Research Report

KSTS/RR-04/008
Sep. 8, 2004

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Moments and Projections of Semistable Probability Measures on p-adic Vector Spaces

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Abstract

In this paper, two topics on semistable probability measures on p-adic vector spaces are studied. One is the existence of absolute moments of operator-semistable probability measures and another is an answer to the question whether one can get semistability of a probability measure from that of all its projections. All results obtained here are extensions of known results for real vector spaces to p-adic vector spaces.

Keywords : semistable probability measure, p-adic vector space, moment of probability measure

Suggested running head:
Semistable Probability Measures on p-adic Vector Spaces

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1 Introduction

Let V be a finite dimensional p-adic vector space, where p is a prime number. Let $\mathcal{P}(V)$ denote the topological semigroup of probability measures on V, with weak topology and convolution '$*$' as the semigroup operation defined with respect to the additive group structure on V. Let $GL(V)$ denote the group of all invertible linear operators on V, namely the set of all bi-continuous automorphisms on V. In the following, $\mathbb{R}, \mathbb{Q}, \mathbb{Z}$ and $\mathbb{N}$ stand for the set of real numbers, rational numbers, integers and natural numbers, respectively.

A probability measure μ on V is said to be *operator-semistable* if there exist $\tau \in GL(V)$, $c \in]0,1[$ and $\{x_t\}_{t\geq 0} \subset V$ such that μ is embeddable in a continuous real one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(V)$ as $\mu = \mu_1$ satisfying $\tau(\mu_t) = \mu_{ct} * \delta_{x_t}$ for all $t \geq 0$, where δ_x denotes the Dirac measure supported on $x \in V$. However, if H is the compact subgroup such that $\mu_0 = \omega_H$, the normalised Haar measure of H, then the image of $\{x_t\}_{t\geq 0}$ on V/H is a continuous real one-parameter semigroup and hence it is trivial. Thus, $\tau(\mu_t) = \mu_{ct}$ for all $t \geq 0$. We call μ and $\{\mu_t\}_{t\geq 0}$ as above (τ, c)-*semistable*; when τ is a scaler automorphism, we say that μ (or $\{\mu_t\}_{t\geq 0}$) is *semistable*.

Let $\|\cdot\|$ be a p-adic vector space norm on V. For $r \in (0, \infty)$, we say that $\mu \in \mathcal{P}(V)$ has an *absolute moment of order* r if $\int_V \|x\|^r \, \mathrm{d}\mu(x) < \infty$. Note that this definition is independent of norm on V, since any two vector space norms on V are equivalent (see, e.g. Cassels [1], Chapter 7, Lemma 2.1).

Operator-semistable probability measures on real vector spaces have been studied extensively. For a complete survey of results on (operator-)semistable probability measures, see Hazod and Siebert [3] and Sato [10]. For some related results for p-adic vector spaces, see Dani and Shah [2], Shah [11], Telöken [17] and Yasuda [19].

In this paper, we discuss the existence of absolute moments of order r of an operator-semistable probability measure μ on a finite dimensional p-adic vector space V as an extension of the corresponding result on real vector spaces (Theorem 1). Using this result, we investigate the relation between semistability of a probability measure μ on V and that of all its one-dimensional projections under certain conditions (Theorem 2).

2 Moments of operator-semistable probability measures on p-adic vector spaces

In this section, we discuss the existence of absolute moments of operator-semistable probability measures on a finite dimensional p-adic vector space V. There exist a lot of operator-semistable probability measures on a p-adic vector space (see Dani and Shah [2], Theorem 4.2).

For a prime number p, let $\mathbb{Q}_p$ denote the topological field of p-adic numbers with the p-adic norm $|\cdot|_p$. Namely, for any rational number $x \in \mathbb{Q}$, if $x = (h/k)p^n$ for some integers $h, k, n \in \mathbb{Z}$, where p does not divide h or k, then $|x|_p = p^{-n}$ and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to this norm. Let $d = \dim V$. Then V is isomorphic to $\mathbb{Q}_p^d$. Let $M(V)$ denote the space of all linear operators on V. Then $M(V)$ (resp. $GL(V)$) is isomorphic to $M_d(\mathbb{Q}_p)$, the vector space of $d \times d$ matrices (resp. $GL_d(\mathbb{Q}_p)$, the group of nonsingular $d \times d$ matrices) with entries in $\mathbb{Q}_p$, having the usual topology as a subset of $\mathbb{Q}_p^{d^2}$. Here, $M(V)$ is a d^2-dimensional p-adic vector space. Given a vector space norm $\|\cdot\|$ on V, we define a vector space norm $\|\cdot\|$ on $M(V)$ as follows: $\|\tau\| = \sup\{\|\tau(x)\| : x \in V, \|x\| = 1\}$ for $\tau \in M(V)$. Here, $\|\tau\tau'\| \le \|\tau\|\,\|\tau'\|$ and $\|\tau(x)\| \le \|\tau\|\,\|x\|$ for $\tau, \tau' \in M(V)$ and $x \in V$. We define the spectral radius $s(\tau) = \lim_{n\to\infty} \|\tau^n\|^{1/n}$ for $\tau \in M(V)$; it is easy to see that the limit exists. Note that $s(\tau) \le \|\tau^n\|^{1/n}$ for all n. Clearly, $s(\tau)$ is independent of the norm defined as above, since any two vector space norms on $M(V)$ are equivalent (see, e.g. Cassels [1], Chapter 7, Lemma 2.1).

We now define a vector space norm $\|\cdot\|_p$ on V (resp. on $M(V)$) as follows: We fix a basis $\{e_1, \ldots, e_d\}$ on V. For $x = (x_1, \ldots, x_d) = \sum_{i=1}^d x_i e_i \in V$, let $\|x\|_p = \max_i |x_i|_p$. Using this norm on V, we can define $\|\cdot\|_p$ on $M(V)$ as follows; for any $\tau \in M(V)$, $\|\tau\|_p = \sup\{\|\tau(x)\|_p : \|x\|_p = 1\}$. Note that if $\tau = (a_{ij}) \in M_d(\mathbb{Q}_p)$ with respect to the basis mentioned above, $\|\tau\|_p = \max_{i,j} |a_{ij}|_p$. Here, for $x = \sum_{i=1}^d x_i e_i$ and $y = \sum_{i=1}^d y_i e_i$ in V, $\|x+y\|_p = \max_i |x_i + y_i|_p = |x_j + y_j|_p \le \max\{|x_j|_p, |y_j|_p\}$ for some j. Then $\|x+y\|_p \le \max\{\|x\|_p, \|y\|_p\}$. Therefore, for any $r > 0$, $\|x+y\|_p^r \le \max\{\|x\|_p^r, \|y\|_p^r\}$, and hence $x \mapsto \|x\|_p^r$ is a continuous subadditive function on V (see its definition below).

A probability measure μ on V is said to be *full* if the support of μ, denoted by $\operatorname{supp}\mu$, is not contained in a proper subspace of V. For $\tau \in GL(V)$, let $C(\tau) = \{x \in V : \tau^n(x) \to 0\}$. Clearly $C(\tau)$ is a τ-invariant vector subspace of V. We say that τ is *contracting* on V if $C(\tau) = V$. Note that, since $C(\tau)$ is closed in V, for any $\mu \in \mathcal{P}(V)$, $\tau^n(\mu) \to \delta_0$ if and only if $\operatorname{supp}\mu \subset C(\tau)$. A measure $\mu \in \mathcal{P}(V)$ is said to be an *idempotent* if $\mu^2 = \mu$, equivalently, if $\mu = \omega_H$, the normalised Haar measure of some compact subgroup H. Clearly, any idempotent has an absolute moment of any order.

The following is a generalization of a result on existence of absolute moments of operator-semistable probability measures on real vector spaces to p-adic vector spaces, (for the result on real vector spaces, see Luzak [5], [6]).

Theorem 1 *Let V be a finite dimensional p-adic vector space. Let $\mu \in \mathcal{P}(V)$ be full, non-idempotent and (τ, c)-semistable for some $\tau \in GL(V)$ and some $c \in]0.1[$. Then μ has an absolute moment of order r if and only if $s(\tau^{-1})^r c < 1$.*

Remark: The fullness condition is not needed for the "if" part of the above theorem. Suppose μ is a non-idempotent probability measure embeddable in a (τ, c)-semistable $\{\mu_t\}_{t\geq 0}$ as $\mu = \mu_1$. Let V_μ be the subspace generated by $\operatorname{supp}\mu$. Since V is totally disconnected, $\operatorname{supp}\mu_t \subset V_\mu$ for all $t \geq 0$, and hence V_μ is τ-invariant. Let τ_μ be the restriction of τ to V_μ. Since μ is full on V_μ, μ has an absolute moment of order r if and only if $s(\tau_\mu^{-1})^r c < 1$, by Theorem 1. Thus, the "if" part of the assertion without the fullness condition follows since $s(\tau_\mu^{-1}) \leq s(\tau^{-1})$.

Before proving the above theorem, let us state a result on subadditive functions on a locally compact (Hausdorff) group G. A function $\phi : G \to [0, \infty[$ (resp. $\phi : G \to]0, \infty[$) is said to be *subadditive* (resp. *submultiplicative*) if $\phi(xy) \leq \phi(x) + \phi(y)$ (resp. $\phi(xy) \leq \phi(x)\phi(y)$) for all $x, y \in G$ and if there exists a positive real number $r = r(\phi)$ such that $U_r = \{x \in G : \phi(x) \leq r\}$ is a neighbourhood of the identity e in G. Note that if a function ϕ is subadditive, then $1 + \phi$ is submultiplicative. The following result (which is perhaps well known) follows from the same result about submultiplicative functions on G, (see Siebert [14], Theorem 1, and Siebert [15], Theorem 5).

Proposition 1 *Let G be a locally compact group with identity e and let $\mathcal{P}(G)$ be the convolution semigroup of probability measures on G. Let $\{\mu_t\}_{t\geq 0}$, $\mu_0 = \delta_e$, be a continuous one-parameter semigroup in $\mathcal{P}(G)$ with the Lévy measure η. Let $\mathcal{U}(e)$ denote the set of all neighbourhoods of the identity e in G. Let ϕ be a subadditive function and let $r = r(\phi) > 0$ and $U_r \in \mathcal{U}(e)$ be as above. Then the following are equivalent:*

(*i*) $\int_G \phi \, \mathrm{d}\mu_t < \infty$ *for some* $t > 0$.

(*ii*) $\sup_{0 \le s \le t} \int_G \phi \, \mathrm{d}\mu_s < \infty$ *for all* $t > 0$.

(*iii*) $\int_{G \setminus U_r} \phi \, \mathrm{d}\eta < \infty$.

Moreover, if ϕ *is continuous, then the above are equivalent to each of the following statements:*

(*iv*) $\int_{G \setminus U} \phi \, \mathrm{d}\eta < \infty$ *for all* $U \in \mathcal{U}(e)$.

(*v*) $\int_{G \setminus U} \phi \, \mathrm{d}\eta < \infty$ *for some compact neighbourhood* $U \in \mathcal{U}(e)$.

Proof of Theorem 1. As in the hypothesis, since μ is (τ, c)-semistable, there exists a (τ, c)-semistable one-parameter semigroup $\{\mu_t\}_{t \ge 0} \subset \mathcal{P}(V)$ with $\mu_1 = \mu$. Now recall that $C(\tau) = \{x \in V : \tau^n(x) \to 0\}$; it is a vector subspace of V. It is shown in Dani and Shah [2] that for all $t \ge 0$, $\mu_t = \mu_t^{(0)} * \omega_H$, $\mu_0 = \omega_H$, for some compact subgroup H such that $\tau(H) = H$, $\{\mu_t^{(0)}\}$ is a (τ, c)-semistable one-parameter semigroup supported on $C(\tau)$, $\mu_0^{(0)} = \delta_0$ and $H \cap C(\tau) = \{0\}$. Since any norm on V satisfies the triangular inequality, it is easy to see that μ has an absolute moment of order r if and only if $\mu_1^{(0)}$ does. Also, since μ, and hence $\mu_1^{(0)}$, is not an idempotent, $\mu_1^{(0)} \ne \delta_0$. Hence $C(\tau) \ne \{0\}$. Let τ_1 be the restriction of τ to $C(\tau)$. Since τ_1 is contracting on $C(\tau)$, it follows that $s(\tau_1^{-1}) > 1$. Now we are going to show that $\mu_1^{(0)}$ is full on $C(\tau)$ and $s(\tau^{-1}) = s(\tau_1^{-1})$.

Let V_H be the subspace of V generated by H. Since $\tau(H) = H$, $\tau(V_H) = V_H$. Moreover, since H is compact, if τ_2 is the restriction of τ to V_H, then it can easily be shown that for some $M > 0$, $\|\tau_2^n\| < M$ for all $n \in \mathbb{Z}$, and hence $s(\tau_2^{-1}) = 1 = s(\tau_2)$. Also, $V_H \cap C(\tau) = \{0\}$. Since $\mu = \mu_1^{(0)} * \omega_H$ is full on V, $V = C(\tau) \oplus V_H$, a direct product, and $\mu_1^{(0)}$ is full on $C(\tau)$. Now, we have $s(\tau^{-1}) = \max\{s(\tau_1^{-1}), s(\tau_2^{-1})\} = s(\tau_1^{-1})$. In particular, $s(\tau^{-1}) > 1$.

Now without loss of generality, we may assume that $\mu = \mu_1^{(0)}$, $V = C(\tau)$ and $\tau = \tau_1$, i.e. μ is embeddable in a (τ, c)-semistable $\{\mu_t\}_{t \ge 0}$ with $\mu_0 = \delta_0$ and τ is contracting on V. Let η be the Lévy measure of $\{\mu_t\}_{t \ge 0}$ on $V \setminus \{0\}$. Then we know that η is finite on $V \setminus U$ for any neighbourhood U of 0 in V and $\tau(\eta) = c\eta$. Since $\mu \ne \delta_0$ and V is totally disconnected, it is easy to show that η is not a zero measure (see also Heyer [4], Theorems 6.2.10 and 6.2.3).

Here, V is isomorphic to $\mathbb{Q}_p^d$, where $d = \dim V$. Since any two norms on V are equivalent, it is enough to consider the norm $\|\cdot\|_p$ defined above.

Since τ is contracting on V, by Lemma 3.3 in Siebert [16] there exist distinct open compact subgroups G_n, $n \in \mathbb{Z}$, $V = \cup_n G_n$, $\cap_n G_n = \{0\}$, and for all n, $G_n \subset G_{n+1}$ and $\tau(G_n) = G_{n-1}$.

We first prove the "if" part. Let $r > 0$ be fixed such that $s(\tau^{-1})^r c < 1$. Since the map $x \mapsto \|x\|_p^r$ is subadditive, by Proposition 1, it is enough to show that $\int_{V\setminus G_0} \|x\|_p^r \,\mathrm{d}\eta(x) < \infty$. Using the relation $\tau(\eta) = c\eta$, we have (for $\tau^0 = I$),

$$\begin{aligned} \int_{V\setminus G_0} \|x\|_p^r \,\mathrm{d}\eta(x) &= \sum_{n=0}^{\infty} \int_{G_{n+1}\setminus G_n} \|x\|_p^r \,\mathrm{d}\eta(x) \\ &= \sum_{n=0}^{\infty} \int_{G_1\setminus G_0} \|\tau^{-n}(x)\|_p^r c^n \,\mathrm{d}\eta(x) \\ &\leq \sum_{n=0}^{\infty} \int_{G_1\setminus G_0} \|\tau^{-n}\|_p^r \, \|x\|_p^r c^n \,\mathrm{d}\eta(x) \\ &\leq M\left(1 + \sum_{n=1}^{\infty} \{(\|\tau^{-n}\|_p^{1/n})^r c\}^n\right), \end{aligned}$$

where $M = \sup\{\|x\|_p^r : x \in G_1\}\eta(G_1 \setminus G_0)$, which is finite. Let $a_n = (\|\tau^{-n}\|_p^{1/n})^r c$, $n \in \mathbb{N}$. Since $\|\tau^{-n}\|_p^{1/n} \to s(\tau^{-1})$, we get that $a_n \to s(\tau^{-1})^r c < 1$, and hence $\sum a_n^n$ converges. Therefore

$$\int_{V\setminus G_0} \|x\|_p^r \,\mathrm{d}\eta(x) < \infty.$$

Hence, μ has an absolute moment of order r if $s(\tau^{-1})^r c < 1$.

For the "only if" part, we need to show that $\int_V \|x\|_p^r \,\mathrm{d}\mu(x) = \infty$ if $s(\tau^{-1})^r c \geq 1$. If possible, suppose this integral is finite for some $r > 0$ satisfying $s(\tau^{-1})^r c \geq 1$. Now from above,

$$\sum_{n=0}^{\infty} \int_{G_1\setminus G_0} \|\tau^{-n}(x)\|_p^r c^n \,\mathrm{d}\eta(x) = \int_{V\setminus G_0} \|x\|_p^r \,\mathrm{d}\eta(x) < \infty.$$

Hence

$$\int_{G_1\setminus G_0} \sum_{n=0}^{\infty} \|\tau^{-n}(x)\|_p^r c^n \,\mathrm{d}\eta(x) < \infty.$$

Let η_0 be the restriction of η to $G_1 \setminus G_0$. Then the above implies that for η_0-almost all x,

$$\sum_{n=0}^{\infty} \|\tau^{-n}(x)\|_p^r c^n < \infty \text{ and hence } \|\tau^{-n}(x)\|_p^r c^n \to 0.$$

Let $t_n = \|\tau^{-n}\|_p$. By definition, $t_n \in \mathbb{Q} \setminus \{0\}$. Let $\psi_n = t_n \tau^{-n}$ for all n. Then $\psi_n \in GL(V)$ and $\|\psi_n\|_p = |t_n|_p \|\tau^{-n}\|_p = t_n^{-1} t_n = 1$ for all n. Also, since $s(\tau^{-1})^r c \geq 1$, $t_n^{-r} \leq c^n$. Now we get for η_0-almost all x,

$$\|\psi_n(x)\|_p^r = t_n^{-r} \|\tau^{-n}(x)\|_p^r \leq c^n \|\tau^{-n}(x)\|_p^r \to 0,$$

This implies that, for η_0-almost all x, $\|\psi_n(x)\|_p \to 0$ and hence $\psi_n(x) \to 0$. Let

$$V' = \{x \in V : \psi_n(x) \to 0\}.$$

Then V' is a vector subspace, hence it is closed and $\operatorname{supp} \eta_0 \subset V'$. Since $\tau \circ \psi_n = \psi_n \circ \tau$ for all n, for any $x \in V'$,

$$\psi_n(\tau(x)) = \tau(\psi_n(x)) \to 0.$$

Therefore, $\tau(V') \subset V'$ and hence $\tau(V') = V'$ as $\tau \in GL(V)$. Since $\eta = \sum_{n \in \mathbb{Z}} c^{-n} \tau^n(\eta_0)$, $\operatorname{supp} \eta \subset V' \setminus \{0\}$. Now we show that $V' = V$. If possible, suppose V' is proper. Let $\pi : V \to V/V'$ be the natural projection. Then $\{\pi(\mu_t)\}_{t \geq 0}$ is a continuous one-parameter semigroup with the Lévy measure $\pi(\eta)$ defined on $(V/V') \setminus \{\pi(0)\}$. From above, $\pi(\eta)$ is a zero measure. Hence, since V/V' is totally disconnected, it is easy to show that $\pi(\mu_t) = \delta_{\pi(0)}$, for all t, (see also Heyer [4], Theorems 6.2.10 and 6.2.3), and hence $\operatorname{supp}\mu_1 \subset V'$. But since $\mu_1 = \mu$ is full on V, we get a contradiction. Hence $V' = V$. That is, $\psi_n(x) \to 0$ for all $x \in V$. But $\|\psi_n\|_p = 1$ for all n, which is a contradiction. Thus, $\int_V \|x\|_p^r \, \mathrm{d}\mu(x)$ must be infinite if $s(\tau^{-1})^r c \geq 1$. This completes the proof. □

We now state two simple lemmas about operator-semistable probability measures which will be used in the next section.

Lemma 1 *Let V be a finite dimensional p-adic vector space. Let $\tau \in GL(V)$ and $c \in \,]0, 1[$.*

(i) If $\mu \in \mathcal{P}(V)$ is (τ, c)-semistable, then for $k_n = [c^{-n}]$, $\tau^n(\mu)^{k_n} \to \mu$.

(ii) *If* $\tau^n(\nu)^{l_n} \to \mu \in \mathcal{P}(V)$ *for some* $\nu \in \mathcal{P}(V)$ *and* $\{l_n\} \subset \mathbb{N}$ *such that* $\tau^n(\nu) \to \delta_0$ *and* $l_n/l_{n+1} \to c$*, then* μ *is* (τ, c)*-semistable.*

Proof. (i) Let $\mu \in \mathcal{P}(V)$ be (τ, c)-semistable. Let $\{\mu_t\}_{t\geq 0}$ be a continuous one-parameter semigroup with $\mu_1 = \mu$ and $\tau(\mu_t) = \mu_{ct}$, for all t. Then for $k_n = [c^{-n}]$, $\tau^n(\mu)^{k_n} = \mu_{c^n k_n} \to \mu_1 = \mu$.

(ii) Let ν, $\{l_n\}$, τ, c be as above. Then the set $T = \{\tau^n(\nu)^m : m \leq l_n,\, n \in \mathbb{N}\}$ is relatively compact (see Shah [13], Theorem 2.1 and Remark following it). Also, since $l_n/l_{n+1} \to c$, we can prove the assertion along the proof of Theorem 4.6 of Telöken [17] using Theorem 2.3 of Telöken [17]. In Telöken [17], the fullness of μ is not assumed. However, it is not necessary here to assume that μ is full, since, for $\tau_n = \tau^n$, we have $\tau_{n+1}\tau_n^{-1} = \tau$ for all n. □

Lemma 2 *Let* V *be a finite dimensional p-adic vector space. Let* $\tau \in GL(V)$ *and* $c \in]0, 1[$*. Let* $\{\psi_m\} \subset K$*, a compact subgroup of* $GL(V)$*, be such that* ψ_m *commutes with* τ *for each* m*. Let* $\{\nu_m\} \subset \mathcal{P}(V)$ *be such that* $\nu_m \to \nu \in \mathcal{P}(V)$*. Suppose each* ν_m *is embeddable in a* $(\psi_m\tau, c)$*-semistable one-parameter semigroup* $\{\mu_t^{(m)}\}_{t\geq 0}$ *as* $\mu_1^{(m)} = \nu_m$*, such that* $\mu_0^{(m)} = \delta_0$*. Then for* $k_n = [c^{-n}]$*,* $\psi^n\tau^n(\nu)^{k_n} * y_n \to \nu$*, for some* $\psi \in K$ *and some sequence* $\{y_n\}$ *in* V*.*

Proof. We may assume, without loss of generality, that $\{\psi_m\}$ itself converges. Let ψ be the limit point of it, then $\psi \in K$ and ψ commutes with τ. Recall that $C(\tau) = \{x \in V : \tau^n(x) \to 0\}$, which is a vector subspace. Moreover, if $\rho \in K$ commutes with τ, then ρ keeps $C(\tau)$ invariant and $C(\rho\tau) = C(\tau)$ (cf. Wang [18], Proposition 2.1). For any $m \in \mathbb{N}$, since $(\psi_m\tau)^n(\nu_m) = \mu_{c^n}^{(m)} \to \delta_0$, we have that each ν_m, and hence ν, is supported on $C(\tau)$. Therefore, without loss of generality, we may assume that $V = C(\tau)$, that is, τ is contracting on V.

Let $n \in \mathbb{N}$ be fixed. Let $s_n = 1 - k_n c^n$. Here, for all m,

$$\nu_m = \mu_1^{(m)} = \mu_{k_n c^n}^{(m)} * \mu_{s_n}^{(m)} = \psi_m^n \tau^n(\nu_m)^{k_n} * \mu_{s_n}^{(m)}.$$

Also, $\mu_{k_n c^n}^{(m)} = \psi_m^n \tau^n(\nu_m)^{k_n} \to \psi^n\tau^n(\nu)^{k_n}$. As $\nu_m \to \nu$, the above implies that $\{\mu_{s_n}^{(m)}\}_{m\in\mathbb{N}}$ is relatively compact (cf. Parthasarathy [9], Chapter III, Theorem 2.1). Let α_n be a limit point of it. Since $s_n < c^n$, $\mu_{s_n}^{(m)}$ is a factor of $\mu_{c^n}^{(m)}$, and since $\mu_{c^n}^{(m)} = \psi_m^n\tau^n(\nu_m) \to \psi^n\tau^n(\nu)$, α_n is a factor of $\psi^n\tau^n(\nu)$.

Here, since $\psi \in K$ and it commutes with τ, $C(\psi\tau) = C(\tau) = V$, and hence $\psi^n\tau^n(\nu) \to \delta_0$. Therefore, it follows that there exists a sequence $\{x_n\} \subset V$ such that $\alpha_n * \delta_{x_n} \to \delta_0$ (cf. Shah [13], Lemma 2.3). Now from the above equation, we have that $\nu = \psi^n\tau^n(\nu)^{k_n} * \alpha_n$, for all n, and hence for $y_n = x_n^{-1}$, $\psi^n\tau^n(\nu)^{k_n} * \delta_{y_n} \to \nu$. This completes the proof. $\square$

3 Semistable probability measures on p-adic vector spaces and their projections

In this section, we compare semistability of a probability measure on a p-adic vector space V with that of all its one-dimensional projections. Clearly, if a probability measure μ on V is semistable then all its projections are also semistable. Conversely, we are interested in finding out conditions under which semistabilty of μ is implied by that of all its one-dimensional projections.

For V isomorphic to $\mathbb{Q}_p^d$ and for $x, y \in V$, $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$, let $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$. It is a continuous bi-linear map from V^2 to $\mathbb{Q}_p$. Any one-dimensional projection of V is of the form $y \mapsto \langle x, y\rangle$ for some $x \in V$. For $x \in V \setminus \{0\}$, let $V_x = \mathrm{Ker}(y \mapsto \langle x, y\rangle)$; it is a subspace of co-dimension 1 in V.

For $\mu \in \mathcal{P}(V)$ and $x \in V$, let $\mu_x = (x, \mu)$ denote the image of μ under the map $y \mapsto \langle x, y\rangle$. We can see by injectivity of Fourier transform that $\mu = \nu$ if and only if $(x, \mu) = (x, \nu)$ for all $x \in V$. Moreover, it follows from Lévy's continuity theorem that $\mu_n \to \mu$ in $\mathcal{P}(V)$ if and only if $(x, \mu_n) \to (x, \mu)$, for all $x \in V$.

In the following, we consider semistable probability measures on V. Here, we identify $a \in \mathbb{Q}_p$ with the map $x \mapsto ax$. Recall that a probability measure μ on V is semistable if it is (a, c)-semistable for some $a \in \mathbb{Q}_p \setminus \{0\}$, $|a|_p < 1$, and some $c \in]0, 1[$, that is, μ is embeddable in a continuous one-parameter semigroup $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(V)$ as $\mu = \mu_1$ and $a(\mu_t) = \mu_{ct}$, for all $t \geq 0$. This automatically implies that $C(a) = V$ and $\mu_0 = \delta_0$. Now for a probability measure μ on V, we denote $\Gamma(\mu) = \{c \in]0, \infty[: \mu$ is embeddable in $\{\mu_t\}_{t\geq 0} \subset \mathcal{P}(V)$ such that $a(\mu_t) = \mu_{ct}, t \geq 0$, for some $a \in \mathbb{Q}_p \setminus \{0\}\}$. Clearly, $\Gamma(\mu)$ is always nonempty as $1 \in \Gamma(\mu)$ and if μ is (a, c)-semistable for some $a \in \mathbb{Q}_p \setminus \{0\}$, then $c, c^{-1} \in \Gamma(\mu)$. Let $\mathbb{Z}_p$ denote the ring of p-adic integers, namely,

$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. Let $\mathbb{Z}_p(1) = \{x \in \mathbb{Z}_p : |x|_p = 1\}$. It is the maximal compact subgroup of the multiplicative group $\mathbb{Q}_p \setminus \{0\}$. For $\mu \in \mathcal{P}(V)$, let $\mathrm{Inv}(\mu) = \{\tau \in GL(V) : \tau(\mu) = \mu\}$. It is a closed subgroup of $GL(V)$.

The following theorem is a generalization (to p-adic vector spaces) of Theorem 1 in Maejima and Samorodnitsky [8] which is for real vector spaces. For semistable probability measures on a real vector space $\mathbb{R}^d$, see Maejima [7] and references cited therein.

Theorem 2 *Let V be a finite dimensional p-adic vector space and let $\mu \in \mathcal{P}(V)$. Suppose that for all one-dimensional projections π of V, $\pi(\mu)$ is semistable and $\Gamma = \cap_\pi \Gamma(\pi(\mu)) \neq \{1\}$. Then there exist unbounded sequences $\{a_n\}$ and $\{d_n\}$ in $\mathbb{N}$ such that $p^{a_n}(\mu)^{d_n} \to \mu$; in particular μ is embeddable. Moreover, if $\mathrm{Inv}(\mu) \cap \mathbb{Z}_p(1) = \{z \in \mathbb{Z}_p(1) : z(\mu) = \mu\}$ is an open subgroup of $\mathbb{Z}_p(1)$, then μ is semistable.*

Remark: 1. Given any subgroup K of $\mathbb{Z}_p(1)$, there exist a lot of semistable probability measures on V which are K-invariant. In the proof of Theorem 4.2 in Dani and Shah [2], for any contracting $\tau \in GL(V)$, (we may choose $\tau = aI$, $a \in \mathbb{Q}_p$, $0 < |a|_p < 1$), and any $c \in]0,1[$, one can construct a Lévy measure λ such that $\tau(\lambda) = c\lambda$ and λ is K-invariant; for this one has to take a K-invariant subgroup H_0 and a K-invariant measure ρ_0 on $H_0 \setminus H_1$, which will imply that the corresponding one-parameter semigroup $\{\mu_t\}_{t\geq 0}$ is such that it is (τ, c)-semistable and each μ_t is K-invariant (see also Yasuda [19]).

2. Any closed infinite subgroup of $\mathbb{Z}_p(1)$ is open, so the additional condition in the above theorem leaves out only those probability measures which have finite invariance subgroups in $\mathbb{Z}_p(1)$.

Proof of Theorem 2. Step 1. We may assume that $\mu \neq \delta_0$. Let W be the subspace generated by $\mathrm{supp}\,\mu$. If $\dim W = 1$, then the statement is trivial. So we assume that $\dim W \geq 2$. Since scaler automorphisms keep any subspace invariant, for all one-dimensional projections π of W, $\pi(\mu)$ is semistable and $\cap_\pi \Gamma(\pi(\mu)) \neq \{1\}$. In view of the above arguments, we may assume that $V = W$. That is, it is enough to prove the theorem for a full probability measure μ on V.

From the hypothesis, $\mu_x = (x, \mu)$ is semistable for all $x \in V$ and $\Gamma = \cap_x \Gamma(\mu_x) \neq \{1\}$. Choose $c \in \Gamma$ such that $c \in]0,1[$. Then for each $x \in V \setminus \{0\}$, μ_x is (a_x, c)-semistable for some $a_x \in \mathbb{Q}_p \setminus \{0\}$, $|a_x|_p < 1$.

Step 2. Suppose for any $x \in V \setminus \{0\}$, μ_x is also (d_x, c)-semistable for some $d_x \in \mathbb{Q}_p \setminus \{0\}$. We show that $|a_x|_p = |d_x|_p$. This can be shown by using Theorem 1, but we give a direct proof here. Let $a_x = b_x p^{m(x)}$, and $d_x = s_x p^{k(x)}$, where $b_x, s_x \in \mathbb{Z}_p(1)$ and $m(x), k(x) \in \mathbb{N} \setminus \{1\}$. Let $k_n = [c^{-n}]$, $n \in \mathbb{N}$. Then by Lemma 1 (i), $a_x^n(\mu_x)^{k_n} \to \mu_x$ and $d_x^n(\mu_x)^{k_n} \to \mu_x$. If possible, suppose $m(x) \neq k(x)$. Without loss of generality, we may assume that $m(x) > k(x)$. Then for $i(x) = m(x) - k(x) \in \mathbb{N}$,

$$\mu_x = \lim_n a_x^n(\mu_x)^{k_n} = \lim_n (a_x d_x^{-1})^n d_x^n(\mu_x)^{k_n} = \lim_n b_x^n s_x^{-n} p^{i(x)n} d_x^n(\mu_x)^{k_n}.$$

Since $\{b_x^n s_x^{-n}\} \subset \mathbb{Z}_p(1)$, which is compact, $p^{i(x)n} \to 0$ in $M(V)$ and since $d_x^n(\mu_x)^{k_n} \to \mu_x$, we have $\mu_x = \delta_0$, and hence $\operatorname{supp} \mu \subset V_x$, which is proper as $x \neq 0$. This is a contradiction as μ is full on V. Therefore $m(x) = k(x)$ and hence $|a_x|_p = p^{-m(x)} = |d_x|_p$.

Step 3. Let $V(1) = \{x \in V : \|x\|_p = 1\}$. Then $V(1)$ is compact. We now define a map $m : V(1) \mapsto \mathbb{N}$ as follows: If μ_x is (a_x, c)-semistable, then $m(x) = -\log_p(|a_x|_p)$, i.e. $a_x = b_x p^{m(x)}$, where $|b_x|_p = 1$. From the above arguments, the map m is well-defined.

For each x, we fix a_x and an (a_x, c)-semistable one-parameter semigroup $\{(x,\mu)_t\}_{t\geq 0}$ with $(x,\mu)_1 = (x,\mu)$.

We now show that the image of m, $F = \{m(x) : x \in V(1)\}$, is a finite subset in $\mathbb{N}$. If possible, suppose that for some sequence $\{x_l\} \subset V(1)$, $m(x_l) \to \infty$. Then $a_{x_l} \to 0$ in $\mathbb{Q}_p$. This implies that $a_{x_l}(x_l, \mu) = (x_l, \mu)_c \to \delta_0$. But (x_l, μ) is a factor of $[(x_l, \mu)_c]^{n_0}$ for some fixed $n_0 \in \mathbb{N}$ with $n_0 c > 1$. From above, $[(x_l, \mu)_c]^{n_0} \to \delta_0$. But since $x_l \in V(1)$, $\{x_l\}$ is relatively compact, and for any limit point x of it, $x \in V(1) \subset V \setminus \{0\}$ and $\mu_x = (x, \mu)$ is a factor of δ_0. Therefore, $\mu_x = \delta_g$ for some $g \in \mathbb{Q}_p$ and hence $\mu_x = \delta_0$ as μ_x is semistable. Now $\operatorname{supp} \mu \subset V_x$, a proper subspace, hence μ is not full, which is a contradiction. This implies that F is finite.

Step 4. We next show that the map m from $V(1)$ to $F \subset \mathbb{N}$ is continuous. Let $\{x_l\} \subset V(1)$, $x_l \to x$ in $V(1)$. Since F is finite, we may assume that $m(x_l) = i_0$ for all l. We have to show that $m(x) = i_0$. Here, (x_l, μ) is (a_{x_l}, c)-semistable. Then by Lemma 2, there exist $b \in \mathbb{Z}_p(1)$ and a sequence $\{y_n\} \subset \mathbb{Q}_p$ such that $b^n p^{i_0 n}(x, \mu)^{k_n} * \delta_{y_n} \to (x, \mu)$. Since (x, μ) is (a_x, c)-semistable, we have that $a_x^n(x, \mu)^{k_n} \to (x, \mu)$. Let $a_x = b_x p^{m(x)}$ as above. If possible, suppose $m(x) < i_0$. Then

$$\lim_n (x,\mu) * \delta_{y_n^{-1}} = \lim_n b^n p^{i_0 n}(x,\mu)^{k_n} = \lim_n (bb_x^{-1})^n p^{(i_0 - m(x))n} a_x^n (x,\mu)^{k_n} = \delta_0,$$

since $\{(bb_x^{-1})^n\} \subset \mathbb{Z}_p(1)$, which is compact, $p^{(i_0-m(x))n} \to 0$ in $M(V)$ and since $a_x^n(x,\mu)^{k_n} \to (x,\mu)$. This implies that $\{y_n^{-1}\}$ is relatively compact and $(x,\mu) = \delta_y$ for some limit point y of $\{y_n\}$ in $\mathbb{Q}_p$. Hence $(x,\mu) = \delta_0$ as it is semistable, Therefore, we have $\operatorname{supp}\mu \subset V_x$, a proper subspace, and this is a contradiction. Therefore, $m(x) \geq i_0$.

If possible suppose $m(x) > i_0$. Then for $z_n = (b_x b^{-1})^n p^{(m(x)-i_0)n}(y_n^{-1})$,

$$(x,\mu) = \lim_n a_x^n (x,\mu)^{k_n} = \lim_n (b_x b^{-1})^n p^{(m(x)-i_0)n} [b^n p^{i_0 n}(x,\mu)^{k_n} * \delta_{y_n}] * \delta_{z_n}.$$

Therefore, arguing as earlier, we get that $(x,\mu) * \delta_{z_n^{-1}} \to \delta_0$. Now we get a contradiction as above. Hence $m(x) = i_0$. That is, the map m from $V(1)$ to F is continuous.

Step 5. Now we show that F consists of only one natural number. If possible, suppose there exist $g_1, g_2 \in V(1)$ such that $m(g_1) \neq m(g_2)$. We may also assume that $m(g_1) > m(g_2)$. Here, $p^{m(g_i)} b_{g_i}(g_i,\mu) = (g_i,\mu)_c$, for $i = 1,2$. Let $\{\rho_n\} \subset \mathbb{Q}_p \setminus \{0\}$ be such that $|\rho_n|_p \to 0$. Let $h_n = \rho_n g_1 + g_2, n \in \mathbb{N}$. Here, $h_n \to g_2$ and $\|g_2\|_p = 1$, we get that, for all large n, $\|h_n\|_p = 1$. Without loss of generality, we may assume that $\|h_n\|_p = 1$ for all n. We know that since μ is full on V, for any $x \in V$, (x,μ) is full on the image space $\{\langle x,y\rangle : y \in V\}$; moreover, since (x,μ) is semistable, it is not an idempotent. Now for $i = 1,2$, let $r_i = -\log c/m(g_i)\log p$. By Theorem 1,

$$\int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(g_i,\mu)(x) = \int_V |\langle g_i, y\rangle|_p^s \,\mathrm{d}\mu(y) < \infty \text{ if and only if } s < r_i.$$

Now for a fixed $s \in [r_1, r_2[$ and a fixed n, using the subadditivity of the function $x \mapsto |x|_p^s$, we get that

$$\begin{aligned}
\int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(h_n,\mu)(x) &= \int_V |\langle h_n, y\rangle|_p^s \,\mathrm{d}\mu(y) \\
&\geq |\rho_n|_p^s \int_V |\langle g_1, y\rangle|_p^s \,\mathrm{d}\mu(y) - \int_V |\langle g_2, y\rangle|_p^s \,\mathrm{d}\mu(y) \\
&= |\rho_n|_p^s \int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(g_1,\mu)(x) - \int_{\mathbb{Q}_p} |x|_p^s \,\mathrm{d}(g_2,\mu)(x) \\
&= \infty,
\end{aligned}$$

as the first integral above with respect to the measure (g_1, μ) is infinite and the second one with respect to (g_2, μ) is finite for $s \in [r_1, r_2[$. Since $h_n \to g_2$ and $h_n \in V(1)$, by continuity of the map m, we get that, for all large n, $m(h_n) = m(g_2)$, i.e. (h_n, μ) is $(b_n p^{m(g_2)}, c)$-semistable and $|b_n|_p = 1$. Hence, from Theorem 1 and above equations, we get that $s \geq -\log c/m(g_2)\log p = r_2$. But $s \in [r_1, r_2[$, which is a contradiction. Therefore, our assumption that $m(g_1) \neq m(g_2)$ is wrong. Hence there exists a unique $u_0 \in \mathbb{N}$ such that $F = \{u_0\}$ and μ_x is (a_x, c)-semistable, where $a_x = b_x p^{u_0}$, $b_x \in \mathbb{Z}_p(1)$, for all $x \in V(1)$. That is, $b_x p^{u_0}(x, \mu)_t = (x, \mu)_{ct}$, for all $x \in V(1)$ and all $t > 0$.

Step 6. Let $m_1 = p - 1$ and let $m_n = m_{n-1}^{p-1}$ for $n \geq 2$. Since $b_x \in \mathbb{Z}_p(1)$, we know that $b_x^{m_n} \to 1$ in $\mathbb{Z}_p(1)$. Let $a_n = u_0 m_n$ and let $d_n = [c^{-m_n}]$. Clearly, $a_n \to \infty$, $d_n \to \infty$ and $c^{m_n} d_n \to 1$. Then for all $x \in V(1)$,

$$\begin{aligned}\lim_{n\to\infty}(x, p^{a_n}(\mu)^{d_n}) &= \lim_{n\to\infty} p^{a_n}(x, \mu^{d_n}) \\ &= \lim_{n\to\infty} b_x^{m_n} p^{u_0 m_n}(x, \mu^{d_n}) \\ &= \lim_{n\to\infty}(x, \mu)_{c^{m_n} d_n} \\ &= (x, \mu).\end{aligned}$$

Since for $x \in V \setminus \{0\}, (x, \mu) = \|x\|_p^{-1}(x', \mu), x' \in V(1)$, the above equation also holds for all $x \in V \setminus \{0\}$. Hence $p^{a_n}(\mu)^{d_n} \to \mu$. By Theorem 1.5 of Shah [12], μ is embeddable.

Step 7. We suppose $K = \mathrm{Inv}(\mu) \cap \mathbb{Z}_p(1)$ is open in $\mathbb{Z}_p(1)$. Then $\mathbb{Z}_p(1)/K$ is a finite group and $K \subset \mathrm{Inv}(\mu_x)$ for all x. Let $q \in \mathbb{N}$ be the order of $\mathbb{Z}_p(1)/K$. Then $b_x^q \in K$ for all $x \in V(1)$. This implies that $p^{u_0 q}(x, \mu) = (x, \mu)_{c^q}$. That is, for $j_0 = u_0 q$ and $r = c^q$, we get that $p^{j_0}(x, \mu) = (x, \mu)_r$, $x \in V(1)$. Now for each n, let $l_n = [r^{-n}]$. Since $0 < r < 1$, $r^n \to 0$, $l_n \to \infty$ and $l_n r^n \to 1$. Then we have that for $x \in V(1)$,

$$\lim_{n\to\infty}(x, p^{j_0 n}(\mu)^{l_n}) = \lim_{n\to\infty} p^{j_0 n}(x, \mu)^{l_n} = \lim_{n\to\infty}(x, \mu)_{l_n r^n} = (x, \mu).$$

Again, since for $x \in V \setminus \{0\}, (x, \mu) = \|x\|_p^{-1}(x', \mu), x' \in V(1)$, the above equation also holds for all $x \in V \setminus \{0\}$. Hence $p^{j_0 n}(\mu)^{l_n} \to \mu$. Since $p^{j_0 n}(\mu) \to \delta_0$, $p^{j_0(n+1)}/p^{j_0 n} = p^{j_0}$ is contracting on V and $l_n/l_{n+1} \to r$, by Lemma 1 (ii), μ is (p^{j_0}, r)-semistable, i.e. μ is semistable. This completes the proof. □

Acknowledgements

This work was done while the second author was visiting Keio University on a long-term Invitation Fellowship from the Japan Society for the Promotion of Science (JSPS). She is grateful to the JSPS for the fellowship and Keio University for the hospitality during her stay. She would like to thank W. Hazod for a useful discussion on subadditive functions. The first author was partially supported by the 21st Century COE program of the Ministry of Education, Culture, Sport, Science and Technology in Japan. The authors would also like to thank the referee for helpful suggestions.

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