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Principal convergents and mediant convergents associated to α-continued fractions

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Abstract

We study some properties of principal and mediant convergents for a class of semi-regular continued fractions, in particular, α-continued fractions, $0 < \alpha \le 1$. We claim that all α-principal convergents are the regular convergents if $\frac{1}{2} \le \alpha < 1$, on the other hand, this is not true in general for $0 \le \alpha < \frac{1}{2}$. We also show that for every x, the set of α-principal and α-mediant convergents of x are identical with that of the regular principal and the regular mediant convergents of x.

1 Regular continued fraction

For an irrational number $x \in (0,1)$, if a non-zero rational number $\frac{p}{q}$, $(p,q) = 1$, satisfies $\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}$, then it is the nth regular principal convergent $\frac{p_n}{q_n}$ for some $n \ge 1$. Here the nth regular principal convergents are defined by

$$\begin{cases} p_{-1} = p_{-1}(x) = 1, & p_0 = p_0(x) = 0 \\ q_{-1} = q_{-1}(x) = 0, & q_0 = q_0(x) = 1 \end{cases}$$

and

$$\begin{cases} p_n = p_n(x) = a_n \cdot p_{n-1} + p_{n-2} \\ q_n = q_n(x) = a_n \cdot q_{n-1} + q_{n-2} \end{cases} \quad \text{for} \quad n \ge 1.$$

with the regular continued fraction expansion of x :

$$x = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \cdots .$$

It is well-known that

$$\frac{p_n}{q_n} = \frac{1|}{|a_1} + \frac{1|}{|a_2} + \cdots + \frac{1|}{|a_n} \quad \text{for} \quad n \ge 1.$$

If $x \in [k, k+1)$ for an integer k, we define its nth regular principal convergent by $\frac{p_n(x-k)}{q_n(x-k)} + k = \frac{p_n(x-k)+k \cdot q_n(x-k)}{q_n(x-k)}$. On the other hand, for some $x \in (0,1)$, there exists $\frac{p}{q}$ with $(p,q) = 1$ and $\left|x - \frac{p}{q}\right| < \frac{1}{q^2}$, which is not the nth regular principal convergent for any $n \geq 0$. However, we can find such a fraction $\frac{p}{q}$ in the set $\{\frac{p_n - p_{n-1}}{q_n - q_{n-1}}, \frac{p_n + p_{n-1}}{q_n + q_{n-1}} : n \geq 1\}$. This leads us the notion of the regular mediant convergents of level n, $\frac{u_{n,t}}{v_{n,t}}$, which is defined by

$$\begin{cases} u_{n,t} = t \cdot p_n + p_{n-1} \\ v_{n,t} = t \cdot q_n + q_{n-1} \end{cases} \quad \text{for} \quad 1 \leq t < a_{n+1},\ n \geq 0.$$

The regular principal and the regular mediant convergents are obtained by the following maps T and F of $[0,1]$, which are called the Gauss map and the Farey map, respectively, see [2] :

$$T(x) = \begin{cases} \frac{1}{x} - [\frac{1}{x}] & \text{if} \quad x \in (0,1] \\ 0 & \text{if} \quad x = 0 \end{cases} \tag{1.1}$$

and

$$F(x) = \begin{cases} \frac{x}{1-x} & \text{if} \quad x \in [0, \frac{1}{2}) \\ \frac{1-x}{x} & \text{if} \quad x \in [\frac{1}{2}, 1], \end{cases}$$

where $[y] = n$ if $y \in [n, n+1)$. We get the coefficients of the regular continued fraction expansion of $x \in [0,1]$ by

$$a_n = a_n(x) = [(T^{n-1}(x))^{-1}], \quad n \geq 1.$$

We refer to Sh.Ito [3] about the relation between F and the regular mediant convergents.

2 α-continued fractions and the α-mediant convergents

We generalize the notion of the mediant convergents to the α-continued fraction expansions introduced by H.Nakada [5]. The notion of α-continued fraction expansions is a generalization of the regular continued fraction expansion and the expansions are induced by the following map T_α of $\mathbf{I}_\alpha = [\alpha - 1, \alpha]$ for $\frac{1}{2} \leq \alpha \leq 1$:

$$T_\alpha(x) = \begin{cases} |\frac{1}{x}| - [\,|\frac{1}{x}|\,]_\alpha & \text{if } x \in \mathbf{I}_\alpha \setminus \{0\} \\ 0 & \text{if} \quad x = 0, \end{cases}$$

where $[\,y\,]_\alpha = n$ if $y \in [n - 1 + \alpha, n + \alpha)$. We note that this definition coincides with (1.1) if $\alpha = 1$. For $n \geq 1$, we put

$$\varepsilon_{\alpha,n} = \varepsilon_{\alpha,n}(x) = \operatorname{sgn} T_\alpha^{n-1}(x),$$
$$c_{\alpha,n} = c_{\alpha,n}(x) = \left[\left| \frac{1}{T_\alpha^{n-1}(x)} \right| \right]_\alpha \quad (\text{or} \quad = \infty \quad \text{if} \quad T_\alpha^{n-1}(x) = 0).$$

Then we have the α-continued fraction expansion of $x \in \mathbf{I}_\alpha$ by

$$x = \frac{\varepsilon_{\alpha,1}|}{|c_{\alpha,1}} + \frac{\varepsilon_{\alpha,2}|}{|c_{\alpha,2}} + \frac{\varepsilon_{\alpha,3}|}{|c_{\alpha,3}} + \cdots, \quad c_{\alpha,n} \geq 1.$$

We define the nth α-principal convergents $\frac{p_{\alpha,n}}{q_{\alpha,n}}$, $n \geq 1$, by

$$\begin{cases} p_{\alpha,-1} = 1,\ p_{\alpha,0} = 0 \\ q_{\alpha,-1} = 0,\ q_{\alpha,0} = 1 \end{cases} \quad \text{and} \quad \begin{cases} p_{\alpha,n} = c_{\alpha,n} \cdot p_{\alpha,n-1} + \varepsilon_{\alpha,n} \cdot p_{\alpha,n-2} \\ q_{\alpha,n} = c_{\alpha,n} \cdot q_{\alpha,n-1} + \varepsilon_{\alpha,n} \cdot q_{\alpha,n-2}. \end{cases}$$

We note that the $\{q_{\alpha,n}\}$ is strictly increasing, see [5]. Also we define the α-mediant convergents of level $n \geq 0$, $\{\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} : 1 \leq t < c_{\alpha,n+1}\}$, by

$$\begin{cases} u_{\alpha,n,t} &= t \cdot p_{\alpha,n} + \varepsilon_{\alpha,n+1} \cdot p_{\alpha,n-1} \\ v_{\alpha,n,t} &= t \cdot q_{\alpha,n} + \varepsilon_{\alpha,n+1} \cdot q_{\alpha,n-1} \end{cases} \quad \text{for} \quad 1 \leq t < c_{\alpha,n+1}. \tag{2.1}$$

Next, we define a map which induces the sequence of the α-principal and the α-mediant convergents for each α, $\frac{1}{2} \leq \alpha \leq 1$. We put $\mathbf{J}_\alpha = [\alpha - 1, \frac{1}{\alpha}]$ and define the map G_α of $\mathbf{J}_\alpha$ by

$$G_\alpha(x) = \begin{cases} -\frac{x}{1+x} & \text{if} \quad x \in [\alpha - 1, 0) := \mathbf{J}_{\alpha,1} \\ \frac{x}{1-x} & \text{if} \quad x \in [0, \frac{1}{1+\alpha}] := \mathbf{J}_{\alpha,2} \\ \frac{1-x}{x} & \text{if} \quad x \in (\frac{1}{1+\alpha}, \frac{1}{\alpha}] := \mathbf{J}_{\alpha,3}. \end{cases}$$

We note that $G_1 = F$. In this sense, G_α is a generalization of the Farey map and is called the α-Farey map. In order to get the α-principal and the α-mediant convergents of $x \in \mathbf{J}_\alpha$ by the iterations of G_α, it is convenient to use the following matrices :

$$V_- = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},\ V_+ = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Since

$$\frac{ax+b}{cx+d} = \frac{u}{v} \quad \text{with} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} xz \\ z \end{pmatrix}$$

for any real numbers x and $z \neq 0$, we denote

$$A(x) = \frac{ax+b}{cx+d} \quad \text{and} \quad A(-\infty) = A(\infty) = \frac{a}{c} \quad \text{for} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence, we write

$$G_\alpha(x) = \begin{cases} V_-^{-1}(x) & \text{if } x \in \mathbf{J}_{\alpha,1} \\ V_+^{-1}(x) & \text{if } x \in \mathbf{J}_{\alpha,2} \\ U^{-1}(x) & \text{if } x \in \mathbf{J}_{\alpha,3}. \end{cases}$$

We put

$$M_n(x) := \begin{cases} V_- & \text{if} \quad (G_\alpha)^{n-1}(x) \in \mathbf{J}_{\alpha,1} \\ V_+ & \text{if} \quad (G_\alpha)^{n-1}(x) \in \mathbf{J}_{\alpha,2} \\ U & \text{if} \quad (G_\alpha)^{n-1}(x) \in \mathbf{J}_{\alpha,3}. \end{cases}$$

Then, we get a sequence of matrices

$$M_1(x),\ M_2(x),\ \ldots$$

from the iterations of G_α for each $x \in \mathbf{J}_\alpha$. Here, all matrices M_n's are of determinants ± 1. We put

$$k_0(x) := 0 \quad \text{and} \quad k_n(x) := \min\{k > k_{n-1}(x) : (G_\alpha)^{k-1}(x) \in \mathbf{J}_{\alpha,3}\},\ n \geq 1.$$

Then we have the following theorem, which connects the map G_α to the α-mediant convergents explicitly.

Theorem 1. *For $x \in \mathbf{I}_\alpha$, we have*
(i) If $l = k_n(x),\ n \geq 1$,

$$M_1(x)M_2(x)\cdots M_l(x) = \begin{pmatrix} p_{\alpha,n-1} & p_{\alpha,n} \\ q_{\alpha,n-1} & q_{\alpha,n} \end{pmatrix} \tag{2.2}$$

(ii) If $l = k_n(x) + t,\ 1 \leq t < c_{\alpha,n+1},\ n \geq 0$,

$$M_1(x)M_2(x)\cdots M_l(x) = \begin{pmatrix} u_{\alpha,n,t} & p_{\alpha,n} \\ v_{\alpha,n,t} & q_{\alpha,n} \end{pmatrix} \tag{2.3}$$

The following is a direct consequence of Theorem 1.

Corollary 1. *We have*

$$(M_1(x)M_2(x)\cdots M_l(x))(\infty) = \begin{cases} \frac{p_{\alpha,n-1}}{q_{\alpha,n-1}} & \text{if} \quad l = k_n(x),\ n \geq 1 \\ \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} & \text{if} \quad l = k_n(x) + t, \\ & \quad\ 1 \leq t < c_{\alpha,n+1}, \quad n \geq 0. \end{cases}$$

Remark. In [3], the regular mediant convergents are obtained as

$$(M_1(x)M_2(x)\cdots M_{l-1}(x))(1).$$

3 The relation of α-convergents and regular convergents

In this section, we describe a relation between the α-convergents and the regular convergents. Here we divide into two cases for α, $0 < \alpha < \frac{1}{2}$ and $\frac{1}{2} \leq \alpha \leq 1$. First we have the following theorem in the case of $\frac{1}{2} \leq \alpha \leq 1$.

Theorem 2 (in the case of $\frac{1}{2} \leq \alpha \leq 1$). *For $x \in \mathbf{I}_\alpha$ we suppose*

$$x = \frac{\varepsilon_{\alpha,1}|}{|c_{\alpha,1}} + \frac{\varepsilon_{\alpha,2}|}{|c_{\alpha,2}} + \frac{\varepsilon_{\alpha,3}|}{|c_{\alpha,3}} + \cdots$$

is the α-continued fraction expansion of x. Then we have the following for any $\frac{1}{2} \leq \alpha < 1$:
(I) $\left\{\frac{p_{\alpha,n}}{q_{\alpha,n}},\ n \geq 1\right\} \subset \left\{\frac{p_m}{q_m},\ m \geq 1\right\}$
(II) If $\frac{p_m}{q_m} \neq \frac{p_{\alpha,n}}{q_{\alpha,n}}$ for any $n \geq 1$, then $m = n + l_n(x)$ for some $n \geq 1$, $\varepsilon_{\alpha,n+1}(x) = -1$, and

$$\frac{u_{\alpha,n-1,c_{\alpha,n}-1}}{v_{\alpha,n-1,c_{\alpha,n}-1}} = \frac{p_m}{q_m} = \frac{u_{\alpha,n,1}}{v_{\alpha,n,1}},$$

where

$$l_n(x) := \sharp\{1 \leq k \leq n : \varepsilon_{\alpha,k}(x) = -1\}$$

$$(III) \quad \left\{\frac{p_{\alpha,n}}{q_{\alpha,n}},\ n \geq 1\right\} \cup \left\{\frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} : 1 \leq t < c_{\alpha,n+1},\ n \geq 0\right\}$$
$$= \left\{\frac{p_n}{q_n},\ n \geq 1\right\} \cup \left\{\frac{u_{n,t}}{v_{n,t}} : 1 \leq t < a_{n+1},\ n \geq 0\right\}$$

We can expand the above theorem to S-algorithm. We give the definition of S-algorithm by C. Kraaikamp [4]. At first, the following is called singularization :

$$\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{1 + \cfrac{1}{a_{n+1} + \ddots}}}$$

$\downarrow$ singularization

$$\ddots + \cfrac{1}{(a_{n-1}+1) + \cfrac{-1}{(a_{n+1}+1) + \ddots}},$$

which follows from

$$\begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1}+1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_{n+1}+1 \end{pmatrix}.$$

Next we define a map $\bar{T}$ on $[0,1] \times [-\infty, -1]$ by

$$\bar{T}(x,y) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \frac{1}{y} - \left[\frac{1}{x}\right]\right)$$
$$= \left(Tx,\ \frac{1}{y} - a_1\right).$$

Then we see

$$\bar{T}^n(x, -\infty)$$
$$= \left(T^n x,\ -\frac{q_n}{q_{n-1}}\right)$$
$$= \left(\cfrac{1}{a_{n+1} + \cfrac{1}{a_{n+2} + \ddots}}, - \left(a_n + \cfrac{1}{a_{n-1} + \ddots + \frac{1}{a_1}} \right) \right).$$

Let S is subset of $[\frac{1}{2}, 1) \times [0,1]$. Then S is called a singularization area if $\bar{m}(\partial S) = 0$ and $S \cap \bar{T}S = \emptyset$, where $\bar{m}$ is 2-dimensional Lebesgue measure.

Definition 1 (S-algorithm).
Let S is a singularization area. Then an algorithm that induces continued fraction expansions is said to be S-algorithm if $\bar{T}^n(x, -\infty) \in S$ induces the singularization at nth coefficients :

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n+2} \end{pmatrix} \cdots$$

$$\downarrow \quad \textit{singularization}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_n + 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & a_{n+2} + 1 \end{pmatrix} \cdots$$

Remark 1 (C. Kraaikamp). *For $\frac{1}{2} \leq \alpha \leq 1$, T_α is S-algorithm.*

We have a generalization of Theorem 2 to S-algorithms.

Theorem 3. *Suppose*

$$x = \frac{\varepsilon_{s,1}|}{|c_{s,1}} + \frac{\varepsilon_{s,2}|}{|c_{s,2}} + \frac{\varepsilon_{s,3}|}{|c_{s,3}} + \cdots$$

is the S-expansion. Then we have the following :
(I) $\left\{ \frac{p_{s,n}}{q_{s,n}},\, n \geq 1 \right\} \subset \left\{ \frac{p_m}{q_m},\, m \geq 1 \right\}$
(II) If $\frac{p_m}{q_m} \neq \frac{p_{s,n}}{q_{s,n}}$ for any $n \geq 1$, then $m = n + l_n(x)$ for some $n \geq 1$, $\varepsilon_{s,n+1}(x) = -1$, and

$$\frac{u_{s,n-1,c_{s,n}-1}}{v_{s,n-1,c_{s,n}-1}} = \frac{p_m}{q_m} = \frac{u_{s,n,1}}{v_{s,n,1}},$$

where

$$l_n(x) := \sharp\{1 \leq k \leq n : \varepsilon_{s,k}(x) = -1\}$$

$$(III) \quad \left\{ \frac{p_{s,n}}{q_{s,n}},\, n \geq 1 \right\} \cup \left\{ \frac{u_{s,n,t}}{v_{s,n,t}} : 1 \leq t < c_{s,n+1},\, n \geq 0 \right\} = \left\{ \frac{p_n}{q_n},\, n \geq 1 \right\} \cup \left\{ \frac{u_{n,t}}{v_{n,t}} : 1 \leq t < a_{n+1},\, n \geq 0 \right\}$$

For $0 < \alpha < \frac{1}{2}$, we see that T_α is not S-algorithm. However we have the following theorem :

Theorem 4 (in the case of $0 < \alpha < \frac{1}{2}$). *For any $0 < \alpha < \frac{1}{2}$, we have the following :*
(I) There exists $x \in [\alpha - 1, \alpha]$ for which

$$\exists n \geq 1 \quad s.t. \quad \frac{p_{\alpha,n}}{q_{\alpha,n}} \neq \frac{p_m}{q_m},\ m \geq 1,$$

that is,

$$\left\{ \frac{p_{\alpha,n}}{q_{\alpha,n}},\, n \geq 1 \right\} \not\subset \left\{ \frac{p_m}{q_m},\, m \geq 1 \right\}$$

(II) There exists $x \in [\alpha - 1, \alpha]$ such that $\frac{p_m}{q_m}$ appears 3-times in the sequence of α-mediant convergents. (If α is small, $\frac{p_m}{q_m}$ appears 4-times, 5-times, ...)

$$(III) \quad \left\{ \frac{p_{\alpha,n}}{q_{\alpha,n}},\, n \geq 1 \right\} \cup \left\{ \frac{u_{\alpha,n,t}}{v_{\alpha,n,t}} : 1 \leq t < c_{\alpha,n+1},\, n \geq 0 \right\} = \left\{ \frac{p_n}{q_n},\, n \geq 1 \right\} \cup \left\{ \frac{u_{n,t}}{v_{n,t}} : 1 \leq t < a_{n+1},\, n \geq 0 \right\}$$

To prove Theorem 4, we use the following.

Lemma 1 (semi-singularization).

$$\begin{pmatrix}0 & \pm1\\ 1 & k\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & t\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & l\end{pmatrix} = \begin{pmatrix}0 & \pm1\\ 1 & k+1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}^{t-1}\begin{pmatrix}0 & -1\\ 1 & l+1\end{pmatrix}$$

$$\begin{pmatrix}0 & \pm1\\ 1 & k\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & t\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & l\end{pmatrix} = \begin{pmatrix}0 & \pm1\\ 1 & k+1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}^{t-1}\begin{pmatrix}0 & 1\\ 1 & l-1\end{pmatrix}$$

4 Some remarks

We recall the notion of semi-regular continued fractions by [4].

Definition 2 (semi-regular continued fractions). *For a real number x,*

$$x = b_0 + \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \ddots}},$$

where b_0 is an integer, b_i $(i \geq 1)$ is a positive integer and $\varepsilon_i = \pm1$ $(i \geq 1)$. Above the continued fraction is called semi-regular

$$\begin{cases} \text{if } \varepsilon_{n+1} + b_n \geq 1 \text{ and } \varepsilon_{n+1} + b_n \geq 2 \text{ infinitely often} \\ \text{(in the case of the infinite continued fraction)} \\ \text{if } \varepsilon_{n+1} + b_n \geq 1 \\ \text{(in the case of the finite continued fraction).} \end{cases}$$

Definition 3 (semi-regular). *An algorithm that induces continued fraction expansions is said to be semi-regular if induced continued fractions are always semi-regular.*

We note the following, see [4].

Remark 2. *Every S-algorithm is semi-regular.*

Remark 3. *If $0 < \alpha < \frac{1}{2}$, then T_α is not S-algorithm, but it is semi-regular.*

Remark 4. *T_0 is not semi-regular*

We have seen that the set of the α-principal and the α-mediant convergents coincides with the set of the regular's. K. Dajani and C. Kraaikamp [1] showed that Lehner fractions induce the set of the regular principal and the regular mediant convergents. They also showed that this set includes all principal convergents arising from S-expansions. In this sense, they called this set "the mother of all semi-regular continued fractions". Our claim is that we can construct the "mother" from any α-continued fractions, $0 < \alpha \leq 1$, by producing the α-mediant convergents.

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