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Algebraic independence of power series generated by linearly independent positive numbers

by

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Abstract

In this paper we establish, using Mahler's method, the algebraic independence of the values at an algebraic number of power series closely related to decimal expansion of linearly independent positive numbers. First we consider a simpler case in Theorem 1 and then generalize it to Theorem 3, which includes Nishioka's result quoted as Theorem 2 of this paper. Lemma 7 plays an essential role in the proof of Theorems 1 and 3.

Key words: Algebraic independence, Mahler's method.
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1 Introduction.

Let $\omega > 0$ and let d be an integer greater than 1. The number ω is expressed as a d-adic expansion

$$\omega = \sum_{i=-l}^{\infty} \varepsilon_i d^{-i}, \quad l = \max\{[\log_d \omega], 0\}, \quad \varepsilon_i \in \{0, 1, \ldots, d-1\},$$

where $[x]$ denotes the largest integer not exceeding the real number x. For those ω having two ways of expression such as $2 = 1.9999\ldots$ (10-adic), we adopt only the left-hand side expression. Then this expansion is uniquely determined. Let

$$a_k = [\omega d^k] \quad (k = 0, 1, 2, \ldots).$$

It is clear that

$$a_k = \sum_{i=-l}^{k} \varepsilon_i d^{k-i},$$

namely the integer a_k is expressed as the d-adic number $\varepsilon_{-l}\varepsilon_{-l+1}\ldots\varepsilon_{k-1}\varepsilon_k$. Hence we see that the sequence $\{a_k\}_{k\geq 0}$ satisfies the recurrence formula

$$a_0 = [\omega], \quad a_k = da_{k-1} + \varepsilon_k \quad (k = 1, 2, 3, \ldots).$$

The author [5] proved that the number $\sum_{k=0}^{\infty} \alpha^{a_k}$ is transcendental for any algebraic number α with $0 < |\alpha| < 1$. In this paper we prove the following algebraic independence result. Let $\omega_1, \ldots, \omega_m > 0$. Define

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m; \; d = 2, 3, 4, \ldots). \tag{1}$$

In what follows, $\mathbb{Q}$ and $\mathbb{R}$ denote the sets of rational and real numbers, respectively.

Theorem 1. *If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the numbers $f_{id}(\alpha)$ $(i = 1, \ldots, m; \; d = 2, 3, 4, \ldots)$ are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$.*

Corollary 1. *If the numbers $\omega_1, \ldots, \omega_m$ are linearly independent over $\mathbb{Q}$, then the functions $f_{id}(z)$ $(i = 1, \ldots, m; \; d = 2, 3, 4, \ldots)$ are algebraically independent over the field $\mathbb{C}(z)$ of rational functions.*

Example. Let

$$f_{1,d}(z) = \sum_{k=0}^{\infty} z^{d^k}, \qquad f_{2,d}(z) = \sum_{k=0}^{\infty} z^{[\sqrt{2}d^k]},$$

$$f_{3,d}(z) = \sum_{k=0}^{\infty} z^{[\sqrt{3}d^k]}, \qquad f_{4,d}(z) = \sum_{k=0}^{\infty} z^{[\pi d^k]} \qquad (d = 2, 3, 4, \ldots).$$

For example we have

$$\begin{aligned} f_{2,10}(z) &= z + z^{14} + z^{141} + z^{1414} + z^{14142} + z^{141421} + \cdots, \\ f_{3,10}(z) &= z + z^{17} + z^{173} + z^{1732} + z^{17320} + z^{173205} + \cdots, \end{aligned}$$

and

$$f_{4,10}(z) = z^3 + z^{31} + z^{314} + z^{3141} + z^{31415} + z^{314159} + \cdots.$$

Then by Theorem 1 the numbers $f_{i,d}(\alpha)$ $(i = 1, \ldots, 4; \; d = 2, 3, 4, \ldots)$ are algebraically independent for any algebraic number α with $0 < |\alpha| < 1$ since the numbers $1, \sqrt{2}, \sqrt{3}$, and π are linearly independent over $\mathbb{Q}$.

Theorem 1 is proved by using the method developed from that of Nishioka used for proving the following:

THEOREM 2 (Nishioka [4, Theorem 1]). *Let*

$$f_d(z) = \sum_{k=0}^{\infty} \sigma_{dk} z^{d^k} \quad (d = 2, 3, 4, \ldots),$$

where the σ_{dk} $(k = 0, 1, 2, \ldots)$ *are in a finite set of nonzero algebraic numbers for every* d. *Then the numbers* $f_d(\alpha)$ $(d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* α *with* $0 < |\alpha| < 1$.

We further obtain the following, which includes both Theorems 1 and 2.

THEOREM 3. *Let* $\omega_1, \ldots, \omega_m > 0$. *Define*

$$f_{id}(z) = \sum_{k=0}^{\infty} \sigma_{idk} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m;\ d = 2, 3, 4, \ldots),$$

where the σ_{idk} $(k = 0, 1, 2, \ldots)$ *are in a finite set of nonzero algebraic numbers for every* i *and for every* d. *If the numbers* $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Q}$, *then the numbers* $f_{id}(\alpha)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* α *with* $0 < |\alpha| < 1$.

Theorem 3 implies the following result, which also includes Theorem 1.

THEOREM 4. *Let* $\omega_1, \ldots, \omega_m > 0$ *and* $\eta_1, \ldots, \eta_m \in \mathbb{R}$. *Define*

$$f_{id}(z) = \sum_{k=0}^{\infty} z^{[\omega_i d^k + \eta_i]} \quad (i = 1, \ldots, m;\ d = 2, 3, 4, \ldots).$$

If the numbers $\omega_1, \ldots, \omega_m$ *are linearly independent over* $\mathbb{Q}$, *then the numbers* $f_{id}(\alpha)$ $(i = 1, \ldots, m;\ d = 2, 3, 4, \ldots)$ *are algebraically independent for any algebraic number* α *with* $0 < |\alpha| < 1$.

REMARK. Concerning the transcendence of a single value of a power series, Corvaja and Zannier [1] proved, as an application of Schmidt's subspace theorem, the following result: Let $\{m_k\}_{k \geq 0}$ be an increasing sequence of positive integers such that $\liminf_{k\to\infty} m_{k+1}/m_k > 1$. Let α be an element of an algebraic number field K with $0 < |\alpha| < 1$ and let $\{\sigma_k\}_{k \geq 0}$ be a sequence of nonzero elements of K satisfying a suitable growth condition on their Weil heights. Then the number $\sum_{k=0}^{\infty}\sigma_k\alpha^{m_k}$ is transcendental. Although this result can treat a wider class of power series, their method does not seem to yield any algebraic independence result of the values of power series.

2 Lemmas.

We prepare the notation for stating the lemmas. For any algebraic number α, we denote by $\lceil\alpha\rceil$ the maximum of the absolute values of the conjugates of α and by $\mathrm{den}(\alpha)$ the smallest positive integer such that $\mathrm{den}(\alpha)\cdot\alpha$ is an algebraic integer. It is easily seen that $\lceil\alpha+\beta\rceil\le\lceil\alpha\rceil+\lceil\beta\rceil$ and $\lceil\alpha\beta\rceil\le\lceil\alpha\rceil\lceil\beta\rceil$ for any algebraic numbers α and β. Furthermore, for any algebraic number α, we define

$$\|\alpha\|=\max\{\lceil\alpha\rceil,\mathrm{den}(\alpha)\}.$$

Then for any $\alpha\neq 0$ we have the inequalities

$$\log|\alpha|\ge -2[\mathbb{Q}(\alpha):\mathbb{Q}]\log\|\alpha\| \tag{2}$$

and

$$\log\left\|\alpha^{-1}\right\|\le 2[\mathbb{Q}(\alpha):\mathbb{Q}]\log\|\alpha\|$$

(cf. [3, Lemma 2.10.2]). If $\Omega=(\omega_{ij})$ is an $n\times n$ matrix with nonnegative integer entries and if $\boldsymbol{z}=(z_1,\dots,z_n)$ is a point of $\mathbb{C}^n$ with $\mathbb{C}$ the set of complex numbers, we define the transformation $\Omega:\mathbb{C}^n\to\mathbb{C}^n$ by

$$\Omega\boldsymbol{z} \;=\; \left(\prod_{j=1}^{n} z_j^{\omega_{1j}},\prod_{j=1}^{n} z_j^{\omega_{2j}},\dots,\prod_{j=1}^{n} z_j^{\omega_{nj}}\right).$$

Let $\{\Omega^{(k)}\}_{k\ge 0}$ be a sequence of $n\times n$ matrices with nonnegative integer entries. We put

$$\Omega^{(k)}=(\omega_{ij}^{(k)})\qquad\text{and}\qquad\Omega^{(k)}\boldsymbol{z}=(z_1^{(k)},\dots,z_n^{(k)}).$$

In what follows, $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and nonnegative integers, respectively. For $\lambda=(\lambda_1,\dots,\lambda_n)\in(\mathbb{N}_0)^n$, we define $\boldsymbol{z}^{\lambda}=z_1^{\lambda_1}\cdots z_n^{\lambda_n}$ and $|\lambda|=\lambda_1+\dots+\lambda_n$. Let K be an algebraic number field. Let $\{f_1^{(k)}(\boldsymbol{z})\}_{k\ge0},\dots,\{f_m^{(k)}(\boldsymbol{z})\}_{k\ge0}$ be sequences of power series in $K[[z_1,\dots,z_n]]$. Let $\chi=(z_1,\dots,z_n)$ be the maximal ideal generated by $z_1,\dots,z_n$ in the ring $K[[z_1,\dots,z_n]]$. In what follows, $c_1,c_2,\dots$ denote positive constants independent of k.

Lemma 1 (cf. Nishioka [4, Theorem 2]). *Assume that*

$$f_i^{(k)}(\boldsymbol{z}) \to f_i(\boldsymbol{z}) \quad \text{as} \quad k \to \infty$$

with respect to the topology defined by χ *for any* i $(1 \le i \le m)$. *Suppose that all the* $f_i^{(k)}(\boldsymbol{z})$ $(k \ge 0)$, $f_i(\boldsymbol{z})$ $(1 \le i \le m)$ *converge in the* n*-polydisc* $\{\boldsymbol{z} = (z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < r \ (1 \le j \le n)\}$. *If* $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ *is a point of* K^n *with* $0 < |\alpha_j| < \min\{1, r\}$ $(1 \le j \le n)$ *and if the following three properties are satisfied, then the values* $f_1^{(0)}(\boldsymbol{\alpha}), \dots, f_m^{(0)}(\boldsymbol{\alpha})$ *are algebraically independent.*

(I) *There exists a sequence* $\{\rho_k\}_{k \ge 0}$ *of positive numbers such that*

$$\lim_{k \to \infty} \rho_k = \infty, \qquad \omega_{ij}^{(k)} \le c_1 \rho_k, \qquad \log |\alpha_j^{(k)}| \le -c_2 \rho_k.$$

(II) *If we put*

$$f_i^{(0)}(\boldsymbol{\alpha}) = f_i^{(k)}(\Omega^{(k)} \boldsymbol{\alpha}) + b_i^{(k)} \quad (1 \le i \le m),$$

then $b_i^{(k)} \in K$ *and*

$$\log \| b_i^{(k)} \| \le c_3 \rho_k \quad (1 \le i \le m).$$

(III) *For any power series* $F(\boldsymbol{z})$ *represented as a polynomial in* $z_1, \dots, z_n, f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z})$ *with complex coefficients of the form*

$$F(\boldsymbol{z}) = \sum_{\lambda,\, \mu = (\mu_1, \dots, \mu_m)} a_{\lambda,\mu} \boldsymbol{z}^{\lambda} f_1(\boldsymbol{z})^{\mu_1} \cdots f_m(\boldsymbol{z})^{\mu_m},$$

where $a_{\lambda,\mu}$ *are not all zero, there exists a* $\lambda_0 \in (\mathbb{N}_0)^n$ *such that if* k *is sufficiently large, then*

$$|F(\Omega^{(k)} \boldsymbol{\alpha})| \ge c_4 |(\Omega^{(k)} \boldsymbol{\alpha})^{\lambda_0}|.$$

Although Theorem 2 of Nishioka [4] requires the assumption that the coefficients of $f_i^{(k)}(\boldsymbol{z})$ are in a finite set $S \subset K$ for all i and k, it can be weakened as in Lemma 1, which is proved by the almost same way as in the proof of Theorem 2 of Nishioka [4]. We state here the proof of Lemma 1 for the sake of the readers. The following lemmas 2 – 5, which are the same as in [4], are necessary for proving Lemma 1.

Lemma 2 (Nishioka [4]). *Let* $f(\boldsymbol{z}) = \sum_{\lambda_1, \dots, \lambda_n} c_{\lambda_1, \dots, \lambda_n} z_1^{\lambda_1} \cdots z_n^{\lambda_n} \in \mathbb{C}[[z_1, \dots, z_n]]$ *converge around the origin. If* $\boldsymbol{z}$ *is sufficiently close to the origin, then*

$$\sum_{\lambda \ge H} |c_{\lambda_1, \dots, \lambda_n}| \cdot |z_1|^{\lambda_1} \cdots |z_n|^{\lambda_n} \le \gamma^{H+1} \max_{1 \le i \le n} |z_i|^H,$$

where γ is a positive constant depending on $f(\boldsymbol{z})$.

LEMMA 3 (Nishioka [4]). (i) *If $f_i^{(k)}(\boldsymbol{z}) - f_i(\boldsymbol{z}) \in \chi^H$, then*

$$|f_i^{(k)}(\Omega^{(k)}\boldsymbol{\alpha}) - f_i(\Omega^{(k)}\boldsymbol{\alpha})| \le c_5^{H+1}e^{-c_2\rho_k H}.$$

(ii) *Using the coefficients $a_{\lambda,\mu}$ given in* (III) *of Lemma* 1, *we put*

$$F^{(k)}(\boldsymbol{z}) = \sum_{\lambda,\,\mu=(\mu_1,\dots,\mu_m)} a_{\lambda,\mu}\boldsymbol{z}^\lambda f_1^{(k)}(\boldsymbol{z})^{\mu_1}\cdots f_m^{(k)}(\boldsymbol{z})^{\mu_m}.$$

Then $F^{(k)}(\Omega^{(k)}\boldsymbol{\alpha}) \neq 0$ if k is sufficiently large.

We assume that $f_1^{(0)}(\boldsymbol{\alpha}),\dots,f_m^{(0)}(\boldsymbol{\alpha})$ are algebraically dependent and deduce a contradiction. There exist a positive integer L and integers τ_μ, not all zero, for $\mu = (\mu_1,\dots,\mu_m)$ with $0 \le \mu_i \le L$ such that

$$\sum_\mu \tau_\mu f_1^{(0)}(\boldsymbol{\alpha})^{\mu_1}\cdots f_m^{(0)}(\boldsymbol{\alpha})^{\mu_m} = 0.$$

Let $w_1,\dots,w_m, y_1,\dots,y_m$, and t_μ $(\mu = (\mu_1,\dots,\mu_m), 0 \le \mu_i \le L)$ be variables and put

$$F^{(k)}(\boldsymbol{z};\boldsymbol{t}) = \sum_\mu t_\mu f_1^{(k)}(\boldsymbol{z})^{\mu_1}\cdots f_m^{(k)}(\boldsymbol{z})^{\mu_m},$$

$$F(\boldsymbol{z};\boldsymbol{t}) = \sum_\mu t_\mu f_1(\boldsymbol{z})^{\mu_1}\cdots f_m(\boldsymbol{z})^{\mu_m},$$

and

$$\sum_\mu t_\mu (w_1+y_1)^{\mu_1}\cdots(w_m+y_m)^{\mu_m} = \sum_\mu T_\mu(\boldsymbol{t};\boldsymbol{y})w_1^{\mu_1}\cdots w_m^{\mu_m}.$$

Then we obtain

$$\begin{aligned} 0 &= F^{(0)}(\boldsymbol{\alpha};\tau) \\ &= \sum_\mu \tau_\mu (f_1^{(k)}(\Omega^{(k)}\boldsymbol{\alpha}) + b_1^{(k)})^{\mu_1}\cdots(f_m^{(k)}(\Omega^{(k)}\boldsymbol{\alpha}) + b_m^{(k)})^{\mu_m} \\ &= \sum_\mu T_\mu(\tau;\boldsymbol{b}^{(k)}) f_1^{(k)}(\Omega^{(k)}\boldsymbol{\alpha})^{\mu_1}\cdots f_m^{(k)}(\Omega^{(k)}\boldsymbol{\alpha})^{\mu_m} \\ &= F^{(k)}(\Omega^{(k)}\boldsymbol{\alpha}; \boldsymbol{T}(\tau;\boldsymbol{b}^{(k)})). \end{aligned}$$

We put $R = K[\boldsymbol{t}] = K[\{t_\mu\}_{\mu=(\mu_1,\dots,\mu_m),\,0\le\mu_i\le L}]$ and

$$V(\tau) = \{Q(\boldsymbol{t}) \in R \mid Q(\boldsymbol{T}(\tau;\boldsymbol{y})) = 0\}.$$

DEFINITION. For $P(\boldsymbol{z};\boldsymbol{t}) = \sum_\lambda P_\lambda(\boldsymbol{t})\boldsymbol{z}^\lambda \in R[[z_1,\ldots,z_n]]$, we define

$$\mathrm{index}P(\boldsymbol{z};\boldsymbol{t}) = \min\{|\lambda| \mid P_\lambda \notin V(\tau)\}.$$

If $P_\lambda \in V(\tau)$ for any λ, then we define $\mathrm{index}P(\boldsymbol{z};\boldsymbol{t}) = \infty$.

LEMMA 4 (Nishioka [4]). *The following two properties are equivalent for any* $P(\boldsymbol{z};\boldsymbol{t}) \in R[\boldsymbol{z}]$.

(i) $P(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)})) = 0$ *for all large* k.

(ii) $\mathrm{index}P(\boldsymbol{z};\boldsymbol{t}) = \infty$.

LEMMA 5 (Nishioka [4]). *Let* p *be a sufficiently large integer. Then there exist polynomials* $P_0(\boldsymbol{z};\boldsymbol{t}),\ldots,P_p(\boldsymbol{z};\boldsymbol{t}) \in K[\boldsymbol{z};\boldsymbol{t}]$ *with* $\deg_{z_j} P_h(\boldsymbol{z};\boldsymbol{t}), \deg_{t_\mu} P_h(\boldsymbol{z};\boldsymbol{t}) \le p$ $(0 \le h \le p)$ *such that the following two properties are satisfied.*

(i) $\mathrm{index}P_0(\boldsymbol{z};\boldsymbol{t}) < \infty$.

(ii) *If we put* $E_p(\boldsymbol{z};\boldsymbol{t}) = \sum_{h=0}^{p} P_h(\boldsymbol{z};\boldsymbol{t})F(\boldsymbol{z};\boldsymbol{t})^h$, *then*

$$\mathrm{index}E_p(\boldsymbol{z};\boldsymbol{t}) \ge c_6(p+1)^{1+1/n}.$$

Now we can complete the proof of Lemma 1. Let $\mathrm{index}E_p(\boldsymbol{z};\boldsymbol{t}) = I$ and let $\gamma_1,\gamma_2,\ldots$ denote positive constants depending only on $E_p(\boldsymbol{z};\boldsymbol{t})$. Let $k \ge \gamma_1$, where γ_1 will be determined below. Let

$$E_p(\boldsymbol{z};\boldsymbol{t}) = \sum_\nu g_\nu(\boldsymbol{z})\boldsymbol{t}^\nu, \qquad g_\nu(\boldsymbol{z}) = \sum_\lambda g_{\nu\lambda}\boldsymbol{z}^\lambda.$$

Then $g_\nu(\boldsymbol{z})$ converges in the n-polydisc $\{\boldsymbol{z} = (z_1,\ldots,z_n) \in \mathbb{C}^n \mid |z_j| < r\ (1 \le j \le n)\}$. Since

$$\lim_{k\to\infty} f_i^{(k)}(\Omega^{(k)}\boldsymbol{\alpha}) = f_i(\boldsymbol{0}),$$

we have

$$|b_i^{(k)}|, |T_\mu(\tau;\boldsymbol{b}^{(k)})| \le c_7.$$

Thus by Lemma 2,

$$|E_p(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)}))| \le \sum_\nu \left(\sum_{|\lambda|\ge I} |g_{\nu\lambda}| \cdot |(\Omega^{(k)}\boldsymbol{\alpha})^\lambda|\right) |\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)})^\nu| \le \gamma_2 \max_{1\le j\le n} |\alpha_j^{(k)}|^I.$$

We choose a positive number θ with $e^{-c_2c_6} < \theta < 1$. By the property (I) we have

$$|E_p(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)}))| \le \frac{1}{2}\theta^{\rho_k(p+1)^{1+1/n}}.$$

We put

$$E_p^{(k)}(\boldsymbol{z};\boldsymbol{t}) = \sum_{h=0}^{p} P_h(\boldsymbol{z};\boldsymbol{t})F^{(k)}(\boldsymbol{z};\boldsymbol{t})^h,$$

and choose a large H satisfying

$$e^{-c_2H} \le \theta \cdot \theta^{(p+1)^{1+1/n}}.$$

If $f_i^{(k)}(\boldsymbol{z}) - f_i(\boldsymbol{z}) \in \chi^H$, by Lemma 3 (i) we have

$$|E_p^{(k)}(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)})) - E_p(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)}))| \le \gamma_3 e^{-c_2H\rho_k}.$$

Then

$$|E_p^{(k)}(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)}))| \le \gamma_3 e^{-c_2H\rho_k} + \frac{1}{2}\theta^{\rho_k(p+1)^{1+1/n}} \le \theta^{\rho_k(p+1)^{1+1/n}}.$$

On the other hand,

$$E_p^{(k)}(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)})) = P_0(\Omega^{(k)}\boldsymbol{\alpha};\boldsymbol{T}(\tau;\boldsymbol{b}^{(k)})) = (say)\ \beta_k \in K.$$

By the properties (I) and (II), we easily see $\|\beta_k\| \le c_8^{\rho_k p}$. Since $\mathrm{index} P_0(\boldsymbol{z};\boldsymbol{t}) < \infty$, by Lemma 4 there are infinitely many k with $\beta_k \ne 0$. For such k, using (2), we have

$$\rho_k(p+1)^{1+1/n}\log\theta \ge \log|\beta_k| \ge -2[K:Q]\log\|\beta_k\| \ge -2[K:Q]\rho_k p\log c_8.$$

Dividing both sides by $\rho_k(p+1)^{1+1/n}$ and letting p tend to ∞, we obtain $\log\theta \ge 0$, a contradiction.

The following lemma is originally due to Masser [2] and improved by Nishioka [4].

LEMMA 6 (Masser [2], Nishioka [4]). *Let $b_1 > \cdots > b_n \ge 2$ be pairwise multiplicatively independent integers. Let $\theta = \log b_1$ and $\theta_j = \theta/\log b_j$ $(1 \le j \le n)$. Suppose that for each α in a finite set A we are given real numbers $\lambda_{1\alpha},\ldots,\lambda_{n\alpha}$, not all zero, and define the sequence*

$$S_\alpha(k) = \sum_{j=1}^{n} \lambda_{j\alpha} b_j^{[\theta_j k]} \quad (k = 0,1,2,\ldots).$$

If $\{k_l\}_{l\geq 1}$ *is an increasing sequence of positive integers with* $\{k_{l+1}-k_l\}_{l\geq 1}$ *bounded, then there exists a positive number* δ *such that*

$$R(\delta) = \{k_l \mid \min_{\alpha\in A} |S_\alpha(k_l)| \geq \delta b_1^{k_l}\} = \{m_l\}_{l\geq 1}, \qquad m_l < m_{l+1},$$

is an infinite set and $\{m_{l+1}-m_l\}_{l\geq 1}$ *is bounded.*

Using Lemma 6, we have the following:

LEMMA 7. *Let* $b_1,\ldots,b_n$ *be integers as in Lemma* 6 *and let* $\theta_1,\ldots,\theta_n$ *be defined in Lemma* 6. *Let* $\omega_1,\ldots,\omega_m > 0$ *be linearly independent over* $\mathbb{Q}$. *Then there exist an infinite set* Λ *of positive integers, a sequence* $\{\delta(l)\}_{l\geq 1}$ *of positive numbers, and a total order* $\succ$ *in* $(\mathbb{N}_0)^{mn}$ *such that if* $\lambda = (\lambda_{ij}) \succ \mu = (\mu_{ij})$ *with* $|\lambda| = \lambda_{11}+\cdots+\lambda_{mn}$, $|\mu| = \mu_{11}+\cdots+\mu_{mn} \leq l$, *then*

$$\sum_{i=1}^m\sum_{j=1}^n \lambda_{ij}[\omega_i b_j^{[\theta_j q]}] - \sum_{i=1}^m\sum_{j=1}^n \mu_{ij}[\omega_i b_j^{[\theta_j q]}] \geq \delta(l) b_1^q$$

for all sufficiently large $q\in\Lambda$. *Moreover, any subset* S *of* $(\mathbb{N}_0)^{mn}$ *has the minimal element with respect to the total order* $\succ$.

Proof. We put

$$A(l) = \{(\lambda,\mu) \mid \lambda,\mu\in(\mathbb{N}_0)^{mn},\ |\lambda|,|\mu|\leq l,\ \lambda\neq\mu\}.$$

For $(\lambda,\mu)\in A(l)$ we set

$$S_{(\lambda,\mu)}(q) = \sum_{i=1}^m\sum_{j=1}^n (\lambda_{ij}-\mu_{ij})[\omega_i b_j^{[\theta_j q]}].$$

We inductively define $\delta(l)$ and $\Lambda(l)$ as follows. First we put $\Lambda(0)=\mathbb{N}$. Letting

$$T_{(\lambda,\mu)}(q) = \sum_{i=1}^m\sum_{j=1}^n (\lambda_{ij}-\mu_{ij})\omega_i b_j^{[\theta_j q]} = \sum_{j=1}^n \left(\sum_{i=1}^m (\lambda_{ij}-\mu_{ij})\omega_i\right) b_j^{[\theta_j q]},$$

we have

$$\begin{aligned} |T_{(\lambda,\mu)}(q) - S_{(\lambda,\mu)}(q)| &= \left|\sum_{i=1}^m\sum_{j=1}^n (\lambda_{ij}-\mu_{ij})(\omega_i b_j^{[\theta_j q]} - [\omega_i b_j^{[\theta_j q]}])\right| \\ &\leq \sum_{i=1}^m\sum_{j=1}^n |\lambda_{ij}-\mu_{ij}| \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{m}\sum_{j=1}^{n}(|\lambda_{ij}|+|\mu_{ij}|)\\
&\le 2l
\end{aligned}
$$

and so

$$|S_{(\lambda,\mu)}(q)| \ge |T_{(\lambda,\mu)}(q)| - 2l.$$

Since $\omega_1,\dots,\omega_m > 0$ are linearly independent over $\mathbb{Q}$, by Lemma 6 there exists a positive number $\varepsilon(l)$ such that

$$\Lambda(l) = \{q \in \Lambda(l-1) \mid \min_{(\lambda,\mu)\in A(l)} |T_{(\lambda,\mu)}(q)| \ge \varepsilon(l) b_1^q > 2l\}$$

is an infinite set and the differences of two consecutive elements of $\Lambda(l)$ are bounded. Letting s_l be the smallest number of $\Lambda(l)$ and choosing $\delta(l)$ such that $0 < \delta(l) \le \varepsilon(l) - 2l b_1^{-s_l}$, we have

$$\min_{(\lambda,\mu)\in A(l)} |S_{(\lambda,\mu)}(q)| \ge \min_{(\lambda,\mu)\in A(l)} |T_{(\lambda,\mu)}(q)| - 2l \ge \varepsilon(l) b_1^q - 2l \ge \delta(l) b_1^q$$

for all $q \in \Lambda(l)$. Noting that $\Lambda(l) \supset \Lambda(l+1)$ for any $l \ge 0$, we can choose a sequence $\{q_l\}_{l\ge1}$ satisfying both $q_l \in \Lambda(l)$ and $q_l < q_{l+1}$. There exists a subsequence $\{q_l^{(1)}\}_{l\ge1}$ of $\{q_l\}_{l\ge1}$ such that the signs of $S_{(\lambda,\mu)}(q_l^{(1)})$ with $|\lambda|,|\mu| \le 1$ are fixed for all $l \ge 1$. There exists a subsequence $\{q_l^{(2)}\}_{l\ge2}$ of $\{q_l^{(1)}\}_{l\ge1}$ such that the signs of $S_{(\lambda,\mu)}(q_l^{(2)})$ with $|\lambda|,|\mu| \le 2$ are fixed for all $l \ge 2$. Continuing this process, we obtain a sequence $\{q_l^{(m)}\}_{l\ge m}$ for every $m \ge 1$. We set

$$\Lambda = \{q_1^{(1)}, q_2^{(2)}, \dots, q_l^{(l)}, \dots\},$$

and for $\lambda,\mu \in (\mathbb{N}_0)^{mn}$ we define $\lambda \succ \mu$ if and only if $S_{(\lambda,\mu)}(q) > 0$ for all large $q \in \Lambda$. The proof of the former part of the lemma is completed by noting that $\Lambda(l) \supset \Lambda(l+1)$ and that $q_l^{(l)} \in \Lambda(l)$. For the latter part, we use the following fact (cf. [3, Lemma 2.6.4]): If S is any subset of $(\mathbb{N}_0)^p$, then there is a finite subset T of S such that for any $(\lambda_1,\dots,\lambda_p) \in S$, there is an element $(\mu_1,\dots,\mu_p) \in T$ with $\mu_i \le \lambda_i$ $(1 \le i \le p)$. If μ is the minimal element of T with respect to the total order $\succ$, we easily see it is also the minimal element of S. This completes the proof of the lemma.

LEMMA 8 (Nishioka [4]). *Let d be an integer greater than 1 and let*

$$f_l(z) = \sum_{h=0}^{\infty} s_h^{(l)} z^{d^{lh}} \quad (l = 1, 2, \dots),$$

where the coefficients $s_h^{(l)}$ *are nonzero complex numbers. Then* $f_l(z)$ $(l = 1, 2, \ldots)$ *are algebraically independent over* $\mathbb{C}(z)$.

3 Proof of Theorems 1, 3, and 4.

Proof of Theorem 1. Let

$$D = \{d \in \mathbb{N} \mid d \neq a^n \ (a, n \in \mathbb{N},\ n \geq 2)\}.$$

Then

$$\mathbb{N} \setminus \{1\} = \bigcup_{d \in D} \{d, d^2, \ldots\},$$

which is a disjoint union since any two distinct elements of D are multiplicatively independent by the definition of D. Let $d_1 > \cdots > d_n$ be elements of D and let $\boldsymbol{z} = (z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn})$, where $z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn}$ are distinct variables. For any i $(1 \leq i \leq m)$ and for any $d_j \in D$ $(1 \leq j \leq n)$, we define the sequence $\{r_k^{(i,j)}\}_{k \geq 0}$ by

$$r_0^{(i,j)} = 1, \quad r_k^{(i,j)} = [\omega_i d_j^k] \quad (k \geq 1) \tag{3}$$

and define

$$f_{ijl0}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \alpha^{r_{lh}^{(i,j)} - d_j^{lh}} z_{ij}^{d_j^{lh}} \quad (1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t).$$

Letting $\boldsymbol{\alpha} = (\alpha, \ldots, \alpha, \ldots, \alpha, \ldots, \alpha)$, we have

$$f_{ijl0}(\boldsymbol{\alpha}) = \sum_{h=0}^{\infty} \alpha^{r_{lh}^{(i,j)}} = \alpha + \sum_{h=1}^{\infty} \alpha^{[\omega_i d_j^{lh}]} = f_{id_j^l}(\alpha) - \alpha^{[\omega_i]} + \alpha,$$

where f_{id} is defined by (1). Hence it suffices to prove the algebraic independency of the values $f_{ijl0}(\boldsymbol{\alpha})$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$. For the purpose we apply Lemma 1.

Put $b_j = d_j^{t!}$, $\theta = \log b_1$, and $\theta_j = \theta / \log b_j$ $(1 \leq j \leq n)$. Noting that

$$0 \leq r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh} \leq d_j^{lh} - 1 \quad (1 \leq i \leq m),$$

we put

$$\Sigma_q = \left(\alpha^{r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh}} \right)_{1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t,\ h \geq 0} \in \prod_{h=0}^{\infty} \prod_{j=1}^{n} \prod_{l=1}^{t} \{1, \alpha, \ldots, \alpha^{d_j^{lh} - 1}\}^m$$

for any $q \in \Lambda$ with the Λ defined in Lemma 7. Since the right-hand side is a compact set, there exists a converging subsequence $\{\Sigma_{q_k}\}_{k\ge 1}$ of $\{\Sigma_q\}_{q\in\Lambda}$, where q_1 will be chosen sufficiently large. Let

$$\lim_{k\to\infty} \Sigma_{q_k} = \left(\alpha^{s_h^{(i,j,l)}}\right)_{1\le i\le m,\ 1\le j\le n,\ 1\le l\le t,\ h\ge 0}$$

and define

$$f_{ijlk}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \alpha^{r_{lh+t![\theta_j q_k]}^{(i,j)} - r_{t![\theta_j q_k]}^{(i,j)} d_j^{lh}} z_{ij}^{d_j^{lh}} \quad (1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ k \ge 1)$$

and

$$f_{ijl}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \alpha^{s_h^{(i,j,l)}} z_{ij}^{d_j^{lh}} \quad (1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t).$$

Then

$$\lim_{k\to\infty} f_{ijlk}(\boldsymbol{z}) = f_{ijl}(\boldsymbol{z}).$$

Define the $mn \times mn$ matrix

$$\Omega^{(k)} = \operatorname{diag}\left([\omega_1 b_1^{[\theta_1 q_k]}], \dots, [\omega_m b_1^{[\theta_1 q_k]}], \dots, [\omega_1 b_n^{[\theta_n q_k]}], \dots, [\omega_m b_n^{[\theta_n q_k]}]\right).$$

We assert first that $\{\Omega^{(k)}\}_{k\ge 1}$, $\boldsymbol{\alpha} = (\alpha, \dots, \alpha, \dots, \alpha, \dots, \alpha)$, and $\rho_k = b_1^{q_k}$ $(k \ge 1)$ satisfy the assumptions (I) and (II) of Lemma 1. Since $b_1 > \cdots > b_n$, we have

$$b_1^{q_k-1} \le b_j^{-1} b_1^{q_k} < b_j^{[\theta_j q_k]} \le b_1^{q_k}$$

and so

$$\frac{1}{2}\left(\min_{1\le i\le m} \omega_i\right) b_1^{q_k-1} \le \left(\min_{1\le i\le m} \omega_i\right) b_1^{q_k-1} - 1 < [\omega_i b_j^{[\theta_j q_k]}] \le b_1^{q_k} \max_{1\le i\le m} \omega_i$$

for any i $(1 \le i \le m)$, j $(1 \le j \le n)$, and for all $k \ge 1$, if q_1 is sufficiently large. Hence the assumption (I) is satisfied.

Let $K = \mathbb{Q}(\alpha)$. Then $f_{ijlk}(\boldsymbol{z}) \in K[[\boldsymbol{z}]]$ $(1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ k \ge 0)$ and

$$f_{ijlk}(\Omega^{(k)}\boldsymbol{\alpha}) = \sum_{h=0}^{\infty} \alpha^{r_{lh+t![\theta_j q_k]}^{(i,j)}} = f_{ijl0}(\boldsymbol{\alpha}) - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha^{r_{lh}^{(i,j)}}$$
$$(1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ k \ge 1).$$

Since $r_{l(k+1)}^{(i,j)} > r_{lk}^{(i,j)}$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$ for all sufficiently large k by the definition, there is a positive constant C such that $\max_{0 \leq h \leq k-1} r_{lh}^{(i,j)} \leq Cr_{lk}^{(i,j)}$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$ for all $k \geq 1$. Hence

$$\begin{aligned}\log \left\| - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \alpha^{r_{lh}^{(i,j)}} \right\| &\leq \log(t!/l)[\theta_j q_k] + \left(\max_{0 \leq h \leq (t!/l)[\theta_j q_k]-1} r_{lh}^{(i,j)} \right) \log \|\alpha\| \\ &\leq \left(1 + C(\max_{1 \leq i \leq m} \omega_i) \log \|\alpha\| \right) \rho_k,\end{aligned}$$

and the assumption (II) is satisfied.

Therefore, if the assumption (III) is also satisfied, the proof is completed. Noting that $z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn}$ are distinct variables, we see by Lemma 8 that the functions $f_{ijl}(\boldsymbol{z})$ $(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t)$ are algebraically independent over $\mathbb{C}(z_{11}, \ldots, z_{m1}, \ldots, z_{1n}, \ldots, z_{mn})$. Let

$$F(\boldsymbol{z}) = \sum_{\mu=(\mu_{ij}),\, \nu=(\nu_{ijl})} a_{\mu,\nu} \boldsymbol{z}^{\mu} f_{111}^{\nu_{111}} \cdots f_{mnt}^{\nu_{mnt}} = \sum_{\lambda=(\lambda_{ij}) \in (\mathbb{N}_0)^{mn}} c_{\lambda} \boldsymbol{z}^{\lambda},$$

where the coefficients $a_{\mu,\nu}$ are not all zero, and let $\lambda_0 = (\lambda_{ij}^{(0)})$ be the minimal element in $(\mathbb{N}_0)^{mn}$ with respect to the total order $\succ$ defined in Lemma 7 among λ with $c_\lambda \neq 0$. Let $l = 2(|\lambda_0| + 1) \left(\left[\dfrac{\max_{1 \leq i \leq m} \omega_i}{\min_{1 \leq i \leq m} \omega_i} \right] + 1 \right) b_1$. If k is sufficiently large, then by Lemma 2

$$\begin{aligned}&\sum_{|\lambda| \geq l} |c_\lambda| \cdot |\alpha|^{\lambda_{11}[\omega_1 b_1^{[\theta_1 q_k]}]} \cdots |\alpha|^{\lambda_{m1}[\omega_m b_1^{[\theta_1 q_k]}]} \cdots |\alpha|^{\lambda_{1n}[\omega_1 b_n^{[\theta_n q_k]}]} \cdots |\alpha|^{\lambda_{mn}[\omega_m b_n^{[\theta_n q_k]}]} \\ \leq\ & \gamma^{l+1} \left(|\alpha|^{\frac{1}{2}(\min_{1 \leq i \leq m} \omega_i) b_1^{q_k - 1}} \right)^l \\ \leq\ & \gamma^{l+1} |\alpha|^{(\max_{1 \leq i \leq m} \omega_i) b_1^{q_k} (|\lambda_0|+1)}.\end{aligned}$$

Since

$$\begin{aligned}&\lambda_{11}^{(0)}[\omega_1 b_1^{[\theta_1 q_k]}] + \cdots + \lambda_{m1}^{(0)}[\omega_m b_1^{[\theta_1 q_k]}] + \cdots + \lambda_{1n}^{(0)}[\omega_1 b_n^{[\theta_n q_k]}] + \cdots + \lambda_{mn}^{(0)}[\omega_m b_n^{[\theta_n q_k]}] \\ \leq\ & |\lambda_0| (\max_{1 \leq i \leq m} \omega_i) b_1^{q_k},\end{aligned}$$

we have

$$\frac{|\sum_{|\lambda| \geq l} c_\lambda (\Omega^{(k)} \boldsymbol{\alpha})^{\lambda}|}{|(\Omega^{(k)} \boldsymbol{\alpha})^{\lambda_0}|} \leq \gamma^{l+1} |\alpha|^{(\max_{1 \leq i \leq m} \omega_i) b_1^{q_k}}$$

if k is sufficiently large. If $|\lambda| < l$ and $\lambda \neq \lambda_0$, then by Lemma 7

$$\frac{|c_\lambda(\Omega^{(k)}\boldsymbol{\alpha})^\lambda|}{|(\Omega^{(k)}\boldsymbol{\alpha})^{\lambda_0}|} \le |c_\lambda| \cdot |\alpha|^{\delta(l)b_1^{q_k}}$$

for all sufficiently large k. Therefore

$$|F(\Omega^{(k)}\boldsymbol{\alpha})/(\Omega^{(k)}\boldsymbol{\alpha})^{\lambda_0} - c_{\lambda_0}| \to 0 \quad (k \to \infty),$$

which implies (III), and the proof of the theorem is completed.

Proof of Theorem 3. We use the same notation as in the proof of Theorem 1. It suffices to prove the algebraic independency of the numbers

$$\zeta_{ijl} = \sum_{h=0}^{\infty} \sigma_{ijlh}\alpha^{[\omega_i d_j^{lh}]} \quad (1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t),$$

where for any $h \ge 0$,

$$\sigma_{ijlh} \in S_{ijl} = \{\beta_1^{(i,j,l)}, \dots, \beta_{p(i,j,l)}^{(i,j,l)}\}$$

with $\beta_1^{(i,j,l)}, \dots, \beta_{p(i,j,l)}^{(i,j,l)}$ nonzero algebraic numbers. Define

$$f_{ijl0}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \sigma_{ijlh}\alpha^{r_{lh}^{(i,j)} - d_j^{lh}} z_{ij}^{d_j^{lh}} \quad (1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t),$$

where the sequence $\{r_k^{(i,j)}\}_{k \ge 0}$ is defined by (3) in the proof of Theorem 1. Letting $\boldsymbol{\alpha} = (\alpha, \dots, \alpha, \dots, \alpha, \dots, \alpha)$, we have

$$f_{ijl0}(\boldsymbol{\alpha}) = \sum_{h=0}^{\infty} \sigma_{ijlh}\alpha^{r_{lh}^{(i,j)}} = \sigma_{ijl0}\alpha + \sum_{h=1}^{\infty} \sigma_{ijlh}\alpha^{[\omega_i d_j^{lh}]} = \zeta_{ijl} - \sigma_{ijl0}\alpha^{[\omega_i]} + \sigma_{ijl0}\alpha.$$

Hence it is enough to prove the algebraic independency of the values $f_{ijl0}(\boldsymbol{\alpha})$ $(1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t)$.

Put

$$\begin{aligned} \Sigma_q &= \left(\sigma_{ijl\ h+(t!/l)[\theta_j q]}\alpha^{r_{lh+t![\theta_j q]}^{(i,j)} - r_{t![\theta_j q]}^{(i,j)} d_j^{lh}}\right)_{1 \le i \le m,\ 1 \le j \le n,\ 1 \le l \le t,\ h \ge 0} \\ &\in \prod_{h=0}^{\infty}\prod_{i=1}^{m}\prod_{j=1}^{n}\prod_{l=1}^{t}\{\beta_\tau^{(i,j,l)}\alpha^s \mid 1 \le \tau \le p(i,j,l),\ 0 \le s \le d_j^{lh} - 1\} \end{aligned}$$

for any $q \in \Lambda$ with the Λ defined in Lemma 7. Since the right-hand side is a compact set, there exists a converging subsequence $\{\Sigma_{q_k}\}_{k\geq 1}$ of $\{\Sigma_q\}_{q\in\Lambda}$, where q_1 will be chosen sufficiently large. Let

$$\lim_{k\to\infty} \Sigma_{q_k} = \left(\beta^{(i,j,l)}_{\tau(i,j,l,h)} \alpha^{s_h^{(i,j,l)}}\right)_{1\leq i\leq m,\ 1\leq j\leq n,\ 1\leq l\leq t,\ h\geq 0}$$

and define

$$f_{ijlk}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \sigma_{ijl\ h+(t!/l)[\theta_j q_k]} \alpha^{r^{(i,j)}_{lh+t![\theta_j q_k]} - r^{(i,j)}_{t![\theta_j q_k]} d_j^{lh}} z_{ij}^{d_j^{lh}}$$
$$(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t,\ k \geq 1)$$

and

$$f_{ijl}(\boldsymbol{z}) = \sum_{h=0}^{\infty} \beta^{(i,j,l)}_{\tau(i,j,l,h)} \alpha^{s_h^{(i,j,l)}} z_{ij}^{d_j^{lh}} \quad (1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t).$$

Then

$$\lim_{k\to\infty} f_{ijlk}(\boldsymbol{z}) = f_{ijl}(\boldsymbol{z}).$$

Define the $mn \times mn$ matrix

$$\Omega^{(k)} = \mathrm{diag}\left([\omega_1 b_1^{[\theta_1 q_k]}], \ldots, [\omega_m b_1^{[\theta_1 q_k]}], \ldots, [\omega_1 b_n^{[\theta_n q_k]}], \ldots, [\omega_m b_n^{[\theta_n q_k]}]\right).$$

Then

$$f_{ijlk}(\Omega^{(k)}\boldsymbol{\alpha}) = \sum_{h=0}^{\infty} \sigma_{ijl\ h+(t!/l)[\theta_j q_k]} \alpha^{r^{(i,j)}_{lh+t![\theta_j q_k]}} = f_{ijl0}(\boldsymbol{\alpha}) - \sum_{h=0}^{(t!/l)[\theta_j q_k]-1} \sigma_{ijlh} \alpha^{r^{(i,j)}_{lh}}$$
$$(1 \leq i \leq m,\ 1 \leq j \leq n,\ 1 \leq l \leq t,\ k \geq 1)$$

and the assumptions (I) and (II) of Lemma 1 are satisfied. The rest of the proof is the same as that of Theorem 1.

Proof of Theorem 4. Define

$$g_{id}(z) = \sum_{k=0}^{\infty} \alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} z^{[\omega_i d^k]} \quad (i = 1, \ldots, m;\ d = 2, 3, 4, \ldots).$$

Then

$$\alpha^{[\omega_i d^k + \eta_i] - [\omega_i d^k]} \in \{\alpha^{[\eta_i]}, \alpha^{[\eta_i]+1}\},$$

since $0 \leq [\omega_i d^k + \eta_i] - [\omega_i d^k] - [\eta_i] \leq 1$ for any i, d, and for all k. By Theorem 3 the numbers $g_{id}(\alpha)$ $(i = 1, \dots, m;\ d = 2, 3, 4, \dots)$ are algebraically independent, which implies the theorem.

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