d, \int)$ together with a homomorphism $\rho : \mathcal{A} \to \Omega^0$. The character χ_n of an n-cycle is defined by:

$$\chi_n(a_0, \cdots, a_n) = \int \rho(a_0)\, d\rho(a_1) \cdots d\rho(a_n) \quad \forall a_i \in \mathcal{A}.$$

Let us also recall that the character of a cycle has the following properties:

1. χ_n is cyclic i.e.
$$\chi_n(a_0, \cdots, a_n) = (-1)^n \chi_n(a_1, \cdots, a_n, a_0), \quad \forall a_i \in \mathcal{A}$$
2. $\chi_n(1, a_1, \cdots, a_n) = 0 \quad \forall a_i \in \mathcal{A}$.
3. If $\Omega = \Omega(\mathcal{A})$ and $\rho = Id$ then $b\,\chi_n = 0$ where b is the Hochschild coboundary associated with the product on $\mathcal{A}$.

Appendix B

Let us check that in the symplectic setting, when T_k satisfies Stokes' property– or equivalently when $\star$ is closed w.r. to T_k– one cannot expect ϕ_{2k} to be a b-cocycle even though we know by Theorem 4 that when T_k satisfies Stokes' property, its antisymmetrized form $\mathrm{Alt}\,\phi_{2k} = \mathrm{Alt}\,\psi_{2k}$ is b-closed. For this, we recall that

Proposition 12 *[CFS], [H] Let $(\mathcal{A}, \cdot)$ be a commutative algebra equipped with a star product $\star$ which is a closed (resp. strongly closed) star product w.r. to some linear map T_k (resp. T) on $\mathcal{A}$. For any $a_0, \cdots, a_{2k+1} \in \mathcal{A}$ we have*

$$b\,\phi_{2k}(a_0, \cdots, a_{2k+1}) = -T_k\left(\theta(a_0, a_1) \star \cdots \star \theta(a_{2k}, a_{2k+1})\right) + T_k\left(\theta(a_{2k+1}, a_0) \star \cdots \star \theta(a_{2k-1}, f_{2k})\right)$$

(resp. $(b+B)\phi = 0$, where we have set, following [H] $\Phi := \sum_{k=0}^{\infty} \frac{\Phi_{2k}}{k!}$.)

Proof: We carry out the proof for Φ_{2k}, since projecting down to the k-th component in the formal expansion in powers of ν easily yields the corresponding result for ϕ_{2_k}. Let us first recall that since 1 is a unit element in $\mathcal{A}$. $\theta(1, a) = \theta(a, 1) = 0$, a fact we use in the course of the proof. The following notation borrowed from [H] is useful for the proof, which mimicks that of [H]:

$$\begin{aligned}\tilde{b}\chi(a_0, \cdots, a_{n+1}) &= \chi(a_0 \star a_1, \cdots, a_j, \cdots, a_{n+1}) \\ &+ \sum_{j=1}^{n} (-1)^j \chi(a_0, \cdots, a_j \cdot a_{j+1}, \cdots, a_{n+1}) + (-1)^{n+1}\chi(a_{n+1} \star a_0, \cdots, a_n).\end{aligned}$$

Since (see [H])

$$\begin{aligned}\tilde{b}\theta(a, b, c) &:= a \star \theta(b, c) - \theta(a \cdot b, c) + \theta(a, b \cdot c) - \theta(a, b) \star c \\ &= a \star (b \star c - b \cdot c) - ((a \cdot b) \star c - a \cdot b \cdot c) \\ &+ (a \star (b \cdot c) - a \cdot b \cdot c) - (a \star b - a \cdot b) \star c \\ &= 0,\end{aligned}$$

it follows that

$$a \star \theta(b, c) - \theta(a \cdot b, c) + \theta(a, b \cdot c) = \theta(a, b) \star c.$$

Using this identity repeatedly combined with the closedness of $\star$ w.r. to T we derive that

$$\begin{aligned}(\tilde{b}\Phi_{2k})(a_0, \cdots, a_{2k+1}) &:= \Phi_{2k}(a_0 \star a_1, \cdots, a_{2k+1}) - \Phi_{2k}(a_0, a_1 \cdot a_2, \cdots, a_{2k+1}) \\ &+ \cdots - \Phi_{2k}(a_0, \cdots, a_{2i}, a_{2k} \cdot a_{2k+1}) + \Phi_{2k}(a_{2k+1} \star a_0, \cdots, a_{2k}) \\ &= T(a_0 \star a_1 \star \theta(a_2, a_3) \star \cdots) - T(a_0 \star \theta(a_1 \cdot a_2, a_3) \star \cdots) \\ &+ T(a_0 \star \theta(a_1, a_2 \cdot a_3) \star \cdots) - T(a_0 \star \theta(a_1, a_2) \star \theta(a_2 \cdot a_3, a_4) \star \cdots) + \cdots \\ &= T(a_0 \star \theta(a_1, a_2) \star a_3 \star \theta(a_4, a_5) \star \cdots) - T(a_0 \star \theta(a_1, a_2) \star \theta(a_2 \cdot a_3, a_4) \star \cdots) + \cdots \\ &= 0.\end{aligned}$$

Since $(\tilde{b}\Phi_{2k})(a_0, \cdots, a_{2k+1}) = 0$ it follows that

$$\begin{aligned}(b\Phi_{2k})(a_0, \cdots, a_{2k+1}) &= \tilde{b}\Phi_{2k}(a_0, \cdots, a_{2k+1}) \\ &- T\left(\theta(a_0, a_1) \star \cdots \star \theta(a_{2k}, a_{2k+1})\right) + T\left(\theta(a_{2k+1}, a_0) \star \cdots \star \theta(a_{2k-1}, f_{2k})\right)\end{aligned}$$

$$
\begin{aligned}
&= -T\left(\theta(a_0,a_1)\star\cdots\star\theta(a_{2k},a_{2k+1})\right) + T\left(\theta(a_{2k+1},a_0)\star\cdots\star\theta(a_{2k-1},f_{2k})\right)\\
&= -\Phi_{2k+1}(1,a_0,a_1,\cdots,a_{2k+1}) + \Phi_{2k+1}(1,a_{2k+1},a_0\cdots,a_{2k})\\
&= -(B_0\Phi_{2k+2})(a_0,a_1,\cdots,a_{2k+1}) + (B_0\Phi_{2k+2})(a_{2k+1},a_0\cdots,a_{2k})\\
&= -\left(\frac{1}{k+1}B\Phi_{2k+2}\right)(a_0,a_1,\cdots,a_{2k+1}),
\end{aligned}
$$

where we have used the cyclicity of T w.r. to $\star$ and the fact that $\theta(1,\cdot)=\theta(\cdot,1)=0$.

The subsequent corollary is a direct consequence of the above theorem.

Corollary 5 *Under the assumptions of the above theorem, let us moreover asume that $\mathcal{A}=C_0^\infty(W)$ for some symplectic manifold (W,ω) and that $\star$ is associated with the symplectic form ω. Then for any $f_0,\cdots,f_{2k+1}\in C_0^\infty(W)$ we have:*

$$b\phi_{2k}(f_0,f_1,\cdots,f_{2k+1}) = T_k(\{f_0,f_1\}\cdots\{f_{2k},f_{2k+1}\}) - T_k(\{f_1,f_2\}\cdots\{f_{2k+1},f_0\}).$$

Now, let us assume that $b\phi_{2k}(f_0,f_1,\cdots,f_{2k+1})=0$ for all $f_0,\cdots,f_{2k+1}\in C_0^\infty(W)$. In particular this would hold true for $f_1=f_2=g_1$, $f_3=f_4=g_2\cdots,f_{2k-1}=f_{2k}=g_k,f_{2k+1}=f_0=g_0$ in which case we would have:

$$T_k(\{g_0,g_1\}\{g_1,g_2\}\cdots\{g_k,g_0\})=0\quad\forall g_i\in C_0^\infty(W).$$

Choosing $g_{2i}=f,g_{2i+1}=g\,\forall i$, this would lead to

$$T_k(\{f,g\}^k)=\int_W\{f,g\}^k\omega^l=0\quad\forall f,g\in C_0^\infty(W)$$

and hence $\{f,g\}=0\quad\forall f,g\in C_0^\infty(W)$ which is not to be expected! Therefore, ϕ_{2k} (and hence Φ_{2k}) cannot be expected to be Hochschild closed.

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