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Noncommutative Cohomological Field Theories and Topological Aspects of Matrix models

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Abstract

We study topological aspects of matrix models and noncommutative cohomological field theories (N.C.CohFT). N.C.CohFT have symmetry under the arbitrary infinitesimal noncommutative parameter θ deformation. This fact implies that N.C.CohFT possess a less sensitive topological property than K-theory, but the classification of manifolds by N.C.CohFT has a possibility to give a new view point of global characterization of noncommutative manifolds. To investigate properties of N.C.CohFT, we construct some models whose fixed point loci are given by sets of projection operators. Particularly, the partition function on the Moyal plane is calculated by using a matrix model. The moduli space of the matrix model is a union of Grassman manifolds. The partition function of the matrix model is calculated using the Euler number of the Grassman manifold. Identifying the N.C.CohFT with the matrix model, we get the partition function of the N.C.CohFT. To check the independence of the noncommutative parameters, we also study the moduli space in the large θ limit and the finite θ, for the Moyal plane case. If the partition function of N.C.CohFT is topological in the sense of the noncommutative geometry, then it should have some relation with K-theory. Therefore we investigate certain models of CohFT and N.C.CohFT from the point of view of K-theory. These observations give us an analogy between CohFT and N.C.CohFT in connection with K-theory. Furthermore, we verify it for the Moyal plane and noncommutative torus cases that our partition functions are invariant under the those deformations which do not change the K-theory. Finally, we discuss the noncommutative cohomological Yang-Mills theory.

1 Introduction

Recent developments in string theory make for a fruitful framework and motivation to study noncommutative field theories for physicists. From the viewpoint of physics, much progress has made by noncommutative geometry. On the other hand, from a point of view of noncommutative space geometry and topology investigated by physical technologies, there are some succeeding cases, for example Kontsevich's deformation quantization is given by some kind of topological string theory [1, 2]. As another example, some kinds of charges that are topological in commutative space are investigated and their results imply the charges have some kind of topological nature [3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

Topology and geometry of "commutative" space is studied by many methods. One of the important ways to investigate them is using quantum (or classical) field theories and string theories. For example, Donaldson theory, Seiberg-Witten theory, Gromov-Witten theory and so on are constructed by cohomological field theories(CohFT). Therefore, it is natural to ask " Can Noncommutative Cohomological field theories (N.C.CohFT) be used for the investigation of noncommutative geometry or topology?" Here we call CohFT naively extended to noncommutative space cases N.C.CohFT. One of the aims of this article is to give the circumstantial evidence for a positive answer to the question.

Noncommutative space is often defined by using an algebraic formulation, for example by using C^* algebras. So its topological discussions are usually done through algebraic K-theory. For example, the rank of K_0 identifies each noncommutative torus T^2_θ that is characterized by the noncommutative parameter θ. In this sense, even if $\theta - \theta'$ is arbitrary small, T^2_θ is distinguished from $T^2_{\theta'}$ without Morita equivalent cases. Meanwhile, some topological charges in commutative space seem to remain "topological" on the noncommutative space, and some do not depend on θ. ("Topological" is used in a slight different sense than the usual topological and its definition is given below.) For example, the Euler number of a noncommutative torus is independent of the noncommutative parameter θ and it is defined as topological invariant by the difference of K_0 and K_1. As another example, it is possible to define the instanton number (the integral of the first Pontrjagin class) as an integer for Moyal space [3, 4], and this fact implies that the instanton number has some kind of "topological" nature even if the base manifold is noncommutative space. (Here, we call Moyal space noncommutative Euclidian space whose commutation relations of the coordinates are given by $[x^\mu, x^\nu] = i\theta^{\mu\nu}$, where $\theta_{\mu\nu}$ is an anti-symmetric constant matrix.) The instanton number does not depend on θ, at least for Moyal space. Also, partition functions of CohFT are one of the such "topological" invariants [13]. These observations show that "topological" charge defined by noncommutative field theory has a tendency of independence from θ. Therefore it is natural to expect the existence of a topological class less sensitive than K-theory but nontrivial. Here, we define an "insensitive topological invariant" as follows: if noncommutative manifolds A and B give the same K-group, then the topological invariant defined on both A and B take the same value, but the inverse of this statement is not always true. In short, if K-theory do not distinguish A from B, then the "insensitive topological invariant" does not classify them. To express thess insensitive topology classes we use "topological" in the above sentences. Some may think that such an insensitive topology is not useful for geometrical classification. Possibly "topological" might not be suitable for the instanton number or the partition function of the N.C.CohFT, because there is some circumstantial evidence but it is not proved. But even if they are not "topological", they have indisputable value from the field theoretical point of view, since it is possible to classify the manifolds by global characters whose equivalent relations are defined by field theories. In this sense, this classification is similar to the mirror of Calabi-Yau manifolds or duality in a physical sense and so on. Therefore, one of the aims of this article is investigating the partition functions of some models of the N.C.CohFT as examples of such "topological" invariants.

As mentioned above, N.C.CohFT have the property of θ-shift invariance and the proof of θ-shift invariance is based on smoothness for θ [13, 14]. In some cases, at the commutative point ($\theta = 0$) theories have singularities, as we know about U(1) instantons and so on. So, we have to note that there are difficulties to connect a noncommutative theory to a commutative theory and the smoothness of θ for the proof should be checked whenever we consider new models. Meanwhile, there is interesting phenomena caused by θ-shift. For examples, when we consider Moyal spaces, derivative terms in the action functional become irrelevant in the large θ limit. Then the theory is determined by the potential of the action and the calculation of the partition function becomes easy. If we can compare the moduli space topology in the large θ limit with the one for finite θ, the θ invariance of the partition function may be checked. We verify this for one model in this article.

Here, we comment on the relation between [13] and this article. As an example of N.C.CohFT, one scalar field theory was investigated and its partition function was calculated in [13]. This model is essentially equivalent to the model that is studied in this article. We found that the partition function was given as the "Euler number" of a moduli space by using the method of the fundamental theorem of Morse theory extended to the operator space. This fact implies that the partition function is still the sum of the Euler numbers even if the base manifold is Noncommutative space. But it is not enough to verify the equivalence of above "Euler number" and usual Euler number defined for commutative manifolds, because we do not know the connection between the usual Euler number and the extension of the fundamental theorem of Morse theory to the operator formalism, in the sense of local geometry. The calculation in [13] is done by choosing some representation of Hilbert space caused from noncommutativity, and choosing the representation can be understood as gauge fixing. The computation of [13] lacks the view point of the local differential geometry of moduli space. Meanwhile, when the moduli spaces are defined as spread commutative manifolds, their Euler number is given by the Chern-Weil theorem, then it is expected that the partition function is obtained by the Chern-Weil theorem. In other words, we will find that the fundamental theorem of Morse theory extended to the operator formulation connects to the usual local geometry or the usual Euler number on commutative space. It is worth verifying this statement. In this article, we do it for an example.

We remark that the operator representation of N.C.field theories can be interpreted as an infinite dimensional matrix model. The partition function of N.C.CohFT is determined by the geometry of the moduli space of the matrix model. In particular, when the noncommutative space is a Moyal space, the matrix model does not include the kinetic terms like the IKKT matrix model in the $\theta \to \infty$ limit, because terms with differential operator in the lagrangian like kinetic terms become infinitesimal. Then, we can calculate the partition functions from only potential terms for the Moyal space in the large θ limit by using the matrix models. This relation between N.C.CohFT and matrix models is also important to the matrix models, because this relation allow us to investigate the topology of their moduli spaces by the N.C.CohFT. Additionally, this correspondence is not for particular cases. Actually the connection between noncommutative cohomological Yang-Mills theory and the IKKT matrix model is discussed in this article. One of the aims of this article is the observation of these relations between the matrix models and N.C.CohFT.

This is the plan of the article: In the next section N.C.CohFT is reviewed. We see that the partition function of the N.C.CohFT is independent from the deformation parameter of $*$ product. In section 3, we introduce a finite size Hermitian matrix model (finite matrix model) as a 0 dimensional cohomological field theory and we calculate its partition function. This partition function is determined by only topological information. In section 4, we construct some models of noncommutative cohomological field theory whose moduli spaces are defined by projection operators. Projection operators play an important role in the topology of noncommutative space because K_0 is made by the Grothendieck construction of equivalent classes of projection operators. The partition function of one of the models is given by the sum of the Euler numbers of moduli space of the projectors spaces. In particular, using the result of finite matrix model in section 3, the partition function of the noncommutative cohomological scalar field theory on Moyal plane is obtained in section 4. Independence from noncommutative parameters is also discussed. The model that contains the derivative terms are investigated for the finite noncommutative parameter case and the large limit case. We see that the topology of the moduli space of both cases are equivalent. In section 5, one model mirrored by N.C.CohFT in section 4 is constructed on COMMUTATIVE space and this model gives the model in section 4 by large N dimensional reduction. We see the connection between the model and the homotopy classification of vector bundles or topological K-theory. Furthermore, from the view point of K_0 we see our partition function of N.C.CohFT is "topological" for the Moyal plane and noncommutative torus cases. In section 6, correspondence between matrix models and N.C.CohFT is investigated for the case of N.C.cohomological Yang-Mills theories. In the last section, we summarize this article.

2 Brief Review of N.C.CohFT

In this section, we give a brief review of cohomological field theory (CohFT) and the nature of its noncommutative version. The CohFT is formulated in several ways [15] [16] but we use only Mathai-Quillen formalism in this article.

2.1 Review of Mathai-Quillen Formalism

Atiyah and Jeffrey give a very elegant approach to CohFT [17]. The Atiyah and Jeffrey approach is an infinite dimensional generalization of Mathai-Quillen formalism that is Gaussian shaped Thom forms [18]. We recall some well known facts here. Details can be found in several lecture notes [19], [20] and [21].

For simplicity we only consider the finite dimensional case in this subsection. Let X be an orientable compact finite dimensional manifold. For a local coordinate x and Grassmann odd variable ψ corresponding to dx, we introduce BRS operator $\hat{\delta}$:

$$\hat{\delta} x_\mu = \psi_\mu, \quad \hat{\delta}\psi_\mu = 0. \tag{1}$$

Let us consider a vector bundle V with $2n$ dimensional fiber and Grassman-odd variables χ_a and Grassmann-even variables H_a, $a = 1, \cdots 2n$. For these variables, we define BRS operator $\hat{\delta}$ transformations:

$$\hat{\delta}\chi_a = H_a, \quad \hat{\delta} H_a = 0. \tag{2}$$

Note that $\hat{\delta}$ is a nilpotent operator. Using some section s and connection A of the vector bundle, the action of the CohFT is defined by BRS-exact form:

$$\begin{aligned} S &= \hat{\delta}\left\{\frac{1}{2}\chi_a(2is^a + A_\mu^{ab}\psi^\mu\chi_b + H^a)\right\} \\ &= \frac{1}{2}|s^a|^2 - \frac{1}{2}\chi_a\Omega_{\mu\nu}^{ab}\psi^\mu\psi^\nu\chi_b - i\nabla_\mu s^a(\psi)^\mu\chi_a. \end{aligned} \tag{3}$$

To get the second equality, we integrate out the auxiliary field H_a. The partition function is defined by

$$Z = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}H \exp\left(-S\right). \tag{4}$$

In the commutative space, Mathai-Quillen formalism tells us that the partition function is a sum of Euler numbers of the vector bundle on the space $\mathcal{M} = \{s_a^{-1}(0)\}$ with sign. We can see this fact as follows. We expand the bosonic part $|s^a|^2$ around the zero section $s^a = 0$ as

$$|s^a|^2 = (\nabla_\mu s^a x^\mu)^2 + \cdots \quad . \tag{5}$$

In general, CohFT is invariant under rescaling the BRS-exact terms, then the exact expectation value is given by Gaussian integral. Gaussian integral of the bosonic parts give

$$1/\sqrt{det|\nabla_\mu s^a|^2}. \tag{6}$$

Note that if connected submanifolds $\mathcal{M}_k$ defined by

$$\bigcup_k \mathcal{M}_k := \{x|s=0\}, \qquad (7)$$
$$\mathcal{M}_i \cap \mathcal{M}_j = \emptyset \text{ for } \ i \neq j$$

have finite dimension, the Gaussian integral is done over $X\backslash\{x|s=0\}$. The fermionic non-zero mode ψ, χ integral is

$$det(\nabla_\mu s^a) \qquad (8)$$

from the fermionic action $\nabla_\mu s^a(\psi)^\mu \chi_a$. From (6) and (8), sign $\epsilon_k = \pm$ is given. Here remaining zero modes of ψ are tangent to $\mathcal{M}_k$ and the zero modes of χ are understood as a section of the vector bundle over $\mathcal{M}_k$. Let ψ_0 and χ_0 be these zero-modes and V_k be the vector bundle over $\mathcal{M}_k$, then the remaining integral over $\mathcal{M}_k$ is expressed as

$$\int_{\mathcal{M}_k} \mathcal{D}\psi_0 \mathcal{D}\chi_0 e^{-\frac{1}{2}\chi_{a0}\Omega^{ab}_{\mu\nu}\psi^\mu_0\psi^\nu_0\chi_{b0}} = \chi(V_k). \qquad (9)$$

Here $\Omega_{\mu\nu}$ is curvature. After using Chern-Weil therem the right hand side is given by the Euler number of the vector bundle V_k. Finally we obtain the partition function

$$Z = \sum_k \epsilon_k \chi(V_k). \qquad (10)$$

The Cohomological field theories are naive extensions of this Mathai-Quillen formalism to the infinitesimal dimensional cases. The transition to the N.C.CohFT is trivially achieved by going over to operator valued objects everywhere or by replacing product by $*$ product everywhere.

2.2 Some Aspects of N.C.CohFT

In this subsection we review some aspects of N.C.CohFT that are investigated in [13, 14].

In this article, we use both $*$ product formulation and operator formulation [22]. We define $*$ product of noncommutative deformation by using the Poisson bracket $\{\ ,\}_\theta$ as follows

$$\phi_1 * \phi_2 = \phi_1\phi_2 + \frac{1}{2}\{\phi_1, \phi_2\}_\theta + (\text{higher order of } \theta), \qquad (11)$$

where ϕ_i (i=1,2) are sections of vector bundles whose base manifold is a Poisson manifold. Note that the Poisson brackets are defined on Poisson manifolds. The $*$ product is frequently expressed by $\hbar$ expansion and this $\hbar$ is distinguished from symplectic form used for definition of the Poisson bracket. But we make no distinction between $\hbar$ and the symplectic form and hereinafter they are collectively called noncommutative parameters θ, for simplicity. The index θ of $\{\ ,\ \}_\theta$ means the set of noncommutative parameters. For example, we will use Moyal product for $\mathbb{R}^{2n}$ and T^2 when we perform concreate calculations in section 4. In these cases, the following Poisson brackets are used,

$$\{\phi_1, \phi_2\}_\theta = \frac{i}{2}\theta^{\mu\nu}(\partial_\mu\phi_1\partial_\nu\phi_2 - \partial_\mu\phi_2\partial_\nu\phi_1), \tag{12}$$

where the noncommutative parameter $\theta^{\mu\nu}$ is a constant anti-symmetric matrix. Then the $*$ product, called the "Moyal product" [23], for $\mathbb{R}^2$ or T^2 is given by

$$\phi_1 * \phi_2(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_\mu\partial'_\nu}\phi_1(x)\phi_2(x')|_{x=x'}. \tag{13}$$

In the following, $*$ is used for both general Poisson manifolds and $\mathbb{R}^2$ or T^2. So, when we consider $\mathbb{R}^2$ or T^2, we write "Moyal product" , "Moyal plane" and so on to distinguish from the general $*$ product.

Let us consider the CohFT on some Poisson manifolds deformed by the $*$ product. Take the lagrangian and the partition function as in the previous subsection but with infinite dimensions. Naively, replacing x, χ and so on by some fields $\phi^i(x)$, $\chi^a(x)$ and so on gives the infinite dimensional extension of the Mathai-Quillen formalism. Since the action functional is defined by an BRS-exact functional like $\hat{\delta}V$, its partition function is invariant under any infinitesimal transformation δ' which commutes (or anti-commutes) with the BRS transformation:

$$\begin{aligned}
\hat{\delta}\delta' &= \pm\delta'\hat{\delta}, \\
\delta'\ Z_\theta &= \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}H\ \ \delta'\left(-\int dx^D\hat{\delta}V\right)\ \exp\left(-S_\theta\right) \\
&= \pm\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\chi\mathcal{D}H\ \ \hat{\delta}\left(-\int dx^D\delta' V\right)\ \exp\left(-S_\theta\right) = 0.
\end{aligned} \tag{14}$$

Let δ_θ be the infinitesimal deformation operator of the noncommutative parameter θ which operates as

$$\delta_\theta\ \theta^{\mu\nu} = \delta\theta^{\mu\nu}, \tag{15}$$

where $\delta\theta^{\mu\nu}$ are some infinitesimal anti-symmetric two form elements. To express the dependence on θ, we use $*_\theta$ as the $*$ product defined by (11) with noncommutative parameter θ in the following discussion. For $*_\theta$, the δ_θ operation is represented as

$$\delta_\theta *_\theta = *_{\theta+\delta\theta} - *_\theta. \tag{16}$$

Then we see that $\hat{\delta}$ commute with δ_θ as follows,

$$\begin{aligned}
\hat{\delta}\delta_\theta(\phi_1 *_\theta \phi_2) &= \hat{\delta}(\phi_1 *_{\theta+\delta\theta}\phi_2 - \phi_1 *_\theta \phi_2) \\
&= (\psi_1 *_{\theta+\delta\theta}\phi_2 + (-1)^{P_{\phi_1}}\phi_1 *_{\theta+\delta\theta}\psi_2) - (\psi_1 *_\theta\phi_2 + (-1)^{P_{\phi_1}}\phi_1 *_\theta\psi_2) \\
&= \delta_\theta(\psi_1 *_\theta \phi_2 + (-1)^{P_{\phi_1}}\phi_1 *_\theta\psi_2) \\
&= \delta_\theta\hat{\delta}(\phi_1 *_\theta \phi_2),
\end{aligned} \tag{17}$$

where $\psi_i = \hat{\delta}\phi_i$ and P_{ϕ_i} is the parity of ϕ_i. This fact shows that the partition function of the N.C.CohFT is invariant under the θ deformation.

If we restrict the models to Moyal spaces, more concrete interesting properties appears from θ shifting. To clarify the character, we introduce the rescaling operator δ_s that satisfies

$$\begin{aligned} x'^{\mu} &= x^{\mu} - \delta_s x^{\mu}, & (18)\\ \delta_s x^{\mu} &= (\frac{1}{2}\delta\theta^{\mu\nu}(\theta^{-1})_{\nu\rho})x^{\rho} & (19)\end{aligned}$$

and

$$(1-\delta_s)[x^{\mu}, x^{\nu}] = [x'^{\mu}, x'^{\nu}] = i(\theta^{\mu\nu} - \delta\theta^{\mu\nu}). \tag{20}$$

The transformation matrix is given as

$$J^{\mu}_{\rho} \equiv \delta^{\mu}{}_{\rho} + \frac{1}{2}\delta\theta^{\mu\nu}(\theta^{-1})_{\nu\rho}, \tag{21}$$

and the integral measure is expressed as

$$dx^D = \det \mathbf{J} dx'^D, \qquad \frac{\partial}{\partial x^{\mu}} = (J^{-1})_{\mu\nu}\frac{\partial}{\partial x'^{\nu}}, \tag{22}$$

where $\det \mathbf{J}$ is the Jacobian.
Using these new variables the Moyal product is rewritten as

$$(1-\delta_s)(*_{\theta}) = \delta_s(\exp(\frac{i}{2}\overleftarrow{\partial}_{\mu}(\theta-\delta\theta)^{\mu\nu}\overrightarrow{\partial}_{\nu})) = *_{\theta-\delta\theta}. \tag{23}$$

These processes are simply changing variables, so the theory is not changed. An action is written before and after this variable change as follows.

$$\begin{aligned} S_{\theta} &= \int dx^D \mathcal{L}(*_{\theta}, \partial_{\mu}) \\ &= \int \det \mathbf{J} dx'^D \mathcal{L}(*_{\theta-\delta\theta}, (J^{-1})^{\mu\nu}\frac{\partial}{\partial x'^{\nu}}), & (24)\end{aligned}$$

where $\mathcal{L}(*_{\theta}, \partial_{\mu})$ is an explicit description to emphasise that the products of fields are the Moyal product and the lagrangian contains derivative terms.

As the next step, we shift the noncommutative parameter θ as follows

$$\theta \to \theta' = \theta + \delta\theta. \tag{25}$$

This deformation changes theories in general. However, the partition function of the N.C.CohFT do not change under this shift as we have seen. After changing of variables (18) and deforming θ (25), the action is expressed as follows.

$$S_{\theta'} = \int \det \mathbf{J} dx'^D \mathcal{L}(*_{\theta}, (J^{-1})^{\mu\nu}\frac{\partial}{\partial x'^{\nu}}). \tag{26}$$

Here $\mathcal{L}(*_\theta, (J^{-1})^{\mu\nu}\frac{\partial}{\partial x'^\nu})$ is a lagrangian in which the multiplication of fields are defined by $*_\theta$ and all differential operators $\frac{\partial}{\partial x^\mu}$ in the original lagrangian are replaced by $(J^{-1})^{\mu\nu}\frac{\partial}{\partial x'^\nu}$ without derivations in $*_\theta$. This action (26) shows that the θ deformation is equivarent to rescaling of x by δ_s, but the Moyal product $*_\theta$ is fixed. Note that $\theta \to \infty$ limit is given by omitting kinetic terms in the action, because the limit $\theta^{\mu\nu} \to \infty$ means $\det \mathbf{J} \to \infty$ in Eq.(26) (see also [24] and [25]). Using this property, we investigate both the large θ limit case and finite θ case for some N.C.CohFT model on the Moyal plane in section 4.

3 Finite Matrix model with Connections

In this subsection, we study a matrix model and its partition function. Several finite or infinite size Hermitian matrix models are important in physics, even a 1-matrix model (see for example [26], [27] and [28]). The model considered here is different from them, but the methods of the analysis done here is applicable to them when we study the geometry of the moduli spaces of them. The matrix model of this section is regarded as operator representation of the N.C. cohomological Scalar model of section 4 with taking the cut off of the Hilbert space. From this fact, the calculations of this section make it possible to determine the partition function of the N.C.CohFT on the Moyal plane in section 4. (This model is given by 0-dimensional reduction of the model in the section 5, too.)

Let M be set of all $N \times N$ Hermitian matrices, then it is a N^2 dim Euclidian manifold $\mathbb{R}^{N^2}$. Let V be rank N^2 (trivial) vector bundle over M. Let $s : M \to V$ denote some given section of a trivial bundle. We adopt the Killing form as a positive-definite inner product.

We construct the finite matrix model as the 0 dimensional CohFT. We take some orthonormal basis of $N \times N$ Hermitian matrices as a canonical coordinate of M, and write $\phi = (\phi^{ab}) \in M$. The other fields (matrices) are introduced by the way of general CohFT. H^{ab} is a bosonic auxiliary field that is a $N \times N$ Hermitian matrix. Fermionic matrices are ψ^{ab} and χ^{ab}, that is the BRS partner of ϕ and H, and these are $N \times N$ Hermitian matrices, too. Their BRS transformation is given as

$$\hat{\delta}\phi = \psi,\ \hat{\delta}\psi = 0,\ \hat{\delta}\chi = H,\ \hat{\delta}H = 0. \tag{27}$$

Let ∇ be a connection $\Gamma(V) \to \Gamma(T^*M \otimes V) = V$, where $\Gamma(V)$ is a set of all sections. Let $A_{ji;mn}^{\ \ \ \ kl}(\phi)$ be a component of connection 1-form in the vector bundle V. Let e_{ij} be a component of local frame field of V. Using e_{ij}, the relation between A and ∇ is written as $\nabla_{ij} e_{kl} = \sum A_{ij;kl}^{\ \ \ \ mn} e_{mn}$. In the following, we take the section of the trivial bundle as $s(\phi) = \phi(1-\phi)$. Then the CohFT action is given by

$$S \ = \ \sum_{i,j} \hat{\delta}\{\chi^{ij}(2[\phi(1-\phi)]_{ji} + i \sum_{m,n,k,l} \chi^{mn} A_{ji,mn}^{\ \ \ \ kl}(\phi)\psi_{kl} - iH_{ij})\}. \tag{28}$$

After Gaussian integral of H_{ij}, the bosonic part of the action becomes

$$\mathrm{Tr}(\phi(1-\phi))^2, \tag{29}$$

and the fermionic part of the action is

$$\mathcal{L}_F \;=\; \mathrm{Tr} i\chi \Big\{ 2(\psi(1-\phi) - \phi\psi) - \sum_{ijklmn} \psi_{ij}\psi_{kl} F(ij,kl;ab,mn)\chi_{mn} \Big\}. \tag{30}$$

Here $F(ij,kl;ab,mn)$ is the curvature defined by

$$F(ij,kl;ab,mn) \equiv \frac{\delta}{\delta\phi_{ij}} A_{kl;ab}{}^{mn} - \frac{\delta}{\delta\phi_{kl}} A_{ij;ab}{}^{mn} + i\sum_{(c,d)} [A_{ij;ab}{}^{cd} A_{kl;cd}{}^{mn} - A_{kl;ab}{}^{cd} A_{ij;cd}{}^{mn}] \tag{31}$$

The fixed points of this action are determined by

$$(\phi(1-\phi)) = 0. \tag{32}$$

Non-zero solutions of ϕ are Projection operators P defined by $P^2 = P$. We denote by P_k the projector that restricts rank N vector space to dimension k vector space. The set of all P_k is connected and

$$\mathcal{M}_{k,N} \equiv \{P_k\} = G_k(N), \tag{33}$$

where $G_k(N)$ is a Grassman manifold ($\frac{U(N)}{U(k)U(N-k)}$) whose dimension is $2k(N-k)$.

Let us investigate the $\mathcal{M}_k$ from a local geometric aspect. At first, we prove the non-degeneracy of s in the normal directions to $\mathcal{M}_k$. The definition of non-degeneracy is as follows. Locally one can pick coordinate e_{ij} (number of combination (i,j) is $N^2 - 2k(N-k)$) in the directions normal to $\mathcal{M}_k$ and a trivialization of V such that

$$s^{ab} = \sum_{i,j} f^{ab}_{ij} e^{ij}, \quad \text{for} \quad (i,j),(a,b) \in \mathbf{N} \tag{34}$$

$$s^{ab} = 0\ , \quad \text{for} \quad (a,b) \in \mathbf{T}. \tag{35}$$

Here $\mathbf{N}$ and $\mathbf{T}$ are sets of indices (i,j) and numbers of their elements are $N^2 - 2k(N-k)$ and $2k(N-k)$. Let us prove this non-degeneracy of $\mathcal{M}_k$. After appropriate coordinate choice, we can take a rank k solution $P_k \in \mathcal{M}_k$ as

$$P_k = \left(\begin{array}{c|c} \mathbf{1}_k & 0 \\ \hline 0 & 0 \end{array} \right), \tag{36}$$

where P is a $N \times N$ matrix valued projection operator and $\mathbf{1}_k$ is the $k \times k$ unit matrix. The (co)tangent vectors at this point are determined by variation of ϕ equation around this solution;

$$\delta\phi(1 - P_k) - P_k\delta\phi = 0. \tag{37}$$

Its solutions are given by

$$\delta\phi_{ij} = 0,\ \delta\phi_{mn} = 0,\ \delta\phi_{in} = \delta\bar{\phi}_{ni}, \quad \text{for } i,j \in \{1,2,\cdots,k\}\ ,\ m,n \in \{k+1,\cdots,N\}. \tag{38}$$

Here $\bar{\phi}$ is complex conjugate of ϕ. We can chose $2(N-k)k$ dim orthonormal basis of $N \times N$ matrices $\delta\phi$:

$$(\phi_R^{(in)}) = \left(\begin{array}{c|c} O & (\delta_{in}) \\ \hline (\delta_{ni}) & O \end{array}\right) \ , \ (\phi_I^{(in)}) = \left(\begin{array}{c|c} O & i(\delta_{in}) \\ \hline -i(\delta_{ni}) & O \end{array}\right) \tag{39}$$

where $i \in \{1, 2, \cdots, k\}$ and $n \in \{k+1, \cdots, N\}$. Let us not confuse "i" of $\sqrt{-1}$ and index in this article. On the other side, basis of normal direction e_{normal} is possible to be chosen as Lie algebra of $U(k) \times U(N-k)$ whose non-zero elements lie only in the block diagonal part i.e. $(e_{normal})_{in} = 0$ for $i \in \{1, 2, \cdots, k\}$ and $n \in \{k+1, \cdots, N\}$. (Note that $\mathrm{Tr}\phi_I^{(in)} e_{normal} = \mathrm{Tr}\phi_R^{(in)} e_{normal} = 0$ shows that the direction of e_{normal} is normal to $\delta\phi$.) A non-degenerate basis of the Lie algebra may be chosen. For example, we can chose non-degenerate $N^2 - 2k(N-k)$ dim basis e_{normal} as,

$$(e_{normal}^{(ij)}) = \begin{cases} \left(\begin{array}{c|c} U^k_{i,j} & O \\ \hline O & O \end{array}\right) & , \text{ for } i \text{ and } j \in \{1, \cdots, k\} \\ \\ \left(\begin{array}{c|c} O & O \\ \hline O & U^{N-k}_{i,j} \end{array}\right) & , \ i \text{ and } j \in \{k+1, \cdots, N\} \end{cases} \tag{40}$$

where $\{U^{N-k}_{i,j;a,b}\}$ is a orthonormal basis of $u(k)$ and $\{U^{N-k}_{i,j;a,b}\}$ is one of $u(N-k)$. We found the local coordinate e_{normal} in the directions normal to $\mathcal{M}_k$ such that (34) holds. This shows non-degeneracy. This discussion for non-degeneracy is parallel to the one in [29].

Let us investigate the mass matrix of fermions near the $\mathcal{M}_k$ and the fermionic zero-modes. The χ equation and the ψ equation are

$$\psi(1-P) - P\psi = 0, \text{ and } \chi(1-P) - P\chi = 0, \tag{41}$$

where we neglect nonlinear terms. Note that f^{ab}_{ij} in (34) is the mass matrix of χ and ψ near $\mathcal{M}_k$. Using the χ equation, we see massless components of ψ are those that are tangent to $\mathcal{M}_k$. There are massless components of χ^{ab} that are regarded as the above trivialization i.e. $(a, b) \in \mathbf{T}$. Furthermore we can understand from the ψ equation that the χ zero-modes are sections of the (co)tangent bundle of $\mathcal{M}_{k,N}$.

Now we evaluate the integral for Z. The mass components integral gives overall factor $(-1)^{k^2} = (-1)^k$ (see [13]). Recall that the moduli space $\{\phi | s = 0\} = \bigcup_k \{P_k\}$ and $\{P_k\} = G_k(N)$. The Poincare polynomial of the Grassman manifold is given as

$$P_t(G_k(N)) = \frac{(1-t^2)\cdots(1-t^{2N})}{(1-t^2)\cdots(1-t^{2(N-k)})(1-t^2)\cdots(1-t^{2k})}.$$

(See for example [30].) Using these results and (10), the partition function is written as

$$Z = \sum_{k=0}^{N} (-1)^k P_{-1}(G_k(N)). \tag{42}$$

When we take $t = \pm 1$, the Poincare polynomial become number of combinations,

$$P_{\pm 1}(G_k(N)) = \frac{N!}{k!(N-k)!} \equiv \begin{pmatrix} N \\ k \end{pmatrix} . \tag{43}$$

The proof of (43) is given as follows.

$$\begin{aligned} P_{\pm 1}(G_k(N)) &= \left. \frac{(1-t^2)\cdots(1-t^{2N})}{(1-t^2)\cdots(1-t^{2(N-k)})(1-t^2)\cdots(1-t^{2k})} \right|_{t=1} \\ &= \left. \frac{(1-t^{2(N-k+1)})\cdots(1-t^{2N})}{(1-t^2)\cdots(1-t^{2k})} \right|_{t=1} . \end{aligned} \tag{44}$$

After replacing t^2 by a positive real number x,

$$\begin{aligned} P_{\pm 1}(G_k(N)) &= \left. \frac{(1-x^{(N-k+1)})\cdots(1-x^N)}{(1-x)\cdots(1-x^k)} \right|_{x=1} \\ &= \left. \frac{\{(1-x)(1+x+\cdots+x^{N-k})\}\cdots\{(1-x)(1+x+\cdots+x^{N-1})\}}{\{(1-x)\}\{(1-x)(1+x)\}\cdots\{(1-x)(1+x+\cdots+x^{k-1})\}} \right|_{x=1} \\ &= \frac{(N-k+1)(N-k+2)\cdots N}{1\cdot 2\cdots k} = \begin{pmatrix} N \\ k \end{pmatrix}. \end{aligned} \tag{45}$$

This is what we want. From (42), (43) and the binomial theorem, the final result is then

$$Z = \sum_{k=0}^{N} (-1)^k 1^{N-k} P_{-1}(G_k(N)) = (1-1)^N = 0. \tag{46}$$

The calculation of the finite matrix model in this section will be used directly in the noncommutative cohomological scalar model in the next section.

4 N.C.Cohomological Scalar model

In this section, we study some N.C.cohomological scalar models and evaluate their partition functions for Moyal space by using the matrix model partition function in the previous section. We also check the θ-shift invariance of Z.

4.1 N.C cohomological scalar model

Let M be a $2n$ dimensional Poisson manifold with Riemannian metric. Let ϕ and H be real scalar fields on M and, ψ and χ be BRS partner fermionic scalar fields of ϕ and H. In other words, (ϕ, H, ψ, χ) are elements of $\Omega^0(M)$ with ghost number $(0, 0, 1, -1)$ and parity (even, even, odd, odd).

We introduce a nilpotent operator $\hat{\delta}$, i.e.

$$\hat{\delta}^2 = 0, \tag{47}$$

as a BRS operator whose transformation is given by

$$\hat{\delta}\phi = \psi,\ \hat{\delta}\chi = H,\ \hat{\delta}\psi = \hat{\delta}H = 0. \tag{48}$$

We consider the deformation quantization defined by some $*$ product. ($*$ product exist on arbitrary Poisson manifolds [1].)

We consider two actions :

$$S_1 = \int_M dx^D \sqrt{g}\mathcal{L} \tag{49}$$

$$S_2 = S_1 + S_{top}, \tag{50}$$

where the lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \hat{\delta}\left(\frac{1}{2}\chi * \left(2(\phi * (1-\phi)) + \frac{2i}{g}\int d^{2n}zd^{2n}y\psi(z)A(z;x,y)\chi(y) - iH\right)\right). \tag{51}$$

Here, g is a coupling constant, $x, y, z \in M$, and $A(z;x,y)$ is some functional of ϕ that should be defined as a connection on the trivial bundle over the set of all ϕ. $A(z;x,y)$ is an anti-symmetric matrix with respect to x and y, and the multiplication between $A(z;x,y), \psi(z)$ and $\chi(y)$ is not $*$ multiplication because trace operation (integral) over z and y was done. (But we can also express their products by $*$ product in the integral.) It looks like some strange non-local interaction, but it is possible to regard this as an integral kernel. Deformation quantization itself is introduced by an integral kernel in many cases, so such non-local interaction is not so strange in noncommutative field theory. The precise definition of $A(z;x,y)$ depends on M and deformation by $*$, so we formally introduce the connection, here. When we consider the $\mathbb{R}^2$ case in the following subsection, it will be verified that $A(z;x,y)$ is a connection and particularly it becomes a nontrivial connection on a submanifold of $\{\phi\}$. Especially in conjunction with the matrix model in previous section, after using the Weyl correspondence, we can regard $A(z;x,y)$ as the usual connection of the (co)tangent vector bundle over some Grassman manifold that appears as a moduli space of ϕ.

The topological action in S_2 is

$$S_{top} = g'\tau_{2n}(\mathcal{F}, \cdots, \mathcal{F}), \tag{52}$$

where g' is coupling constant and $\mathcal{F}$ is defined by

$$\mathcal{F}_{ij} = [\phi\partial_i\phi, \phi\partial_j\phi]. \tag{53}$$

This action is not topological itself but in our case the ϕ is replaced by projection operators. In such case, we can regard S_{top} as Connes's Chern character. Connes's Chern character homomorphism is;

$$ch_{2n} : K_0(\mathcal{A}) \to HC_{2n}(\mathcal{A})$$

$$ch_{2n}(p) = \sum_{n=0}^{\infty}\tau_{2n}(f, \cdots, f) \tag{54}$$

where $f_{ij} = [p\partial_i p, p\partial_j p]$. It is worth emphasizing that S_{top} is not invariant under changing the noncommutative parameter θ in general because it is not a BRS exact action. Indeed $ch_{2n}(p)$ depends on θ apparently for noncommutative torus example. Therefore S_2 is not suitable if we are interested in only constructing the θ-shift invariant theory. But there is another motivation to construct the N.C.CohFT, that is to construct some "topological" invariant. In the commutative case, we often add a topological action to the BRS exact one, and the topological terms play important roles. In analogy with commutative CohFT, it seems useful to consider the both S_1 and S_2 case.

The Lagrangian $\mathcal{L}$ without the S_{top} part is divided into a bosonic part $\mathcal{L}_B$ and fermionic part $\mathcal{L}_F$:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F, \tag{55}$$

$$\mathcal{L}_B = |\phi * (1-\phi)|^2 \ , \tag{56}$$

$$\mathcal{L}_F = i\chi * \Big\{ 2(\psi * (1-\phi) - \phi * \psi) - \frac{i}{2g}\int d^n z d^n w d^n y \psi(z)\psi(w) F(z,w;x,y)\chi(y) \Big\}.$$

Here $F(z,w;x,y)$ is defined by

$$\frac{\delta A(z;x,y)}{\delta\phi(w)} - \frac{\delta A(w;x,y)}{\delta\phi(z)} + \frac{i}{g}\int d^n u \big(\ A(z;x,u)A(w;u,y) - \ A(w;x,u)A(z;u,y)\big), \tag{57}$$

and it corresponds to the curvature.

From the general argument of the Mathai-Quillen formalism and a parallel analysis of the previous section, the partition function of this theory is given by the sum of the Euler numbers of the solution space of ϕ. From Eq.(56), fixed point loci of ϕ are given by the set of all projection operators P, i.e. $P * P = P$, and they are called GMS soliton [24]. We denote by $\mathcal{M}_k$ the set of projections distinguished by index k. An example of the index k is given by rank of projections when we can define the rank by a discrete number. If there is ghost number anomaly, the partition function vanishes in general. But in our case there is no ghost number anomaly as we saw in section 3, then we get some nontrivial partition functions for S_1 and S_2:

$$Z_1 = \sum_k \epsilon_k \chi(\mathcal{M}_k), \tag{58}$$

$$Z_2 = \sum_k \epsilon_k \chi(\mathcal{M}_k) e^{g'\tau_{2n}(k)} \tag{59}$$

where $\chi(\mathcal{M}_k)$ is the Euler number of $\mathcal{M}_k$ and ϵ_k gives a sign $\pm$.

When we consider the noncommutative theory from the topological view point, the most important operators are projectors and unitary operators because they define K_0 and K_1. This partition function is a sum of integer valued Euler numbers of the sets of all projections that construct the K_0 elements when the moduli space is a manifold. So it is natural to expect the partition function is "topological".

Concrete calculation of the partition function will be done for the Moyal plane case, soon. We are interested in whether the "topological" quantity is invariant under the continuous changing of the noncommutative parameter. For the S_1 model of this section, it is clear that the partition function is invariant under the θ changing as far as there is no singularity. A more interesting case is when the lagrangian has kinetic terms. To investigate the behavior of the partition function whose lagrangian contains kinetic terms, we slightly deform our models in the following subsections.

4.2 N.C. cohomological scalar model with kinetic terms

Let M be a $2n$ dimensional Poisson manifold with a Riemannian metric. Let ϕ and H be real scalar fields on M and, ψ and χ be ϕ and H's BRS partner fermionic scalar fields. Let B_μ and H_μ be real vector fields and ψ_μ and χ_μ be BRS partner fermionic vector fields of B_μ and H_μ. In other words, (ϕ, H, ψ, χ) are elements of $\Omega^0(M)$ with ghost number $(0, 0, 1, -1)$ and parity (even, even, odd, odd). $(B_\mu, H_\mu, \psi_\mu, \chi_\mu)$ are elements of $\Omega^1(M)$ with ghost number $(0, 0, 1, -1)$ and parity (even, even, odd, odd). The BRS operator transformation is given by

$$\hat{\delta}\phi = \psi,\ \hat{\delta}\chi = H,\ \hat{\delta}\psi = \hat{\delta}H = 0,\ \hat{\delta}B^\mu = \psi^\mu,\ \hat{\delta}\chi^\mu = H^\mu,\ \hat{\delta}\psi^\mu = \hat{\delta}H^\mu = 0. \tag{60}$$

One of our interests is to investigate the behavior of the partition function of N.C.CohFT under changing of the noncommutative parameter. It is difficult to study the general case of deformation quantization. Therefore, we put an assumption in this subsection such that terms including derivatives like kinetic terms become irrelevant in the large noncommutative parameter limit ($\theta \to \infty$) as far as evaluating perturbative contribution is concerned. For example, when we consider the deformation of $\mathbb{R}^d$ by the Moyal product, only the potential terms become relevant in the $\theta \to \infty$ limit [13, 24]. Note that we make this assumption only for simplicity of calculation, however, the invariance under changing of θ is essential and this is not affected by our assumption.

Similar to the previous subsection, we consider two types of action :

$$\begin{aligned} S_1 &= \int_M dx^D \sqrt{g}\mathcal{L} && (61)\\ S_2 &= S_1 + S_{top}, && (62) \end{aligned}$$

where lagrangian is slightly different from (51),

$$\begin{aligned} \mathcal{L} &= \hat{\delta}\left(\frac{1}{2}\chi * \left(2(\phi*(1-\phi) - \partial_\mu B^\mu) + \frac{2i}{g}\int d^n z d^n y \psi(z) A(z;x,y)\chi(y) - iH\right)\right) \\ &\quad + \hat{\delta}\left(\frac{1}{2}\chi^\mu * (2(\partial_\mu\phi + B_\mu) - iH_\mu)\right). \end{aligned} \tag{63}$$

As noted in the previous subsection, the topological term S_{top} have noncommutative parameter θ dependence in general. For example, the noncommutative torus have θ dependence. On the other hand, the Moyal plane theory does not depend on the θ. When we construct θ independent "topological" invariant Z, we find whether we can add S_{top} to the action S_1 from the K-theory (cyclic cohomology) information of the base manifold.

The Lagrangian $\mathcal{L}$ without S_{top} part is divided into bosonic part and fermionic part:

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F, \tag{64}$$

$$\mathcal{L}_B = |\phi * (1-\phi) - \partial_\mu B^\mu|^2 + |\partial_\mu \phi + B_\mu|^2, \tag{65}$$

$$\mathcal{L}_F = i\chi * \Big\{ 2(\psi * (1-\phi) - \phi * \psi - \partial_\mu \psi^\mu) - \\ -\frac{i}{2g} \int d^n z d^n w d^n y \psi(z)\psi(w) F(z,w;x,y)\chi(y) \Big\} + i\chi^\mu * \{2\partial_\mu \psi + 2\psi_\mu\}. \tag{66}$$

Note that this theory is invariant under arbitrary A deformation ($A \to A + \delta A$) and coupling constant g deformation. In the following subsections, we investigate moduli space deformation and invariance of partition function under changing θ. If we observe $\theta \to \infty$ in the Moyal plane case by using scaling method discussed in section 2, $F(z,w;x,y)$ contribution to the partition function becomes bigger than the other terms because each integral measure d^2z, d^2w and d^2y is of order θ. Then, the surviving terms in the limit are not BRS exact terms. Therefore, we have to tune other parameters in such limit to use usual convenience methods of CohFT. From the fact that the partition function has symmetry under arbitrary g and A variation, we can fit g without changing Z for surviving terms being BRS exact terms in $\theta \to \infty$.

4.3 Moyal plane case in $\theta \to \infty$

In this subsection, the partition function of the N.C.cohomological Scalar model is calculated. To calculate it concretely, we consider the two dimension Moyal plane. There are two reasons to choose the Moyal plane here. The first reason is the Moyal plane satisfies the assumption given in the previous subsection that derivative terms like kinetic terms in the lagrangian become irrelevant in $\theta \to \infty$. The other reason is that the rank of a projection operator is defined by an integer. From this, the solution space of ϕ is given by a Grassmann manifold whose properties are well known. In particular, if we represent our theory by operator representation, the theory is regarded as an infinite dimensional matrix model. It is possible to represent noncommutative Euclidian plane by a Hilbert space and we can chose some set of eigenvectors with discrete eigenvalues as the basis of the Hilbert space, for example a fock state. So if we take cut-off for the Hilbert space, we can regard our model as the finite matrix model appearing in section 3.

We have used $*$ product representation of noncommutative field theory, but the operator representation is used in this subsection because it is convenient to see the relation between the finite matrix model and large θ N.C. cohomological scalar model.

In $\theta \to \infty$, we can ignore the terms including derivative as we saw in section 2. Then the surviving action in operator formalism is

$$S_\infty = \sum_{i,j} \hat{\delta}\{\chi^{ij}(2[\phi(1-\phi)]_{ji} + i(\sum_{m,n,k,l} \chi^{mn} A_{ji,mn}^{\ \ kl}(\hat{\phi})\psi_{kl}) - iH_{ij})\} \qquad (67)$$
$$+Tr\hat{\delta}\{\hat{\chi}^\mu(2\hat{B}_\mu - i\hat{H}_\mu)\},$$

where $\hat{\phi},\ \hat{\psi},\ \cdots$ are operator representation of $\phi,\ \psi,\ \cdots$ that have infinite dimensional matrix representation $\hat{\phi} = \sum_{ij} |i\rangle\phi_{ij}\langle j|$ with some complete system $\{|i\rangle\}$.

We introduce some cut-off to restrict the Hilbert space into the finite N dimension vector space. Let $\{|i\rangle | i = 1, \cdots, N\}$ be a set of orthonormal basis. Using this representation, the operators $\hat{\phi},\ \hat{H},\ \hat{\psi},\ \cdots$ are expressed by $N \times N$ Hermitian matrices, i.e. $\hat{\phi} \to (\phi_{ij})$ and so on. After integrating out B_μ, H_μ, χ_μ and ψ_μ, we will find this model is equivalent to the finite dimensional matrix model appearing in section 3.

The Bosonic part of the action is

$$Tr\{(\hat{\phi}(1-\hat{\phi}))^2 + \hat{B}_\mu \hat{B}^\mu\}. \qquad (68)$$

The fixed point locus is determined by $(\hat{\phi}(1-\hat{\phi})) = 0$ and $\hat{B}_\mu = 0$. The solution is given by $\phi = P$, where P is an arbitrary projection operator, which is called the GMS soliton. The moduli space is obtained as a set of Grassman manifolds $\{G_k(N)\} := \{\frac{U(N)}{U(k)\times U(N-k)}\}$ because the rank k projection operator determines the subspace whose codimension is $N-k$. This solution of rank k projector is interpreted as symmetry breaking from $U(N)$ to $U(k) \times U(N-k)$.

On the other hand, the integration of Fermionic part generate the Euler numbers of the Grassmann manifolds that is given in section 3. For the Moyal plane, topological term S_{top} (ch_2) is $g'k$ when the solution of $\hat{\phi}$ is given by rank k projection operator. Note that ch_2 value is independent of θ for the Moyal plane (see for example [32]).

Using the Euler number of the Grassman manifolds and contribution from the topological term, the partition function is then

$$Z_2 = \lim_{N\to\infty} \sum_{k=0}^{N} P_{-1}(G_k(N)) e^{g'k} (-1)^{N-k}$$
$$= \lim_{N\to\infty} (1-e^{g'})^N, \qquad (69)$$

where we take $N \to \infty$ after using the result from the finite matrix model.

If we take S_1 as the total action of the theory, the partition function is given by (69) with the condition $g' = 0$, then

$$Z_1 = 0. \qquad (70)$$

It is worth commenting here on taking cut-off above analyses. As is a well known fact, some kind of properties of noncommutative field theories only come from the characteristic nature of infinite dimensional Hilbert space. For example, the trace of a commutation

Tr$(AB - BA)$ does not vanish in noncommutative theories in general. This phenomena does not exist in the finite matrix model. So one might think we have to add some collection of the effect from infinite dimension to the above partition functions. But there are some reasons that we do not have to collect the partition function. At first, we consider the real scalar field ϕ and its fixed point is given by a projector in this case. If the solution is given by a shift operator like the complex scalar field case in [33, 34], then the calculation is not closed in the finite size matrices even though the trace operation is done. Meanwhile, our solutions are given by projection operators in this case, then the calculation is possible to be closed in the finite Hilbert space. Additionally, even if we treat the shift operator, there is a way of computation to take the infinite dimension effect into account. The way is to put the cut-off only for the initial states and final states to define the trace operation for finite matrices. On the other hand, intermediate states are not restricted by cut-off, (see [3, 4] for detail). Using such methods we can estimate effects of infinite dimension like the shift operator by finite size computation. The other reason is that we should discuss partition function in the terms of the weak topology because the trace operation is done in the partition function calculation. So it is difficult to distinguish $U(\mathcal{H})$ from $U(\infty) = \lim_{N\to\infty} U(N)$ by our calculation. From these facts, it is reasonable that we can evaluate the partition function by using the finite matrix model.

4.4 finite θ

One of our aims is to confirm that the partition function does not change under a changing of the noncommutative parameter. The proof of the invariance under the θ-shift is based on the smoothness for θ. So, we have to check the smoothness for each models. In the previous subsection, we considered the $\theta \to \infty$ case and we calculated the partition function of the N.C.cohomological scalar model on the 2-dimensional Moyal space by using the result of the finite matrix model. Obeying the general property of N.C.CohFT, for finite θ, we expect that the partition function takes the same value as Eq.(69). This statement is realized when the moduli space smoothly deform and its topology does not change under the θ changing. Therefore, let us compare the moduli space of large θ limit with finite θ in this subsection.

It is difficult to analyze the arbitrary finite θ case because derivative terms and nonlinear terms are intertwining, so we analyze moduli space deformation from large θ limit perturbatively. Let ϕ_0 and $B_{\mu 0}$ be large θ limit solutions of ϕ and B_μ i.e. $\phi_0 = P,\ B_{\mu 0} = 0$. We consider that the fields belong to $C^\infty(\mathbb{R}^2)[[1/\sqrt{\theta}]]$. ϕ and B_μ are expanded as

$$\phi = \phi_0 + \frac{1}{\sqrt{\theta}}\phi_1 + \cdots,\ B_\mu = B_{\mu 0} + \frac{1}{\sqrt{\theta}} B_{\mu 1} + \cdots, \tag{71}$$

and we substitute them into the action. The leading order bosonic action is then

$$\frac{1}{\theta}\mathrm{Tr}|\phi_1(\phi_0 - 1) + \phi_0\phi_1|^2 + \frac{1}{\theta}\mathrm{Tr}|\partial_\mu\phi_0 + B_{\mu 1}|^2 \tag{72}$$

$$= \frac{1}{\theta}\mathrm{Tr}\left\{|\phi_1(P-1)|^2 + |P\phi_1|^2 + |\partial_\mu P + B_{\mu 1}|^2\right\} \tag{73}$$

Let $|P,i\rangle$ be a eigenvector of projector P with eigenvalue 1 i.e. $P|P,i\rangle = |P,i\rangle$. Using this vector, $\sum_{i,j} |1-P,i\rangle a_{ij}\langle P,j| + h.c.$ is a solution of ϕ_1, where (a_{ij}) is a Hermitian matrix. But deformation of the moduli space from $\{a_{1,ij}\}$ is trivial and retractable. Meanwhile, $B_{\mu 1} = -\partial_\mu P$. B_μ is deformed but it is determined completely by the given P. Therefore the moduli space topology is not changed at all. In other words, we can deform the moduli space smoothly. This result is consistent with the expectation, then the partition function is invariant under θ deformation.

5 K-theory and Cohomological Scalar model

We discuss the relation between our theory and K-theory in this section.

5.1 Commutative CohFT and Homotopy of Vector bundle

The relation between some model of CohFT on a COMMUTATIVE space and the homotopy of classifying map of a vector bundle is studied in this subsection. The model is deeply related to the N.C.CohFT models that appeared in section 4. Using the model, an analogy of correspondence between our N.C.cohomological scalar model and algebraic K-theory will be found in the correspondence between CohFT and topological K-theory.

Let M be a n dim Riemannian Manifold, V be a rank N trivial vector bundle.

$$\begin{array}{rcl} \phi : M & \to & H \\ x & \mapsto & \phi^{ab}(x) \in H,\ a,b \in \{1,\cdots,N\} \end{array} \tag{74}$$

where H is set of all $N \times N$ Hermitian matrices i.e. $H \equiv \{h | h^{ab} = \bar{h}^{ba}\}$. In other words, ϕ is a $N \times N$ Hermitian matrix valued scalar field on M. $N \times N$ Hermitian matrix valued scalar fields $\phi^{ab}(x)$ and $H^{ab}(x)$ have the ghost number 0 and fermionic BRS partners $\psi^{ab}(x)$ and $\chi^{ab}(x)$ have ghost number 1 and -1. The BRS transformation is similar to the previous one but there is difference caused by $U(N)$ gauge symmetry. [1] The BRS operator is nilpotent up to gauge transformation δ_g, i.e. $\hat{\delta}^2 = \delta_g$. When we denote $c(x)$ as scalar field corresponding to a local gauge parameter with ghost number 2, the explicit BRS transformation is given by

$$\begin{array}{lcl} \hat{\delta}\phi(x) = \psi(x),\ \hat{\delta}\chi(x) = H(x) & , & \hat{\delta}c(x) = 0 \\ \hat{\delta}\psi(x) = \delta_g\phi(x) = i[c(x),\phi(x)] & , & \hat{\delta}H(x) = \delta_g\chi(x) = i[c(x),\chi(x)]. \end{array} \tag{75}$$

We introduce the following action;

$$S = S_0 + S_p + S_g \tag{76}$$

[1] The theory of this subsection has $U(N)$ gauge symmetry. But gauge symmetry is not main subject in this subsection. So, we do not discuss some technical problems caused by gauge symmetry.

$$S_0 \;=\; \int_M tr\hat{\delta}\{\frac{1}{2}\chi(2\phi(1-\phi)-iH)\}, \tag{77}$$

where S_0 has $U(N)$ gauge symmetry and we have to project out the pure gauge degrees of freedom. So we introduce S_p for the projection to the gauge horizontal part and S_g for the gauge fixing action.

After the Gaussian integral the bosonic part of S_0 is

$$(\phi(1-\phi))^2, \tag{78}$$

and the fermionic action is

$$\chi(-2\psi(1-\phi)+2\phi\psi-[c,\chi]). \tag{79}$$

The fixed point is determined by $(\phi(1-\phi))=0$. If this ϕ is not matrix valued, then the only nontrivial solution which is a smooth function is $\phi=1$. But if $N>1$ then some projection operator P which restricts rank N vector space to dimension k for each point in M is a solution. In other words, the solution of ϕ is a classification map to the $G_k(N)$ whose homotopy class classifies the vector bundle.

Following the general method of cohomological gauge theory [21], we can construct S_{pro}. Let us introduce anti-ghost $\bar{c}$ whose ghost number is -2 and its BRS partner η. Then, S_{pro} is given as

$$S_{pro} = i\int Tr\hat{\delta}((C^\dagger\psi)\bar{c}). \tag{80}$$

Here $C^\dagger$ is adjoint operator of C. C is defined by $\delta_g\phi(x)=i[c,\phi]=Cc(x)$ i.e. $C=i[\ ,\phi]$. (More precisely speaking, we define a group action of $U(N)$ for some point p in principal bundle P over the base manifold M. Then we can define C as the differential of the group action on the point p; $C: u(N)\rightarrow T_pP$. The image of C is the vertical tangent space of p.)

$$S_{pro} = \int Tr\{i[\phi,[c,\phi]]\bar{c}+[\psi,\psi]\bar{c}-[\psi,c]\eta\} \tag{81}$$

When we consider the theory near the rank k solution, the gauge symmetry $U(N)$ is broken to $U(k)\times U(N-k)$. Note that for a rank k projection operator ϕ there are c satisfying $C^\dagger Cc=[\phi,[c,\phi]]=\phi c(\phi-1)-(1-\phi)c\phi=0$ i.e. if c is a generator of the gauge group of $U(N-k)\times U(k)$ then the first term of the right hand side of (81) vanishes. This zero mode causes other type problems that should be solved by inserting observables and choosing a good gauge. To inquire further into the matter would lead us into that specialized area, and such a digression would obscure the outline of our argument. In the following discussion, $1/C^\dagger C$ operate non-zeromodes and we assume there are some methods to deal with the zero modes. It is a well known fact of Cohomological gauge theory, that from the $\bar{c}$ equation of motion c is given as the curvature of the moduli space. But this discussion is not possible to adopt to our case because our case with non-trivial solution of ϕ cause symmetry breaking. The moduli space is the coset space whose equivalent relation is given by left gauge symmetry.

$$\mathcal{M}_{k,N} = \{\phi \mid M \to G_k(N)\}/\mathcal{G}_{k,N}, \tag{82}$$

where $\mathcal{G}_{k,N}$ is a group of gauge transformations with gauge group of $U(N-k) \times U(k)$. Meanwhile, from the $\bar{c}$ equation of motion,

$$c = -\frac{1}{C^\dagger C}[\psi, \psi]. \tag{83}$$

Unlike the usual case, we can not regard c as the curvature on the principal bundle whose base manifold is the moduli space.

Let us consider fermionic zermo-modes of χ and ψ. Similar to N.C.CohFT and the finite matrix model, the equations of motion of ψ and χ without nonlinear terms are

$$\psi(1-P) - P\psi = 0, \text{ and } \chi(1-P) - P\chi = 0. \tag{84}$$

Note that the solution of both equations represent the cotangent vector of the solution space of ϕ. As far as these equations are concerned, the number of the zero modes of ψ is equal to the one of χ and there is no ghost number anomaly. After nonzeromode integration that produce some sign factor $\epsilon_{k,N} = \pm 1$, zero-modes integral remains as

$$\mathcal{E}_{k,N} := \int_{\mathcal{M}_{k,N}} \mathcal{D}\phi_0 \mathcal{D}\chi_0 e^{-\int \frac{1}{C^\dagger C}[\psi_0,\psi_0][\chi_0,\chi_0]}. \tag{85}$$

Now we recall that our theory has a symmetry that allows arbitrary infinitesimal ϕ deformation i.e. $\phi \to \phi + \delta\phi$, where $\delta\phi$ is arbitrary infinitesimal $N \times N$ Hermite matrix valued scalar field. This is the since we can regard the BRS exact action as gauge fixing action of this local symmetry. This symmetry means that the partition function is homotopy invariant. Therefore, the equivalent class of this symmetry corresponds to homotopy equivalent class of ϕ. So the zero-mode integral (85) is summed up by the homotopy class $[M, G_k(N)]$.

In the end, the partition function is given as

$$Z \sim \sum_{[M,G_k(N)]} \epsilon_{k,N}\ \mathcal{E}_{k,N} \tag{86}$$

To interpret this partition function from the point of view of classifying homotopy of vector bundles, note that ϕ is a classifying map for complex vector bundles when N is enough large (see [30]). (Note that there are no non-trivial vector bundle with fiber space whose dimension is larger than $n+2$.)

We introduce homotopy class $Vect_k(M) = [M, BU(k)]$, where

$$BU(k) \equiv \bigcup_{m=k+n+1}^{\infty} Gr_k(m); m > k+n,$$

and consider the case when N is sufficiently large. Using this, the partition function is represented as

$$Z \sim \sum_{Vect_k(M)} \epsilon_{k,N}\ \mathcal{E}_{k,N}. \tag{87}$$

Note that this homotopy class is related to the $K'(M)$ group whose virtuarl dimmension is 0 where $K'(M) = [M, BU(\infty)]$ (see for example [31]). In particular, when M is connected $K(M) = \mathbb{Z} \oplus \tilde{K}$ and $K' = \tilde{K}$. For stable range $k > \frac{1}{2}\dim M$, we can put the relation between the homotopy class and $K'(M)$ as $K'(M) = [M, BU(k)]$. Therefore the partition function is in proportion to the sum of $\epsilon_{k,N}\ \mathcal{E}_{k,N}$ over the $K'(M)$ elements for large enough N. This is analogous to the N.C.CohFT partition function which is given as a sum over the elements of the algebraic K-group. (See also the next subsection.)

To compare with the noncommutative theory with kinetic terms, we consider the model (77) with kinetic terms and investigate its large scale limit and finite scale case. The lagrangian is similar to the N.C.CohFT in section 4.3;

$$\begin{aligned} \mathcal{L} &= \hat{\delta}\left(\frac{1}{2}\chi\left(2(\phi(1-\phi) - \partial_\mu B^\mu) - iH\right)\right) \\ &\quad + \hat{\delta}\left(\frac{1}{2}\chi^\mu(\partial_\mu \phi + B_\mu - iH_\mu)\right). \end{aligned} \tag{88}$$

Since $U(N)$ gauge symmetry is not main subject, so we break gauge symmetry here, i.e. we do not introduce gauge fields and gauge covariant derivatives. In the N.C.CohFT case, we take large θ limit. We can introduce a similar discussion by scaling

$$g_{\mu\nu} \to (1+\epsilon^2) g_{\mu\nu}, \quad g^{\mu\nu} \to (1-\epsilon^2) g^{\mu\nu}. \tag{89}$$

Since the partition function is invariant under this transformation, when we take the large scale limit the kinetic terms become irrelevant and B_μ becomes an auxiliary field. After integrating out, the theory is equivalent to the one with above action (77). This observation is similar to the N.C.CohFT case in $\theta \to \infty$.

The N.C.CohFT in the previous section is naive extension of the model dealt with in this section. If we consider the noncommutative deformation of the model of this subsection, after renumbering the U(N) indices and Hilbert space indices so that we do not distinguish these indices, then we can identify this model with the N.C.CohoFT model of section 4. Alternatively, the N.C.CohFT model is obtained by dimentional reduction to zero dimention and large N limit.

5.2 K_0 and N.C.CohFT

In this subsection, we disscuss the correspondence with K_0-theory. As mentioned in section 1, one of our purposes is to construct a less sensitive topology than K-theory, where the term "topology" is used as vacuum expectation value of the field theory is invariant under continuous deformation of the theory. It is natural to expect that our partition function is invariant under deformations which do not change the K-theory. In a sense, θ independence of the partition function implies this fact. To see this closely, we consider not the general case, but the Moyal plane and noncommutative torus.

For the Moyal plane, as we saw in the previous section, the partition function (69) is expressed as summation over the projection operators that are identified by their rank. The rank of the projection can be identified $\tau_0(P_k) = k$ or $\tau_2(P_k) = k$ (see for example [35] and [32]). Furthermore the Euler number of the Grassmann manifolds is determined essentially only by k because we take $N \to \infty$ in the end. Therefore, the partition function is determined by K_0 data alone.

Next, we consider the N.C.torus T^2_θ. The classification by Morita equivalence corresponds to one by the K-theory and the equivalence is determined by a noncommutative parameter θ up to SL(2,$\mathbb{Z}$) transformation. If T^2_θ and $T^2_{\theta'}$ are Morita equivalence, θ' should be written as

$$\theta' = \frac{a\theta + b}{c\theta + d} \ , \ ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \tag{90}$$

For arbitrary θ we can transform T^2_θ to a non-Morita equivalent noncommutative torus by infinitesimal θ deformation. So, the θ shift changes the K-group. On the other hand, the model whose action is given by (49) or (61) is invariant under the θ shift when there is no singular point. (Note that the one with the action (50) or (62) is not invariant under the arbitally θ deformation but it is invariant under SL(2,$\mathbb{Z}$) transformation.) At least, if some deformation of noncommutative manifolds does not change K-theory, it is expected that the partition function of N.C.CohFT will not change. This fact implies that the partition function satisfies the condition of the object of our desire, that is less sensitive topological invariant than K-theory.

6 N.C. Cohomological Yang-Mills Theory

In this section, Cohomological Yang-Mills theories on noncommutative manifolds are discussed. If there is gauge symmetry, BRS-like symmetry is slight different from (48). The BRS-like symmetry is not nilpotent but

$$\delta^2 = \delta_{g,\theta}, \tag{91}$$

where $\delta_{g,\theta}$ is gauge transformation operator deformed by the star product $*_\theta$. The partition function of the N.C.CohFT is invariant under changing noncommutative parameter when the BRS transformation is nilpotent, because the BRS transformation δ and θ deformation δ_θ commute. Conversely, when definition of BRS-like operator (91) depends on the noncommutative parameter θ, then δ and δ_θ do not commute;

$$\delta_\theta \delta \neq \delta \delta_\theta \Rightarrow \delta_\theta \delta = \delta' \delta_\theta, \tag{92}$$

where δ' is BRS-like operator that generates the same transformations as the original BRS-like operator δ without the square

$$\delta'^2 = \delta_{g,\theta+\delta\theta}. \tag{93}$$

This fact makes a little complex problem to prove the θ-shift invariance of N.C.cohomological Yang-Mills theory.

After deformation from θ to θ', the action functional becomes not δ-exact but δ'-exact. Then the partition function is invariant when its path integral measure is invariant under both δ and δ', because we can regard δ' as a redefined BRS-like operator. We can prove the invariance of the measure by direct observation. Furthermore, the gauge transformation itself is changed to $\delta_{g,\theta+\delta\theta}$, but it is possible to define the path integral measure to be invariant under both $\delta_{g,\theta}$ and $\delta_{g,\theta+\delta\theta}$ transformation. So changing the gauge transformation does not break the symmetry generated by δ_θ. Therefore, N.C.cohomological Yang-Mills theory is invariant under the θ deformation, as similar to the N.C.CohFT like one appearing in the section 4. [2]

From applying this fact for several physical models, some interesting information can be found. For example, the partition function of the N.C. Cohomological Yang-Mills Theory on 10-dim Moyal space and the partition function of the IKKT matrix model have a correspondence, because the IKKT matrix model is constructed as dimensional reduction of the 10 dimensional super $U(N)$ Yang-Mills theory with large N limit [36] [37]. This dimensional reduction is regarded as the large noncommutative parameter limit ($\theta \to \infty$ in section 4). Taking the large N limit of the matrix model is equivalent to considering the Yang-Mills theories on noncommutative Moyal space, i.e. matrices are regarded as linear transformation of the Hilbert space caused from noncommutativity in similar manner to the case of N.C.CohFT on the Moyal plane. Particularly, the Noncommutative Cohomological Yang-Mills model on 10 dimensional Moyal space in the large θ limit is almost the same as the model of Moore, Nekrasov and Shatashvili [38]. Moore et al. show that the partition function is calculated by the chomological matrix model in [38] and related works are seen in [39, 40, 41]. We can be fairly certain that we can reproduce their result by using N.C.cohomological Yang-Mills theories.

Another example is an application to N=4 d=4 Vafa-Witten theory [29]. The theory is constructed as balanced CohFT (see [42] and [43]). The partition function of Vafa-Witten theory is given by the sum of the Euler numbers of the instanton moduli space over all instanton numbers, if the vanishing theorem is true. Here the vanishing theorem guaranties the fixed point locus of the theory is the instanton moduli space. On commutative manifolds, one of the conditions for vanishing theorem being true is that there is no $U(1)$ instanton. On the other hand existence of $U(1)$ instantons is well-known in noncommutative Moyal space [44, 45], so it is likely that $U(1)$ instantons exist on the other noncommutative manifolds even if the manifolds do not have $U(1)$ instantons before noncommutative deformation. Therefore if we consider the Vafa-Witten theory on

[2] More details will be given by the author of this article.

noncommutative manifolds, the $U(1)$ instanton effect appears as difference to the commutative manifold case. The results is a sum of Euler numbers of instanton moduli spaces and moduli space deformed by $U(1)$ instanton effect. In this case, it is expected that its partition function on a commutative manifold is computed by the matrix theory calculation like [38]. By comparing this partition function, it is reasonable to suppose that the Euler number of deformed moduli space is given, and we obtain a partition function on manifold that does not satisfy the vanishing theorem. Such difference from CohFT on the commutative manifold will emphasize that N.C.CohFT is non-trivial though it is less sensitive than K-theory.

In this way, there are many interesting subjects to be studied by using N.C.chomological Yang-Mills theory. With all of these subjects, concrete analysis and calculations are left for our future work.

7 Summary

Let us summarize this article. We have studied topological aspects of N.C.CohFT and matrix models. At first, we reviewed the N.C.CohFT and its properties. Particularly, through this article we have used the property that the N.C.CohFT have symmetry under the arbitrary infinitesimal noncommutative parameter deformation. This symmetry implies that the partition function of N.C.CohFT is an insensitive "topological" invariant. In section 3, we introduced a Hermitian finite size matrix model of CohFT and calculated its partition function. The calculation was done by using only topological information of its moduli space. The partition function was given as the sum of the Euler numbers of Grassman manifolds with sign and we showed that the partition function vanished. This calculation of the partition is the first example of determining its partition function by only moduli space topology of a matrix model. The scalar field models of N.C.CohFT were discussed in section 4. The variations of the models are caused by adding kinetic terms or topological action that correspond to Connes's Chern character. The fixed point loci of the scalar fields were given by the set of all projection operators on the noncommutative manifold. From the analogy of the finite size matrix model, we introduced a connection functional in these N.C.CohFT models. Using curvature obtained from the connections, the partition functions were represented as sum of Euler numbers of the set of all projection operators. As an example, we calculated the partition function of the model including kinetic terms on the Moyal plane in the large noncommutative parameter limit. Through the operator formulation, this calculation boiled down to the calculation of Hermitian finite size matrix model of CohFT in section 3. Additionally, to confirm the independence of the noncommutative parameter of the N.C.CohFT we studied moduli space for finite θ. If the partition function of CohFT is "topological", then it should have some relation with K-theory and the partition function should not change under deformation that do not change the K-group. Therefore we investigated the models of CohFT and N.C.CohFT from the point of view of K-theory. At first, one CohFT was constructed. This model and N.C.CohFT model in section 4 are related by dimensional reduction or noncommutative deformation. The partition function is invariant under scaling and this scaling is similar to the θ-shift. In the large scaling limit, kinetic terms become irrelevant and the fixed point loci are given by a classifying map. The partition function was given by sum of topological invariants with sign. This sum is taken over all the homotopy equivalent classes of the classifying map of the vector bundle. This homotopy class is regarded as K'. From comparing the connection between the CohFT model and K-theory with the relation between the N.C.cohomological scalar model and algebraic K-theory, we found an analogy. Furthermore, we studied the correspondence with the K_0-theory for the Moyal plane and noncommutative torus. It was verified that our partition function is invariant under deformations which do not change the K_0, at least for the Moyal plane and noncommutative torus. Finally, we considered the noncommutative cohomological Yang-Mills theory. The noncommutative parameter independence is non trivial for noncommutative gauge theory but it is possible to prove. Therefore we can remove kinetic terms in the large θ limit on the Moyal spaces as same as N.C.CohFT studied in section 4. The observations of the N.C.CohFT of scalar models give us a general correspondence between N.C.CohFT and Matrix models. As an example the connection between the IKKT matrix model and noncommutative cohomological Yang-Mills theory was discussed. For another example, we considered the Vafa-Witten theory. The contribution from noncommutative solitons like $U(1)$ instantons may make expectation value of N.C.CohFT different from expectation value of CohFT on a commutative manifold. In such case, N.C.CohFT gives a different topological invariant from commutative topological invariant and that is less sensitive than algebraic K-theory. In other words, there will be new nontrivial global characterization of the geometry though its classification is less sensitive than K-theory. It is likely that the Vafa-Witten theory is one of such examples. A detailed analysis of similar variations for noncommutative cohomological Yang-Mills theory corresponding to matrix model will be carried out in future work.

The other unsettled question is as follows. As we have seen, there is evidence to suggest the partition function of N.C.CohFT is insensitive but nontrivial topological invariant. But a more strict topological discussion about N.C.CohFT for the general case should be done, because there are many ambiguous problems concerning the relation to the K-theory. This subject is also left for future works.

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