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Limit Theorems Related to a Class of Operator Semi-Selfsimilar Processes *†

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Abstract

We construct a class of operator semi-selfsimilar processes with stationary increments as limiting processes of scaled partial sums of some random walks in random scenery, studied by Kesten and Spitzer for selfsimilar processes' case. In the present paper, we generalize previous works related to problems of random walks in random scenery in the following two senses; we assume (i) random sceneries belong to the domain of partial normal attraction of strictly operator semi-stable distributions, and (ii) random walks belong to the domain of partial normal attraction of strictly semi-stable distributions. To consider such a problem, we also study local limit theorems related to semi-stable Lévy processes.

1 Introduction

In the fields of statistical physics and mathematical finance, many applications of selfsimilar processes are studied, and operator selfsimilar processes are useful to describe a model of multivariate phenomena such that each of components has different scalings in different directions and are possibly dependent of others. Semi-selfsimilarity is an extension of selfsimilarity and expected to offer higher flexibility to stochastic modeling of random phenomena because of its weaker scaling property. In such a reason, Maejima and Sato, and Becker-Kern studied operator semi-selfsimilar processes (whose definition is given later) with independent increments in [MaSa03] and [Be04]. On the other hand, as mentioned in [KS79], it is important to find selfsimilar processes which have stationary increments. It is also the case for semi-selfsimilar processes. However as far as authors know, operator semi-selfsimilar processes with stationary increments are not founded in the literature. The aim of the present paper is to obtain an integral

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representation of such processes and to show that some of them can be constructed as limiting processes from scaled partial sums of some random walks in random sceneries studied by Kesten and Spitzer in [KS79].

An $\mathbf{R}^d$-valued stochastic process $\{Y(t), t \geq 0\}$ is called *operator semi-selfsimilar with exponent* H if there exist $a > 1$ and a linear operator H on $\mathbf{R}^d$ such that

$$\{Y(at), t \geq 0\} \stackrel{f.d.}{=} \{a^H Y(t), t \geq 0\}, \tag{1.1}$$

where $\stackrel{f.d.}{=}$ means equality for all finite dimensional distributions and

$$a^H = \exp\{(\log a)H\} = \sum_{k=0}^{\infty} \frac{1}{k!}\{(\log a)H\}^k.$$

Set

$$s = \inf\{a > 1 : (1.1) \text{ satisfies.}\}.$$

In the case where $s = 1$, this process is nothing but operator selfsimilar process. We thus assume $s > 1$ and we use a notation, *operator* (s, H)*-semi-selfsimilar process*. When $H = hI$ for some $h > 0$, we omit the term "operator".

A probability measure μ on $\mathbf{R}^d$ is said to be full if its support is not contained in any $(d-1)$-dimensional hyperplane. We call a full probability measure μ on $\mathbf{R}^d$ *operator semi-stable*, if its characteristic function $\widehat{\mu}$ satisfies

$$\widehat{\mu}(z)^a = \widehat{\mu}(a^{B^*} z)e^{i\langle z,c\rangle}, z \in \mathbf{R}^d, \tag{1.2}$$

for some $a > 1$, an invertible linear operator B on $\mathbf{R}^d$ and $c \in \mathbf{R}^d$. Here B^* is an adjoint operator of B and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbf{R}^d$. If we can take $c = 0$, then we call μ strictly operator semi-stable. Such an operator B is not determined uniquely but its eigenvalues are determined. For real parts of eigenvalues of B we denote by T_B and τ_B their maximum one and minimum one, respectively. In the case where (1.2) is satisfied, we have $\tau_B \geq 1/2$, and if $\tau_B > 1/2$, then μ is purely non-Gaussian. We consider non-Gaussian case, since Gaussian case can be handled similarly to [KS79] and [Bo89]. Set

$$r = \inf\{a > 1 : (1.2) \text{ holds.}\}.$$

In the case where $r = 1$, μ is nothing but operator stable. We thus assume $r > 1$ and call μ in (1.2) and r above an *operator* (r, B)*-semi-stable distribution* and its *span*, respectively. When $B = \frac{1}{\alpha}I$, μ is called a d-dimensional α-semi-stable distribution, and we call it (r, α)*-semi-stable distribution*. It is known that semi-stable distributions can be characterized as certain limits of normalized partial sums of independent and identically distributed random variables. See Chapter 7 in [MS01] for more details about operator (semi-)stable distributions. For a full semi-stable distribution μ, we define the domain of partial normal attraction of μ as follows:

Definition 1.1 *Let* $\{X_i, i \in \mathbf{N}\}$ *be independent and identically distributed* $\mathbf{R}^d$*-valued random variables.* X_i*'s belong to the domain of partial attraction of operator* (r, B)*-semi-stable distribution* μ *with span* $r > 1$*, if there exist a sequence* $\{k_n\}$ *satisfying* $\lim_{n\to\infty} k_{n+1}/k_n = r^{n_0}$ *with some* $n_0 \in \mathbf{N}$*, a sequence of invertible linear operators* $\{A_n\}$ *on* $\mathbf{R}^d$ *and* $c_n \in \mathbf{R}^d$ *such that*

$$A_n^{-1}\left\{\sum_{i=1}^{k_n} X_i - c_n\right\} \xrightarrow{d} \mu, \tag{1.3}$$

where $\xrightarrow{d}$ *denotes weak convergence. And if* $c_n \equiv 0$*, then* $\{X_i\}$ *is said to belong to the domain of partial "normal" attraction of operator semi-stable distribution.*

Remark 1.1 *When* $B = \frac{1}{\alpha}I$*, we can take* $A_n = a_n I$ *for some* $a_n > 0$*.*

Kesten and Spitzer considered "Random walks in random scenery" in [KS79] as follows: Let $\mathbf{Z}$-valued random variables X_i's and $\mathbf{R}$-valued random variables $\xi(k)$'s belong to the domain of normal attraction of strictly α-stable ($\alpha \in (1, 2]$) distribution and that of strictly β-stable ($\beta \in (0, 2]$) distribution, respectively. Assume that they are independent and $E[X_1] = 0$. We set

$$W_l = \sum_{k=0}^{l} \xi(S_k), \tag{1.4}$$

where $S_k = \sum_{i=1}^{k} X_i$ and $S_0 = 0$. We define $W(t)$ by

$$W(t) = W_l + (t - l)(W_{l+1} - W_l) \quad l < t < l + 1. \tag{1.5}$$

Asymptotic behavior of $\{W_n\}$ is determined by two kinds of randomness, random walks $\{S_n\}$ and random scenery $\{\xi(k)\}$, and they imply an interesting selfsimilarity for a scaled limiting process of $\{W(t)\}$.

Certain extensions of the pioneer work of [KS79] have been considered, for example higher dimensional cases, [Sh95] and [Ma96]; and semi-selfsimilar case, [A01]. We follow [Ma96] and [A01], namely we consider the case where random sceneries belong to the domain of partial normal attraction of strictly operator semi-stable distributions. Maejima assumed that random sceneries belong to the domain of normal attraction of strictly operator stable distribution in [Ma96], and Arai considered the case where they belong to the domain of partial normal attraction of strictly semi-stable distribution in [A01]. Each of scaled limiting processes of $\{W(t)\}$ converges weakly to a operator selfsimilar process or a semi-selfsimilar process along full or subsequence, respectively. In each case, it is assumed that random walks belong to the domain of normal attraction of strictly stable distributions. Now we consider the case where not only sceneries but also random walks have semi-stability, namely we deal with two kinds of semi-stable randomness. Semi-stable distribution is a natural extension of stable one, and hence we consider such a case. However in our case, each semi-stability has own span, which would obstacle to deal with some pairs of two randomness. In the present paper, we give a sufficient condition to such a situation.

To describe our main theorem, we need more notations. We assume that X_i's and $\xi(k)$'s belong to the domains of partial normal attraction of strictly (r_1, α)-semi-stable ($\alpha \in (1, 2]$) and that of strictly operator (r_2, B)-semi-stable distribution, respectively. Semi-stable distributions are infinitely divisible, and to any infinitely divisible distribution there corresponds a Lévy process, which is a process having independent and stationary increments, stochastic continuity, and starting at 0. Lévy processes whose distributions at $t = 1$ coincide with semi-stable distributions are called *semi-stable Lévy processes*. We show that a suitably scaled process of $\{W(t)\}$ converges weakly to a process, which is expressed by a strictly operator (r_2, B)-semi-stable Lévy process $\{Z(x)\}$ and a local time of strictly (r_1, α)-semi-stable Lévy process $\{Y(t)\}$, $\{L_t(x)\}$ such that

$$\int_{-\infty}^{\infty} L_t(x) dZ(x), \tag{1.6}$$

along a subsequence.

However, as far as authors know, a construction of semi-stable Lévy processes from random walks, local limit theorems of such random walks and the existence of their local times have not appeared in the literature. In such reasons, we study these topics in Section 2. In Section 3, we construct operator semi-selfsimilar processes with stationary increments obtained from the technique of subordination like (1.6). This is an extension of selfsimilar case studied in [V87]. In Section 4, we consider the problems of random walks in random scenery.

2 (r, α)-semi-stable Lévy processes

As mentioned in the previous section, semi-stable distributions can be characterized as certain limits of normalized partial sums of independent and identically distributed random variables, and we show that (r, α)-semi-stable Lévy processes can also be constructed from sums of such random variables. In this section, we consider the case of (r, α)-semi-stable distributions. By using suitable slowly varying functions l_1, l_2 at ∞, we can take subsequences $\{k_n\}$ and $\{a_n\}$ in (1.3) such that

$$k_n = r^n l_1(r^n) \quad \text{and} \quad a_n = k_n^{1/\alpha} l_2(k_n), \tag{2.1}$$

respectively. In the following, we always assume that $\alpha \in (1, 2)$ unless specified. We also assume that C will be an absolute positive constant, which may differ with other C's.

2.1 Construction of semi-stable Lévy process from random walks

Let $\mathrm{D} = \mathrm{D}([0, \infty); \mathbf{R}^d)$ be the space of $\mathbf{R}^d$-valued right continuous functions on $(0, \infty)$ with left limits with the Skorohod topology. Then for (r, α)-semi-stable distributions we have the following:

Theorem 2.1 *Given independent and identically distributed $\mathbf{R}^d$-valued random variables $\{X_i\}_{i\in\mathbf{N}}$ on a probability space $(\Omega, \mathcal{F}, P)$. Assume that X_i's belong to the domain of partial attraction of strictly (r,α)-semi-stable distribution with $c_n = a_n E[a_n^{-1}X_1 I[|a_n^{-1}X_1| \le L]]$ for a fixed $L > 0$. Let $Y^n(t)$ be a function in* D *with a value*

$$Y^n(t) = \frac{1}{a_n}\sum_{i=1}^{[k_n t]} X_i - c_n \frac{[a_n t]}{a_n} \tag{2.1}$$

at t, where $[t]$ is the largest integer less than or equal to the real number t. We set $Y(0) = 0$. Then the random function defined by (2.1) *converges weakly to an (r,α)-semi-stable Lévy process $\{Y(t), t \ge 0\}$ starting at* 0 *in* D.

In the present paper, we consider more general cases in Section 4 for showing tightness of the family of random walks in random scenery and give outline of its proof here. We consider the case where $c_n \equiv 0$. Since X_i's belong to the domain of partial normal attraction of strictly (r,α)-semi-stable distribution, for each $t > 0$ we obtain

$$Y^n(t) = \frac{1}{a_n}\sum_{i=1}^{[k_n t]} X_i \xrightarrow{d} Y(t),$$

and this implies convergence of all finite dimensional distributions. We next consider the tightness of the family of $\{Y^n(t), t \ge 0, n \in \mathbf{N}\}$ in D. Since X_i's are assumed to be independent and identically distributed, it is enough to show that

$$\lim_{\delta\to 0}\limsup_{n\to\infty} P\left\{\sup_{0\le t\le\delta}\{|Y^n(t) - Y^n(0)|\} > \varepsilon\right\} = 0. \tag{2.2}$$

If (2.2) would be shown, then we can show that for each $T > 0$, any ε and a sufficiently small δ, there exists a constant $C > 0$ such that

$$\limsup_{n\to\infty} P\left\{\sup_{\substack{0\le t_1,t,t_2\le T\\ |t_2-t_1|\le\delta}}\{|Y^n(t) - Y^n(t_1)| \wedge |Y^n(t_2) - Y^n(t)|\} > \varepsilon\right\} \le \left(1 + \frac{1}{\delta}\right)(C\delta)^2$$

in the same way as that of α-stable Lévy process in Chapter 9 of [GS69], which completes the proof of the theorem.

To show (2.2), for fixed $L > 0$ and $n \in \mathbf{N}$ we let

$$X_j' = \begin{cases} X_j & \text{for} \quad |X_j| \le a_n L, \\ 0 & \text{for} \quad |X_j| > a_n L, \end{cases} \qquad X_j'' = \begin{cases} 0 & \text{for} \quad |X_j| \le a_n L, \\ X_j & \text{for} \quad |X_j| > a_n L. \end{cases}$$

Using them and Lemma 4.3 in [A01], we estimate a probability, $P\left\{\sup_{0\le t\le\delta}\{|Y^n(t) - Y^n(0)|\} > \varepsilon\right\}$.

2.2 Local limit theorems of semi-stable Lévy processes

We study properties of asymptotic behavior of $\mathbf{Z}$-valued random walks $\{S_n\}$ satisfying

- $E[X_i] = 0$
- for subsequences $\{k_n\}$ and $\{a_n\}$, $\{a_n^{-1} \sum_{i=1}^{k_n} X_i\}$ converges weakly to a strictly (r, α)-semi-stable random variable Y_α with $\alpha \in (1, 2]$.

The purpose of this subsection is to show the following local limit theorems.

Theorem 2.2 *We have the following:*

(i) $P\{S_l = 0\} = O\left(l^{-1/\alpha}\right)$ *for all large* $l \in \mathbf{N}$.

(ii) $\sum_{k=0}^{\infty} \{P\{S_k = 0\} - P\{S_k = u\}\} = O\left(|u|^{\alpha-1}\right)$ *for all large* $|u| \in \mathbf{N}$.

In the case where $\alpha = 2$ strictly (r, α)-semi-stable distribution is nothing but Gaussian, and this is already known (cf. Chapter 4 in [IL71]). Hence we consider the case where $1 < \alpha < 2$. To prove (i) of Theorem 2.2 we firstly calculate a characteristic function of X_i (we denote by $\lambda(z)$), which belong to the domain of partial normal attraction of strictly (r, α)-semi-stable distribution. Secondly, using the characteristic function λ, we prove local limit theorems of random walks along subsequences. Lastly, we prove for full sequence's case. Here we use Lévy-Khintchin representation of characteristic function of strictly (r, α)-semi-stable distribution (here we denote by φ) and the distribution function of X_i's (here we denote it by $F(x)$) given in [Me00] as follows:

∘ For $z \in \mathbf{R}$,

$$\varphi(z) = \exp\left\{\int_{-\infty}^{0} \left(e^{izx} - 1 - \frac{izx}{1+x^2}\right) d\left(\frac{M_L(x)}{|x|^\alpha}\right) + \int_0^\infty \left(e^{izx} - 1 - \frac{izx}{1+x^2}\right) d\left(\frac{M_R(x)}{x^\alpha}\right)\right\},$$

where M_L on $(-\infty, 0)$ and M_R on $(0, \infty)$ are non-negative, bounded, one of them has a strictly positive infimum and the other one either has a strictly positive infimum or is identically 0, and satisfy $M_L(r^{1/\alpha}x) = M_L(x)$ and $M_R(r^{1/\alpha}x) = M_R(x)$.

∘ For all large $|x|$,

$$F(x) = \begin{cases} (-x)^{-\alpha}\tilde{l}(-x)\{M_L(x) + h_L(-x)\} & x < 0, \\ 1 - x^{-\alpha}\tilde{l}(x)\{M_R(x) + h_R(x)\} & x > 0, \end{cases}$$

where $\tilde{l}$ is right continuous and slowly varying at ∞ defined by

$$x^{-\alpha}\tilde{l}(x) := \sup\{u : u^{-1/\alpha} l_2(u) > x\} \quad x > 0, \tag{2.3}$$

(recall that l_2 is the slowly varying function for the subsequence $\{a_n\}$ in (2.1)), and error functions h_L and h_R are right continuous and

$$h_L(a_n x_0) \to 0 \quad \text{and} \quad h_R(a_n x_0) \to 0 \quad \text{as } n \to \infty \tag{2.4}$$

at every continuity point x_0 of each of M_L and M_R, respectively.

To show Theorem 2.2, we require the following:

Lemma 2.1 *If X_i's belong to the domain of the partial normal attraction of strictly (r,α)-semi-stable distribution with sequences $\{k_n\}$ and $\{a_n\}$, then their characteristic function $\lambda(z)$ in the neighborhood of the origin is represented as*

$$|\lambda(z)| = \exp\{-\eta(z)|z|^{\alpha}\tilde{l}(1/|z|)\}, \tag{2.5}$$

where $\eta(z)$ is a nonnegative bounded continuous function satisfying $\eta(r^{1/\alpha}z) = \eta(z)$ and $\tilde{l}(\cdot)$ is a slowly varying function at ∞, which is determined by a representation of the distribution function of X_i.

Proof.

We follow the proof of α-stable case in Section 2.6 of [IL71]. In the neighborhood of the origin, we have

$$\log\lambda(z) = \log\{1 + (\lambda(z) - 1)\} = \{\lambda(z) - 1\} + O(|\lambda(z) - 1|^2) \quad \text{as } z \to \infty,$$

and we thus need to calculate $\lambda(z) - 1$. Set $F_-(x) = F(-x)$ with taking left continuous version of F. We assume $z > 0$, and in the case where $z < 0$ we can calculate similarly.

Recall that $E[X_i] = 0$. For a sufficiently small z there exists a $k \in (0,1)$ such that

$$\begin{aligned}
\lambda(z) - 1 &= \int_{-\infty}^{\infty}(e^{izx} - 1 - izx)dF(x) \\
&= -\int_0^{\infty}(e^{izx} - 1 - izx)d(1 - F(x)) - \int_0^{\infty}(e^{-izx} - 1 + izx)dF_-(x) \\
&= iz\int_0^{\infty}(e^{izx} - 1)(1 - F(x))dx - iz\int_0^{\infty}(e^{-izx} - 1)F_-(x)dx \\
&= i\int_0^{\infty}(e^{ix} - 1)(1 - F(x/z))dx - i\int_0^{\infty}(e^{-ix} - 1)F_-(x/z)dx \\
&= i\left\{\int_{z^k}^{\infty}(e^{ix} - 1)(1 - F(x/z))dx + \int_0^{z^k}(e^{ix} - 1)(1 - F(x/z))dx\right\} \\
&\qquad -i\left\{\int_{z^k}^{\infty}(e^{-ix} - 1)(F_-(x/z))dx + \int_0^{z^k}(e^{-ix} - 1)(F_-(x/z))dx\right\} \\
&\sim iz^{\alpha}\int_0^{\infty}(e^{ix} - 1)\frac{\tilde{l}(x/z)(M_R(x/z) + h_R(x/z))}{x^{\alpha}}dx \\
&\qquad -iz^{\alpha}\int_0^{\infty}(e^{-ix} - 1)\frac{\tilde{l}(x/z)(M_L(-x/z) + h_L(x/z))}{x^{\alpha}}dx \quad \text{as } z \to 0.
\end{aligned}$$

For a general slowly varying function $l(x)$ at ∞, the following fact is known:

Proposition 2.1 (*cf. Section* 2.6 *in* [IL71]) *We assume that $l(x)$ is a positive slowly varying function at ∞ and $x^{-\alpha}l(x)$ is monotone decreasing. For $1 < \alpha < 2$ we have*

$$\lim_{z\downarrow 0}\int_0^{\infty}\frac{e^{\pm ix} - 1}{x^{\alpha}}l(x/z)dx = \lim_{z\downarrow 0} l(1/z)\int_0^{\infty}\frac{e^{\pm ix} - 1}{x^{\alpha}}dx$$

$$= \lim_{z\downarrow 0} l(1/z)\exp\left\{\pm\frac{1}{2}i\pi(\alpha-1)\right\}\Gamma(1-\alpha).$$

In our case, $\tilde{l}(x/z)(M_L(-x/z)+h(x/z))$ and $\tilde{l}(x/z)(M_R(x/z)+h(x/z))$ satisfy the conditions for $l(x)$ above, and except on Lebesgue measure 0 sets $\lim_{z\to 0} h_L(x/z) = \lim_{z\to 0} h_R(x/z) = 0$. Thus we have, for positive z in the neighborhood of origin,

$$|\lambda(z)| = \exp\left\{-\eta(z)|z|^{\alpha}\tilde{l}(1/|z|)\right\},$$

where

$$\eta(z) = \{M_L(-1/z) + M_R(1/z)\}\cos\frac{\pi\alpha}{2}\Gamma(1-\alpha). \tag{2.6}$$

Since M's are periodic, (2.6) proves Lemma 2.1. ∎

We next prove local limit theorems for random walks along subsequences as follows:

Lemma 2.2 *Let $g_{k_n}(x)$ be the density of strictly (r,α)-semi-stable distribution μ, that is,*

$$g_{k_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixz}\varphi(z)dz.$$

Then

$$\lim_{n\to\infty}\sup_{u\in\mathbf{Z}} |a_n P\{S_{k_n} = u\} - g_{k_n}(u/a_n)| = 0.$$

Proof.

The characteristic function of S_{k_n} is given by

$$\lambda(z)^{k_n} = \sum_{u\in\mathbf{Z}} e^{iuz} P\{S_{k_n} = u\}.$$

This implies

$$\begin{aligned} P\{S_{k_n} = u\} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iuz}\lambda(z)^{k_n}dz \\ &= \frac{1}{2\pi a_n}\int_{-\pi a_n}^{\pi a_n} e^{-iuz/a_n}\lambda(z/a_n)^{k_n}dz. \end{aligned}$$

For any $u\in\mathbf{Z}$ we have

$$|a_n P\{S_{k_n} = u\} - g_{k_n}(u/a_n)| \le \frac{1}{2\pi}(I_1+I_2+I_3+I_4),$$

where

$$\begin{aligned} I_1 &= \int_{-A}^{A} \left|\lambda(z/a_n)^{k_n} - \varphi(z)\right| dz, \\ I_2 &= \int_{A\le|z|\le\varepsilon a_n} |\lambda(z/a_n)|^{k_n} dz, \\ I_3 &= \int_{\varepsilon a_n\le|z|\le\pi a_n} |\lambda(z/a_n)|^{k_n} dz, \\ I_4 &= \int_{|z|>A} |\varphi(z)|\, dz, \end{aligned}$$

and constants A and ε are determined later.

We turn now the estimation of each integral.

(I_1): Since X_i's belong to the domain of partial normal attraction of strictly α-semi-stable distribution, I_1 converges to zero as $n \to \infty$.

(I_3): Since X_i's are $\mathbf{Z}$-valued, Theorem 1.4.2 of [IL71] implies that $|\lambda(z)| < 1$ for $0 < z < 2\pi$, and thereby a positive constant c such that $|\lambda(z)| \le e^{-c}$ for $\varepsilon \le |z| \le 2\pi$ can be taken. This implies

$$\begin{aligned} I_3 &= \int_{\varepsilon a_n}^{\pi a_n} |\lambda(z/a_n)|^{k_n}\, dz \\ &\le 2\pi e^{-ck_n} a_n \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

(I_4): $|\varphi(z)|$ is integrable on $\mathbf{R}$, and this implies $\lim_{A\to\infty} I_4 = 0$.

(I_2): By Karamata's theorem there exists a function $\varepsilon(u) \to 0$ as $u \to \infty$ such that as $n \to \infty$

$$\begin{aligned} \frac{\tilde{l}(a_n/|z|)}{\tilde{l}(a_n)} &= \exp\left\{ -\int_{a_n}^{a_n/|z|} \frac{\varepsilon(u)}{u} du \right\} (1 + o(1)) \\ &\le |z|^{\varepsilon_1}(1 + o(1)) \end{aligned}$$

with a small ε_1. Since (2.3) implies that $\lim_{n\to\infty} k_n a_n^{-\alpha} \tilde{l}(a_n) = 1$, for sufficiently large k_n and a_n and $\delta \le \alpha - \varepsilon_1$ there exists a positive constant $c(\delta)$ not depending on n such that

$$|\lambda(z/a_n)|^{k_n} = \exp\left\{ -\frac{\eta(z) k_n}{a_n^{\alpha}} \tilde{l}(a_n) |z|^{\alpha} \frac{\tilde{l}(a_n/|z|)}{\tilde{l}(a_n)} \right\} \le \exp\{-c(\delta)|z|^{\delta}\}.$$

These arguments imply that for a sufficiently large k_n such that for sufficient small $\varepsilon > 0$

$$I_2 \le \int_{A \le |z| \le \varepsilon a_n} \exp\{-c(\alpha/2)|z|^{\alpha/2}\} dz \le \int_{|z|\ge A} \exp\{-c(\alpha/2)|z|^{\alpha/2}\} dz$$

can be shown, and this implies that $I_2 \to 0$ as $A \to \infty$. Hence we have shown that each integral can be made arbitrarily small, and (2.2) follows. ∎

Remark 2.1 *In the case where $\alpha \in (0, 1]$, the assertions in Lemmas* 2.1 *and* 2.2 *are also valid. They are shown in similar ways to those for showing the lemmas above and Theorems* 2.6.5 *and* 4.2.1 *in* [IL71].

To show the full sequence's case, we need the following lemma:

Lemma 2.3 *Let μ and $\widetilde{\mu}$ be strictly (r, α)-semi-stable distributions as scaled limit of sums of X_i's with a pair of subsequences in* (1.3), *$\{k_n\}, \{a_n\}$ and $\{\widetilde{k}_n\}, \{\widetilde{a}_n\}$, respectively. Denote by $\varphi(z)$ and $\widetilde{\varphi}(z)$ the characteristic functions of μ and $\widetilde{\mu}$, respectively. If $\lim \widetilde{k}_n/k_n = \theta < \infty$, then there exists a positive constant $\widetilde{\theta} = \lim a_n/\widetilde{a}_n$ such that*

$$\widetilde{\varphi}(z) = \varphi(\widetilde{\theta} z)^{\theta}. \tag{2.7}$$

Proof.

Recall that to the distribution μ there corresponds a strictly (r, α)-semi-stable Lévy process, which we denote by $\{Y(t)\}$, namely, $a_n^{-1} \sum_{i=1}^{[k_n t]} X_i$ converges weakly to $\{Y(t)\}$ in $D([0, \infty); \mathbf{R})$. Then we have

$$\frac{1}{\widetilde{a}_n} \sum_{i=1}^{\widetilde{k}_n} X_i = \frac{a_n}{\widetilde{a}_n} \frac{1}{a_n} \sum_{i=1}^{\widetilde{k}_n} X_i \xrightarrow{d} \widetilde{\theta} Y(\theta),$$

and this implies $\widetilde{\mu}$ coincides with the distribution of $\widetilde{\theta} Y(\theta)$. Since $Y(t)$ is a Lévy process, whose distribution at each $t > 0$ can be represented by t-convolution of μ. This implies (2.7). ∎

Proof of (i) *of Theorem* 2.2.

From Lemma 2.2, the following estimation is satisfied:

$$P\{S_{k_n} = u\} = O\left(\frac{g_{k_n}(u)}{a_n}\right) \quad \text{as } n \to \infty,$$

and Lemma 2.3 implies that the estimation above is also valid for other subsequences. Now the characteristic function $\varphi(z)$ belongs to $L^1(\mathbf{R})$, and general theory of Fourier transformation implies that $g_{k_n}(x)$ is uniformly continuous and that $g_{k_n}(0)$ is bounded. (2.7) implies that for each subsequence $\{\widetilde{k}_n\}$ $g_{\widetilde{k}_n}(0)$ is also bounded. These arguments show (i) of Theorem 2.2. ∎

Proof of (ii) *of Theorem* 2.2.

Let $a(u)$ be a potential kernel of random walk S_k defined by

$$a(u) = \sum_{k=0}^{\infty} \left\{ P\{S_k = 0\} - P\{S_k = u\} \right\}.$$

Since $\{S_n\}$ is recurrent for $\alpha > 1$, the following is satisfied for all large u (see Section 28 P4 in [Sp76]):

$$a(u) + a(-u) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1 - \cos zu}{1 - \lambda(z)} dz.$$

Recall (2.5), which is the representation of $\lambda(z)$ in the neighborhood of the origin. Then there exists a slowly varying function $l'(1/|z|)$ at ∞, which is determined by $\tilde{l}$ such that

$$|1 - \lambda(z)| = |z|^{\alpha} |\eta(z) l'(1/|z|)|,$$

where $\eta(z)$ is in (2.6). Hence there exist some constants C and C' such that for a sufficiently small ε

$$\begin{aligned} |a(u) + a(-u)| &= \left| \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \frac{1 - \cos zu}{1 - \lambda(z)} dz \right| + C \\ &\leq \frac{1}{\pi} \int_{-\varepsilon}^{\varepsilon} \left| \frac{1 - \cos zu}{1 - \lambda(z)} \right| dz + C \\ &= \frac{1}{\pi} \int_{-1/u}^{1/u} \frac{z^2 u^2}{|z|^{\alpha} |\eta(z) l'(1/|z|)|} dz + C' \\ &= O\left(u^{\alpha - 1}\right) \quad \text{as } u \to \infty. \end{aligned}$$

This implies (ii) of Theorem 2.2. ∎

2.3 Local times of semi-stable Lévy processes

We study properties of local times for (r, α)-semi-stable Lévy processes, and consider the case where $d = 1$ in this subsection. It is known that a strictly α-stable Lévy process has a local time at x, $L(t, x)$ with $\alpha > 1$, and we can take a version of $L(t, x)$ which is jointly continuous in (x, t) with $\alpha > 1$ (see [GK72]). In the case of (r, α)-semi-stable Lévy processes, we have the following.

Theorem 2.3 *An α-semi-stable Lévy process $\{Y(t), t \geq 0\}$ has a local time at x, $L(t, x)$ with $\alpha > 1$, and there exists a jointly continuous version of $L(t, x)$. If $\alpha = 1$ and $\{Y(t), t \geq 0\}$ is not strictly 1-semi-stable, then it has a local time but does not have its continuous version.*

From now on we denote by $L_t(x)$ a continuous version of such a local time.

Proof.

Theorem 7.5 in [Sa99a] implies that almost all sample functions of $\{Y(t)\}$ have the following properties: (i) 0 is regular for $\{0\}$ (see Section 7 in [Sa99a] or Section 43 in [Sa99b] for the definition of "regular"), (ii) for all x, y we have $P^x\{Y(t) = y \text{ for some } t \geq 0\} > 0$, where P^x is the law of $Y(t)$ starting at x. They ensure that, for each x, a local time $L(t, x)$ exists (see [GK72]), and at any fixed point it is continuous as a function of t almost surely.

We next show its joint continuity. Proposition 14.9 and 24.20 in [Sa99b] imply that for each $t > 0$ $Y(t)$ is determined as follows:

$$E[\exp\{izY(t)\}] = e^{-t\phi(z)}, \quad \phi(z) = |z|^\alpha\{\eta_1(z) + i\eta_2(z)\} - icz, \tag{2.8}$$

where $z \in \mathbf{R}$, $\eta_1(z)$ is bounded from below by a positive constant and continuous in $\mathbf{R} \setminus \{0\}$ satisfying $\eta_1(r^{1/\alpha}z) = \eta_1(z)$, $\eta_2(z)$ is a real function continuous in $\mathbf{R}\setminus\{0\}$ satisfying $\eta_2(r^{1/\alpha}z) = \eta_2(z)$. By Theorem 4 in [GK72], it is enough to show that

$$\sum_{n=1}^{\infty}\{\delta(2^{-n})\}^{1/2} < \infty, \tag{2.9}$$

where

$$\delta(u) = \sup_{|x|\leq u} \frac{1}{\pi}\int_{-\infty}^{\infty}(1 - \cos xz)\mathrm{Re}\left\{\frac{1}{1 + \phi(z)}\right\}dz.$$

To show (2.9), we use a similar way to that for proving Theorem 7.4 in [Sa99a]. It is known that there exist positive constants k_1 and k_2 such that $k_1 \leq \eta_1(z) \leq k_2$. Using them, we obtain

$$\begin{aligned}\mathrm{Re}\left\{\frac{1}{1 + \phi(z)}\right\} &= \frac{1 + |z|^\alpha\eta_1(z)}{\{1 + |z|^\alpha\eta_1(z)\}^2 + \{|z|^\alpha\eta_2(z) - cz\}^2} \\ &\leq \frac{1 + k_2|z|^\alpha}{k_1^2|z|^{2\alpha}}.\end{aligned}$$

This implies that for each $x > 0$

$$\begin{aligned} & \frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos xz)\mathrm{Re}\left\{\frac{1}{1+\phi(z)}\right\}dz \\ \leq\ & \frac{x^2}{\pi}\int_{-1/x}^{1/x} z^2\frac{1+k_2|z|^\alpha}{k_1^2|z|^{2\alpha}}dz + \frac{2}{\pi}\int_{|z|>1/x}\frac{1+k_2|z|^\alpha}{k_1^2|z|^{2\alpha}}dz \\ =\ & O(x^{\alpha-1}) \qquad \text{as } x\to 0, \end{aligned}$$

and we can take a continuous version of a local time.

In the case of not strictly 1-semi-stable Lévy process $\eta_1(z)$ in (2.8) is same, but $\eta_2(z)$ is not; $\eta_2(z)$ is given by

$$\eta_2(r^{1/\alpha}z) = \eta_2(z) + \mathrm{sgn} z\int_{1<|x|\leq r^{1/\alpha}} x\nu(dx),$$

where $\nu(dx)$ is Lévy measure and $M := \int_{1<|x|\leq r^{1/\alpha}} x\nu(dx) \neq 0$. Remark that the constant which is larger than $\eta_2(z)$ for any $z \in \mathbf{R}$ does not exist. In this case, we have $|\eta_2(z)| \sim \frac{|M|}{\log r^{1/\alpha}}\log|z|$ as $|z| \to \infty$ (see page 312 in [Sa99b]). To show the assertion, it is enough to show that

$$\limsup_{\kappa\to\infty}(\log\kappa)\int_{-\infty}^{\infty}\mathrm{Re}\left\{\frac{1}{\kappa+\phi(z)}\right\}dz > 0 \tag{2.10}$$

(see Theorem 4 in [GK72]). Hence

$$\begin{aligned} \mathrm{Re}\left\{\frac{1}{\kappa+\phi(z)}\right\} &\geq \frac{k_1|z|}{(\kappa+|z|k_2)^2+(|z|\eta_2(z)-cz)^2} \\ &\sim \frac{k_1(\log r^{1/\alpha})^2}{M^2}\frac{1}{z(\log|z|)^2} \quad \text{as } |z|\to\infty, \end{aligned}$$

and we have

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{Re}\left\{\frac{1}{\kappa+\phi(z)}\right\}dz &\geq \int_{|z|\geq\kappa}\frac{k_1|z|}{(\kappa+|z|k_2)^2+(|z|\eta_2(z)-cz)^2} \\ &\sim \frac{k_1(\log r^{1/\alpha})^2}{M^2}\frac{1}{\log\kappa} \quad \text{as } \kappa\to\infty, \end{aligned}$$

which concludes (2.10). ∎

3 Semi-selfsimilar integrated processes

As mentioned in [V87], many selfsimilar processes with stationary increments are obtained from basic $\mathbf{R}^d$-valued selfsimilar processes $\{Y(s)\}$ by the following integral:

$$D(t) = \int_{-\infty}^{\infty} K(t,x)dY(x) \quad t\geq 0, \tag{3.1}$$

where $\{K(t,s)\}$ is a deterministic or random function on $[0,\infty)\times\mathbf{R}$ with values in $[-\infty,\infty]$, provided that the integral can be defined in some stochastic integration. If K is random, it is usually assumed to be independent of X.

An $\mathbf{R}$-valued stochastic process $\{K(t,x)\}$, regarded as random functions of s and t, is called (h_1, h_2)-semi-selfsimilar if there exists $a > 1$ such that

$$\{K(at, a^{h_2}x)\} \stackrel{f.d.}{=} \{a^{h_1}K(t,x)\}. \tag{3.2}$$

Set

$$s = \inf\{a > 1 : (3.2) \text{ satisfies.}\},$$

and we use a notation $(s,(h_1,h_2))$-*semi-selfsimilar process*. Following theorem corresponds to Theorem 7.1 in [V87], which is the case of selfsimilar processes. Since we consider 1-dimensional random function $K(t,x)$ and d-dimensional process $Y(x)$, we define a d-dimensional process $D(t)$ in (3.1) by its component,

$$D^{(i)}(t) = \int_{-\infty}^{\infty} K(t,x)dY^{(i)}(x).$$

Theorem 3.1 *We assume the following:*

(i) $\{K(t,x)\}$ *is* $(s_1,(h_1,h_2))$*-semi-selfsimilar,*

(ii) $\{Y(x)\}$ *is operator* (s_2, H_3)*-semi-selfsimilar,*

(iii) $h_2 \log s_1/\log s_2 \in \mathbf{Q}$,

(iv) $\{K(t,x)\}$ *and* $\{Y(x)\}$ *are independent.*

Then there exists $s_0 = s_0(h_2, s_1, s_2)$ *such that* $\{D(t)\}$ *in* (3.1) *is an operator* $(s_0,(h_1I + h_2H_3))$*-semi-selfsimilar. We say that* $\{K(t,x)\}$ *has stationary increments if there are random variables* $w(b)$ *and* t *such that for* $b, t \geq 0$ *and* $s \in \mathbf{R}$*,*

$$K(b+t,x) - K(b,x) \stackrel{d}{\sim} K(t, x + w(b)), \tag{3.3}$$

where $\stackrel{d}{\sim}$ *means the equality of the marginal distribution. If we also assume that* $\{K\}$ *and* $\{Y\}$ *have stationary increments, then* $\{D\}$ *has stationary increments.*

Proof.

Semi-selfsimilarity Definition of K implies the following semi-selfsimilarity:

$$\{K(t,x)\} \stackrel{f.d.}{=} \left\{s_1^{-h_1}K(s_1t, s_1^{h_2}x)\right\}.$$

Then we have

$$\{D(t), t \geq 0\} \stackrel{f.d.}{=} \left\{s_1^{-h_1}\int_{-\infty}^{\infty} K(s_1t, s_1^{h_2}x)dY(x), t \geq 0\right\}.$$

Assumption (iii) implies that there exists an irreducible fraction q/p such that $s_2^q = s_1^{ph_2}$, and it can be shown that

$$\begin{aligned}\{Y(x), x \in \mathbf{R}\} &\overset{f.d.}{=} \{s_2^{-qH_3} Y(s_2^q x), x \in \mathbf{R}\} \\ &= \{s_1^{-ph_2 H_3} Y(s_1^{ph_2} x), x \in \mathbf{R}\}.\end{aligned}$$

Setting $s_0 = s_1^p$, we obtain

$$\begin{aligned}\{D(t), t \geq 0\} &\overset{f.d.}{=} \left\{\int_{-\infty}^{\infty} s_1^{-ph_1 I} K(s_1^p t, s_1^{ph_2} x) s_1^{-ph_2 H_3} dY(s_1^{ph_2} x), t \geq 0\right\} \\ &\overset{f.d.}{=} \left\{s_0^{-(h_1 I + h_2 H_3)} D(s_0 t), t \geq 0\right\}.\end{aligned}$$

Stationary increments From increments of $K(t,x)$ and $Y(x)$ are stationary, we prove stationary increments of $\{D(t)\}$,

$$\begin{aligned}\{D(b+t) - D(b)\} &= \left\{\int_{-\infty}^{\infty} (K(b+t,x) - K(b)) dY(x)\right\} \\ &\overset{f.d.}{=} \left\{\int_{-\infty}^{\infty} K(t, x + w(b)) dY(x)\right\} \quad \text{(by (3.3))} \\ &= \left\{\int_{-\infty}^{\infty} K(t, x') dY(-w(b) + x')\right\} \quad \text{(setting } x' = x + w(b)) \\ &\overset{f.d.}{=} \left\{\int_{-\infty}^{\infty} K(t, x') dY(x')\right\} \quad \text{(stationary increment of } \{Y(x)\}) \\ &= \{D(t)\}.\end{aligned}$$

This implies the process $\{D(t)\}$ has stationary increments. ∎

An example of random function in (3.1) is a local time of a strictly α-semi-stable process. In the next section, we consider a problem for the case.

4 Random walks in random scenery

In this section, we assume that slowly varying functions in (2.1) are $l_i \equiv 1, i = 1, 2$. Let $\{S_k, k = 0, 1, 2, \ldots\}$ be a $\mathbf{Z}$-valued random walks such that $\{r_1^{-n/\alpha} S_{[r_1^n t]}, t \geq 0\}$ converges weakly to a strictly (r_1, α)-semi-stable Lévy process $\{Y(t), t \geq 0\}$ with $1 < \alpha \leq 2$ and $E[Y(1)] = 0$. We also let $\{\xi(u), u \in \mathbf{Z}\}$ independent identically distributed $\mathbf{R}^d$-valued random variables, independent of $\{S_k\}$, belonging to the domain of partial normal attraction of strictly operator (r_2, B)-semi-stable random variable Z_B, namely $r_2^{-nB} \sum_{k=1}^{[r_2^n]} \xi(k)$ converges weakly to Z_B. We use the following representation of the characteristic function of purely non-Gaussian operator (r, B)-semi-stable distribution $\widehat{\mu}$ in (1.2) given in [Ch87]:

$$\widehat{\mu}(z) = \exp\left\{\int_{S_B} \gamma(dx) \int_0^{\infty} \left[e^{i\langle z, s^B x\rangle} - 1 - i\langle z, s^B x\rangle I[s^B x \in D]\right] d\left(-\frac{H_x(s)}{s}\right) + i\langle z, c\rangle\right\}, \qquad (4.1)$$

where $S_B = \{x \in \mathbf{R}^d : \|x\| = 1, \|t^B x\| > 1 \text{ for any } t > 1\}$ with Euclidean norm $\|\cdot\|$, $D = \{x \in \mathbf{R}^d : \|x\| \leq 1\}$, γ is a finite measure on S_B, and $H_x(s)$ is a non-negative function such that

(1) $H_x(s)/s$ is non-increasing in s for each x,

(2) $H_x(s)$ is right-continuous in s for each x and measurable in x for each s,

(3) $H_x(1) = 1$,

(4) $H_x(rs) = H_x(s)$.

As mentioned in Section 1, we assumed that Z_B is purely non-Gaussian. When for real parts of eigenvalues of B satisfy $\tau_B \leq 1 \leq T_B$, we need "symmetry condition", that is, the distribution of $\xi(0)$ is same as that of $-\xi(0)$.

For two kinds of randomness, we consider a strongly dependent sequence $\{\xi(S_n)\}$, its partial sum $W_l = \sum_{k=0}^{l} \xi(S_k)$ and the process $\{W(t), t \geq 0\}$ in (1.5). Let $\{Z(x), x \in \mathbf{R}\}$ be an $\mathbf{R}^d$-valued strictly operator (r_2, B)-semi-stable Lévy process, whose distribution of $Z(1)$ is the same as that of Z_B, independent of strictly (r_1, α)-semi-stable process $\{Y(t)\}$. By Theorem 2.3, we can take a version of local time of $\{Y(t)\}$, which is continuous in (t, x), and denote by $L_t(x)$. Hence we can define a stochastic integral

$$\Delta(t) = \int_{-\infty}^{\infty} L_t(x) dZ(x).$$

Then we have the following theorem.

Theorem 4.1 *Let $H = (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}B$. If $\log r_1/(\alpha \log r_2) \in \mathbf{Q}$, then there exists $r_0 = r_0(r_1, r_2, \alpha)$ such that $\{r_0^{-nH} W(r_0^n t), t \geq 0\}$ converges weakly in $\mathrm{C}([0,\infty); \mathbf{R^d})$ to the operator (r_0, H)-semi-selfsimilar process $\{\Delta(t), t \geq 0\}$, which has stationary increments.*

Remark 4.1 *In the case where $0 < \alpha < 1$ $\{S_n\}$ is transient, and we omit the case (see page 9 in* [KS79]*). In the case where $\alpha = 1$ $\{L_t(x)\}$ does not have its continuous version, and we cannot define $\Delta(t)$ by a stochastic integral. On the other hand $\{S_n\}$ is recurrent, and this is the case for* [Bo89].

Since $\{Y(t)\}$ is a strictly (r_1, α)-semi-stable Lévy process, it has semi-selfsimilarity, $\{Y(r_1^n t)\} \overset{\mathrm{d}}{=} \{r_1^{n/\alpha} Y(t)\}$. We thus have the following semi-selfsimilarity of its occupation time in the space interval $[a, b)$, $\Gamma_t(a,b) = \int_0^t \mathbf{1}_{[a,b)}(Y(s)) ds$:

$$\{\Gamma_{r_1^n t}(a,b), t \geq 0\} \overset{d}{=} \left\{ r_1^n \Gamma_t(r_1^{-n/\alpha} a, r_1^{-n/\alpha} b), t \geq 0 \right\} \qquad \text{for each } n \in \mathbf{Z},$$

where $\overset{d}{=}$ denotes the equality of all finite dimensional distributions with respect to the probability measure of $\{Y(t)\}$ on $\mathrm{D}([0,\infty); \mathbf{R})$, and through a simple calculate its local time $\{L_t(x)\}$ has the following $(r_1, (1-$

$1/\alpha, 1/\alpha)$)-semi-selfsimilarity:

$$\{L_t(x), t \geq 0\} \overset{d}{=} \left\{r_1^{-n(1-1/\alpha)} L_{r_1^n t}(r_1^{n/\alpha} x), t \geq 0\right\} \qquad \text{for each } n \in \mathbf{Z}.$$

Hence we can take an r_0 in the same way to that of taking s_0 in Theorem 3.1. Namely, if we denote the irreducible fraction $q/p := \log r_1/(\alpha \log r_2)$ and set $r_0 = r_1^p = r_2^{\alpha q}$, then $\{\Delta(t)\}$ has $(r_0, (1 - \frac{1}{\alpha})I + \frac{1}{\alpha}B)$-semi-selfsimilarity. Moreover, since $\{Y(t)\}$ has stationary independent increments and a spatially homogeneous transition function, we can show that $\{L_t(x)\}$ has stationary increments (see (3.3) for its definition) in a similar way to that of showing for the case of α-stable Lévy process in [L85]. They imply that $\{\Delta(t)\}$ has stationary increments.

We prove the rest of Theorem 4.1 by showing the following propositions under the same assumption as those in the theorem.

Proposition 4.1

$$\left\{r_0^{-nH} W(r_0^n t), t \geq 0\ \right\} \overset{\mathcal{L}}{\Longrightarrow} \{\Delta(t), t \geq 0\} \qquad \text{as } n \to \infty,$$

where $\overset{\mathcal{L}}{\Longrightarrow}$ *denotes convergence of all finite dimensional distributions with respect to the product measure between the probability measures of* $\{Y(t)\}$ *and* $\{Z_B(x)\}$.

Proposition 4.2 *The family* $\left\{r_0^{-nH} W(r_0^n t), t \geq 0,\ n \in \mathbf{N}\right\}$ *is tight in* $\mathrm{C}([0, \infty); \mathbf{R}^d)$.

4.1 Proof of Proposition 4.1

Let $N(l, u)$ be the number of visits of the random walk $\{S_k\}$ to the point $u \in \mathbf{Z}$ in the time interval $[0, l]$. Using this, we can represent W_l as

$$W_l = \sum_{k=0}^{l} \xi(S_k) = \sum_{u \in \mathbf{Z}} N(l, u)\xi(u). \tag{4.2}$$

For the occupation time $N(l, u)$ we consider their linear interpolation:

$$N_t(u) = N(l, u) + (t - l)\{N(l + 1, u) - N(l, u)\}, \quad l < t < l + 1.$$

Using (i) of Theorem 2.2, we can show the following properties of N_t in the same way as that in [KS79]:

Lemma 4.1 (i) *For each* $p \geq 1$, $\sup_{u \in \mathbf{Z}} E[N_t(u)^p] = O(t^{p(1-1/\alpha)})$.

(ii) *For* $t > 0$, $P\{N_t(u) > 0$ *for some* u *with* $|u| > A(t + 1)^{1/\alpha}\} \leq \varepsilon(A)$, *where* $\varepsilon(A) \to 0$ *as* $A \to \infty$ *and* $\varepsilon(A)$ *is independent of* t.

For $-\infty < a < b < \infty$ we set

$$T_t^n(a,b) = r_0^{-n} \sum_{a \le r_0^{-n/\alpha} u < b} N_{r_0^n t}(u), \quad \Gamma_t(a,b) = \int_a^b L_t(x)dx. \tag{4.3}$$

Then for each $k \in \mathbf{N}$ and $t_1, t_2, t_3, \ldots, t_k > 0$, Theorems 2.1 and 2.3 imply the following convergence with respect to the measure of $\{Y(t)\}$ (see Section 2 in [KS79]):

$$\{T_{t_j}^n(a_j, b_j), j \in \mathbf{N}\} \xrightarrow{d} \{\Gamma_{t_j}(a_j, b_j), j \in \mathbf{N}\}. \tag{4.4}$$

In this section we consider two kinds of randomness, hence redenote $\widehat{\mu}$ by $\varphi_B(z)$, and let $f(z) = \log \varphi_B(z)$. This $f(z)$ has the following property, which is shown in the same way as that for showing Lemma 4 in [Ma96] or Lemma 3.1 in [A01]:

Let $\beta = 1$ when $T_B < 1$, and let $0 < \beta < 1/T_B$ when $T_B \ge 1$. Then for any $z_1, z_2 \in \mathbf{R}^d$, there exists some constant $K > 0$ such that

$$|f(z_1) - f(z_2)| \le K \left\{ \|z_1 - z_2\|(1 + \|z_1\| + \|z_2\|) + \|z_1 - z_2\|^\beta \right\}. \tag{4.5}$$

Since $\{Z_B(x)\}$ is a Lévy process, we can show the following equality related to joint distributions of $\Delta(t)$ in the same way as that showing Lemma 4 in [Ma96]:

For any $k \in \mathbf{N}, t_1, t_2, t_3, \ldots, t_k > 0$ and $z_1, z_2, z_3, \ldots, z_k \in \mathbf{R}^d$,

$$E\left[\exp\left\{ i \sum_{j=1}^k \langle z_j, \Delta(t_j)\rangle \right\}\right] = E\left[\exp\left\{ \int_{-\infty}^{\infty} f\left(\sum_{j=1}^k L_{t_j}(u) z_j\right) du \right\}\right]. \tag{4.6}$$

We next prepare for showing convergence of all finite dimensional distributions. The following is shown in the same way as those for showing Lemma 6 in [KS79], Lemma 5 in [Ma96] and Lemma 2.5 in [A01].

Lemma 4.2 *For any $k \in \mathbf{N}$, $t_1, t_2, t_3, \ldots, t_k > 0$ and $z_1, z_2, z_3, \ldots, z_k \in \mathbf{R}^d$,*

$$\sum_{u \in \mathbf{Z}} f\left(r_0^{-nH^*} \sum_{j=1}^k N_{r_0^n t_j}(u) z_j \right) \xrightarrow{d} \int_{-\infty}^{\infty} f\left(\sum_{j=1}^k L_{t_j}(u) z_j\right) du,$$

where H^* is an adjoint operator of H.

Outline of proof.

Since $\varphi_B(z)^{r_0^n} = \varphi_B(r_0^{nB^*} z)$ and $r_0^{-nH^*} = r_0^{-n(1-1/\alpha)} \cdot r_0^{-\frac{n}{\alpha}B^*}$ for any $z \in \mathbf{R}$, we have

$$\sum_{u \in \mathbf{Z}} f\left(r_0^{-nH^*} \sum_{j=1}^k N_{r_0^n t_j}(u) z_j \right) = \sum_{u \in \mathbf{Z}} {r_0}^{-n/\alpha} f\left(r_0^{-n(1-1/\alpha)} \sum_{j=1}^k N_{r_0^n t_j}(u) z_j \right).$$

Hence it is enough to show that

$$\sum_{u \in \mathbf{Z}} {r_0}^{-n/\alpha} f\left(r_0^{-n(1-1/\alpha)} \sum_{j=1}^k N_{r_0^n t_j}(u) z_j \right) \xrightarrow{d} \int_{-\infty}^{\infty} f\left(\sum_{j=1}^k L_{t_j}(u) z_j\right) du. \tag{4.7}$$

Fixing small $\tau > 0$ and large M, we define

$$\begin{aligned} A_{n,l} &:= \left\{ u \in \mathbf{Z} : l\tau {r_0}^{n/\alpha} \le u < (l+1)\tau {r_0}^{n/\alpha} \right\}, \quad l \in \mathbf{Z}. \\ g_j &:= N_{r_0^n t_j}. \\ h_j &:= \frac{1}{\tau r_0^{n/\alpha}} \sum_{y \in A_{n,l}} N_{r_0^n t_j}(y). \end{aligned}$$

Using (ii) of Theorem 2.2, we obtain

$$\max_{-M \le l < M} \max_{u \in A_{n,l}} E\left[|g_j - h_j|^2 \right] \le C\tau^{\alpha-1} (r_0^n)^{2-2/\alpha} \tag{4.8}$$

in the same way as that for showing (3.9) in [KS79], and divide the left hand side of (4.7) into some parts, which are determined by τ and M above, and show the convergence with using Lemma 4.1, (4.5), (4.8), and continuity and compact support properties of local times of strictly (r_1, α)-semi-stable Lévy processes. ∎

Proof of Proposition 4.1.

By (4.2) we have

$$I_n := E\left[\exp\left\{ i \sum_{j=1}^{k} \langle z_j, r_0^{-nH} W_{r_0^n t_j} \rangle \right\} \right] = E\left[\prod_{u \in \mathbf{Z}} \lambda_B \left(r_0^{-nH^*} \sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j \right) \right].$$

To show its convergence, we need more preparations. The following convergence is shown in a similar way to that for showing Lemma 6 in [Ma96] by using (i) and (ii) of Lemma 4.1:

$$\lim_{s \to \infty} \sup_{u \in \mathbf{Z}} N_s(u) s^{-H^*} z = 0 \quad \text{in probability.} \tag{4.9}$$

Recall that for any $z \in \mathbf{R}^d$, $\varphi_B(z)$ denotes the characteristic function of Z_B and we denote by $\lambda_B(z)$ the characteristic function of $\xi(x)$. By respectively replacing r and n/β in Lemma 2.6 in [A01] with r_2^{-1} and nB, with a simple calculation we obtain that

$$\lim_{z \to 0} \frac{\log \lambda_B(z)}{\log \varphi_B(z)} = 1. \tag{4.10}$$

(4.9) and (4.10) imply that

$$\begin{aligned} \lim_{n \to \infty} I_n &= \lim_{n \to \infty} E\left[\prod_{u \in \mathbf{Z}} \varphi_B \left(r_0^{-nH^*} \sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j \right) \right] \\ &= \lim_{n \to \infty} E\left[\exp\left\{ \sum_{u \in \mathbf{Z}} f\left(r_0^{-nH^*} \sum_{j=1}^{k} N_{r_0^n t_j}(u) z_j \right) \right\} \right] \\ &= E\left[\exp\left\{ \int_{-\infty}^{\infty} f\left(\sum_{j=1}^{k} L_{t_j}(u) z_j \right) du \right\} \right] \quad \text{(by Lemma 4.2)} \\ &= E\left[\exp\left\{ i \sum_{j=1}^{k} \langle z_j, \Delta(t_j) \rangle \right\} \right] \quad \text{(by (4.6)).} \end{aligned}$$

This completes the proof of Proposition 4.1. ∎

4.2 Proof of Proposition 4.2

Recall (4.2), and for each $t \geq 0$ and $n \in \mathbf{N}$ we set

$$D_n(t) := r_0^{-nH} W(r_0^n t) = r_0^{-nH} \sum_{u \in \mathbf{Z}} N_{r_0^n t}(u) \xi(u).$$

To show the tightness of $\{D_n(t), t \geq 0, n \in \mathbf{N}\}$ in $\mathrm{C}([0,\infty); \mathbf{R}^d)$, we need to show the following estimation (see [Bi99]): For each $T < \infty$ and any $\eta > 0$,

$$\lim_{\delta \to \infty} \limsup_{n \to \infty} P \left\{ \sup_{\substack{0 \leq t_1, t_2 \leq T \\ |t_2 - t_1| \leq \delta}} \{ \|D_n(t_2) - D_n(t_1)\| \geq \eta \} \right\} = 0. \tag{4.11}$$

To show this convergence, as in [KS79], [Ma96] and [A01], we first approximate $D_n(t)$ by $D_n^{'}(t)$ plus a linear function $E_n t$ such that $D_n^{'}(t)$ has the second moments, E_n are bounded and

$$\limsup_{n \to \infty} P \left\{ \sup_{t \leq T} \|D_n(t) - D_n'(t) - E_n t\| \geq \frac{1}{2}\eta \right\} \leq \frac{\varepsilon}{2},$$

and then use Kolmogorov's criteria for $D_n'(t)$.

The following lemma is shown in a similar way to that for showing (3.19) in [KS79] by using (i) and (ii) of Lemma 4.1:

Lemma 4.3 *For any $\varepsilon > 0$, there exists an $A = A(\varepsilon)$ such that*

$$P\{N_{r_0^n t}(u) > 0 \textit{ for some } |u| > A r_0^{n/\alpha} \textit{ and } t \leq T\} \leq \frac{\varepsilon}{4}.$$

To simplify notation, we use ξ instead of $\xi(0)$ (recall that ξ's are identically distributed). Let

$$\begin{aligned} c_n(G) &:= r_2^n P\{\|r_2^{-nB}\xi\| \in G\}, \quad G \in \mathcal{B}((0,\infty)). \\ M(F) &:= \int_{S_B} \gamma(dx) \int_0^\infty \mathbf{1}_F(s^B x) d\left(-\frac{H_x(s)}{s}\right), \quad F \in \mathcal{B}(\mathbf{R}^d \backslash \{0\}). \\ c(G) &:= M(\{x : \|x\| \in G\}), \quad G \in \mathcal{B}((0,\infty)). \end{aligned}$$

By using Theorem 3.3.8 in [MS01], which is a general central limit theorem for independent and infinitely divisible distributed random variables, it is shown that

$$r_2^n P\left\{r_2^{-nB}\xi \in F\right\} \longrightarrow M(F) \tag{4.12}$$

for every Borel set F, which is bounded away from zero and $M(\partial F) = 0$. Using this convergence, we obtain that for any $y > 0$ such that $c(\{y\}) = 0$ and

$$c_n([y,\infty)) \longrightarrow c([y,\infty)) \tag{4.13}$$

in a similar way to show Lemma 8 in [Ma96]. Recall that $r_0 = r_2^{\alpha q}$ with some $q \in \mathbf{N}$. By (4.13) we obtain for ρ with $c(\{\rho\}) = 0$ such that

$$\begin{aligned} r_0^{n/\alpha} P\left\{\left\|r_0^{-\frac{n}{\alpha}B}\xi\right\| > \rho\right\} &= r_2^{qn} P\left\{\left\|r_2^{-qnB}\xi\right\| > \rho\right\} \\ &= c_{qn}([\rho,\infty)) \longrightarrow c([\rho,\infty)), \end{aligned}$$

which implies the following:

Lemma 4.4 *We can find a ρ such that for all large n*

$$(2Ar_0^{n/\alpha}+1)P\left\{\left\|r_0^{-\frac{n}{\alpha}B}\xi\right\| > \rho\right\} \le \frac{\varepsilon}{4}.$$

Using ρ above, we introduce the following notations,

$$\begin{aligned} \xi_n(u) &:= \xi(u) I\left[\left\|r_0^{-\frac{n}{\alpha}B}\xi(u)\right\| \le \rho\right]. \\ E_n &:= r_0^{-nH} E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\xi_n(u)\right], \\ D_n'(t) &:= r_0^{-nH}\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\{\xi_n(u) - E[\xi_n(u)]\}, \end{aligned}$$

and divide the variation of $\{D_n\}$ into the following:

$$\|D_n(t_2)-D_n(t_1)\| \le \|D_n(t_2)-D_n'(t_2)-E_nt_2\|+\|D_n(t_1)-D_n'(t_1)-E_nt_1\|+\|E_n\||t_2-t_1|+\|D_n'(t_2)-D_n'(t_1)\|,$$

and estimate each part.

For notational simplicity, we write ξ_n for $\xi_n(0)$ again. By using properties of the measures c_n, M and c above, the following estimation of expectation of ξ_n can be shown in the same way as those for showing Lemma 14 in [Ma96] and inequality (4.9) in [A01]:

$$\left\|E\left[r_0^{-\frac{n}{\alpha}B}\xi_n\right]\right\| = O\left(r_0^{-n/\alpha}\right), \tag{4.14}$$

provided that ξ is symmetric when $\tau_B \le 1 \le T_B$.

Proof of Proposition 4.2.

(4.14) implies that

$$\|E_n\| = \left\|r_0^{-n(1-1/\alpha)}E\left[r_0^{-\frac{n}{\alpha}B}\xi_n\right]E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\right]\right\| = O(1).$$

We have and set

$$\begin{aligned} & D_n(t) - D_n'(t) - E_nt \\ = \quad & r_0^{-nH}\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\{\xi(u)-\xi_n(u)\} + r_0^{-nH}\left\{\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)E[\xi_n(u)] - E\left[\sum_{u\in\mathbf{Z}} N_{r_0^n}(u)\xi_n(u)\right]t\right\} \\ =: \quad & r_0^{-nH}\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\{\xi(u)-\xi_n(u)\} + Q_n(t). \end{aligned}$$

By using (4.14), it is shown that for each $t \le T$,

$$\begin{aligned}\|Q_n(t)\| &= \left\|r_0^{-nH}E[\xi_n]\{r_0^n t + 1 - (r_0^n+1)t\}\right\| \\ &\le Tr_0^{-n(1-1/\alpha)}\left\|E\left[r_0^{-\frac{n}{\alpha}B}\xi_n\right]\right\| = O(r_0^{-n}),\end{aligned}$$

and by using Lemma 4.3 and 4.4, it is shown that

$$\begin{aligned}& P\left\{\sum_{u\in\mathbf{Z}} N_{r_0^n t}(u)\{\xi(u)-\xi_n(u)\} \ne 0 \text{ for some } t \le T\right\} \\ \le\ & P\left\{\xi(u)\ne\xi_n(u) \text{ for some } |u| \le Ar_0^{n/\alpha}\right\} + P\left\{N_{r_0^n t}(u) > 0 \text{ for some } |u| > Ar_0^{n/\alpha}\right\} \\ \le\ & \left(2Ar_0^{n/\alpha}+1\right)P\left\{\left\|r_0^{-\frac{n}{\alpha}B}\xi\right\| > \rho\right\} + \frac{\varepsilon}{4} \ \le\ \frac{\varepsilon}{2}.\end{aligned}$$

Hence for any $\eta > 0$ we have

$$\limsup_{n\to\infty} P\left\{\sup_{t\le T}\|D_n(t) - D_n'(t) - E_n t\| \ge \frac{1}{2}\eta\right\} \le \frac{\varepsilon}{2}, \tag{4.15}$$

and need to show

$$E[\|D_n'(t) - D_n'(s)\|^2] \le C(t-s)^{2-1/\alpha}. \tag{4.16}$$

If (4.16) is satisfied, with the respective replacements of $D_n(t)$ and η by $D_n'(t)$ and $\eta/2$, the relation (4.11) is also satisfied, and this together with (4.15) imply (4.11). We have

$$\begin{aligned}E[\|D_n'(t)-D_n'(s)\|^2] &= E\left[\left\|r_0^{-nH}\sum_{u\in\mathbf{Z}}(N_{r_0^n t}(u) - N_{r_0^n s}(u))(\xi_n(u) - E(\xi_n(u)))\right\|^2\right] \\ &= r_0^{-2n(1-1/\alpha)}E\left[\left\|r_0^{-\frac{n}{\alpha}B}\{\xi_n - E[\xi_n]\}\right\|^2\right]\sum_{u\in\mathbf{Z}}E\left[\left\{N_{r_0^n t}(u) - N_{r_0^n s}(u)\right\}^2\right] \\ &\le r_0^{-2n(1-1/\alpha)}E\left[\left\|r_0^{-\frac{n}{\alpha}B}\xi_n\right\|^2\right]\sum_{u\in\mathbf{Z}}E\left[\left\{N_{r_0^n t}(u) - N_{r_0^n s}(u)\right\}^2\right].\end{aligned} \tag{4.17}$$

Using a property of Lévy measure, we obtain that

$$\begin{aligned}\sup_n r_0^{n/\alpha}E\left[\left\|r_0^{-\frac{n}{\alpha}B}\xi_n\right\|^2\right] &= \sup_n r_0^{n/\alpha}E\left[\left\|r_0^{-\frac{n}{\alpha}B}\xi\right\|^2 I\left[\left\|r_0^{-\frac{n}{\alpha}B}\xi\right\| \le \rho\right]\right] \\ &= \sup_n \int_0^\rho y^2 c_{qn}(dy) < \infty.\end{aligned} \tag{4.18}$$

On the other hand, using (i) of Theorem 2.2, we can show that $\sum_{u\in\mathbf{Z}}E[N_t(u)^2] = O(t^{2-1/\alpha})$ and

$$E\left[\sum_{u\in\mathbf{Z}}\{N_{r_0^n t_2}(u) - N_{r_0^n t_1}(u)\}^2\right] \le C\{r_0^n(t_2-t_1)\}^{2-1/\alpha} \tag{4.19}$$

in the same way as those for showing Lemmas 1 and 3 in [KS79] by using (i) of Theorem 2.2. Thus (4.16) is shown by (4.17), (4.18) and (4.19), and the proof is completed. ∎

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References

[A01] Arai, T., A class of semi-selfsimilar processes related to random walks in random scenery, Tokyo J.Math., **24**, 69–85, 2001.

[Be04] Becker-Kern, P., Random integral representation of operator-semi-self-similar processes with independent increments, Stochastic Process. Appl., **109**, 327–344, 2004.

[Bi99] Billingsley, P., *Convergence of Probability Measures, Second edition*, Wiely, 1999.

[Bo89] Bolthausen, E., A central limit theorem for two-dimensional random walks in random sceneries, Ann. Probab., **17**, 108–115, 1989.

[C94] Choi, G.S., Criteria for recurrence and transience of semistable processes, Nagoya Math. J., **134**, 91–106, 1994.

[Ch87] Chorny, V., Operator-semistable distributions on $\mathbf{R}^d$, Theory Probab. Appl., **31** 703–705, 1987.

[GK72] Getoor, R.K. and Kesten, H., Continuity of local times for Markov processes, Composito Math., **24**, 277–303, 1972.

[GS69] Gikhman, I.I. and Skorohod, A.V., *Introduction to the Theory of Random Processes*, Saunders, 1969.

[IL71] Ibragimov, I.A. and Linnik, Yu.V., *Independent and Stationary Sequences of Random Variables*, Wolters-Noordhoff Publishing, 1971.

[KS79] Kesten, H. and Spitzer, F., A limit theorem related to a new class of self similar processes, Zeit. Wahrsch. Verw. Gebiete, **50**, 5–25, 1979.

[L85] Lou, J.H., Some properties of a special class of self-similar processes, Zeit. Wahrsch. Verw. Gebiete, **68**, 493–502, 1985.

[Ma96] Maejima, M., Limit theorems related to a class of operator-self-similar processes, Nagoya Math. J., **142**, 161–181, 1996.

[MaSa99] Maejima, M. and Sato, K., Semi-selfsimilar processes, J. Theoret. Probab., **12**, 347–373, 1999.

[MaSa03] Maejima, M. and Sato, K., Semi-Levy processes, semi-selfsimilar additive processes, and semi-stationary Ornstein-Uhlenbeck type processes, J. Math. Kyoto, **43**, 609–639, 2003.

[MS01] Meerschaert, M.M. and Scheffler H.P., *Limit Distributions for Sums of Independent Random Vectors*, Wiley, 2001.

[Me00] Megyesi, Z., A probabilistic approach to semistable laws and their domains of partial attraction, Acta Sci. Math. (Szeged), **66**, 404–434, 2000.

[Sa99a] Sato, K., Semi-stable processes and their extensions, Trends in Probability and Related Analysis, Proc. SAP'98, 129–145, World Scientific, 1999.

[Sa99b] Sato, K., *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Univ. Press, 1999.

[Sh95] Shieh, N.R., Some self-similar processes related to local times, Statist. Probab. Lett., **24**, 213–218, 1995.

[Sp76] Spitzer, F., *Principles of Random Walks, Second edition*, Springer-Verlag, 1976.

[V87] Vervaat, W., Properties of general self-similar process, Proc. the 46th Session of ISI, 199–216, Bull. Inst. Internat. Statist. **52**, 1987.

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