On the group extension of the transformation associated to non-archimedean continued fractions

by
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1 Introduction

Let

$$\Gamma(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod k \right\}$$

for a fixed integer $k \geq 2$. We put

$$\tilde{\Gamma}(k) = \Gamma(k) \setminus SL(2,\mathbb{Z}).$$

In 1982, R. Moeckel [4] proved the following result by using the ergodicity of geodesic flows over the modular surfaces :

$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \begin{pmatrix} p_n(x) \\ q_n(x) \end{pmatrix} \equiv \pm \begin{pmatrix} r \\ s \end{pmatrix} \bmod k \right\} = \frac{k}{|\tilde{\Gamma}(k)|} \quad \text{(a.e. } x)$$

for any $0 \leq r,\, s < k$ with $(r,s,k) = 1$, where $p_n(x)$ and $q_n(x)$ are the numerator and the denominator of the nth convergent of a real number x. Later in 1988, H. Jager and P. Liardet [3] got an analogous result :

$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \leq n \leq N : \begin{pmatrix} p_n(x) \\ q_n(x) \end{pmatrix} \equiv \begin{pmatrix} r \\ s \end{pmatrix} \bmod k \right\} = \frac{k}{|\tilde{\Gamma}_*(k)|} \quad \text{(a.e. } x)$$

for any $0 \leq r,\, s < k$ with $(r,s,k) = 1$, where $\tilde{\Gamma}_*(k) = \Gamma_*(k) \setminus GL(2,\mathbb{Z})$ with

$$\Gamma_*(k) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod k \right\}.$$

To prove their result, they made use of the group extension of the continued fraction transformation. A similar discussion for $(q_{n-1}(x), q_n(x))$, instead of

$(p_n(x),\ q_n(x))$,was done by P. Szusz [6] in 1962. Recently M. Fuchs [2] followed P. Szusz's idea and discussed an analogue of such results, mostly convergence in distribution, for the case of non-archimedean continued fractions. Our aim of this paper is to show similar results but almost everywhere convergence, in other words the strong law of large numbers, for non-archimedean continued fractions by the ergodicity of the group extension following H. Jager and P. Liardet's. First we prove the ergodicity of the group extension of the non-archimedean continued fraction transformation with some applications and then show its continued fraction mixing property. The latter result leads to further metric properties of non-archimedean continued fractions.

We start with some defintions and notations. Let $\mathbb{F}_q$ be a finite field with q elements and define the following :

$$\mathbb{F}_q[X] = \{a_n\, X^n + a_{n-1}\, X^{n-1} + \cdots + a_1\, X + a_0 : a_i \in \mathbb{F}_q,\ 0 \le i \le n,\ n \in \mathbb{Z}\}$$

: the ring of polynomials with $\mathbb{F}_q$-coefficients,

$$\mathbb{F}_q(X) = \left\{ \frac{P}{Q} : P, Q \in \mathbb{F}_q[X],\ Q \neq 0 \right\}$$

: the fraction field of $\mathbb{F}_q[X]$,

$$\mathbb{F}_q((X^{-1})) = \{a_n\, X^n + a_{n-1}\, X^{n-1} + \cdots : a_i \in \mathbb{F}_q,\ i \le n,\ n \in \mathbb{Z}\}$$

: the field of formal Laurent power series with $\mathbb{F}_q$-coefficients.

In this paper, we assume that $q > 2$. When $q = 2$, we see that $SL(2,\ \mathbb{F}_q[X]) = SL_{\pm}(2,\ \mathbb{F}_q[X])$, in §2 for the definition, and get the same result with a simple modification. We denote by 0 and 1 the additive and the multiplicative units of $\mathbb{F}_q$, respectively. We may regard $\mathbb{F}_q[X]$, $\mathbb{F}_q(X)$, and $\mathbb{F}_q((X^{-1}))$ as the set of integers, of rational numbers, and of real numbers, respectively. We note the natural inclusions $\mathbb{F}_q \subset \mathbb{F}_q[X] \subset \mathbb{F}_q(X) \subset \mathbb{F}_q((X^{-1}))$. For an element $f \in \mathbb{F}_q((X^{-1}))$, we define

$$\deg f = \begin{cases} n & \text{if} \quad f = a_n X^n + a_{n-1} X^{n-1} + \cdots \ \text{ with } a_n \neq 0, \\ -\infty & \text{if} \quad f = 0 \end{cases}$$

and

$$|f| = q^{\deg f}.$$

Note that $|a| = 1$ for any $a \in \mathbb{F}_q$ with $a \neq 0$ and $|0| = 0$. We also define

$$[f] = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$$

for

$$f = a_n X^n + a_{n-1} X^{n-1} + \cdots, \quad n \ge 0$$

and put

$$\mathbb{L} = \{ f = a_{-1}X^{-1} + a_{-2}X^{-2} + \cdots : a_i \in \mathbb{F}_q,\ i \le -1 \},$$

where $\mathbb{L}$ corresponds to the unit interval $\mathbf{I} = [0,1)$ for the case of real numbers. Since $\mathbb{L}$ is a compact abelian group with the addition and the metric $d(f,g) = |f-g|$, there exists a unique normalized Haar measure m. If we identify $\mathbb{L}$ with $\prod_1^\infty \mathbb{F}_q$, we can identify m with the product probability measure $\prod_1^\infty \delta_{\mathbb{F}_q}$ where $\delta_{\mathbb{F}_q}$ denotes the measure on $\mathbb{F}_q$ that consists of the equal point mass $\frac{1}{q}$ for each $a \in \mathbb{F}_q$.

In §2, we define the continued fraction transformation T on $\mathbb{L}$ and give some fundamental facts on continued fraction expansions derived from T. We denote by $\frac{P_n}{Q_n}$ the n-th principal convergent of the continued fraction expansion of $f \in \mathbb{L}$. Then in §3, we prove the following. We fix $R \in \mathbb{F}_q[X]$ and denote by $C(R)$ the number of pairs (U, V) such that $0 \le \deg U,\ \deg V < \deg R$ and $(U, V, R) = 1$.

Theorem 3.
For any $P, Q \in \mathbb{F}_q[X]$ with $(P,Q,R) = 1$ and deg P, deg $Q <$ deg R, and any integer $l \ge 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le n \le N : \begin{pmatrix} P_n \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \textit{ mod } R,\ |T^n(f)| < \frac{1}{q^l} \right\} = \frac{1}{q^l \cdot C(R)} \quad (m\text{-a.e.}).$$

As mentioned before, we use the group extension of the continued fraction transformation of $\mathbb{L}$ to get Theorem 3. We prove the ergodicity of this extension at the first half of §3. It is easy to see that the above group extension is not mixing, but its 2-fold power can be mixing if we restrict it to a proper subset. In §4, we discuss a strong mixing property of this 2-fold power and show some applications. Indeed, we prove the continued fraction mixing property with exponential decay rate and then have the following :

Theorem 3' *For any $P, Q \in \mathbb{F}_q[X]$ with $(P,Q,R) = 1$ and deg P, deg $Q <$ deg R, and any $l \ge 0$, we have*

$$\frac{1}{N} \sharp \left\{ 1 \le n \le N : \begin{pmatrix} P_n \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \textit{ mod } R,\ |T^n(f)| < \frac{1}{q^l} \right\}$$
$$= \frac{1}{q^l \cdot C(R)} + O(N^{1/2} (\log N)^{3/2+\epsilon}) \quad (m\text{-a.e.})$$

2 Continued fractions for $\mathbb{L}$

We define the continued fraction transformation T on $\mathbb{L}$ by the following

$$T(f) = \begin{cases} \frac{1}{f} - \left[\frac{1}{f}\right] & \text{if } f(\neq 0) \in \mathbb{L} \\ 0 & \text{if } f = 0. \end{cases}$$

We put

$$A_n = A_n(f) = [(T^{n-1}(f))^{-1}] \quad (= 0 \quad \text{if} \quad T^{n-1}(f) = 0) \quad \text{for } n \geq 1$$

and get the expansion

$$f = \cfrac{1}{A_1 + \cfrac{1}{A_2 + \ddots}} = [0\,;\, A_1, A_2, \ldots].$$

As usual, we define $P_n = P_n(f)$ and $Q_n = Q_n(f)$ by

$$\begin{pmatrix} P_n \\ Q_n \end{pmatrix} = \begin{pmatrix} P_{n-2} & P_{n-1} \\ Q_{n-2} & Q_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ A_n \end{pmatrix} \quad \text{for} \quad n \geq 1$$

with

$$\begin{pmatrix} P_{-1} & P_0 \\ Q_{-1} & Q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$\frac{P_n}{Q_n} = [0\,;\, A_1, A_2, \ldots, A_n],$$

which we call the n-th convergent of f. The following hold (see [1]):

(i) $\left| f - \frac{P_n}{Q_n} \right| = \frac{1}{|Q_n||Q_{n+1}|}$

(ii) If $\left| f - \frac{P}{Q} \right| < \frac{1}{|Q|^2}$ with $\deg Q \geq 1$, then $\frac{P}{Q} = \frac{P_n}{Q_n}$ for some $n \geq 1$.

For any $A \in \mathbb{F}_q[X]$, we see that the restriction of T to $\langle A \rangle = \{f \in \mathbb{L} : A_1(f) = A\}$ is one-to-one, and

$$T|_{\langle A \rangle} \langle A \rangle = \mathbb{L},$$

and that its Radon-Nikodym derivative $\frac{dmT|_{\langle A \rangle}}{dm}$ is $q^{2\deg A}$ (m-a.e.). From these properties, it is possible to show that m is an invariant probability measure for T, that is,

$$m(T^{-1}\mathbf{M}) = m(\mathbf{M}) \quad \text{for any Borel subsets } \mathbf{M} \subset \mathbb{L},$$

and $\{A_n : n \geq 1\}$ is independent and identically distributed sequence as random variables defined on $(\mathbb{L}, \mathcal{B}, m)$, where $\mathcal{B}$ denotes the set of Borel subsets of $\mathbb{L}$. For any $B_1, B_2, \ldots, B_l \in \mathbb{F}_q[X]$, we define

$$\langle B_1, B_2, \ldots, B_l \rangle = \{f \in \mathbb{L} : A_1(f) = B_1,\, A_2(f) = B_2, \ldots, A_l(f) = B_l\}.$$

Then we have the following by induction.
(i) $T|_{\langle B_1, B_2, \ldots, B_l \rangle}$ is one-to-one and onto $\mathbb{L}$,
(ii)

$$\frac{dmT|_{\langle B_1, B_2, \ldots, B_l \rangle}}{dm} = q^{2(\sum_{i=1}^{l} \deg(B_i))} \quad \text{(a.e.)}. \tag{1}$$

Since $Tf = \frac{1}{f} - A_1$ is the linear fractional transformation associated to

$$\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}^{-1} = \begin{pmatrix} -A_1 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} P_{n-2} & P_{n-1} \\ Q_{n-2} & Q_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & A_n \end{pmatrix}, \tag{2}$$

it is natural to deal with 2×2 matrices of $\mathbb{F}_q[X]$-entries for the arithmetic discussion of the convergents $\{\frac{P_n}{Q_n} : n \geq 0\}$. Let

$$SL_{\pm}(2,\, \mathbb{F}_q[X]) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A,\, B,\, C,\, D \in \mathbb{F}_q[X],\, AD - BC = \pm 1 \right\}$$

and

$$SL(2,\, \mathbb{F}_q[X]) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A,\, B,\, C,\, D \in \mathbb{F}_q[X],\, AD - BC = 1 \right\}.$$

For a fixed $R \in \mathbb{F}_q[X]$ with $\deg R \geq 1$, we define

$$G(R) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2,\, \mathbb{F}_q[X]) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod R \right\}.$$

We see that $G(R)$ is a normal subgroup of $SL_{\pm}(2,\, \mathbb{F}_q[X])$ and also of $SL(2,\, \mathbb{F}_q[X])$. Then $\gamma_1\gamma_2^{-1} \in G(R)$ if and only if $\gamma_1 \equiv \gamma_2 \bmod R$ for any $\gamma_1,\, \gamma_2 \in SL_{\pm}(2,\, \mathbb{F}_q[X])$ (or $SL(2,\, \mathbb{F}_q[X])$). Moreover if $\gamma_1 \equiv \gamma_2 \bmod R$ and $\det \gamma_1 = 1$ imply $\det \gamma_2 = 1$ for $\gamma_1,\, \gamma_2 \in SL_{\pm}(2,\, \mathbb{F}_q[X])$. We denote by $\tilde{G}(R)$ and $\tilde{G}_{+}(R)$ the factor group $G(R) \backslash SL_{\pm}(2,\, \mathbb{F}_q[X])$ and $G(R) \backslash SL(2,\, \mathbb{F}_q[X])$, respectively. Since $SL(2,\, \mathbb{F}_q[X])$ is a subgroup of $SL_{\pm}(2,\, \mathbb{F}_q[X])$, $\tilde{G}_{+}(R)$ can be regarded as a subset of $\tilde{G}(R)$. We put

$$\tilde{G}_{-}(R) = \tilde{G}(R) - \tilde{G}_{+}(R).$$

Thus we see $\tilde{\gamma} \in \tilde{G}_+(R)$ or $\tilde{\gamma} \in \tilde{G}_-(R)$ if and only if its representative is of determinant 1 or -1, respectively. We write by $\widetilde{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}$ an element of $\tilde{G}(R)$ for which $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a representative.

Lemma 1. *There exists a positive integer $k \geq 1$ such that the following (i) and (ii) hold :*
(i) For any $\tilde{\gamma} \in \tilde{G}_+(R)$ there exists a sequence $B_1, B_2, \dots, B_{2k} \in \mathbb{F}_q[X]$, deg $B_i \geq 1$, such that

$$\widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2k} \end{pmatrix}} = \tilde{\gamma}. \tag{3}$$

(ii) For any $\tilde{\gamma} \in \tilde{G}_-(R)$ there exists a sequence $B_1, B_2, \dots, B_{2k}, B_{2k+1} \in \mathbb{F}_q[X]$, deg $B_i \geq 1$, such that

$$\widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2k} \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2k+1} \end{pmatrix}} = \tilde{\gamma}. \tag{4}$$

Proof. We only show that there exists $k \geq 1$ such that (i) holds since (ii) follows from (i) immediately. We choose a representative of $\tilde{\gamma} \in \tilde{G}_+(R)$ as $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL(2, \mathbb{F}_q[X])$ with deg$C <$ deg$D$. Then by the Euclidean algorithm we can find a sequence $W_1, W_2, \dots, W_s \in \mathbb{F}_q[X]$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} W_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} W_s & 1 \\ 1 & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} a^{-1} & z_1 \\ 0 & a \end{pmatrix} \\ \text{or} \\ \begin{pmatrix} -a^{-1} & z_1 \\ 0 & a \end{pmatrix} \end{cases}$$

for some $a \in \mathbb{F}_q$ and $z_1 \in \mathbb{F}_q[X]$. Suppose that the first case occurs. Then s is even and

$$\begin{pmatrix} a^{-1} & z_1 \\ 0 & a \end{pmatrix} \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{-1}z_1 + a^{-1} - 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus we see that

$$\begin{aligned} \begin{pmatrix} A & B \\ C & D \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a^{-1}z_1 - a^{-1} + 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -a \end{pmatrix} \\ &\qquad \begin{pmatrix} 0 & 1 \\ 1 & -W_s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -W_{s-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -W_1 \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & 1 \\ 1 & R+1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & R-1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & R - a^{-1}z_1 - a^{-1} + 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & R+a \end{pmatrix} \\ &\qquad \begin{pmatrix} 0 & 1 \\ 1 & R-a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & R-a \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -W_s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -W_{s-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & -W_1 \end{pmatrix} \pmod{R}. \end{aligned}$$

This implies (3). A similar calculation also holds for the second case with odd s. Since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & R \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & R \end{pmatrix} \quad \text{mod } R, \ l \geq 1$$

and $|\tilde{G}_+(R)| < \infty$, we can choose the same length $2k$ for any $\tilde{\gamma} \in \tilde{G}_+(R)$. □

For our purpose, we also need the following lemma :

Lemma 2. *For any $U, V \in \mathbb{F}_q[X]$ with $(U, V, R) = 1$, there exist $A, B, C, D, B', D' \in \mathbb{F}_q[X]$ such that*

$$det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 1, \ det \begin{pmatrix} A & B' \\ C & D' \end{pmatrix} = -1, and \begin{pmatrix} A \\ C \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix} \ mod \ R.$$

Proof. It is easy to see that there exist $A, C \in \mathbb{F}_q[X]$ such that

$$\begin{pmatrix} A \\ C \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix} \quad \text{mod } R, \text{ and } (A, C) = 1.$$

We can assume that $\deg A < \deg C$. Then by the Euclidean algorithm, there exist $Z_1, Z_2 \in \mathbb{F}_q[X]$ such that

$$AZ_1 + cZ_2 = a \neq 0 \in \mathbb{F}_q[X]$$

This shows

$$\det \begin{pmatrix} A & -a^{-1}Z_2 \\ C & a^{-1}Z_1 \end{pmatrix} = 1 \text{ and } \det \begin{pmatrix} A & a^{-1}Z_2 \\ C & -a^{-1}Z_1 \end{pmatrix} = -1.$$

□

Remark 1. *Because of (3), (4) and the above proof of Lemma 2, we can find* $\begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *mod R (or* $\begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix} = \begin{pmatrix} A & B' \\ C & D' \end{pmatrix}$ *mod R).*

Lemma 3. *For any $U, V \in \mathbb{F}_q[X]$ with $(U, V, R) = 1$, we have*

$$\sharp \left\{ \tilde{\gamma} \in \tilde{G}_+(R) \, : \, \tilde{\gamma} = \widetilde{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}, \begin{pmatrix} A \\ C \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix} \ mod \ R \right\} \tag{5}$$

$$= \sharp \left\{ \tilde{\gamma} \in \tilde{G}_-(R) \, : \, \tilde{\gamma} = \widetilde{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}, \begin{pmatrix} A \\ C \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix} \ mod \ R \right\} \tag{6}$$

$$= q^{deg \ R}.$$

Proof. From Lemma 2, there exists $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $AD - BC = 1$ and $\begin{pmatrix} A \\ C \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix}$ mod R. Then from Lemma 1, there exists a sequence $A_1, A_2, \ldots, A_n \in \mathbb{F}_q[X]$, deg $A_i \geq 1$, such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & A_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_n \end{pmatrix} \quad \text{mod } R,$$

which implies

$$\frac{A}{C} = [0\,;\, A_1, A_2, \ldots, A_n] \quad \text{with } A = P_n, C = Q_n.$$

For any E, $E' \in \mathbb{F}_q[X]$ with $E \not\equiv E'$ mod R, we see that

$$\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & E \end{pmatrix} = \begin{pmatrix} P_{2n} & EP_{2n} + P_{2n-1} \\ Q_{2n} & EQ_{2n} + Q_{2n-1} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & A_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & E' \end{pmatrix} = \begin{pmatrix} P_{2n} & E'P_{2n} + P_{2n-1} \\ Q_{2n} & E'Q_{2n} + Q_{2n-1} \end{pmatrix}$$

are not equivalent mod R. Hence we have (5) (or (6)) is greater than or equal to $q^{\deg R}$. On the other hand, if $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \not\equiv \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$ mod R with $\begin{pmatrix} A \\ C \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix}$ mod R, $AD - BC = 1$ (or -1) and $\begin{pmatrix} A' \\ C' \end{pmatrix} \equiv \begin{pmatrix} U \\ V \end{pmatrix}$ mod R, $A'D' - B'C' = 1$ (or -1), then

$$\begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \not\equiv \begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix}^{-1} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \quad \text{mod } R,$$

where we define $\begin{pmatrix} P_{n+1} \\ Q_{n+1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & E \end{pmatrix}$ with a fixed $E \in \mathbb{F}_q[X]$, $\deg E \geq 1$. Moreover we see

$$\begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix}^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \equiv \begin{pmatrix} 1 & E_1 \\ 0 & 1 \end{pmatrix} \quad \text{mod } R$$

and

$$\begin{pmatrix} P_n & P_{n+1} \\ Q_n & Q_{n+1} \end{pmatrix}^{-1} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \equiv \begin{pmatrix} 1 & E_2 \\ 0 & 1 \end{pmatrix} \quad \text{mod } R$$

for some $E_1, E_2 \in \mathbb{F}_q[X]$, $E_1 \not\equiv E_2$ mod R. This concludes that (5) (or (6)) is less than or equal to $q^{\deg R}$. □

Remark. From the above lemma, we have

$$|\tilde{G}_+(R)| = |\tilde{G}_-(R)| = q^{\deg R} \cdot C(R), \tag{7}$$

where we recall that $C(R)$ is the number of pairs (U, V) such that $0 \le \deg U,\ \deg V < \deg R$ and $(U, V, R) = 1$.

3 Group extension

We fix $R \in \mathbb{F}_q[X]$ and consider $\mathbb{L} \times \tilde{G}(R)$ with $m_{\tilde{G}} = m \times \delta_{\tilde{G}(R)}$, where $\delta_{\tilde{G}(R)}$ denotes the probability measure that consists of the equal point mass on $\tilde{G}(R)$. We define a map $T_{\tilde{G}}(= T_{\tilde{G}(R)})$ of $\mathbb{L} \times \tilde{G}(R)$ onto itself, which we call a group extension of T, by

$$T_{\tilde{G}}(f, \tilde{\gamma}) = \left(T(f),\ \tilde{\gamma} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}} \right).$$

It is clear that $m_{\tilde{G}}$ is an invariant probability measure for $T_{\tilde{G}}$. The following lemma is essential for applications of $T_{\tilde{G}}$.

Lemma 4.

$$T_{\tilde{G}}^n \left(f,\ \widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \right) = \left(T^n(f),\ \widetilde{\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix}} \right)$$

Proof. This follows from (2). □

Next theorem is also essential.

Theorem 1. *$(T_{\tilde{G}}, m_{\tilde{G}})$ is ergodic.*

Proof. Since (T, m) is ergodic, it is easy to see that

$$\{f \in \mathbb{L} : (f,\ \tilde{\gamma}) \in \mathbf{E} \ \text{ for some } \tilde{\gamma} \in \tilde{G}(R)\} = \mathbb{L} \quad (m\text{-a.e.}),$$

for any $T_{\tilde{G}}$-invariant set $\mathbf{E}$ of positive ($m_{\tilde{G}}$-) measure. This implies that there exist measurable sets

$$\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_t \subset \mathbb{L} \times \tilde{G}(R)$$

such that

$$\begin{cases} T_{\tilde{G}}^{-1}\mathbf{E}_i = \mathbf{E}_i & \text{for} \quad 1 \le i \le t \\ m_{\tilde{G}}(\mathbf{E}_i) > 0 & \text{for} \quad 1 \le i \le t \\ \cup_{i=1}^t \mathbf{E}_i = \mathbb{L} \times \tilde{G}(R) & (m_{\tilde{G}}\text{-a.e.}). \end{cases}$$

We note that $1 \leq t \leq |\tilde{G}(R)|$, that is, the number of ergodic components of $T_{\tilde{G}}$ is at most $|\tilde{G}(R)|$. Thus there exists a measurable set $\mathbf{E}_0$ such that

$$\begin{cases} T_{\tilde{G}}^{-1}\mathbf{E}_0 = \mathbf{E}_0, \\ m_{\tilde{G}}(\mathbf{E}_0) > 0 \\ m_{\tilde{G}}\left(\mathbf{E}_0 \cap \left(\mathbb{L} \times \left\{\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right\}\right)\right) > 0. \end{cases}$$

Then by the density theorem, we have the following : For $\{\varepsilon_j\}_{j=1}^{\infty} \searrow 0$, there exist $\{n_j\}_{j=1}^{\infty} \nearrow \infty$, $B_1, B_2, \cdots \in \mathbb{F}_q[X]$, and $\tilde{\gamma} \in \tilde{G}(R)$ such that

$$\frac{m_{\tilde{G}}\left(\langle B_1, B_2, \ldots, B_{n_l}\rangle \times \{\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\} \cap \mathbf{E}_0\right)}{m_{\tilde{G}}\left(\langle B_1, B_2, \ldots, B_{n_l}\rangle \times \{\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\}\right)} > 1 - \varepsilon_l \quad \text{for any } l \geq 1$$

and

$$\tilde{\gamma} = \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{n_l} \end{pmatrix}}.$$

Since

$$T^{n_l}\langle B_1, B_2, \ldots, B_{n_l}\rangle = \mathbb{L}$$

and

$$T_{\tilde{G}}^{n_l}\mathbf{E}_0 = \mathbf{E}_0,$$

we have

$$(\mathbb{L} \times \{\tilde{\gamma}\}) \cap \mathbf{E}_0 = \mathbb{L} \times \{\tilde{\gamma}\} \quad (m_{\tilde{G}}\text{-a.e.}).$$

For any $\tilde{\gamma}' \in \tilde{G}(R)$, from (3) and (4) there exists a sequence $B_1', B_2', \ldots, B_{l'}' \in \mathbb{F}_q[X]$ such that

$$\widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1' \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2' \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{l'}' \end{pmatrix}} = \tilde{\gamma}^{-1}\tilde{\gamma}'.$$

Hence

$$\mathbf{E}_0 \supset T_{\tilde{G}}^{l'}(\mathbb{L} \times \{\tilde{\gamma}\}) \supset \mathbb{L} \times \{\tilde{\gamma}'\} \quad (m_{\tilde{G}}\text{-a.e.}),$$

which implies

$$\mathbf{E}_0 = \mathbb{L} \times \tilde{G}(R) \quad (m_{\tilde{G}}\text{-a.e.}).$$

Thus we get the assertion of the theorem. □

Remark 2. *We should note that*

$$T_{\tilde{G}}(\mathbb{L} \times \tilde{G}_{\pm}(R)) = \mathbb{L} \times \tilde{G}_{\mp}(R),$$

which means $T_{\tilde{G}}$ is not mixing. Actually $\mathbb{L} \times \tilde{G}_{+}(R)$ is an invariant set of $T_{\tilde{G}}$. In next section, we show that the restriction $T^2_{\tilde{G}}$ to this set has a strong mixing property.

As a consequence of Theorem 1, we have the following.

Theorem 2. *For any $\tilde{\gamma} \in \tilde{G}(R)$ and any integer $l \geq 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{1 \leq n \leq N : \widetilde{\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix}} = \tilde{\gamma}, |T^n(f)| < \frac{1}{q^l}\} = \frac{1}{2 \cdot q^{degR+l} \cdot C(R)}$$

for m-almost every $f \in \mathbb{L}$.

Proof. For a fixed $\tilde{\gamma} \in \tilde{G}_{+}(R)$, we put

$$\mathbf{D}_{\tilde{\gamma}} = \left\{(f, \tilde{\gamma}) : f \in \mathbb{L},\ |f| < \frac{1}{q^l}\right\}.$$

Then from (7), we have

$$m_{\tilde{G}}(\mathbf{D}_{\tilde{\gamma}}) = \frac{1}{2 \cdot q^{\deg R} \cdot C(R)} \cdot \frac{1}{q^l}.$$

From the individual ergodic theorem and the ergodicity of $T_{\tilde{G}}$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{N=1}^{\infty} \mathbf{1}_{\mathbf{D}_{\tilde{\gamma}}}(T^n_{\tilde{G}}(f,\tilde{\gamma})) = \frac{1}{2 \cdot q^{\deg R} \cdot C(R)} \cdot \frac{1}{q^l} \quad (m_{\tilde{G}}\text{-a.e.}).$$

Since for any measurable set $\mathbf{M} \subset \mathbb{L}$, $m_{\tilde{G}}\left(\mathbf{M} \times \left\{\widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right\}\right) = 0$ if and only if $m(\mathbf{M}) = 0$, the assertion of the theorem follows from Lemma 4. □

Remark 3. *As the special case $l = 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sharp\left\{1 \leq n \leq N : \widetilde{\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix}} = \tilde{\gamma}\right\} = \frac{1}{2 \cdot q^{degR} \cdot C(R)} \quad (m\text{-a.e.})$$

From Theorem 2, we have the following.

Theorem 3. *For any $P, Q \in \mathbb{F}_q[X]$ with $(P,Q,R) = 1$ and $deg P$, $deg Q < deg R$, and any $l \geq 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{1 \leq n \leq N : \begin{pmatrix} P_n \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \bmod R,\ |T^n(f)| < \frac{1}{q^l}\} = \frac{1}{q^l \cdot C(R)} \quad (m\text{-a.e.})$$

for almost every $f \in \mathbb{L}$.

Proof. To use Lemma 3 and (7), we consider $\begin{pmatrix} P_{n-1} \\ Q_{n-1} \end{pmatrix}$ instead of $\begin{pmatrix} P_n \\ Q_n \end{pmatrix}$. Then we have the assertion of this theorem directly from Theorem 2. □

Remark 4. *As before, $l = 0$, we see*

$$\lim_{N\to\infty} \frac{1}{N} \sharp \left\{ 1 \le n \le N : \begin{pmatrix} P_n \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \text{ mod } R \right\} = \frac{1}{C(R)} \quad \text{(m-a.e.)}$$

Now we discuss the strong ergodic property, called exactness, of $T_{\tilde{G}}^2$ on $\mathbb{L} \times \tilde{G}_+(R)$, where the invariant probability measure $m_{\tilde{G}_+}$ is the normalized measure of $m_{\tilde{G}}$ restricted to $\mathbb{L} \times \tilde{G}_+(R)$ with an ergodic transformation T defined on a probability space $(\Omega, \mathbf{B}, m)$ is said to be exact if its tail σ-field is trivial, which means $\cap_{n=1}^{\infty} T^{-n}\mathbf{B}$ only consists of sets of measure 0 and 1.

Theorem 4. *$T_{\tilde{G}}^2|_{\mathbb{L}\times\tilde{G}_+(R)}$ is exact.*

Proof. To prove the exactness it is enough to show that

$$\lim_{n\to\infty} m_{\tilde{G}_+(R)}(T_{\tilde{G}}^{2n} A) = 1$$

for any $A \subset \mathbb{L} \times \tilde{G}_+(R)$ of positive measure. From (1) and the density theorem, it is sufficient to show that

$$T_{\tilde{G}}^{2(k+l)}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}\rangle) = \mathbb{L} \times \tilde{G}_+(R).$$

for any $B_1, B_2, \ldots, B_{2l} \in \mathbb{F}_q[X]$ and $\tilde{\gamma} \in \tilde{G}_+(R)$, where

$$\langle B_1, B_2, \ldots, B_l; \tilde{\gamma}\rangle = \{(f, \tilde{\gamma}) : f \in \langle B_1, B_2, \ldots, B_l\rangle\},$$

for $B_1, B_2, \ldots, B_l \in \mathbb{F}_q[X]$ and $\tilde{\gamma} \in \tilde{G}(R)$. Let

$$\tilde{\gamma}_0 = \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2l} \end{pmatrix}}.$$

Then

$$T_{\tilde{G}}^{2l}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}\rangle) = \mathbb{L} \times \{\tilde{\gamma}\tilde{\gamma}_0\}.$$

For any $\tilde{\gamma}_1 \in \tilde{G}_+(R)$, from (3) we choose $B_1', B_2', \ldots, B_{2k}' \in \mathbb{F}_q[X]$ so that

$$\widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1' \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2' \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2k}' \end{pmatrix}} = \tilde{\gamma}_0^{-1}\tilde{\gamma}^{-1}\tilde{\gamma}_1.$$

Then we have

$$T_{\tilde{G}}^{2k}(\langle B_1', B_2', \ldots, B_{2k}'; \tilde{\gamma}\tilde{\gamma}_0\rangle) = \mathbb{L} \times \{\tilde{\gamma}_1\}.$$

Thus

$$T_{\tilde{G}}^{2(l+k)}(\langle B_1, B_2, \ldots, B_{2l}, B_1', B_2', \ldots, B_{2k}'; \tilde{\gamma}\rangle) = \mathbb{L} \times \{\tilde{\gamma}_1\}$$

for any $\tilde{\gamma}_1 \in \tilde{G}_+(R)$. So we have

$$T_{\tilde{G}}^{2(l+k)}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}\rangle) = \mathbb{L} \times \tilde{G}_+(R)$$

□

4 Continued fraction mixing property

In this section, we consider the continued fraction mixing property for the stochastic process arising from $T_{\tilde{G}}^2$. We consider $\mathbb{L} \times \tilde{G}_+(R)$ with $m_{\tilde{G}_+}(= 2m_{\tilde{G}})$ as a probability space and the stochastic process $\{X_n : n \geq 1\}$ is defined by

$$X_n(f, \tilde{\gamma}) = (A_{2n-1}(f), A_{2n}(f), \tilde{\gamma}_{2n-1}) \quad \text{for} \quad (f, \tilde{\gamma}) \in \mathbb{L} \times \tilde{G}_+(R)$$

with

$$\tilde{\gamma}_{2n-1} = \begin{cases} \tilde{\gamma} & \text{if} \quad n = 1 \\ \tilde{\gamma} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & A_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & A_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & A_{2n-2} \end{pmatrix}} & \text{if} \quad n \geq 2. \end{cases}$$

Here the stochastic process $\{X_n\}$ is said to be ψ-mixing if

$$\psi(n) = \sup_{j \geq 1} \sup_{\substack{\mathcal{A} \in \mathbb{F}_1^j,\, \mathcal{B} \in \mathbb{F}_{j+n}^\infty \\ m_{\tilde{G}_+}(\mathcal{A}) \neq 0,\, m_{\tilde{G}_+}(\mathcal{B}) \neq 0}} \frac{|m_{\tilde{G}_+}(\mathcal{A} \cap \mathcal{B}) - m_{\tilde{G}_+}(\mathcal{A}) \cdot m_{\tilde{G}_+}(\mathcal{B})|}{m_{\tilde{G}_+}(\mathcal{A}) \cdot m_{\tilde{G}_+}(\mathcal{B})} \to 0 \quad (n \to \infty)$$

where $\mathbb{F}_k^l$ and $\mathbb{F}_k^\infty$ are the smallest σ-algebras for which $\{X_j : k \leq j \leq l\}$ and $\{X_j : j \geq k\}$ are measurable, respectively. Moreover $\{X_n\}$ is said to be continued fraction mixing if $\{X_n\}$ is ψ-mixing and $\psi(1) < \infty$. We start with proving the continued fraction mixing property for $\{X_n\}$.

Theorem 5. *The stochastic process $\{X_n : n \geq 1\}$ defined in the above is continued fraction mixing with exponential decay.*

Proof. First we show that there exists a constant ρ, $0 < \rho < 1$, such that for any $B_1, B_2, \ldots, B_{2l}, C_1, C_2, \ldots, C_m \in \mathbb{F}_q[X]$ and $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \tilde{G}_+(R)$,

$$\begin{aligned} &m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1\rangle \cap T_{\tilde{G}}^{-2(l+n)}\langle C_1, C_2, \ldots, C_m; \tilde{\gamma}_2\rangle) \\ &\quad = m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1\rangle) \cdot m_{\tilde{G}_+}(\langle C_1, C_2, \ldots, C_m; \tilde{\gamma}_2\rangle)(1 + o(\rho^{2n})) \quad (8) \end{aligned}$$

holds. This means that $\{X_n\}$ is ψ-mixing. It is easy to see that

$$\begin{aligned} & m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1\rangle \cap T_{\tilde{G}_+}^{-2(l+n)}\langle C_1, C_2, \ldots, C_m; \tilde{\gamma}_2\rangle) \\ &= \sum_{(B_1', B_2', \ldots, B_{2n}')} m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}, B_1', B_2', \ldots, B_{2n}', C_1, C_2, \ldots, C_m; \tilde{\gamma}_1\rangle), \end{aligned} \tag{9}$$

where $(B_1', B_2', \ldots, B_{2n}')$ runs over all $2n$-polynomials such that

$$\tilde{\gamma}_1 \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2l} \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1' \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2' \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2n}' \end{pmatrix}} = \tilde{\gamma}_2.$$

By the independence of $A_i(\cdot)$, the right hand side of (9) is equal to

$$m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1\rangle) \sum_{(B_1', B_2', \ldots, B_{2n}')} m(\langle B_1', B_2', \ldots, B_{2n}'\rangle) \cdot m(\langle C_1, C_2, \ldots, C_m\rangle). \tag{10}$$

Now we use the following proposition.

Proposition 1. *There exists a constant ρ, $0 < \rho < 1$ such that for any $\tilde{\eta}_1$, $\tilde{\eta}_2 \in \tilde{G}_+(R)$*

$$\sum_{(B_1', B_2', \ldots, B_{2n}')} m(\langle B_1', B_2', \ldots, B_{2n}'\rangle) = \frac{1}{q^{\deg R} \cdot C(R)} (1 + o(\rho^{2n})),$$

where $(B_1', B_2', \ldots, B_{2n}')$ runs all $2n$ polynomials such that

$$\tilde{\eta}_1 \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1' \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2' \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2l}' \end{pmatrix}} = \tilde{\gamma}_2$$

Proof. For any $\tilde{\eta}_1$, $\tilde{\eta}_2 \in \tilde{G}_+(R)$, we put

$$p_{\tilde{\eta}_1 \tilde{\eta}_2} = \sum_{(B_1', B_2')} m(\langle B_1', B_2'\rangle)$$

where (B_1', B_2') runs all pairs of polynomials such that

$$\tilde{\eta}_1 \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1' \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2' \end{pmatrix}} = \tilde{\eta}_2.$$

Then we have a stochastic matrix $\mathbb{P} = (p_{\tilde{\eta}_1 \tilde{\eta}_2})$. By (3) we see $\mathbb{P}^{2k} = (p_{\tilde{\eta}_1 \tilde{\eta}_2}^{(2k)})$ is a positive matrix (i.e. all components $p_{\tilde{\eta}_1 \tilde{\eta}_2}^{(2k)}$ are positive). Moreover, since $m \times \delta_{\tilde{G}_+(R)}$ is the ergodic invariant probability measure for $T_{\tilde{G}}$ (and so $T_{\tilde{G}}^2$), it is easy to see that the stochastic vector

$$\mathbf{p} = \left(\frac{1}{q^{\deg R} \cdot C(R)}, \frac{1}{q^{\deg R} \cdot C(R)}, \ldots, \frac{1}{q^{\deg R} \cdot C(R)} \right)$$

is the left invariant vector of $\mathbb{P}$, i.e. $\mathbf{p}\,\mathbb{P} = \mathbf{p}$. By the Frobenius-Perron theorem, we see that there exists ρ, $0 < \rho < 1$, such that

$$p_{\tilde{\eta}_1\tilde{\eta}_2} = \frac{1}{q^{\deg R} \cdot C(R)} \, (1 + o(\rho^{2n})).$$

By the definition of $p_{\tilde{\eta}_1\tilde{\eta}_2}$, we have the assertion of the proposition. □

(*Proof of Theorem 5 – continued*)

We choose $\tilde{\eta}_1 = \tilde{\gamma}_1 \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_{2l} \end{pmatrix}}$ and $\tilde{\eta}_2 = \tilde{\gamma}_2$. Since

$$m_{\tilde{G}_+}(\langle C_1, C_2, \ldots, C_m \rangle) = \frac{1}{q^{\deg R} \cdot C(R)} \cdot m(\langle C_1, C_2, \ldots, C_m \rangle),$$

we see (8) from (9), (10) and Proposition 1. Thus we have the ψ-mixing property with exponential decay for $\{X_n\}$. Next, we show $\psi(1) < \infty$. It is enough to estimate the following.

$$m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1 \rangle \cap T_{\tilde{G}}^{2l} \langle C_1, C_2, \ldots, C_m; \tilde{\gamma}_1 \rangle)$$

If

$$\tilde{\gamma}_1 \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_l \end{pmatrix}} \neq \tilde{\gamma}_2,$$

then the above measure is equal to 0, since it is empty. On the other hand, if

$$\tilde{\gamma}_1 \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_1 \end{pmatrix}} \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 \end{pmatrix}} \cdots \widetilde{\begin{pmatrix} 0 & 1 \\ 1 & B_2 l \end{pmatrix}} = \tilde{\gamma}_2,$$

then we have

$$\begin{aligned} & m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l} : \tilde{\gamma}_1 \rangle) m(\langle C_1, C_2, \ldots, C_m \rangle) \\ &= m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l} : \tilde{\gamma}_1 \rangle) m_{\tilde{G}_+}(\langle C_1, C_2, \ldots, C_m : \tilde{\gamma}_1 \rangle) \cdot q^{\deg R} C(R). \quad (11) \end{aligned}$$

This shows $\psi(1) < \infty$. Consequently, we have the assertion of this theorem. □

Remark 5. *By the same way, we have*

$$\begin{aligned} & m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1 \rangle \cap T_{\tilde{G}}^{-2(l+n)-1} \langle C_1, C_2, \ldots, C_m; \tilde{\gamma}_2 \rangle) \\ &\quad = m_{\tilde{G}_+}(\langle B_1, B_2, \ldots, B_{2l}; \tilde{\gamma}_1 \rangle) \cdot m_{\tilde{G}_-}(\langle C_1, C_2, \ldots, C_m; \tilde{\gamma}_2 \rangle)(1 + o(\rho^{2n})) \end{aligned}$$

for $\tilde{\gamma}_1 \in \tilde{G}_+$ and $\tilde{\gamma}_2 \in \tilde{G}_-$, where $m_{\tilde{G}_-}$ is the normalized measure of the restriction of $m_{\tilde{G}}$ to $\mathbb{L} \times \tilde{G}_-(R)$.

Now we put $\tilde{\gamma}_1 = \widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$ and sum up all $m_{\tilde{G}}(\langle B_1, B_2, \dots, B_l; \tilde{\gamma}_1 \rangle)$ for a fixed l ($l = 1$ is enough). Then we have the following :

Proposition 2. *For any $\tilde{\gamma} \in \tilde{G}(R)$ and $l \geq 0$, we have*

$$m\left(\left\{f \in \mathbb{L} : T_{\tilde{G}}^n\left(f, \widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right) = (T^n f, \tilde{\gamma}) \text{ and } |T^n f| < \frac{1}{q^l}\right\}\right) = \frac{1}{q^{\deg R} \cdot C(R)} \cdot \frac{1}{q^l}(1 + o(\rho^n))$$

holds for any

$$\begin{cases} \text{even} & n \text{ and } \tilde{\gamma} \in \tilde{G}_+(R) \\ \text{odd} & n \text{ and } \tilde{\gamma} \in \tilde{G}_-(R) \end{cases}$$

Take all $\widetilde{\begin{pmatrix} U & P \\ V & Q \end{pmatrix}}$ such that $UQ - PV = \pm 1$ and have the following :

Corollary 1. *For any $l \geq 0$, we have*

$$m\left(\left\{f \in \mathbb{L} : |T^n(f)| < \frac{1}{q^l} \text{ and } \begin{pmatrix} P_n \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \bmod R\right\}\right) = \frac{1}{C(R)} \cdot \frac{1}{q^l}(1 + o(\rho^n)).$$

Moreover if we fix $B_1, B_2, \dots, B_l \in \mathbb{F}_q[X]$, then we have the following.

Corollary 2. *For any $l \geq 0$, we have*

$$m\left(\left\{f \in \mathbb{L} : A_1(f) = B_1, \dots, A_l(f) = B_l, |T^{n+l}(f)| < \frac{1}{q^l} \text{ and } \begin{pmatrix} P_{n+l} \\ Q_{n+l} \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \bmod R\right\}\right) = \frac{1}{C(R)} \cdot \frac{1}{q^{2(\sum_{i=1}^{l} \deg B_i) + l}}(1 + o(\rho^n)).$$

If we choose $\tilde{\gamma}_1 = \widetilde{\begin{pmatrix} U & V \\ P & Q \end{pmatrix}}$ and sum up all possible (U, V)'s. then we see that the same assertions hold with $(Q_{n-1}, Q_n) \equiv (P, Q) \bmod R$ (see M. Fuchs [2]).

For any $n \geq 1$, we choose $l_n \geq 0$ and

$$\begin{cases} \tilde{\gamma}_n \in \tilde{G}_+ & \text{if } n \text{ is even} \\ \tilde{\gamma}_n \in \tilde{G}_- & \text{if } n \text{ is odd .} \end{cases}$$

We define

$$\mathbf{D}_n = \left\{f \in \mathbb{L} : T_{\tilde{G}}^n\left(f, \widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right) = (T^n f, \tilde{\gamma}_n) \text{ and } |T^n f| < \frac{1}{q^{l_n}}\right\}$$

Since $\{A_n(\cdot)\}$ is an i.i.d sequence, we see that for any $0 \le n \le m$ and l_n, $l_m > 0$,

$$m\left(\left\{f \in \mathbb{L} : |T^n f| < \frac{1}{q^{l_n}}\right\} \cap \left\{f \in \mathbb{L} : |T^m f| < \frac{1}{q^{l_m}}\right\}\right)$$
$$= m\left(\left\{f \in \mathbb{L} : |T^n f| < \frac{1}{q^{l_n}}\right\}\right) \cdot m\left(\left\{f \in \mathbb{L} : |T^m f| < \frac{1}{q^{l_m}}\right\}\right). \quad (12)$$

Now by the above theorem we get the following

Theorem 6. *If* $\sum_{n=1}^{\infty} \frac{1}{q^{l_n}} = \infty$, *then*

$$\sharp\{1 \le n \le N : f \in \mathbf{D}_n\} = \Lambda(N) + O(\Lambda(N)^{1/2}(\log \Lambda(N))^{3/2+\varepsilon}) \quad (m-a.e.),$$

where $\Lambda(N) = \sum_{n=1}^{N} m(\mathbf{D}_n)$

We note that $\sum_{n=1}^{\infty} \frac{1}{q^{l_n}} = \infty$ if and only if $\sum_{n=1}^{\infty} m(\mathbf{D}_n) = \infty$.

Proof. We divide positive integers into even and odd integers. Obviously either $\sum_{n=1}^{\infty} \frac{1}{q\ l_{2n}}$ or $\sum_{n=1}^{\infty} \frac{1}{q^{l_{2n+1}}}$ is ∞. Suppose that $\sum_{n=1}^{\infty} \frac{1}{q^{l_{2n}}} = \infty$. By the continued fraction mixing property of $\{X_n\}$ it follows that

$$m(\mathbf{D}_n \cap \mathbf{D}'_n) = m(\mathbf{D}_n) \cdot m(\mathbf{D}'_n) \cdot (1 + o(\rho^{n'-n}))$$

for any even integers $n < n'$, which implies that the quantitative Borel-Cantelli lemma (see W.Philipp [5]) holds for $\{\mathbf{D}_n$: even $n\}$. The same holds for odd numbers if $\sum_{1}^{\infty} \frac{1}{q^{l_{2n+1}}}$. Thus we get the assertion of the theorem. □

In particular if we take $l_n = l$, $n \ge 1$, we have Theorem 2 and 3 with an estimate of the remainder term.

Theorem 2' *For any* $\tilde{\gamma} \in \tilde{G}(R)$ *and any integer* $l \ge 0$, *we have*

$$\frac{1}{N} \sharp \left\{1 \le n \le N : \widetilde{\begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix}} = \tilde{\gamma},\ |T^n(f)| < \frac{1}{q^l}\right\}$$
$$= \frac{1}{2 \cdot q^{degR+l} \cdot C(R)} + O(N^{1/2}(\log N)^{3/2+\varepsilon}) \quad (m\text{-}a.e.)$$

Theorem 3' *For any* $P, Q \in \mathbb{F}_q[X]$ *with* $(P, Q, R) = 1$ *and* $deg P, deg Q < deg R$, *and any integer* $l \ge 0$, *we have*

$$\frac{1}{N} \sharp \left\{1 \le n \le N : \begin{pmatrix} P_n \\ Q_n \end{pmatrix} \equiv \begin{pmatrix} P \\ Q \end{pmatrix} \bmod R,\ |T^n(f)| < \frac{1}{q^l}\right\}$$
$$= \frac{1}{q^l \cdot C(R)} + O(N^{1/2}(\log N)^{3/2+\varepsilon}) \quad (m\text{-}a.e.)$$

For any subsequence of positive integers $n_1 < n_2 < \cdots < n_i < n_{i+1} < \cdots$, we choose non-negative integers $l_1, l_2, \ldots, l_i, \ldots$, and $\tilde{\gamma}_i \in \tilde{G}_+$ if n_i is even and $\tilde{\gamma}_i \in \tilde{G}_-$ if n_i is odd, respectively. We put

$$\mathbf{D}'_i = \left\{ f \in \mathbb{L} \,:\, T_{\tilde{G}}^{n_i}\left(f, \widetilde{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}\right) = (T^{n_i} f, \tilde{\gamma}_i) \text{ and } |T^{n_i} f| < \frac{1}{q^{l_i}} \right\}.$$

Then we have the following by the same proof in the above.

Corollary 3. *If* $\sum_{i=1}^{\infty} \frac{1}{q^{l_i}} = \infty$, *then*

$$\sharp\{1 \le i \le N \,:\, f \in \mathbf{D}'_{n_i}\} = \sum_{i=1}^{N} m(\mathbf{D}'_{n_i}) + O(\Lambda'(N)^{1/2}(\log \Lambda'(N))^{3/2+\varepsilon}) \quad (m - a.e.),$$

where $\Lambda'(N) = \sum_{i=1}^{N} m(\mathbf{D}'_{n_i})$.

References

[1] V. Berthé and H .Nakada, *On continued fraction expansions in positive characteristic : equivalence relations and some metric properties*, *Exposition. Math.* **18** (2000), 257–284.

[2] M. Fuchs, *An analogue of a theorem of Szüsz for formal Laurent series over finite fields*, *J. Number Theory* **101** (2003), no. 1, 105–130.

[3] H. Jager and P. Liardet, *Distributions arithmétiques des dénominateurs de convergents de fractions continues*, *Nederl. Akad. Wetensch. Indag. Math.* **50** (1988), no. 2, 181–197.

[4] R. Moeckel, *Geodesics on modular surfaces and continued fractions*, *Ergodic Theory Dynamical Systems* **2** (1982), no. 1, 69–83.

[5] W. Philipp, *Some metrical theorems in number theory*, *Pacific J. Math.* **20** (1967), 109–127.

[6] P. Szüsz, *Verallgemeinerung und Anwendungen eines Kusminschen Satzes*, *Acta Arith.* **7** (1961/1962), 149–160.