Research Report

KSTS/RR-04/004
Jul. 27, 2004

Does the Cantor set contain irrational algebraic numbers?

by

Michael Keane

Michael Keane
Department of Mathematics and Computer Science
Wesleyan University

Department of Mathematics
Faculty of Science and Technology
Keio University

3-14-1 Hiyoshi, Kohoku-ku, Yokohama, 223-8522 Japan

KSTS/RR-04/004
july 27, 2004

Does the Cantor set contain irrational algebraic numbers?

Michael Keane

Abstract

To my point of view, one of the central objectives of the Integrative Mathematical Sciences program is, as expressed by Yoshiaki Maeda in his Kickoff Meeting Report, to encourage young researchers internationally. Therefore I was expecially pleased by the request of Hitoshi Nakada to give a research level lecture for young participants; I hope that my remarks on the unsolved and interesting question of the title will lead them to new results and international collaboration in the future.

1 Coefficient dynamics for continued fractions

Let me begin with an idea which I developed during my first visiting professorship at Keio, some years ago. For almost two hundred years, perhaps more, mathematicians have been interested in developments of numbers into continued fractions. If α is an irrational number in the unit interval, the continued fraction of α is a sequence

$$n_1, n_2, n_3, \cdots$$

of positive integers such that

$$\alpha = \lim_{k\to\infty} \frac{1}{n_1 + \frac{1}{n_2 + \frac{1}{\cdots + \frac{1}{n_k}}}}$$

According to the modern, integrative point of view, this sequence is defined algorithmically using an iterative procedure, as follows. For α as above, define

$$n(\alpha) := [\frac{1}{\alpha}],$$

the integer part of α, and

$$T(\alpha) := \{\frac{1}{\alpha}\},$$

the fractional part of α. It then follows immediately that

$$\alpha = \frac{1}{[\frac{1}{\alpha}] + \{\frac{1}{\alpha}\}} = \frac{1}{n(\alpha) + T(\alpha)}.$$

Thus, iteratively, we set for $k \geq 1$

$$n_k := n(T^{k-1}\alpha)$$

to define the continued fraction of α. The young researcher (or the experienced mathematician) not familiar with this approach will certainly find it interesting to prove that, under this definition, the above limit holds, and that the correspondence between irrationals in the unit interval and sequences of positive integers is bijective.

From these remarks it should be clear that study of the transformation T holds interest, and many articles have been dedicated to this subject. The idea I wish to explain is what I call coefficient dynamics, and runs as follows. Suppose now that a, b, and c are three real numbers, and that α is a root of the quadratic equation

$$ax^2 + bx - c = 0.$$

(The minus sign is simply for convenience later.)

We also assume that $a > 0$, changing all three signs if necessary. So in some sense, the number α can be "represented" by the triple (a, b, c). If now

$$\beta = T(\alpha),$$

so that

$$\alpha = \frac{1}{n + \beta}$$

(where we have put $n = n_1$, dropping the index), then substitution of

$$x = \frac{1}{n + y}$$

into the quadratic equation teaches us that $\beta = T(\alpha)$ is a root of the equation

$$a(\frac{1}{n+y})^2 + b(\frac{1}{n+y}) - c = 0$$

or, after changing signs and multiplying by $(n+y)^2$,

$$cy^2 + (2nc - b)y - (a + nb - n^2c) = 0.$$

That is, the new triple

$$(c, 2nc - b, a + nb - n^2c) =: T(a, b, c)$$

"represents" $\beta = T(\alpha)$.

As in real life, this change of viewpoint has both advantages and disadvantages - we lose something and we gain something.

Loss:

In changing from α to (a, b, c), we introduce ambiguity in two different ways. First, another root must be introduced, and second, even then, (a, b, c) is determined only up to a multiplicative constant.

This already seems to be enough to convince us to abandon the idea. However, the advantage is also considerable.

Gain:

If we know n, the mapping T is linear in the triple (a, b, c) and has determinant one. Moreover, if

$$D(a, b, c) := b^2 + 4ac$$

(for the suspicious, remember that we have inserted a minus sign, so that this is just the familiar discriminant), then a simple calculation yields

$$D(T(a, b, c)) = D(a, b, c).$$

In my article [K], I exploit this point of view, coefficient dynamics, to give a simple derivation of Gauss' invariant measure for T, and to show that under this measure, T has the optimal ergodic property of weak Bernoullicity. Both results become elementary using coefficent dynamics; previously, only complicated treatments were known. The advantage of dealing with a linear situation outweighs the disadvantage of ambiguity.

2 Lagrange's theorem

As another, more recent illustration suggested by P. Arnoux (oral communication), let me mention that the idea gives a nice proof of the classical theorem of Lagrange, according to which continued fractions of quadratic algebraic numbers are eventually periodic. If α is such a number, say, in the unit interval, then it satisfies a quadratic equation of the above type in which the coefficients a, b, and c are all integers. It is very easy to see that after a finite number of iterations of T, these coefficients, which remain integers, satisfy the conditions $a > 0$ and $c > 0$; this corresponds to what one commonly calls the reduced case. Now simply note that there are only finitely many triples (a, b, c) for a given value of the discriminant $b^2 + 4ac$; since this discriminant is invariant under T we obtain periodicity. The reader is invited to compare this reasoning with the classical proofs, which are in some way similar but involve more complicated case reduction and ideas which are less clear.

3 A conjecture

Here, I would also like to suggest that this advantage may be present in treating the question of the title. As I cannot yet prove what I would like, my treatment will be incomplete, but I hope that it will serve to illustrate the point.

I now want to develop coefficient dynamics corresponding to the mapping

$$T(x) = 3x \bmod 1$$

on the unit interval. The function which gives the ternary expansion for x is simply a step function taking the values 0, 1, and 2 on the respective intervals $(0, 1/3)$, $(1/3, 2/3)$, and $(2/3, 1)$. Let me denote this function by m. Ignoring triadic rational values, if we set for $k \geq 1$

$$m_k := m(T^{k-1}(x)),$$

then

$$x = \sum_{k=1}^{\infty} m_k \frac{1}{3^k}.$$

Important is the simple fact that x belongs to the classical Cantor set if and only if no m_k is equal to 1.

To keep the presentation simple, I want to restrict attention to the countable set

$$S := \{(b, c) : b, c \in \mathbb{N}, 1 \leq c \leq b\}.$$

For each $(b, c) \in S$, the function

$$g(x) = x^2 + bx - c$$

satisfies

$$g(0) = -c < 0$$

and

$$g(1) = 1 + b - c > 0;$$

it therefore has exactly one zero $\alpha = \alpha(b, c)$ between 0 and 1, which must be irrational. An easy way to see this is to note that

$$b^2 < b^2 + 4c \leq b^2 + 4b < (b + 2)^2,$$

so that only

$$(b + 1)^2 = b^2 + 4c$$

would be possible if $b^2 + 4c$ is to be a perfect square; this leads to $2b + 1 = 4c$ which is impossible, as $2b + 1$ is odd and $4c$ is even. Of course, this also follows from more general considerations, as α is an algebraic integer.

Our next task is to determine the partition $\{S_0, S_1, S_2\}$ of our countable set s according to the values of $m(\alpha)$. It should be clear from a small sketch of the function $g(x)$ that whether $m(\alpha)$ is 0, 1, or 2 depends on the values $g(1/3)$ and $g(2/3)$:

– if $g(1/3) > 0$ then $m(\alpha) = 0$

– if $g(1/3) < 0$ and $g(2/3) > 0$ then $m(\alpha) = 1$

– otherwise $m(\alpha) = 2$.

Now,

$$g(1/3) = (\frac{1}{3})^2(1 + 3b - 9c)$$

and

$$g(2/3) = (\frac{1}{3})^2(4 + 6b - 9c)$$

so that

$$g(1/3) > 0 \iff 3c \leq b$$

$$g(1/3) > 0 \text{ and } g(2/3) > 0 \iff b < 3c \leq 2b + 1$$

$$\text{otherwise} \iff 2b + 1 < 3c \leq 3b.$$

Hence we define

$$S_0 = \{(b, c) \in S : 3c \leq b\}$$

$$S_1 = \{(b, c) \in S : b < 3c \leq 2b + 1\}$$

$$S_2 = \{(b, c) \in S : 2b + 1, 3c \leq 3b\};$$

for $(b, c) \in S_i$ we have then $m(\alpha(b, c)) = i$.

Finally, we produce a mapping

$$T : S \to S$$

corresponding to the ternary expansion algorithm T. The definition separates into three parts:

Part 0: If $(b, c) \in S_0$, we have for $\beta = T(\alpha)$ the relation

$$\beta = 3\alpha.$$

Thus we substitute $x = \frac{y}{3}$ into $g(x)$, yielding after multiplication by 9,

$$9((\frac{y}{3})^2 + b(\frac{y}{3}) - c) = y^2 + 3by - 9c,$$

and we see that on S_0,

$$T(b, c) = (3b, 9c).$$

Part 1: If $(b, c) \in S_1$, we have

$$\beta = 3\alpha - 1;$$

substituting $x = \frac{y+1}{3}$ then yields

$$9((\frac{y+1}{3})^2 + b(\frac{y+1}{3}) - c) = y^2 + (3b + 2)y - (9c - 3b - 1)$$

and

$$T(b, c) = (3b + 2, 9c - 3b - 1).$$

Part 2: If $(b, c) \in S_2$, we have

$$\beta = 3\alpha - 2;$$

then $x = \frac{y+2}{3}$ yields

$$9((\frac{y+2}{3})^2 + b(\frac{y+2}{3}) - c) = y^2 + (3b + 4)y - (9c - 6b - 4)$$

and

$$T(b, c) = (3b + 4, 9c - 6b - 4).$$

If now $(b, c) \in S$ would give rise to a quadratic irrational $\alpha = \alpha(b, c)$ belonging to the Cantor set, then the iterates under T of (b, c) would never fall into the subset S_1. This indeed seems to be impossible for any element of S, but I can't yet prove it.

Conjecture: For each $(b, c) \in S$, there exists a nonnegative integer n such that

$$T^n(b, c) \in S_1.$$

Finally, note that our transformation on S mimics the action of the ternary expansion transformation, and is in a certain sense expanding and hyperbolic, but it acts on a countable set. An interesting general question would be then whether one can find general conditions of mappings enjoying similar properties for which we can verify or disprove corresponding conjectures. Another interesting question concerns continued fractions of algebraic numbers. If we have a cubic algebraic number, a similar transference of the continued fraction transformation to the four coefficients (three, if we assume that we are dealing with an algebraic integer) yields a partition of a countable set into infinitely many subsets, each subset corresponding to a value for a partial quotient in the continued fraction expansion. Again, all mappings are (piecewise) linear and the regions S_i are also given by linear inequalities. If we could show that the corresponding transformation always exits from, say, the first two regions (or a finite number of regions), then we could show that cubic algebraic numbers have at least one partial quotient larger than two (or unbounded partial quotients). This is a classical unsolved problem in number theory.

References

[K] M. Keane, A continued fraction titbit, Fractals 3, 641-650 (1995).

M.S.Keane
Department of Mathematics and Computer Science
Wesleyan University
Middletown, CT 06459, USA
mkeane@@wesleyan.edu