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A further study of t-distributions on spheres

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A further study of t-distributions on spheres

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Abstract

We propose new distributions on spheres which possess some analogous properties to the t-distribution on Euclidean space. Actually, the skew t-, the offset t-, the wrapped t-distributions on the circle, and the t-distribution on the unit complex sphere are studied. Two t-distributions on the cylinder are also proposed. These distributions are obtained by calculating a scale mixture of normal distributions, or conditioning or projecting multivariate t-distributions, or wrapping univariate t-distributions. Some properties of the proposed distributions are also studied. In particular, it is shown that the t-distributions converge to the corresponding 'normal' distributions on the circle, on the unit complex sphere, and on the cylinder as degrees of freedom tend to infinity.

Key words: Bingham distribution; distributions on the cylinder; offset normal; von Mises distribution; wrapped normal.

1 Introduction

What distribution can be viewed as the 'normal' distribution when we think of the distributions on the manifolds except for Euclidean space? The von Mises distribution (1,2) on the circle or more generally, the von Mises-Fisher (or Langevin) distribution (1,2) on an arbitrary dimensional unit sphere is a probable answer since it is known as a distribution which possesses some analogous properties to the normal distribution on the real line or on Euclidean space. On the complex sphere, the Bingham distribution (3) can be considered to be a normal distribution. On the cylinder, Mardia and Sutton (4) proposed a distribution, which should be called the normal distribution on the cylinder. It is obtained by conditioning a trivariate normal distribution on Euclidean space. These distributions can be considered to be the 'normal' distributions because of their importance and similarities to the normal distribution on Euclidean space.

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The t-distribution on the real line has importance in the the theory of estimation and it is also usable as a parent distribution. Shimizu and Iida (5) proposed the (symmetric) t-distribution on an arbitrary dimensional sphere and gave some properties. The distribution is provided by calculating a scale mixture of normal distributions and converges to the von Mises-Fisher distribution as degrees of freedom tend to infinity. In this paper we propose the t-distributions on three manifolds: on the circle, on the unit complex sphere, and on the cylinder. They converge to the corresponding 'normal' distributions as degrees of freedom tend to infinity.

Subsequent sections are organized as follows. In Section 2, 3 and 4, we propose different t-distributions on the circle. Actually, the skew t-distribution in Section 2, the offset t in Section 3, and the wrapped t in Section 4. Each distribution converges to each 'normal' distribution on the circle as degrees of freedom tend to infinity. The skew t-distribution converges to the skew normal or Rukhin's (6) distribution, the offset t to the offset normal, and the wrapped t to the wrapped normal. In Section 5 the t-distribution on the unit complex sphere is proposed, which includes the complex Bingham distribution as a limit. In Section 6 we propose two t-distributions on the cylinder. They are obtained by different methods. However, they converge to the same normal distribution on the cylinder as degrees of freedom tend to infinity.

2 Skew t-distribution on the circle

2.1 Probability density function

The skew t-distribution with n degrees of freedom on the circle is defined by the probability density function (p.d.f.)

$$f(\theta) = C\left[1 - \frac{2}{n}\left\{\kappa_n \cos(\theta-\mu) + \lambda_n \cos\left(2(\theta-\nu)\right)\right\}\right]^{-(n+2)/2}, \quad 0 \le \theta < 2\pi, \tag{2.1}$$

where κ_n represents a concentration parameter with $\kappa_n = \kappa/\left(1-2c/n\right)$, $\kappa \ge 0$, $c < 0$, and λ_n a skewness parameter with $\lambda_n = \lambda/\left(1-2c/n\right)$, $\lambda \ge 0$, $\lambda + \kappa < (n-2c)/2$. Directional parameters are μ and ν with $0 \le \mu, \nu < 2\pi$, and C is the normalizing constant

$$\begin{aligned} C^{-1} &= 2\pi\Bigg[F_4\left(\frac{n}{4}+\frac{1}{2}, \frac{n}{4}+1; 1, 1; \left(\frac{2\kappa_n}{n}\right)^2, \left(\frac{2\lambda_n}{n}\right)^2\right) \\ &\quad + 2\sum_{r=1}^{\infty} \frac{(\kappa_n/n)^{2r}\,(\lambda_n/n)^r\,(n/2+1)_{3r}}{r!\quad(2r)!}\cos(2r(\mu-\nu)) \\ &\quad \times F_4\left(\frac{n}{4}+\frac{3r}{2}+\frac{1}{2}, \frac{n}{4}+\frac{3r}{2}+1; 2r+1, r+1; \left(\frac{2\kappa_n}{n}\right)^2, \left(\frac{2\lambda_n}{n}\right)^2\right)\Bigg], \end{aligned}$$

where $(a)_r$ denotes Pochhammer's symbol which is defined by

$$(a)_r = \begin{cases} a(a+1)(a+2)\cdots(a+r-1), & r \ge 1 \\ 1, & r = 0 \end{cases}$$

and F_4 Appell's double hypergeometric function given by

$$F_4(a, b; c, c'; x, y) = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_{r+s}}{(c)_r(c')_s}\frac{x^r y^s}{r!s!}.$$

This series converges for $|\sqrt{x}|+|\sqrt{y}|<1$. As for the normalizing constant of (2.1), the region of convergence is $\lambda+\kappa<(n-2c)/2$.

This p.d.f. can be used to represent symmetric or asymmetric, unimodal or bimodal shapes depending on the choice of $\kappa_n, \lambda_n, \mu, \nu$. When $\lambda=0$ and thus $\lambda_n=0$, (2.1) reduces to the symmetric t-distribution on the circle, a special case of the t-distribution (5) on an arbitrary dimensional sphere. The p.d.f. of the symmetric t-distribution on the circle is given by

$$f(\theta)=\frac{\{1-2\kappa_n\cos(\theta-\mu)/n\}^{-n/2-1}}{2\pi\ {}_2F_1\left(n/4+1/2, n/4+1; 1; (2\kappa_n/n)^2\right)}, \quad 0\le\theta<2\pi, \tag{2.2}$$

where $\kappa_n=\kappa/(1-2c/n),\ 0\le\mu<2\pi,\ \kappa\ge 0,\ c<0$, and ${}_2F_1$ denotes the Gauss hypergeometric function

$${}_2F_1(a,b;c;z)=\sum_{r=0}^{\infty}\frac{(a)_r(b)_r}{(c)_r}\frac{z^r}{r!}.$$

As $n\to\infty$, from Stirling's formula, (2.1) approaches Rukhin's (6) distribution, which is a skew normal distribution on the circle. Rukhin's distribution is defined by the density

$$f(\theta)=C\exp\{\kappa\cos(\theta-\mu)+\lambda\cos(2(\theta-\nu))\}, \quad 0\le\theta<2\pi,$$

where $\kappa,\lambda\ge 0,\ 0\le\mu,\nu<2\pi$ and C is the normalizing constant

$$C^{-1}=2\pi\left\{I_0(\kappa)I_0(\lambda)+2\sum_{r=1}^{\infty}I_r(\lambda)I_{2r}(\kappa)\cos(2r(\mu-\nu))\right\},$$

where I_r denotes the modified Bessel function of the first kind and order r which is given by

$$I_r(z)=\sum_{j=0}^{\infty}\frac{1}{\Gamma(r+j+1)j!}\left(\frac{z}{2}\right)^{2j+r}.$$

When $\lambda=0$, Rukhin's distribution reduces to the von Mises distribution.

2.2 Genesis of skew t-distribution on the circle

Rukhin's distribution is originally obtained by considering distributions which have nontrivial sufficient statistics for location parameters on the circle with positive and continuous density. It is also generated through conditioning a bivariate normal distribution without any restriction on the parameters. To make a more general distribution on the circle, we consider here the bivariate t-distribution, which includes the bivariate normal distribution with mean vector zero as a limit.

Let $\boldsymbol{X}=(X_1,X_2)'$ be a random vector which obeys the bivariate t-distribution with n degrees of freedom on $\mathbf{R}^2$. It has p.d.f.

$$f(\boldsymbol{x})=\frac{1}{2\pi|\Sigma|^{1/2}}\left(1+\frac{\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}}{n}\right)^{-(n+2)/2}, \quad \boldsymbol{x}\in\mathbf{R}^2, \tag{2.3}$$

where Σ is a positive definite matrix of order 2:

$$\Sigma=\begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}, \quad \sigma_1,\sigma_2>0,\ -1<\rho<1.$$

Using the scale mixture of the bivariate normal distribution, (2.3) can be also expressed as

$$f(\boldsymbol{x}) = \int_0^\infty \frac{1}{2\pi|S|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{x}'S^{-1}\boldsymbol{x}\right) dG(\sigma), \quad \boldsymbol{x} \in \mathbf{R}^2, \tag{2.4}$$

where $G(\sigma)$ is

$$\frac{d}{d\sigma}G(\sigma) = \frac{2^{1-n/2}n^{n/2}}{\Gamma(n/2)}\sigma^{-1-n}\exp\left(-\frac{n}{2\sigma^2}\right), \tag{2.5}$$

and S is defined by $S = \sigma\Sigma,\ \sigma > 0$. Let $\boldsymbol{Y} = (Y_1, Y_2)'$ be a random vector defined by $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{\eta}$, where $\boldsymbol{\eta} = (\xi\cos\eta, \xi\sin\eta)'$ $(\xi \geq 0,\ 0 \leq \eta < 2\pi)$ is a constant vector. We transform the bivariate random vector $\boldsymbol{Y} = (Y_1, Y_2)'$ into polar co-ordinates $(R, \Theta)'$ by putting $(Y_1, Y_2)' = (R\cos\Theta, R\sin\Theta)'$. Then, the joint p.d.f. of $(R, \Theta)'$ is given by

$$\begin{aligned} f(r,\theta) &= \frac{r}{2\pi|\Sigma|^{1/2}}\left[1 + \frac{1}{n(1-\rho^2)}\left\{\frac{(r\cos\theta - \xi\cos\eta)^2}{\sigma_1^2}\right.\right. \\ &\quad \left.\left. - \frac{2\rho(r\cos\theta - \xi\cos\eta)(r\sin\theta - \xi\sin\eta)}{\sigma_1\sigma_2} + \frac{(r\sin\theta - \xi\sin\eta)^2}{\sigma_2^2}\right\}\right]^{-(n+2)/2} \\ &= \frac{r}{2\pi|\Sigma|^{1/2}}\left[1 + \frac{1}{n(1-\rho^2)}\left\{\frac{r^2}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2}\right)\cos 2\theta - \frac{\rho r^2}{\sigma_1\sigma_2}\sin 2\theta\right.\right. \\ &\quad + 2\left(\frac{-r\xi\cos\eta}{\sigma_1^2} + \frac{\rho r\xi\sin\eta}{\sigma_1\sigma_2}\right)\cos\theta + 2\left(\frac{-r\xi\sin\eta}{\sigma_2^2} + \frac{\rho r\xi\cos\eta}{\sigma_1\sigma_2}\right)\sin\theta \\ &\quad \left.\left. + \frac{\xi^2\cos^2\eta}{\sigma_1^2} - 2\rho\frac{\xi^2\cos\eta\sin\eta}{\sigma_1\sigma_2} + \frac{\xi^2\sin^2\eta}{\sigma_2^2} + \frac{r^2}{2}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\right\}\right]^{-(n+2)/2}, \end{aligned} \tag{2.6}$$

$$r > 0,\quad 0 \leq \theta < 2\pi.$$

Now, we define new parameters $\kappa, \lambda, \mu, \nu$ and c as

$$\lambda = \frac{1}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\sqrt{\left(\frac{r^2}{2}(\sigma_2^2 - \sigma_1^2)\right)^2 + \rho^2\sigma_1^2\sigma_2^2 r^4},$$

$$\kappa = \frac{2r\xi}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\sqrt{(\rho\sigma_1\sigma_2\sin\eta - \sigma_2^2\cos\eta)^2 + (\rho\sigma_1\sigma_2\cos\eta - \sigma_1^2\sin\eta)^2},$$

$$\cos\mu = -\frac{2r\xi(\rho\sigma_1\sigma_2\sin\eta - \sigma_2^2\cos\eta)}{2\sigma_1^2\sigma_2^2(1-\rho^2)\kappa}, \quad \sin\mu = \frac{-2r\xi(\rho\sigma_1\sigma_2\cos\eta - \sigma_1^2\sin\eta)}{2\sigma_1^2\sigma_2^2(1-\rho^2)\kappa},$$

$$\cos 2\nu = \frac{-r^2(\sigma_2^2 - \sigma_1^2)/2}{2\sigma_1^2\sigma_2^2(1-\rho^2)\lambda}, \quad \sin 2\nu = \frac{\rho\sigma_1\sigma_2 r^2}{2\sigma_1^2\sigma_2^2(1-\rho^2)\lambda},$$

$$c = \frac{-1}{2\sigma_1^2\sigma_2^2(1-\rho^2)}\left(\sigma_2^2\xi^2\cos^2\eta - 2\rho\sigma_1\sigma_2\xi^2\cos\eta\sin\eta + \sigma_1^2\xi^2\sin^2\eta + \frac{r^2}{2}(\sigma_1^2 + \sigma_2^2)\right).$$

It is clear that $\lambda, \kappa \geq 0,\ 0 \leq \mu, \nu < 2\pi,\ c < 0$. Equation (2.6) is simply expressed as

$$\begin{aligned} f(r,\theta) &= \frac{r}{2\pi|\Sigma|^{1/2}}\left[1 - \frac{2}{n}\{\kappa\cos(\theta - \mu) + \lambda\cos(2(\theta - \nu)) + c\}\right]^{-(n+2)/2} \\ &= \frac{r}{2\pi|\Sigma|^{1/2}}\left(1 - \frac{2c}{n}\right)^{-(n+2)/2}\left[1 - \frac{2}{n}\{\kappa_n\cos(\theta - \mu) + \lambda_n\cos(2(\theta - \nu))\}\right]^{-(n+2)/2}, \end{aligned} \tag{2.7}$$

where $\kappa_n = \kappa/(1-2c/n)$ and $\lambda_n = \lambda/(1-2c/n)$. Integrating (2.7) with respect to θ, we have the marginal p.d.f. of the distance R:

$$\begin{aligned}
f(r) &= \int_0^{2\pi} f(r,\theta)d\theta = \int_0^{2\pi}\int_0^{\infty} \frac{r}{2\pi\sigma^2|\Sigma|^{1/2}} \exp\left\{\frac{1}{\sigma^2}(\kappa\cos(\theta-\mu)\right. \\
&\quad \left. +\lambda\cos(2(\theta-\nu))+c)\right\} dG(\sigma)d\theta \\
&= \frac{n^{n/2}r}{2^{n/2}\pi\Gamma(n/2)|\Sigma|^{1/2}} \int_0^{\infty} \sigma^{-n-3} \int_0^{2\pi} \exp\left\{\frac{1}{\sigma^2}(\kappa\cos(\theta-\mu)\right. \\
&\quad \left. +\lambda\cos(2(\theta-\nu))+c-\frac{n}{2}\right\} d\theta d\sigma \\
&= \frac{n^{n/2}r}{2^{n/2}\pi\Gamma(n/2)|\Sigma|^{1/2}} \int_0^{\infty} \sigma^{-n-3} \exp\left(-\frac{n-2c}{2\sigma^2}\right) \left[2\pi\left\{I_0\left(\frac{\kappa}{\sigma^2}\right) I_0\left(\frac{\lambda}{\sigma^2}\right)\right.\right. \\
&\quad \left.\left. +2\sum_{r=1}^{\infty} I_r\left(\frac{\lambda}{\sigma^2}\right) I_{2r}\left(\frac{\kappa}{\sigma^2}\right) \cos\left(2r(\mu-\nu)\right)\right\}\right] d\sigma, \qquad (2.8)
\end{aligned}$$

where $G(\sigma)$ is (2.5). Using Appell's double hypergeometric function, the integral part of each term of (2.8) can be expressed as

$$\begin{aligned}
&\int_0^{\infty} \sigma^{-n-3} \exp\left(-\frac{n-2c}{2\sigma^2}\right) I_r\left(\frac{\kappa}{\sigma^2}\right) I_{2r}\left(\frac{\lambda}{\sigma^2}\right) d\sigma \\
&= \sum_{i,j=0}^{\infty} \frac{2^{-3r-2(i+j)}\kappa^{2i+2r}\lambda^{2j+r}}{(2r+i)!i!(r+j)!j!} \int_0^{\infty} \sigma^{-n-3-6r-4(i+j)} \exp\left(-\frac{n-2c}{2\sigma^2}\right) d\sigma \\
&= \frac{2^{n+3r} n^{(-n-2-6r)/2} {\kappa_n}^{2r} {\lambda_n}^{r} (n-2c)^{(-n-2)/2}}{\sqrt{\pi}\ (2r)!\ r!\ n^{3r}} \\
&\quad \times F_4\left(\frac{n}{4}+\frac{3r}{2}+\frac{1}{2}, \frac{n}{4}+\frac{3r}{2}+1; 2r+1, r+1; \left(\frac{2}{n}\kappa_n\right)^2, \left(\frac{2}{n}\lambda_n\right)^2\right), \quad r=0,1,2,\ldots
\end{aligned}$$

Then $f(r)$ is calculated as

$$\begin{aligned}
f(r) &= \frac{r(n-2c)^{(-n-2)/2} n^{n/2+1}}{|\Sigma|^{1/2}} \\
&\quad \times \left[F_4\left(\frac{n}{4}+\frac{1}{2}, \frac{n}{4}+1; 1, 1; \left(\frac{2}{n}\kappa_n\right)^2, \left(\frac{2}{n}\lambda_n\right)^2\right)\right. \\
&\quad +2\sum_{j=1}^{\infty} \frac{(\kappa_n/n)^{2j} (\lambda_n/n)^j (n/2+1)_{3j}}{j!\ (2j)!} \cos\left(2j(\mu-\nu)\right) \\
&\quad \left. \times F_4\left(\frac{n}{4}+\frac{3j}{2}+\frac{1}{2}, \frac{n}{4}+\frac{3j}{2}+1; 2j+1, j+1; \left(\frac{2}{n}\kappa_n\right)^2, \left(\frac{2}{n}\lambda_n\right)^2\right)\right].
\end{aligned}$$

The skew t-distribution on the circle is obtained by the conditional distribution of Θ given $R=r$, whose density is

$$\begin{aligned}
f(\theta|r) &= \frac{f(\theta,r)}{f(r)} \\
&= C\left\{1-\frac{2}{n}\left(\kappa_n\cos(\theta-\mu)+\lambda_n\cos(2(\theta-\nu))\right)\right\}^{-(n+2)/2}, \quad 0 \le \theta < 2\pi,
\end{aligned}$$

where C is the normalizing constant given by

$$\begin{aligned} C^{-1} &= 2\pi\left[F_4\left(\frac{n}{4}+\frac{1}{2},\frac{n}{4}+1;1,1;\left(\frac{2\kappa_n}{n}\right)^2,\left(\frac{2\lambda_n}{n}\right)^2\right)\right. \\ &\quad +2\sum_{j=1}^{\infty}\frac{(\kappa_n/n)^{2j}\,(\lambda_n/n)^j\,(n/2+1)_{3j}}{j!\quad(2j)!}\cos(2j(\mu-\nu)) \\ &\quad \left.\times F_4\left(\frac{n}{4}+\frac{3j}{2}+\frac{1}{2},\frac{n}{4}+\frac{3j}{2}+1;2j+1,j+1;\left(\frac{2\kappa_n}{n}\right)^2,\left(\frac{2\lambda_n}{n}\right)^2\right)\right]. \end{aligned}$$

Thus, we get (2.1).

It is remarked that different distributions from the skew t on the circle can be generated through replacing the weight function of scale mixtures in (2.4).

3 Offset t-distribution on the circle

3.1 Probability density function

We define the offset t-distribution with n degrees of freedom on the circle with p.d.f.

$$\begin{aligned} f(\theta) &= \frac{1}{(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})}\left[\frac{1}{2\pi|\Sigma|^{1/2}}\left(1+\frac{1}{n}\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}\right)^{-n/2}\right. \\ &\quad +\frac{|\Sigma|^{-1/2}D(\theta)\Gamma((n+1)/2)}{2\sqrt{2\pi n}\Gamma(n/2)}\left\{1+\frac{1}{n}|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2\right\}^{-(n+1)/2} \\ &\quad +\frac{|\Sigma|^{-1/2}(D(\theta))^2}{2\pi}\left\{1+\frac{1}{n}|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2\right\}^{-(n+2)/2} \\ &\quad \left.\times\,{}_2F_1\left(\frac{1}{2},\frac{n}{2}+1;\frac{3}{2};-(D(\theta))^2\left\{|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2+n\right\}^{-1}\right)\right], \end{aligned} \tag{3.1}$$

where $\boldsymbol{\mu}=(\mu_1,\mu_2)'$, $-\infty<\mu_1,\mu_2<\infty$, Σ is a positive definite matrix of order 2, and

$$D(\theta)=\frac{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{t}}{(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{1/2}},\quad \boldsymbol{t}=(\cos\theta,\sin\theta)',\quad \boldsymbol{\mu}\wedge\boldsymbol{t}=\mu_1\sin\theta-\mu_2\cos\theta.$$

Here we suppose $(D(\theta))^2<|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2+n$, so that the Gauss hypergeometric function ${}_2F_1$ converges. As n tends to infinity, from Stirling's formula, (3.1) converges to the offset or projected normal distribution (1,2).

3.2 Genesis of offset t-distribution on the circle

The offset normal distribution on the circle is generated through projecting the bivariate normal distribution. The offset t-distribution proposed here is obtained by projecting the bivariate t-distribution.

Suppose that $\boldsymbol{X}$ denotes a random vector having the bivariate t-distribution with parameter Σ, where Σ is a positive definite matrix of order 2. Let $\boldsymbol{Y}$ be another random vector defined by $\boldsymbol{Y}=\boldsymbol{X}+\boldsymbol{\mu}$, where $\boldsymbol{\mu}=(\mu_1,\mu_2)'$, $-\infty<\mu_1,\mu_2<\infty$. We here transform the random vector $\boldsymbol{Y}=(Y_1,Y_2)'$ into polar coordinates $(R,\Theta)'$, $(R>0,\ 0\le\Theta<2\pi)$ with the relationship $(Y_1, Y_2)' = (R\cos\Theta, R\sin\Theta)'$. Then the marginal distribution of Θ is the offset t-distribution on the circle.

Using the fact that the bivariate t-distribution is a scale mixture of the bivariate normal distribution, the offset t-distribution on the circle can be expressed as a scale mixture of the offset normal distribution. It has p.d.f.

$$\begin{aligned} f(\theta) &= \int_0^\infty \frac{\sigma^2}{\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t}} \Big[\phi(\mu_1,\mu_2;\mathbf{0},\sigma^2\Sigma) \\ &\quad + \frac{D(\theta)}{\sigma^3|\Sigma|^{1/2}} \Phi\left(\frac{1}{\sigma}D(\theta)\right) \phi\left(\frac{\boldsymbol{\mu}\wedge\boldsymbol{t}}{\sigma|\Sigma|^{1/2}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{1/2}}\right)\Big]\, dG(\sigma), \end{aligned} \tag{3.2}$$

where $\phi(\cdot,\cdot;\boldsymbol{\eta},\Xi)$ is the p.d.f. of $N_2(\boldsymbol{\eta},\Xi)$, and $\phi(\cdot),\Phi(\cdot)$ are the p.d.f. and distribution function of $N(0,1)$, respectively, and $G(\sigma)$ is given by (2.5). The first term of (3.2) is calculated as

$$\int_0^\infty \frac{\sigma^2}{\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t}}\phi(\mu_1,\mu_2;\mathbf{0},\sigma^2\Sigma)dG(\sigma) = \frac{1}{2\pi|\Sigma|^{1/2}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})}\left(1+\frac{1}{n}\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}\right)^{-n/2}.$$

On the condition that $(D(\theta))^2\left\{|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2+n\right\}^{-1}<1$, the second term of (3.2) is

$$\begin{aligned} &\int_0^\infty \frac{\sigma^2}{\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t}}\frac{D(\theta)}{\sigma^3|\Sigma|^{1/2}}\Phi\left(\frac{1}{\sigma}D(\theta)\right)\phi\left(\frac{\boldsymbol{\mu}\wedge\boldsymbol{t}}{\sigma|\Sigma|^{1/2}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{1/2}}\right)dG(\sigma) \\ &= \frac{|\Sigma|^{-1/2}D(\theta)2^{1-n/2}n^{n/2}}{(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})\Gamma(n/2)\sqrt{2\pi}}\int_0^\infty \sigma^{-2-n}\left\{\frac{1}{2}+\frac{D(\theta)}{\sqrt{2\pi}\sigma}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};-\frac{(D(\theta))^2}{2\sigma^2}\right)\right\} \\ &\quad\times\exp\left[-\frac{1}{2\sigma^2}\left\{|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2+n\right\}\right]d\sigma \\ &= \frac{|\Sigma|^{-1/2}D(\theta)}{(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})\Gamma(n/2)}\left\{1+\frac{1}{n}|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2\right\}^{-n/2} \\ &\quad\times\left[\frac{\Gamma((n+1)/2)}{2\sqrt{\pi n}}\left\{1+\frac{1}{n}|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2\right\}^{-1/2}\right. \\ &\quad+\frac{D(\theta)\Gamma((n+2)/2)}{\pi n}\left\{1+\frac{1}{n}|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2\right\}^{-1} \\ &\quad\left.\times\ {}_2F_1\left(\frac{1}{2},\frac{n}{2}+1;\frac{3}{2};-\frac{(D(\theta))^2}{|\Sigma|^{-1}(\boldsymbol{t}'\Sigma^{-1}\boldsymbol{t})^{-1}(\boldsymbol{\mu}\wedge\boldsymbol{t})^2+n}\right)\right], \end{aligned}$$

where ${}_1F_1$ denotes the confluent hypergeometric function. Hence, the p.d.f. of the offset t-distribution is given by (3.1).

4 Wrapped t-distribution on the circle

4.1 Probability density function

The wrapped t-distribution with n degrees of freedom on the circle is defined to have p.d.f.

$$f(\theta) = \frac{1}{2\pi}\left[1+2\sum_{r=1}^{\infty}\alpha_r\cos(r(\theta-\mu))\right], \quad 0<\theta\le 2\pi, \tag{4.1}$$

with

$$\alpha_r = \frac{2}{\Gamma(n/2)}\left(\frac{|r|\sqrt{n}}{2}\right)^{n/2} K_{n/2}(|r|\sqrt{n}),$$

where $K_{n/2}$ denotes the modified Bessel function of the second kind and order $n/2$, which can be expressed as

$$\begin{aligned} K_{n/2}(xz) &= \frac{\Gamma((n+1)/2)(2z)^{n/2}}{\sqrt{\pi}x^{n/2}} \int_0^\infty \frac{\cos(xt)}{(t^2+z^2)^{(n+1)/2}}dt, \\ &\quad \mathrm{Re}\, n > -1, \quad x > 0, \quad |\arg z| < \pi/2. \end{aligned}$$

As $n \to \infty$, (4.1) approaches the p.d.f. of the wrapped normal distribution. The wrapped t-distribution with 1 degree of freedom is also called the wrapped Cauchy distribution (1,2) with density

$$\begin{aligned} f(\theta) &= \frac{1}{2\pi}\left[1 + 2\sum_{r=1}^{\infty} e^{-r}\cos(r(\theta-\mu))\right] \\ &= \frac{1}{2\pi}\frac{1-e^{-2}}{1+e^{-2}-2e^{-1}\cos(\theta-\mu)}, \quad 0 \le \theta < 2\pi. \end{aligned}$$

Let Θ be a random variable having the wrapped t-distribution on the circle. Using the modified Bessel function of the second kind, the rth central trigonometric moment of Θ is given by

$$\phi_n^*(r) = E\left[e^{ir(\Theta-\mu)}\right] = E\left[\cos(r(\Theta-\mu))\right] = \frac{2}{\Gamma(n/2)}\left(\frac{|r|\sqrt{n}}{2}\right)^{n/2} K_{n/2}(|r|\sqrt{n}).$$

When the degrees of freedom are odd, i.e., $n = 2m+1$ $(m = 0, 1, 2, \ldots)$, from the residue theorem, the kth central trigonometric moment of Θ can be expressed more simply as a finite series

$$\begin{aligned} \phi_{2m+1}^*(r) &= \frac{1}{B(1/2, m+1/2)} \int_{-\infty}^{\infty} \frac{\exp(i|r|\sqrt{n}z)}{(1+z^2)^{m+1}}dz \\ &= \frac{\pi e^{-|r|\sqrt{n}}}{B(1/2, m+1/2)2^{2m}m!} \sum_{j=0}^{m} \frac{(2m-j)!}{(m-j)!j!}(2|r|\sqrt{n})^j, \end{aligned}$$

where B denotes the beta function.

4.2 Genesis of wrapped t-distribution on the circle

The wrapped t-distribution is obtained by wrapping the t-distribution on the real line around the circle. The idea has been already remarked by Mardia (7). Suppose that X is a random variable having the t-distribution on the real line. Let Y be a radom variable defined by $Y = X + \mu$ where $-\infty < \mu < \infty$.

Then the distribution of Θ $(\equiv Y \pmod{2\pi})$ is the wrapped t-distribution. Through the Fourier expansion (1,2), the p.d.f. of Θ is given by

$$f(\theta) = \frac{1}{2\pi}\sum_{r=-\infty}^{\infty} \phi_n^*(r)e^{-ir(\theta-\mu)}$$

$$= \frac{1}{2\pi}\left[1+2\sum_{r=1}^{\infty}\left(\alpha_r^*\cos(r(\theta-\mu))+\beta_r^*\sin(r(\theta-\mu))\right)\right]$$

$$= \frac{1}{2\pi}\left[1+2\sum_{r=1}^{\infty}\frac{2}{\Gamma(n/2)}\left(\frac{|r|\sqrt{n}}{2}\right)^{n/2}K_{n/2}(|r|\sqrt{n})\cos(r(\theta-\mu))\right], \quad 0\le\theta<2\pi,$$

where $\phi_n^*(r)$ $(=\alpha_r^*+i\beta_r^*,\ \alpha_r^*,\beta_r^*\in\mathbf{R})$ denotes the rth central trigonometric moment of Θ. Thus, we get (4.1).

Pewsey (8) proposed the wrapped skew-normal distribution. Similarly, wrapped skew-t distributions are obtainable if skew-t distributions on the real line are defined.

5 t-distribution on the complex sphere

5.1 Probability density function

In this section we propose the t-distribution on the unit complex sphere, which may be usable in shape analysis (9). The t-distribution with n degrees of freedom on the $(k-1)$-dimensional unit complex sphere is defined to have p.d.f.

$$f(\boldsymbol{z}) = C\left(1-\frac{2}{n}\boldsymbol{z}^*A\boldsymbol{z}\right)^{-(n+k)/2}, \quad \boldsymbol{z}\in\mathbf{C}S^{k-1}, \tag{5.1}$$

where A represents an Hermitian matrix of order k, $\boldsymbol{z}^*=\overline{\boldsymbol{z}'}$, and $\mathbf{C}S^{k-1}=\{\boldsymbol{z}\in\mathbf{C}^k|\boldsymbol{z}^*\boldsymbol{z}=1\}$. Let $\lambda_1<\lambda_2<\cdots<\lambda_k=0$ denote the eigenvalues of A, and $b_j=\prod_{\substack{i=1\\ i\neq j}}(\lambda_j-\lambda_i)$, $(n-k)/2+1>0$. Then, the normalizing constant is given by

$$C=\left\{2\pi^k\left(\frac{n}{2}\right)^{k-1}\frac{\Gamma((n-k)/2+1)}{\Gamma((n+k)/2)}\sum_{j=1}^{k}b_j\left(1-\frac{2}{n}\lambda_j\right)^{-((n-k)/2+1)}\right\}^{-1}.$$

As n tends to infinity, (5.1) approaches the complex Bingham distribution (3), whose density is given by

$$f(\boldsymbol{z})=\left\{2\pi^k\sum_{j=1}^{k}b_j\exp\lambda_j\right\}^{-1}\exp\left(\boldsymbol{z}^*A\boldsymbol{z}\right), \quad \boldsymbol{z}\in\mathbf{C}S^{k-1}.$$

5.2 Genesis of t-distribution on the complex sphere

The t-distribution on the unit complex sphere is generated through a scale mixture of the complex Bingham distribution. The p.d.f. of the t-distribution on the unit complex sphere is obtained by

$$\begin{aligned} f(\boldsymbol{z}) &= c(A)^{-1}\int_0^{\infty}\frac{1}{\sigma^k}\exp\left(\frac{1}{\sigma^2}\boldsymbol{z}^*A\boldsymbol{z}\right)dG(\sigma), \quad \boldsymbol{z}\in\mathbf{C}S^{k-1}, \\ &= \left\{2\pi^k\left(\frac{n}{2}\right)^{k-1}\frac{\Gamma((n-k)/2+1)}{\Gamma((n+k)/2)}\sum_{j=1}^{k}b_j\left(1-\frac{2}{n}\lambda_j\right)^{-((n-k)/2+1)}\right\}^{-1}\left(1-\frac{2}{n}\boldsymbol{z}^*A\boldsymbol{z}\right)^{-(n+k)/2}, \end{aligned}$$

where $G(\sigma)$ is (2.5) and $c(A)$ is the constant which satisfies

$$c(A)^{-1}\int_{\mathbf{C}S_{k-1}}\int_0^\infty \frac{1}{\sigma^k}\exp\left(\frac{1}{\sigma^2}\boldsymbol{z}^*A\boldsymbol{z}\right)dG(\sigma)d\boldsymbol{z}=1.$$

Thus, we get (5.1).

6 t-distribution on the cylinder

6.1 Probability density function

In this section we propose the t-distribution on the cylinder. The t-distribution on the cylinder with n degrees of freedom is defined by the p.d.f.

$$f(x,\theta)=C\left[1+\frac{1}{n}\left\{\frac{(x-\mu(\theta))^2}{\sigma_n^2(1-\rho^2)}-2\kappa_n\cos(\theta-\mu_0)\right\}\right]^{-(n+3)/2},\quad -\infty<x<\infty,\ \ 0\le\theta<2\pi, \tag{6.1}$$

where $\mu(\theta)=\mu+\sigma\sqrt{\kappa}\{\rho_1(r\cos\theta-\xi\cos\mu_0)+\rho_2(r\sin\theta-\xi\sin\mu_0)\}$, $\sigma_n^2=\sigma^2\left\{1+\kappa(r^2+\xi^2)/n\right\}$, $\kappa_n=\kappa r\xi/\left\{1+\kappa(r^2+\xi^2)/n\right\}$, $\rho=\sqrt{\rho_1^2+\rho_2^2}$, $\xi\ge0$, $0\le\mu_0<2\pi$, $-\infty<\mu<\infty$, $\kappa\ge 0$, $0\le\rho<1$, $-1<\rho_1,\rho_2<1$, $\sigma>0$, and C is the normalizing constant:

$$C=\frac{(n+1)!!(n^2-4\kappa_n^2)^{n/4+1/2}}{2\pi n!!\sigma_n\sqrt{n(1-\rho^2)}P_{n/2}\left(n/\sqrt{n^2-4\kappa_n^2}\right)},$$

where $(2m+1)!!=(2m+1)(2m-1)\cdots3\cdot1$, $(2m)!!=(2m)(2m-2)\cdots4\cdot2$, and $P_{n/2}$ denotes the Legendre function of the first kind which is given by

$$P_{n/2}(x)={}_2F_1\left(-\frac{n}{2},\frac{n}{2}+1;1;\frac{1-x}{2}\right),\quad x\in\mathbf{R}.$$

As $n\to\infty$, (6.1) converges to the normal distribution on the cylinder (1,4), whose p.d.f. is given by

$$f(x,\theta)=\frac{1}{2\pi I_0(\kappa)}\exp\{\kappa\cos(\theta-\mu_0)\}\frac{1}{\sqrt{2\pi\sigma^2(1-\rho^2)}}\exp\left\{-\frac{(x-\mu(\theta))^2}{2\sigma^2(1-\rho^2)}\right\}, \tag{6.2}$$

$$-\infty<x<\infty,\ \ 0\le\theta<2\pi.$$

6.2 Genesis of t-distribution on the cylinder

The t-distribution on the cylinder is constructed by the method of conditioning the trivariate t-distribution on $\mathbf{R}^3$. Suppose that $\boldsymbol{X}$ is distributed as the trivariate t-distribution with p.d.f.

$$f(\boldsymbol{x})=\frac{\Gamma((n+3)/2)}{(\pi n)^{3/2}\Gamma(n/2)|\Sigma|^{1/2}}\left\{1+\frac{1}{n}\boldsymbol{x}'\Sigma^{-1}\boldsymbol{x}\right\}^{-(n+3)/2},\quad \boldsymbol{x}\in\mathbf{R}^3,$$

where

$$\Sigma=\begin{pmatrix}\sigma^2 & \rho_1\sigma\sqrt{1/\kappa} & \rho_2\sigma\sqrt{1/\kappa}\\ \rho_1\sigma\sqrt{1/\kappa} & 1/\kappa & 0\\ \rho_2\sigma\sqrt{1/\kappa} & 0 & 1/\kappa\end{pmatrix},$$

with $\kappa, \sigma > 0$, $-1 < \rho_1, \rho_2 < 1$. Let $\boldsymbol{Y}$ be a trivariate random vector which is defined by $\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{\mu}$ where $\boldsymbol{\mu} = (\mu, \xi \cos \mu_0, \xi \sin \mu_0)'$ with $\xi \geq 0$, $0 \leq \mu_0 < 2\pi$, $-\infty < \mu < \infty$. We transform the trivariate random vector $\boldsymbol{Y} = (Y_1, Y_2, Y_3)'$ into $(Y_1, R, \Theta)'$ by the relationship $(Y_1, Y_2, Y_3)' = (Y_1, R\cos\Theta, R\sin\Theta)'$ $(R > 0,\ 0 \leq \Theta < 2\pi)$. The conditional density of $(Y_1, \Theta)'$ given $R = r$ is thus the t-distribution on the cylinder.

The joint p.d.f. of $(Y_1, R, \Theta)'$ is given by

$$\begin{aligned} f(y_1, r, \theta) &= \frac{\Gamma\left((n+3)/2\right) r}{(\pi n)^{3/2}\Gamma(n/2)|\Sigma|^{1/2}} \left[1 + \frac{1}{n}\left\{\frac{(y_1 - \mu(\theta))^2}{\sigma^2(1-\rho^2)} - 2\kappa r \xi \cos(\theta - \mu_0) + \kappa(r^2 + \xi^2)\right\}\right]^{-(n+3)/2} \\ &= \frac{\Gamma\left((n+3)/2\right) r}{(\pi n)^{3/2}\Gamma(n/2)|\Sigma|^{1/2}} \left(1 + \frac{\kappa(r^2+\xi^2)}{n}\right)^{-(n+3)/2} \\ &\quad \times \left[1 + \frac{1}{n}\left\{\frac{(y_1 - \mu(\theta))^2}{\sigma_n^2(1-\rho^2)} - 2\kappa_n \cos(\theta - \mu_0)\right\}\right]^{-(n+3)/2}, \end{aligned}$$

$$-\infty < y_1 < \infty, \quad r > 0, \quad 0 \leq \theta < 2\pi,$$

where $\sigma_n^2 = \sigma^2\left\{1 + \kappa(r^2 + \xi^2)/n\right\}$, $\kappa_n = \kappa r \xi / \left\{1 + \kappa(r^2+\xi^2)/n\right\}$. The marginal p.d.f. of R is

$$\begin{aligned} f(r) &= \int_0^{2\pi}\int_{-\infty}^{\infty} f(y_1, r, \theta) dy_1 d\theta \\ &= \frac{\Gamma\left((n+3)/2\right) r}{(\pi n)^{3/2}\Gamma(n/2)|\Sigma|^{1/2}} \left(1 + \frac{\kappa(r^2+\xi^2)}{n}\right)^{-(n+3)/2} \\ &\quad \times \int_0^{2\pi}\int_{-\infty}^{\infty} \left[1 + \frac{1}{n}\left\{\frac{(y_1 - \mu(\theta))^2}{\sigma_n^2(1-\rho^2)} - 2\kappa_n \cos(\theta - \mu_0)\right\}\right]^{-(n+3)/2} dy_1 d\theta \\ &= \frac{\Gamma\left((n+3)/2\right) r}{(\pi n)^{3/2}\Gamma(n/2)|\Sigma|^{1/2}} \left(1 + \frac{\kappa(r^2+\xi^2)}{n}\right)^{-(n+3)/2} \frac{n!!\sigma_n\sqrt{n(1-\rho^2)}}{(n+1)!!} \\ &\quad \times \int_0^{2\pi} \frac{1}{(n - 2\kappa_n \cos(\theta - \mu_0))^{n/2+1}} d\theta \\ &= \frac{\Gamma\left((n+3)/2\right) r}{(\pi n)^{3/2}\Gamma(n/2)|\Sigma|^{1/2}} \left(1 + \frac{\kappa(r^2+\xi^2)}{n}\right)^{-(n+3)/2} \frac{n!!\sigma_n\sqrt{n(1-\rho^2)}}{(n+1)!!} \\ &\quad \times \frac{2\pi}{(n^2 - 4\kappa_n^2)^{n/4+1/2}} P_{n/2}\left(\frac{n}{\sqrt{n^2 - 4\kappa_n^2}}\right). \end{aligned}$$

Then the p.d.f. of $(Y_1, \Theta)'$ given $R = r$ is calculated as

$$\begin{aligned} f(y_1, \theta | r) &= \frac{f(y_1, \theta, r)}{f(r)} \\ &= \frac{(n+1)!!(n^2 - 4\kappa_n^2)^{n/4+1/2}}{2\pi n!!\sigma_n\sqrt{n(1-\rho^2)} P_{n/2}\left(n/\sqrt{n^2 - 4\kappa_n^2}\right)} \\ &\quad \times \left[1 + \frac{1}{n}\left\{\frac{(x - \mu(\theta))^2}{\sigma_n^2(1-\rho^2)} - 2\kappa_n \cos(\theta - \mu_0)\right\}\right]^{-(n+3)/2}. \end{aligned}$$

Thus, we get (6.1).

6.3 Alternative t-distribution on the cylinder

It is known that the normal distribution on the cylinder (1,4) has the property that the marginal distribution of Θ is the von Mises distribution with mean direction μ_0 and concentration κ, where $0 \le \mu_0 < 2\pi$, $\kappa \ge 0$, and the conditional distribution of X given $\Theta = \theta$ is a normal distribution $N\left(\eta(\theta), \sigma^2(1-\rho^2)\right)$, where $\eta(\theta) = \mu + \sigma\rho\sqrt{\kappa}\cos(\theta - \nu)$, $(0 \le \rho < 1,\ -\infty < \mu < \infty,\ 0 \le \nu < 2\pi)$. As a similar manner to the fact, we can consider an alternative t-distribution on the cylinder. It has p.d.f.

$$\begin{aligned} f(x,\theta) &= \frac{\Gamma((n+1)/2)}{2\pi^{3/2}\sigma\sqrt{n(1-\rho^2)}\Gamma(n/2)} \left({}_2F_1\left(\frac{n}{4}+\frac{1}{2}, \frac{n}{4}+1; 1; \left(\frac{2}{n}\kappa_n\right)^2\right)\right)^{-1} \\ &\quad \times \left(1 - \frac{2}{n}\kappa_n \cos(\theta - \mu_0)\right)^{-(n+2)/2} \left(1 + \frac{(x-\eta(\theta))^2}{n\sigma^2(1-\rho^2)}\right)^{-(n+1)/2}, \qquad (6.3) \\ &\qquad -\infty < x < \infty,\ 0 \le \theta < 2\pi. \end{aligned}$$

As n tends to infinity, (6.3) also converges to the normal distribution on the cylinder (1,4), whose density is given by (6.2).

This distribution with p.d.f. (6.3) is obtained as follows. Let Θ be a random variable which obeys the t-distribution on the circle whose p.d.f. is given by (2.2). Suppose that $(X - \eta(\theta))/\sigma\sqrt{1-\rho^2}$ given $\Theta = \theta$ is a random variable having the t-distribution on the real line. Then the joint distribution of $(X, \Theta)'$ is (6.3).

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